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WEAK ERROR EXPANSION OF THE IMPLICIT EULER SCHEME

OMAR ABOURA

Abstract. In this paper, we extend the Talay Tubaro theorem to the implicit Euler scheme.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) a probability space and \(T > 0\) a fixed time. \(W\) will be a Brownian motion in \(\mathbb{R}\) with respect to his own filtration \(\mathcal{F}_t\). We will consider the following stochastic differential equation

\[
X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,
\]

where \(x \in \mathbb{R}\), \(b\) and \(\sigma\) are real functions defined on \(\mathbb{R}\). It is well known that, under Lipschitz conditions on \(b\) and \(\sigma\), this equation admits a unique strong solution.

For various reasons, including mathematical finance or partial differential equations, the approximation of \(E f(X_T)\) is of importance. One way to do this is to use an Euler scheme and to study the speed of convergence. There is a vast literature on this subject and one of the pioneering work is the paper of D. Talay and L. Tubaro [7].

Let \(\mathbb{N}^*\) and \(h := T/N\). Consider \((t_k)_{0 \leq k \leq N}\) the uniform subdivision of \([0, T]\) defined by \(t_k := kh\). In their paper [7] the authors deal with the explicit Euler scheme \((\bar{X}_t)_k\) defined as: \(\bar{X}_t_0 = x\) and for \(k = 0, \ldots, N - 1\),

\[
\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(\bar{X}_{t_k}) h + \sigma(\bar{X}_{t_k}) \Delta W_{k+1},
\]

where \(\Delta W_{k+1} := W_{t_{k+1}} - W_{t_k}\). They study the weak error \(E f(\bar{X}_T) - E f(X_T)\).

Here, we will use the implicit Euler scheme defined as follow: \(X^N_t = x\) and for \(k = 0, \ldots, N - 1\),

\[
X^N_{t_{k+1}} = X^N_{t_k} + b(X^N_{t_{k+1}}) h + \sigma(X^N_{t_k}) \Delta W_{k+1}.
\]

Despite the fact that this implicit scheme cannot be implemented in most cases, it has been studied in [5] but, to the best of our knowledge, its weak error expansion has not been given. The main reason of this study is that we believe it would be a step in order to study a weak convergence error for SPDEs. So far in that framework only few cases have been studied in [2]-[4] for the stochastic heat or Schrödinger equation.

Notations. Let \(n \in \mathbb{N}\) and \(v, w : [0, T] \times \mathbb{R} \to \mathbb{R}\) be smooth functions. We will denote by \(\partial^n v(t, x)\) the \(n^{th}\) derivative of \(v\) with respect to the space variable \(x\), except for the second derivative denoted \(\Delta v(t, x)\) as usual. Moreover, by an abuse of notation, for a function \(v : \mathbb{R} \to \mathbb{R}\) and \(w : [0, T] \times \mathbb{R} \to \mathbb{R}\), we will write \((vw)(t, x) := v(x)w(t, x)\).

Given \(p \in \mathbb{N}\), \(C_p\) will denote a constant that depends on \(p, T\) and the coefficients \(b\) and \(\sigma\), but does not depend on \(N\). As usual, \(C_p\) may change from line to line.

For \(h\) small enough, we denote by \(S_h\) the functions defined on \(\mathbb{R}\) by

\[
S_h(x) := 1/(1 - hb'(x)).
\]

It is similar to the map used by Debussche in [3].
2. THE MAIN RESULT

Let $u$ the (classical) solution of the following pde, called the Kolmogorov equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + b(x)\partial u(t,x) + \frac{1}{2}\sigma^2(x)\Delta u(t,x) &= 0, \\
u(T,x) &= f(x).
\end{aligned}
\tag{2.1}
\]

The properties of $u$ will be given in the next section. Let us mention that for $b$ and $\sigma$ smooth enough, $u$ is smooth too. We define the function $\psi_i : [0,T] \times \mathbb{R} \to \mathbb{R}$, where $i$ stands for implicit, as follows for a smooth enough function $u$:

\[
\psi_i := \frac{1}{2}b\partial (b\partial u) + \frac{1}{4}\sigma^2\Delta (b\partial u) - \frac{1}{2}b^2\Delta u + \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{4}b\partial (\sigma^2\Delta u) - \frac{1}{8}\sigma^2\Delta (\sigma^2\Delta u).
\]

We are now in position to state the main result of this paper.

**Theorem 2.1.** Let $b, \sigma, f$ be $C^\infty$-functions with bounded derivatives.

(i) The implicit Euler scheme (1.3) is of weak order 1, that is, there exists a constant $C$, such that for $h$ small enough $|Ef(X^N_T) - Ef(X_T)| \leq Ch$.

(ii) The weak error can be expanded as

\[
Ef(X^N_T) - Ef(X_T) = hE\int_0^T \psi_i(t,X_t)dt + O(h^2).
\]

We have not given the minimal hypothesis; indeed we want to focus on the ideas and not on the best set of assumptions. The proof of this theorem is quite long; it uses intensively the Kolmogorov equation (2.1), the Itô and Clark-Ocone formulas. It will be proved in the next section. We at first compare our result with that of Talay Tubaro. In their paper [7], the authors introduce the following function

\[
\psi_i = \frac{1}{2}b^2\Delta u + \frac{1}{2}b\sigma^2\partial^3 u + \frac{1}{8}\sigma^4\partial^4 u + \frac{1}{2}b\partial (\sigma^2\Delta u) + \frac{1}{2}\sigma^2\partial\Delta u,
\]

and prove the following result (see [7] page 489).

**Theorem 2.2.** Let $(X_{t_k})_{k=0,...,N}$ denote the explicit Euler scheme defined by (1.2). Then weak error has the following expansion

\[
Ef(X_T) - Ef(X_T) = hE\int_0^T \psi_e(t,X_t)dt + O(h^2).
\]

Applying $\frac{\partial^2}{\partial t^2}$, $b\partial$ and finally $\frac{1}{2}\sigma^2\Delta$ to (2.1) and summing these equations we have

\[
\frac{\partial^2}{\partial t^2}u + 2b\partial\frac{\partial}{\partial t}u + \sigma^2\Delta\frac{\partial}{\partial t}u = -b\partial (b\partial u) - \frac{1}{2}b\partial (\sigma^2\Delta u) - \frac{1}{2}\sigma^2\Delta (b\partial u) - \frac{1}{4}\sigma^2\Delta (\sigma^2\Delta u)
\]

So we can rewrite the function $\psi_e$ as

\[
\psi_e = \frac{1}{2}b^2\Delta u + \frac{1}{2}b\sigma^2\partial^3 u + \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{2}b\partial (b\partial u) - \frac{1}{4}b\partial (\sigma^2\Delta u) - \frac{1}{4}\sigma^2\Delta (b\partial u) - \frac{1}{8}\sigma^2\Delta (\sigma^2\Delta u)
\]

For $b = 0$, we have $\psi_e = \psi_i = \frac{1}{8}\sigma^2\partial^4 u - \frac{1}{4}\sigma^2\Delta (\sigma^2\Delta u)$ as expected since in this case the explicit and the implicit Euler scheme coincide. We can notice that $\psi_i = \psi_e - b^2\Delta u + \frac{1}{2}\sigma^2\Delta (b\partial u) + b\partial (b\partial u) - \frac{1}{2}b\sigma^2\partial^3 u$.

3. PROOF THEOREM 2.1

Here is a sketch of the proof: After proving some property of the scheme, we introduce a continuous interpolation of this scheme. Finally, after decomposing the weak error, we study a remainder term.
3.1. Some tools.

Proposition 3.1 (Property of u). Let \( \left( X_{s}^{t,x} \right)_{s \in [t,T]} \) denote the stochastic flow, that is the solution of (1.1) starting from \( x \) at time \( t \) and let \( u(t,x) = E f \left( X_{T}^{x} \right) \). Then \( u \) belongs to \( C^{\infty, \infty} \left( [0,T] \times \mathbb{R} \right) \) and satisfies the Kolmogorov equation (2.1). Moreover, for any \( n, p \in \mathbb{N} \), there exists constants \( C \) and \( k \) such that

\[
\left| \frac{\partial^n}{\partial t^n} \partial^p u(t,x) \right| \leq C \left( 1 + |x|^k \right).
\]


Now we recall several results from Malliavin Calculus that will be used in the sequel. For a detailed introduction, we send the reader to D. Nualart's book [6].

Proposition 3.2 (Clark-Ocone formula). Let \( t \in [0,T] \) and \( F \in L^2(\mathcal{F}_t) \cap D^{1,2} \); then we have for all \( s \in [0,t] \)

\[
F = E(F|\mathcal{F}_s) + \int_s^t E(D_r F|\mathcal{F}_r) \, dW_r.
\]

Lemma 3.3. Let \( F, G \in D^{1,2} \).

(i) If \( F \) and \( DF \) are bounded, then \( FG \in D^{1,2} \) and \( D(FG) = FDG + GDF \).

(ii) Let \( f \in C^1 \) with a bounded derivative; then \( f(F) \in D^{1,2} \) and \( D f(F) = f'(F) DF \).

(iii) Let \( (s,t) \in [0,T]^2 \) such that \( s < t \) and let \( F \in D^{1,2} \cap L^2(\mathcal{F}_s) \). Then \( F(W_s - W_t) \in D^{1,2} \) and

\[
D_r \left[ F(W_t - W_s) \right] = D_r F(W_t - W_s) + F1_{\{s \leq r \leq t\}}.
\]

(iv) Let \( \{H_n, n \geq 1\} \) be a sequence of random variables in \( D^{1,2} \) that converges to \( H \) in \( L^2(\Omega) \) and such that \( \sup_n E \left( \|DH_n\|_{L^2(\Omega)}^2 \right) < \infty \). Then \( H \) belongs to \( D^{1,2} \).

For a proof of (iii), see [6] Lemma 1.3.4. Now we state some technical lemmas that will be useful in the sequel. The following discrete Gronwall lemma is classical.

Lemma 3.4 (Gronwall’s lemma). For any nonnegative sequences \( (a_k)_{0 \leq k \leq N} \) and \( (b_k)_{0 \leq k \leq N} \) satisfying \( a_{k+1} \leq (1+Ch)a_k + b_{k+1} \), with \( C > 0 \). Then we have \( a_k \leq e^{C(T-t_k)} \left( a_0 + \sum_{i=1}^{k} b_i \right) \).

Lemma 3.5. Let \( L > 0 \); then for \( h^* \) small enough (more precisely \( Lh^* < 1 \)) there exists \( \Gamma := \frac{L}{1-Lh^*} > 0 \) such that for all \( h \in (0,h^*) \) we have \( \frac{1}{1-Lh} < 1 + \Gamma h \).

Proof. Let \( h \in (0,h^*) \); then we have \( 1 - Lh > 1 - Lh^* > 0 \). Hence \( \frac{L}{1-Lh} < \frac{L}{1-Lh^*} = \Gamma \), so that \( Lh < \Gamma h(1-Lh) \), which yields \( 1 + \Gamma h - Lh - \Gamma Lh^2 = (1 + \Gamma h)(1-Lh) > 1 \). This concludes the proof.

Lemma 3.6 (Generalization of Young’s lemma). For an integer \( p \geq 1 \) and \( \epsilon > 0 \), we have

\[
(a + \epsilon)^{2p} \leq (1 + \epsilon)^{2p-1} a^{2p} + \left( 1 + \frac{1}{\epsilon} \right)^{2p-1} b^{2p}.
\]

Proof. We use an induction argument. The inequality is true for \( p = 1 \), that is \( (a + \epsilon)^2 \leq (1 + \epsilon)a^2 + (1 + \frac{1}{\epsilon})b^2 \). Now, suppose that it is true until \( p \) and will prove it for \( p + 1 \); indeed the induction hypothesis yields

\[
(a + \epsilon)^{2p+1} \leq (1 + \epsilon)^{2p-1} a^{2p} + \left( 1 + \frac{1}{\epsilon} \right)^{2p-1} b^{2p} \geq (1 + \epsilon) \left( 1 + \frac{1}{\epsilon} \right)^{2p-1} \left| a^{2p} \right|^2 + \left( 1 + \frac{1}{\epsilon} \right)^2 \left| b^{2p} \right|^2.
\]
This concludes the proof.

\section*{3.2. Property of the implicit Euler scheme.}

\textbf{Lemma 3.7} (Existence of the scheme). \textit{For small} $h$, \textit{the implicit Euler scheme (1.3) is well defined. Moreover, for all} $k = 0, \ldots, N$, \textit{we have} $X_{t_k}^N \in L^2(\mathcal{F}_{t_k})$.

We will denote by $N_0$ the smallest integer such that the scheme is well defined.

\textit{Proof.} For $k = 0$, we have $X_{t_0}^N = x \in L^2(\mathcal{F}_{t_0})$. Suppose that for all $j = 0, \ldots, k$, $X_{t_j}^N$ is well defined and belongs to $L^2(\mathcal{F}_j)$; we prove this for $j = k + 1$. We define $\xi_{k+1} := X_{t_k}^N + \sigma (X_{t_k}^N) \Delta W_{k+1}$. By independence of $\Delta W_{k+1}$ and $\mathcal{F}_k$ and the linear growth of $\sigma$, we have that $\xi_{k+1} \in L^2(\Omega)$. Let $F_{k+1} : L^2(\Omega) \to L^2(\Omega)$ be defined by

$$F_{k+1}(X) := \xi_{k+1} + b(X)h,$$

\hspace{1cm} (3.1)

for all $X \in L^2(\Omega)$. Using the Lipschitz property of $b$ we have $E |F_{k+1}(X) - F_{k+1}(Y)|^2 \leq ||b'||\|h||E|X - Y|^2$. So by the fixed point theorem, if $||b'||\|h < 1$ there exist an unique element of $L^2(\Omega)$, noted $X_{t_{k+1}}^N$, such that $X_{t_{k+1}}^N = F_{k+1}(X_{t_k}^N)$. The measurability of $X_{t_{k+1}}^N$ with respect to $\mathcal{F}_{t_{k+1}}$ is obvious.

\textbf{Lemma 3.8} (Malliavin derivability). \textit{Let} $h > 0$ \textit{small enough; then for all} $k = 0, \ldots, N$, \textit{we have} $X_{t_k}^N \in D^{1,2}$. Moreover, for all $t \in [t_k, t_{k+1})$, we have $D_t X_{t_k}^N = S_h (X_{t_{k+1}}^N) \sigma (X_{t_k}^N)$, where $S_h$ is defined by (1.4).

\textit{Proof.} It is true for $k = 0$, since $X_{t_0}^N = x$. Now suppose that for all $j = 1, \ldots, k$, $X_{t_j}^N \in D^{1,2}$ and prove that $X_{t_{k+1}}^N \in D^{1,2}$. First, we define the following sequence in $L^2(\Omega)$:

$$X_{t_{k+1}}^N(0) = 0 \text{ and for } i \geq 0, X_{t_{k+1}}^N(i+1) = F_{k+1}(X_{t_{k+1}}^N(i)) \text{ where } F_{k+1} \text{ is defined by (3.1).}$$

Using the Lipschitz property of $F_{k+1}$, since $X_{t_{k+1}}^N$ is a fixed point of $F_{k+1}$, we have

$$E \left| X_{t_{k+1}}^N(i+1) - X_{t_{k+1}}^N(i) \right|^2 \leq \||b'||\|h\|^2 E \left| X_{t_{k+1}}^N(i+1) - X_{t_{k+1}}^N(i) \right|^2 \leq \left( \||b'||\|h\|^2 \right)^{i+1} E \left| X_{t_{k+1}}^N \right|^2.$$

So $X_{t_{k+1}}^N(i)$ converge to $X_{t_{k+1}}^N$ in $L^2(\Omega)$ if $||b'||\|h < 1$. Using the induction hypothesis, the assumptions on $\sigma$ and Lemma 3.3 (ii) and (iii), we deduce that $\xi_{k+1} = X_{t_k}^N + \sigma (X_{t_k}^N) \Delta W_{k+1}$ belongs to $D^{1,2}$. Finally, since $b$ is Lipschitz, we deduce by induction that for all $i \geq 0$ $X_{t_k}^N(i) \in D^{1,2}$. Moreover we have $D X_{t_k}^N(i+1) = D \xi_{k+1} + b'(X_{t_k}^N(i)) D X_{t_k}^N(i)$ and $D X_{t_k}^N(0) = 0$, so that

$$\|DX_{t_k}^N(i+1)\|_{L^2(0,T)}^2 \leq 2 \|D \xi_{k+1}\|_{L^2(0,T)}^2 + 2h^2 \|b'|\|_{\infty} \|DX_{t_k}^N(i)\|_{L^2(0,T)}^2.$$

An induction argument yields for $i \geq 1$ and $2h^2 \|b'|\|_{\infty} < 1$,

$$\|DX_{t_k}^N(i)\|_{L^2(0,T)}^2 \leq 2 \frac{\|D \xi_{k+1}\|_{L^2(0,T)}^2}{1 - 2h^2 \|b'|\|_{\infty}^2}.$$ 

Finally, we have sup \( \|DX_{t_k}^N(i)\| < \infty \). Lemma 3.3 (iv) proves that $X_{t_{k+1}}^N \in D^{1,2}$.

Finally, let $t \in [t_k, t_{k+1})$; applying the Malliavin derivative $D$ to (1.3) and using Lemma 3.3 we have

$$D_t X_{t_k}^N = h b'(X_{t_k}^N) D_t X_{t_k}^N + \sigma (X_{t_k}^N);$$

which concludes the proof.  \hfill \Box

The following result gives a bound of $p^{th}$ moments of the implicit scheme.
Lemma 3.10. For all $k \geq 0$, then for $N_0$ large enough, there exists a constant $C(p) > 0$ such that

$$
\sup_{N \geq N_0} \max_{k=0, \ldots, N} E \left| X_{t_k}^N \right|^p \leq C(p).
$$

Proof. Holder’s inequality shows that it suffices to consider moments which are power of 2, that is to check $\sup_{N \geq N_0} \max_{k=0, \ldots, N} E \left| X_{t_k}^N \right|^{2p} \leq C_p$, for every integer $p \geq 1$. Using the generalized Young Lemma 3.6 the independence between $\Delta W_{k+1}$ and $F_{t_k}$, and the fact that for all $j \in \mathbb{N}$, $E (\Delta W_{k+1})^{2j+1} = 0$, we have for $h \in (0, h^*)$ and some constant $C_p$ depending on $h^*

\begin{align*}
E \left| X_{t_{k+1}}^N \right|^{2p} &\leq (1 + h)^{2p-1} E \left| X_{t_k}^N + \sigma (X_{t_k}^N) \Delta W_{k+1} \right|^{2p} \\
&+ \left( 1 + \frac{1}{h} \right)^{2p-1} h^{2p} E \left| b \left( X_{t_{k+1}}^N \right) \right|^{2p} \\
&\leq (1 + C_p h) E \left| X_{t_k}^N \right|^{2p} + (1 + C_p h) \sum_{j=1, \ldots, 2p-1} \binom{2p}{2j} E \left( \left| \left[ X_{t_k}^N \right]^{2p-2j} \right| \sigma \left( X_{t_k}^N \right) \right)^{2j} E \left| \Delta W_{k+1} \right|^{2j} \\
&+ C_p h \left( 1 + \frac{1}{h} \right)^{2p-1} h^{2p} E \left| b \left( X_{t_{k+1}}^N \right) \right|^{2p}.
\end{align*}

Using the identity $E \left| \Delta W_{k+1} \right|^{2j} = C(2j)h^j$ and the linear growth of $\sigma$ we deduce for $h < 1$

\begin{align*}
E \left| X_{t_{k+1}}^N \right|^{2p} &\leq (1 + C_p h) E \left| X_{t_k}^N \right|^{2p} + C_p h + C_p h E \left| X_{t_{k+1}}^N \right|^{2p} \\
&+ (1 + C_p h) C_p h \sum_{j=1, \ldots, 2p-1} E \left( \left| \left[ X_{t_k}^N \right]^{2p-2j} \right| \left( 1 + \left| X_{t_k}^N \right|^{2j} \right) \right).
\end{align*}

Using the inequality: $a^{2p+1-2j} \leq a^{2p+1} + 1$ valid for any $a > 0$, we get for some constant $C_p > 0$ and $h < 1$

\begin{align*}
E \left| X_{t_{k+1}}^N \right|^{2p} &\leq (1 + C_p h) E \left| X_{t_k}^N \right|^{2p} + C_p h + C_p h E \left| X_{t_{k+1}}^N \right|^{2p}.
\end{align*}

Provided that $h$ is small enough, the Gronwall Lemma 3.4 and Lemma 3.5 conclude the proof. $\square$

3.3. Some martingales and related process: $\beta_{t, k}^N, \gamma_{t, k}^N, \delta_{t, k}^N$ and $\eta_{t, k}^N$. Let $k \in \{0, \ldots, N - 1\}$ be fixed; in the sequel, we will use the following processes defined for $t \in [t_k, t_{k+1}]

\begin{align*}
\beta_{t, k}^N &:= E \left( b \left( X_{t_{k+1}}^N \right) \right| F_t), \\
\gamma_{t, k}^N &:= \sigma \left( X_{t_k}^N \right) + (t - t_k) \gamma_{t, k-1}^N, \\
\delta_{t, k}^N &:= E \left( D_t b \left( X_{t_{k+1}}^N \right) \right| F_t), \\
\eta_{t, k}^N &:= \sigma \left( X_{t_k}^N \right) + E \left( D_t \left( \sigma (S_{h'} b') \right) \left( X_{t_{k+1}}^N \right) \right| F_t).
\end{align*}

The following lemma describes the time evolution of these processes.

Lemma 3.10. For all $k = 0, \ldots, N - 1$, and for $t \in [t_k, t_{k+1}]$, we have the following relation

\begin{align*}
d\beta_{t, k}^N &= \gamma_{t, k-1}^N dW_t, \\
d\gamma_{t, k}^N &= \eta_{t, k}^N dW_t, \\
d\delta_{t, k}^N &= \gamma_{t, k-1}^N dt + \eta_{t, k}^N (t - t_k) dW_t, \\
d\eta_{t, k}^N &= \sigma \left( X_{t_k}^N \right) \mid E \left( D_t \left( \sigma \left( S_{h'} b' \right) \right) \left( X_{t_{k+1}}^N \right) \mid F_t \right) dW_t.
\end{align*}
Proof. Let $k = 0, \ldots, N - 1$ and let $t \in [t_k, t_{k+1}]$. Lemmas 3.3 (ii), 3.7 and 3.8 and the bounds of $\|b''\|_\infty$ imply that $b \left(X_{t_{k+1}}^N\right) \in L^2(F_{t_{k+1}}) \cap D^{1,2}$ and

$$D_t b \left(X_{t_{k+1}}^N\right) = b' \left(X_{t_{k+1}}^N\right) D_t X_{t_{k+1}}^N = (S_h b') \left(X_{t_{k+1}}^N\right) \sigma \left(X_{t_{k+1}}^N\right).$$

(3.4)

So the Clark-Ocone formula in Proposition 3.2 yields $\beta_{t_k}^{k, N} = b \left(X_{t_{k+1}}^N\right) - \int_{t_k}^{t_{k+1}} z_s^{k,N} dW_s$ and hence $d\beta_{t_k}^{k, N} = z_{t_k}^{k,N} dW_t$; where $z_{t_k}^{k,N} = E \left(D_u b \left(X_{t_{k+1}}^N\right) | F_s\right)$; using (3.4)

$$z_{t_k}^{k,N} = \sigma \left(X_{t_{k+1}}^N\right) E \left(\left(S_h b'\right) \left(X_{t_{k+1}}^N\right) | F_s\right).$$

(3.5)

So taking conditional expectation with respect to $F_t$, we have (3.5). Since $b''$ is bounded and $b'$ Lipschitz we have that for $h$ small enough, $S_h b' = \frac{b'}{1-hb'} \in C_b^1$. So we can conclude that $(S_h b') \left(X_{t_{k+1}}^N\right) \in D^{1,2}$ and using the Clark-Ocone formula we deduce that for $s \in [t_k, t_{k+1})$,

$$\left(S_h b'\right) \left(X_{t_{k+1}}^N\right) = E \left(S_h b' \left(X_{t_{k+1}}^N\right) | F_s\right) + \int_{t_k}^{t_{k+1}} E \left(D_u \left(S_h b' \left(X_{t_{k+1}}^N\right)\right) | F_u\right) dW_u,$$

and hence

$$dz_{t_k}^{k,N} = \sigma \left(X_{t_{k+1}}^N\right) E \left(D_s \left[ \frac{b'}{1-hb'} \left(X_{t_{k+1}}^N\right)\right] \right) | F_s\right) dW_s = \eta_{t_k}^{k,N} dW_s.$$

The differential of $\gamma_{t_k}^{k,N}$ is a consequence of the previous result and Itô’s formula. Finally, since $S_h b' \in C_b^1$ and $(S_h b')' = S_h^2 b''$, Lemma 3.8 implies that $D_t (S_h b') \left(X_{t_{k+1}}^N\right) = \sigma \left(X_{t_k}^N\right) (S_h^2 b'') \left(X_{t_{k+1}}^N\right)$ and then

$$\eta_{t_k}^{k,N} = |\sigma \left(X_{t_{k+1}}^N\right)|^2 E \left(\left(S_h^2 b''\right) \left(X_{t_{k+1}}^N\right) | F_t\right).$$

(3.6)

Applying once more the Clark-Ocone formula in Proposition 3.2, we deduce

$$E \left(S_h^2 b'' \left(X_{t_{k+1}}^N\right) | F_t\right) = (S_h^2 b'') \left(X_{t_{k+1}}^N\right) - \int_{t_k}^{t_{k+1}} E \left(D_s (S_h^2 b'') \left(X_{t_{k+1}}^N\right) | F_s\right) dW_s.$$

Multiplying this by $|\sigma \left(X_{t_{k+1}}^N\right)|^2$ and using (3.6), we conclude the proof. \qed

The next lemma provides uniform moment estimates of the above processes.

Lemma 3.11. Let $p \in \mathbb{N}$; then for $N_0$ large enough there exists a constant $C_p$ such that for $N \geq N_0$,

$$\max_{k=0, \ldots, N-1} \sup_{t_k \leq t \leq t_{k+1}} E \left\{ |\beta_{t_k}^{k,N}|^p + |z_{t_k}^{k,N}|^p + |\gamma_{t_k}^{k,N}|^p + |\eta_{t_k}^{k,N}|^p \right\} \leq C_p.$$

Proof. Using Jensen’s inequality, the Lipschitz property of $b$ and Lemma 3.9 we have

$$E \left|\beta_{t_k}^{k,N}\right|^p \leq E \left|b \left(X_{t_{k+1}}^N\right)\right|^p \leq C_p \left(1 + E \left|X_{t_{k+1}}^N\right|^p\right) \leq C_p.$$

The identity (3.5), Jensen’s inequality, the growth property of $\sigma$ and the upper estimate $b'/(1-hb') \geq C$ for small $h$, Schwarz’s inequality and Lemma 3.9 yield

$$E \left|z_{t_k}^{k,N}\right|^p \leq E \left|\sigma \left(X_{t_{k+1}}^N\right) \left(\gamma_{t_k}^{k,N} \right) \left(S_h b'\right) \left(X_{t_{k+1}}^N\right)\right|^p \leq C_p.$$

Using the definition of $\gamma_{t_k}^{k,N}$ in (3.3) and the previous upper estimates we deduce

$$E \left|z_{t_k}^{k,N}\right|^p \leq C_p E \left|\sigma \left(X_{t_{k+1}}^N\right)\right|^p + C_p h^p E \left|z_{t_k}^{k,N}\right|^p \leq C_p.$$
Finally (3.6), the Jensen inequality, the growth condition on $\sigma$, the upper bounds of $b'$ and $b''$, Lemma 3.9 and Schwarz’s inequality yield
\[ E \left| \eta_{tk}^{b,N} \right|^p \leq E \left| \sigma \left( X_{tk}^N \right) \right|^{2p} \left( S^p b'' \right) \left( X_{tk+1}^N \right)^p \leq C_p \]
This concludes the proof. \( \square \)

3.4. Continuous interpolation. As usual we need to introduce a continuous process that interpolates the implicit Euler scheme (1.3). With an abuse of notation, let \((X_t^N)_{t \in [0,T]}\) be the process defined as follow: \(X_0^N = x_0\) and for \(k = 0, \ldots, N - 1\) and \(t_k \leq t \leq t_{k+1}\)
\[ X_t^N := X_{tk}^N + E \left( b \left( X_{tk+1}^N \right) \right) \left( \mathcal{F}_t \right) (t - t_k) + \sigma \left( X_{tk}^N \right) (W_t - W_{tk}) \tag{3.7} \]
This process satisfies the following

Lemma 3.12. The process \((X_t^N)_{t \in [0,T]}\) is continuous \(\mathcal{F}_t\)-adapted and is an interpolation of the scheme (1.3). Moreover, for \(k \in \{0, \ldots, N - 1\}\) and \(t \in [t_k, t_{k+1}]\) we have
\[ dX_t^N = \beta_t^{b,N} dt + \gamma_t^{b,N} dW_t, \]
where the process \((\beta_t^{b,N})\) and \((\gamma_t^{b,N})\) are defined by (3.3).

Remark 3.13. (1) If \(b = 0\), (3.7) corresponds to the classical interpolation given by Talay Tubaro [7], since the explicit and implicit Euler scheme are the same.

(2) If \(b\) is linear, this continuous process differs from that used by Debussche in [3]. Indeed, the finite dimensional analog of the interpolation corresponding to the process \(dX_t = -\beta_t dt + \sigma (X_t) dW_t\), is defined by
\[ X_t^D = X_{tk}^N + \int_{t_k}^t -\beta_s X_s^N \frac{ds}{1 + h\beta} + \int_{t_k}^t \frac{\sigma (X_s^N)}{1 + h\beta} dW_s; \]
for \(t \in [t_k, t_{k+1}]\) (see [3] page 96 equation (3.2)). In this particular case, our interpolation is given by
\[ X_t^N = X_{tk}^N + \int_{t_k}^t E \left( -\beta_s X_s^N \mid \mathcal{F}_s \right) ds + \int_{t_k}^t \left( \sigma (X_s^N) + (s - t_k) E \left( -D_s \beta X_{tk+1}^N \mid \mathcal{F}_s \right) \right) dW_s. \]

Proof of Lemma 3.12. The fact that \((X_t^N)\) is an \((\mathcal{F}_t)\)-adapted process which interpolates the scheme (1.3) is a consequence of (3.7). The continuity is a consequence of the fact that the map \((t \rightarrow E (X_t \mid \mathcal{F}_t))\) has a continuous modification. So, applying Itô’s formula and Lemma 3.10, we obtain
\[ d\beta_t^{b,N} (t - t_k) = (t - t_k) z_t^{b,N} dW_t + \beta_t^{b,N} dt, \]
and hence
\[ dX_t^N = \beta_t^{b,N} dt + \left( \sigma (X_t^N) + (t - t_k) z_t^{b,N} \right) dW_t. \]
This concludes the proof. \( \square \)

We next give moment estimates of the interpolation process \(X_t^N\).

Lemma 3.14. Let \(p \geq 1\) and \(h^* > 0\) be small enough. There exits a constant \(C_p > 0\) depending on \(h^*\) such that
\[ \sup_{N \geq N_0} \sup_{t \in [0,T]} E \left| X_t^N \right|^p < C_p. \]

Proof. Using Lemma 3.9, Jensen’s inequality and the independence of \(W_t - W_{tk}\) and \(X_{tk}^N\), we have for \(t \in [t_k, t_{k+1}]\):
\[ E \left| X_t^N \right|^p \leq C_p E \left| X_{tk}^N \right|^p + C_p E \left| b \left( X_{tk+1}^N \right) \right|^{2p} |t - t_k|^p + C_p E \left| \sigma \left( X_{tk}^N \right) \right|^p |W_t - W_{tk}|^p \]
\[ \leq C_p + C_p h^p E \left| b \left( X_{tk+1}^N \right) \right|^p + E \left| \sigma \left( X_{tk}^N \right) \right|^p E |W_t - W_{tk}|^p. \]
Using the growth condition on \(b\) and \(\sigma\), moments of the normal law and Lemma 3.9, we deduce the result. \( \square \)
The following is a straightforward consequence of Lemmas 3.11 and 3.14

**Corollary 3.15.** Let \( v : [0,T] \times \mathbb{R} \to \mathbb{R} \) be a function with polynomial growth, and let \( n_1, \ldots, n_6 \) non negative integers. Then there exists a constant \( C \) independent of \( h^* \) such that for \( r \in [t_k, t_{k+1}] \) and \( h \in (0, h^*) \)

\[
E \left( \sum_{r}^{N+1} \left| b^k_{r,N} \right|^2 \partial u_r(t_k) \right) \leq C. 
\]

3.5. **Local decomposition.** Now we return to the proof of the main theorem. Let \( u \) be the solution to the Kolmogorov equation (2.1). Using (2.1), we decompose the weak error into a sum of local errors. Let \( \delta_k^N := Eu \left( t_{k+1}, X_{t_{k+1}}^N \right) - Eu \left( t_{k}, X_{t_k}^N \right) \); we deduce

\[
Ef \left( X_t^N \right) - Ef \left( X_T \right) = Eu \left( T, X_T^N \right) - Eu \left( 0, x \right) = \sum_{k=0}^{N-1} \delta_k^N. \tag{3.8}
\]

We introduce, for \( t_k \leq t \leq t_{k+1} \),

\[
\mathcal{I}_k^N(t) := E \left[ \left( \beta_t^{k,N} - b \left( X_t^N \right) \right) \partial u_r(t, X_t^N) \right], \quad \mathcal{J}_k^N(t) := E \left[ \left( \left| \gamma_t^{k,N} \right|^2 - \sigma^2 \left( X_t^N \right) \right) \Delta u_r(t, X_t^N) \right].
\]

Since \( u \in C^{1,2} \), using Itô’s formula, Lemma 3.11 and the Kolmogorov equation (2.1) at the point \((t, X_t^N)\), we obtain

\[
\delta_k^N = Ef \left( X_t^N \right) - Ef \left( X_T \right) = Eu \left( T, X_T^N \right) - Eu \left( 0, x \right) = \sum_{k=0}^{N-1} \delta_k^N. \tag{3.9}
\]

We introduce, for \( t_k \leq t \leq t_{k+1} \),

\[
\mathcal{I}_k^N(t) := E \left[ \left( \beta_t^{k,N} - b \left( X_t^N \right) \right) \partial u_r(t, X_t^N) \right], \quad \mathcal{J}_k^N(t) := E \left[ \left( \left| \gamma_t^{k,N} \right|^2 - \sigma^2 \left( X_t^N \right) \right) \Delta u_r(t, X_t^N) \right].
\]

Since \( u \in C^{1,2} \), using Itô’s formula, Lemma 3.11 and the Kolmogorov equation (2.1) at the point \((t, X_t^N)\), we obtain

\[
\delta_k^N = Ef \left( X_t^N \right) - Ef \left( X_T \right) = Eu \left( T, X_T^N \right) - Eu \left( 0, x \right) = \sum_{k=0}^{N-1} \delta_k^N. \tag{3.9}
\]

Now for \( k = 0, \ldots, N-1 \), we introduce the following quantities for \( s \in [t_k, t_{k+1}] \):

\[
i_k^N(s) := \frac{\partial}{\partial s} \left( b \partial u_r(s, X_s^N) + \beta_s^{k,N} \partial (b \partial u_r)(s, X_s^N) \right) \tag{3.11}
\]

\[
+ \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta (b \partial u_r)(s, X_s^N) - \beta_s^{k,N} \frac{\partial}{\partial s} \partial u_r(s, X_s^N)
\]

\[
- \left( \left| \beta_s^{k,N} \right|^2 + \gamma_s^{k,N} \beta_s^{k,N} \right) \Delta u_r(s, X_s^N) - \frac{1}{2} \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^2 u_r(s, X_s^N),
\]

\[
j_k^N(s) := \left| \gamma_s^{k,N} \right|^2 \frac{\partial}{\partial s} \Delta u_r(s, X_s^N) + \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^2 u_r(s, X_s^N) \tag{3.12}
\]

\[
+ \frac{1}{2} \left| \gamma_s^{k,N} \right|^4 \frac{\partial^4 u_r(s, X_s^N) + 2 \gamma_s^{k,N} \gamma_s^{k,N} \Delta u_r(s, X_s^N)}{\partial s} \| \gamma_s^{k,N} \|^2 \partial^2 u_r(s, X_s^N)
\]

\[
+ \left| s - t_k \right|^2 \left| \eta_s^{k,N} \right|^2 \Delta u_r(s, X_s^N) + 2 \left( s - t_k \right) \left| \gamma_s^{k,N} \right|^2 \eta_s^{k,N} \partial^2 u_r(s, X_s^N)
\]

\[
- \frac{\partial}{\partial s} \left( \sigma^2 \Delta u_r(s, X_s^N) - \beta_s^{k,N} \partial (\sigma^2 \Delta u_r)(s, X_s^N) - \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta (\sigma^2 \Delta u_r)(s, X_s^N) \right).
\]

The next two lemmas explain that, up to some sign, \( \mathcal{I}_k^N \) (resp. \( \mathcal{J}_k^N \)) can be viewed as an antiderivative of \( i_k^N \) (resp. \( j_k^N \)).

**Lemma 3.16.** For all \( k = 0, \ldots, N-1 \), we have \( \mathcal{I}_k^N(t) = Ef \int_t^{t_k+1} i_k^N(s)ds \) for \( t \in [t_k, t_{k+1}] \).

**Proof.** If we denote by \( A := E \left[ \beta_t^{k,N} \partial u_r(t, X_t^N) \right] - E \left[ b \left( X_{t_k+1}^N \right) \partial u \left( t_{k+1}, X_{t_{k+1}}^N \right) \right] \) and by \( B := E \left[ b \left( X_{t_k+1}^N \right) \partial u \left( t_{k+1}, X_{t_{k+1}}^N \right) \right] - E \left[ b \left( X_t^N \right) \partial u_r(t, X_t^N) \right] \) we can write \( \mathcal{I}_k^N(t) = A + B. \)
Lemma 3.11 enables us to apply Itô’s formula: Let \( v : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of class \( C^{1,2} \); Itô’s formula yields

\[
dv(t, X_t^N) = \left\{ \frac{\partial}{\partial t} v + \beta_t^{k,N} \frac{\partial}{\partial r} v + \frac{1}{2} |\gamma_t^{k,N}|^2 \Delta v \right\} (t, X_t^N) \, dt + \gamma_t^{k,N} \frac{\partial}{\partial r} v (t, X_t^N) \, dW_t. \tag{3.13} \]

Using this equation with Lemma 3.10 we have for \( v \in C^{1,2} \)

\[
d [\beta_r^{k,N} v (r, X_r^N)] = \left\{ \beta_r^{k,N} \frac{\partial}{\partial r} v + |\beta_r^{k,N}|^2 \frac{\partial}{\partial r} v + \frac{1}{2} |\beta_r^{k,N}|^2 \Delta v + \left( \gamma_r^{k,N} \right)^2 \frac{\partial}{\partial r} v + \gamma_r^{k,N} \gamma_r^{k,N} \Delta v \right\} (r, X_r^N) \, dr \]

\[
+ \left\{ \beta_r^{k,N} \gamma_r^{k,N} \frac{\partial}{\partial r} v + \gamma_r^{k,N} \gamma_r^{k,N} \Delta v \right\} (r, X_r^N) \, dW_r. \tag{3.14} \]

The function \( \Delta u \) has polynomial growth; hence corollary 3.15 implies that

\[ E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \gamma_s^{k,N} \Delta u + z_s^{k,N} \frac{\partial}{\partial s} u \right\} (s, X_s^N) \, ds = 0. \]

Using equation (3.14) with \( v = \frac{\partial}{\partial u} \), integrating between \( t \) and \( t_{k+1} \), using the fact that \( \beta_{k+1}^{k,N} = b \left( X_{t_{k+1}}^N \right) \) and taking expectation we obtain

\[
A = -E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \frac{\partial}{\partial s} u + |\beta_s^{k,N}|^2 \Delta u + \frac{1}{2} \beta_s^{k,N} \gamma_s^{k,N} \Delta u \right\} (s, X_s^N) \, ds. \tag{3.15} \]

Similarly, Corollary 3.15 implies that

\[ E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \gamma_s^{k,N} \Delta (b \frac{\partial}{\partial u}) \right\} (s, X_s^N) \, ds = 0. \]

Using (3.13) with \( v = b \frac{\partial}{\partial u} \), integrating between \( t \) and \( t_{k+1} \) and taking expectation yields

\[
B = E \int_t^{t_{k+1}} \left\{ \frac{\partial}{\partial s} (b \frac{\partial}{\partial u}) + \beta_s^{k,N} \gamma_s^{k,N} \Delta (b \frac{\partial}{\partial u}) \right\} (s, X_s^N) \, ds. \]

The stochastic integral is centered by Corollary 3.15. This identity combined with (3.15) concludes the proof.

\[ \square \]

Lemma 3.17. For all \( k = 0, \ldots, N - 1 \), we have \( \mathcal{J}_k^N (t) = E \int_{t_k}^{t} j_k^N (s) \, ds \) for \( t \in [t_k, t_{k+1}] \).

Proof. Using (3.3) we clearly deduce that \( \mathcal{J}_k^N (t) = C + D \) where

\[
C := E \left[ \sigma \left( X_{t_k}^N \right) \left( t - t_k \right) \gamma_t^{k,N} \right] \frac{\partial}{\partial u} \Delta u \left( t, X_t^N \right) \right] - E \left[ \sigma^2 \left( X_{t_k}^N \right) \Delta u \left( t, X_t^N \right) \right],
\]

\[
D := E \left[ \sigma^2 \left( X_{t_k}^N \Delta u \left( t, X_t^N \right) \right) \right] - E \left[ \sigma^2 \left( X_{t_k}^N \Delta u \left( t, X_t^N \right) \right) \right].
\]

We at first rewrite the term \( D \): using (3.13) with \( v = \sigma^2 \Delta u \), integrating between \( t_k \) and \( t \) and taking expectation, we obtain:

\[
D = -E \int_{t_k}^{t} \left\{ \frac{\partial}{\partial t} \left( \sigma^2 \frac{\partial}{\partial u} \right) + \beta_s^{k,N} \gamma_s^{k,N} \Delta (\sigma^2 \Delta u) \right\} (s, X_s^N) \, ds,
\]

since \( \sigma^2 \Delta u \) has polynomial growth which implies that the stochastic integral is centered using Corollary 3.15. Itô’s formula and Lemma 3.10 yield for \( r \in [t_k, t_{k+1}] \)

\[
d \left| \gamma_r^{k,N} \right|^2 = \left\{ 2 \gamma_r^{k,N} \gamma_r^{k,N} + \left| \eta_r^{k,N} \right|^2 \right\} \left| r - t_k \right|^2 \, dr + 2 \gamma_r^{k,N} \eta_r^{k,N} \left( r - t_k \right) dW_r. \tag{3.16} \]

Using equation (3.17) with \( v = \Delta u \), integrating between \( t_k \) and \( t \), using the identity \( \gamma_{r,k}^{N,N} = \sigma \left( X_{t_k}^N \right) \) and taking expectation, we deduce

\[
C = E \int_{t_k}^t \left\{ |\gamma_{s,k}^{N,N}|^2 \frac{\partial}{\partial s} \Delta u + \beta_{s}^{k,N} |\gamma_{s,k}^{N,N}|^2 \partial^3 u + \frac{1}{2} |\gamma_{s,k}^{N,N}|^4 \partial^4 u \right. \]

\[
+ 2\gamma_{s,k}^{k,N} z_{s,k}^{k,N} \Delta u + |s - t_k|^2 |\eta_{s,k}^{k,N}|^2 \Delta u + 2(s - t_k) |\eta_{s,k}^{k,N}|^2 \partial^3 u \left. \right\} (s, X_s^N) \, ds.
\]

Indeed, once more Corollary 3.15 and the polynomial growth of \( \partial \Delta u \) and \( \Delta u \) implies that the corresponding stochastic integral is centered. This concludes the proof. \( \Box \)

Plugging the results of Lemmas 3.16 and 3.17 into (3.10) we obtain

\[
Ef \left( X_T^N \right) - Ef \left( X_T^N \right) = \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} \left\{ \int_{t_k}^{t_{k+1}} \nu_i^N(s) \, ds + \frac{1}{2} \int_{t_k}^{t_{k+1}} \nu_i^N(s) \, ds \right\} dt. \tag{3.18}
\]

**Note:** Thanks to Corollary 3.15 and the assumptions growth or boundedness on the coefficients, all the stochastic integrals appearing in the next section, are centered.

3.6. **Upper estimate of** \( I_k^N(t) \). We next upper estimate the difference \( \phi_i(s) - \phi_i(t_{k+1}) \), where \( \phi_i \) is one of the seven terms in the right hand side of (3.11)

3.6.1. The term \( \phi_1(s) = \frac{\partial}{\partial s} (b \partial u) \left( s, X_s^N \right) \). Using (3.13) with \( v = \frac{\partial}{\partial s} (b \partial u) \), integrating from \( s \) to \( t_{k+1} \) and taking expected value we deduce

\[
E \frac{\partial}{\partial s} (b \partial u) \left( s, X_s^N \right) = E \frac{\partial}{\partial s} (b \partial u) \left( t_{k+1}, X_{t_{k+1}}^N \right) + R_1(s),
\]

where

\[
R_1(s) := -E \int_s^{t_{k+1}} \left\{ \frac{\partial^2}{\partial s^2} (b \partial u) + \beta_{r}^{k,N} \frac{\partial}{\partial s} (b \partial u) + \frac{1}{2} \left| \gamma_{r}^{k,N} \right|^2 \Delta \frac{\partial}{\partial s} (b \partial u) \right\} \left( r, X_r^N \right) \, dr.
\]

Furthermore, Lemmas 3.14 and 3.11 and the polynomial growth of the functions involved imply that \( |R_1(s)| \leq Ch \).

3.6.2. The term \( \phi_2(s) = \beta_{s}^{k,N} \Delta (b \partial u) \left( s, X_s^N \right) \). Using (3.14) with \( v = \partial (b \partial u) \), integrating between \( s \) and \( t_{k+1} \) and taking expectation we obtain

\[
E \left[ \beta_{s}^{k,N} \Delta (b \partial u) \left( s, X_s^N \right) \right] = E \left[ b \left( X_{t_{k+1}}^N \right) \partial (b \partial u) \left( t_{k+1}, X_{t_{k+1}}^N \right) \right] + R_2(s),
\]

where

\[
R_2(s) := -E \int_s^{t_{k+1}} \left[ \beta_{r}^{k,N} \frac{\partial}{\partial s} (b \partial u) + \beta_{r}^{k,N} \Delta (b \partial u) + \frac{1}{2} \left| \gamma_{r}^{k,N} \right|^2 \partial^3 (b \partial u) \right. \]

\[
+ \left. |\gamma_{r}^{k,N}|^2 |\gamma_{r}^{k,N}|^2 \Delta (b \partial u) \right\} \left( r, X_r^N \right) \, dr.
\]

The polynomial growth of the functions and Lemmas 3.14 and 3.11 imply that \( |R_2(s)| \leq Ch \).
3.6.3. The term \( \phi_3(s) = \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta (b \partial u) (s, X_s^N) \). Let

\[
R_3(s) := -\frac{1}{2} E \int_s^{t_{k+1}} \left\{ \left| \gamma_r^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta (b \partial u) + \left| \gamma_r^{k,N} \right|^2 \frac{\partial^3}{\partial b \partial u} + \frac{1}{2} \left| \gamma_r^{k,N} \right|^4 \frac{\partial^4}{\partial u} \\
+ \left| \gamma_r^{k,N} \eta_r^{k,N} \right|^2 |r - t_k|^2 \Delta (b \partial u) + \left| \gamma_r^{k,N} \eta_r^{k,N} (r - t_k) \right| \frac{\partial^3}{\partial b \partial u} \right\} (r, X_r^N) \, dr.
\]

Using (3.17) with \( v = \frac{1}{2} \Delta (b \partial u) \), integrating between \( s \) and \( t_{k+1} \), and taking expectation give us

\[
\frac{1}{2} E \left[ \left| \gamma_s^{k,N} \right|^2 \Delta (b \partial u) (s, X_s^N) \right] = \frac{1}{2} E \left[ \left| \gamma_{t_{k+1}}^{k,N} \right|^2 \Delta (b \partial u) (t_{k+1}, X_{t_{k+1}}^N) \right] + R_3(s),
\]

with \( |R_3(s)| \leq Ch \).

3.6.4. The term \( \phi_4(s) = \beta_s^{k,N} \frac{\partial}{\partial s} u (s, X_s^N) \). Let

\[
R_4(s) := E \int_s^{t_{k+1}} \left[ \beta_r^{k,N} \frac{\partial^2}{\partial s^2} \partial u + \left| \beta_r^{k,N} \eta_r^{k,N} \right|^2 \frac{\partial}{\partial s} \partial u + \frac{1}{2} \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \Delta \frac{\partial}{\partial s} \partial u \\
+ \left| \gamma_r^{k,N} \eta_r^{k,N} \right|^2 \frac{\partial}{\partial s} \partial u \right] (r, X_r^N) \, dr.
\]

Using (3.14) for \( v = \frac{\partial}{\partial s} \partial u \) and integrating between \( s \) and \( t_{k+1} \), we obtain

\[
- E \left[ \beta_s^{k,N} \frac{\partial}{\partial t} \partial u (s, X_s^N) \right] = - E \left[ b (X_{t_{k+1}}^N) \frac{\partial}{\partial t} \partial u (t_{k+1}, X_{t_{k+1}}^N) \right] + R_4(s),
\]

with \( |R_4(s)| \leq Ch \).

3.6.5. The term \( \phi_5(s) = \left| \beta_s^{k,N} \right|^2 \Delta u (s, X_s^N) \). Using Itô’s formula and Lemma 3.10 we have

\[
d \left| \beta_r^{k,N} \right|^2 = \left| \gamma_r^{k,N} \right|^2 \, dr + 2 \beta_r^{k,N} \gamma_r^{k,N} dW_r.
\]

Using this equation, (3.13) and Itô’s formula we obtain

\[
d \left[ \beta_r^{k,N} \right]^2 \Delta u (r, X_r^N) = \left[ \left| \beta_r^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u + \left( \beta_r^{k,N} \right)^3 \frac{\partial^3}{\partial u} + \frac{1}{2} \left| \beta_r^{k,N} \right|^2 \left| \gamma_r^{k,N} \right|^2 \frac{\partial^4}{\partial u} \\
+ \left| \gamma_r^{k,N} \right|^2 \left[ \frac{\partial^2}{\partial s^2} \partial u + \frac{2 \beta_r^{k,N} \gamma_r^{k,N} \eta_r^{k,N} \frac{\partial}{\partial u} \right] (r, X_r^N) \right] \, dr + dM_r,
\]

where \( dM_r = \left\{ 2 \beta_r^{k,N} \gamma_r^{k,N} \Delta u + \left| \gamma_r^{k,N} \right|^2 \gamma_r^{k,N} \frac{\partial}{\partial u} \right\} (r, X_r^N) \, dW_r \) and \( M_t \) is a square integrable martingale. Let

\[
R_5(s) := E \int_s^{t_{k+1}} \left[ \left| \beta_r^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u + \left( \beta_r^{k,N} \right)^3 \frac{\partial^3}{\partial u} + \frac{1}{2} \left| \beta_r^{k,N} \right|^2 \left| \gamma_r^{k,N} \right|^2 \frac{\partial^4}{\partial u} \\
+ \left| \gamma_r^{k,N} \right|^2 \left[ \frac{\partial^2}{\partial s^2} \partial u + \frac{2 \beta_r^{k,N} \gamma_r^{k,N} \eta_r^{k,N} \frac{\partial}{\partial u} \right] (r, X_r^N) \right] \, dr.
\]

Integrating between \( s \) and \( t_{k+1} \) and taking expectation we have

\[
- E \left[ \beta_s^{k,N} \right]^2 \Delta u (s, X_s^N) = - E \left[ \left| \beta_{t_{k+1}}^{k,N} \right|^2 \Delta u (t_{k+1}, X_{t_{k+1}}^N) \right] + R_5(s),
\]

with \( |R_5(s)| \leq Ch \).
3.6.6. The term $\phi_6(s) = z_{s}^{k,N} \gamma_{s}^{k,N} \Delta u (s, X_{s}^{N})$. Applying Itô’s formula to the product of $z_{r}^{k,N} \gamma_{r}^{k,N}$ and (3.13), and Lemma 3.10, we obtain for $r \in [t_k, t_{k+1}]
\begin{align*}
d \left[ z_{r}^{k,N} \gamma_{r}^{k,N} v (r, X_{r}^{N}) \right] &= \left\{ \frac{k,N}{r} \frac{\partial}{\partial r} v + \frac{k,N}{r} \gamma_{r}^{k,N} \frac{\partial}{\partial r} \gamma_{r}^{k,N} \Delta v + \frac{1}{2} \frac{k,N}{r} \gamma_{r}^{k,N} \Delta v + \left| \eta_{r}^{k,N} \right|^2 (r - t_k) v + \frac{k,N}{r} \gamma_{r}^{k,N} \eta_{r}^{k,N} \Delta v + \left| \gamma_{r}^{k,N} \right|^2 \eta_{r}^{k,N} \Delta v \right\} (r, X_{r}^{N}) dr + dM_r
\end{align*}
(3.19)
\end{equation}

where $dM_r = \left\{ \frac{k,N}{r} \gamma_{r}^{k,N} \Delta u (s, X_{s}^{N}) \right\} (r, X_{r}^{N}) dW_r$ and $M_r$ is a square integrable martingale. Using equation (3.19) with $v = \Delta u$, integrating between $s$ and $t_{k+1}$ and taking expectation give us
\begin{equation}
- E \left[ z_{s}^{k,N} \gamma_{s}^{k,N} \Delta u (s, X_{s}^{N}) \right] = - E \left[ z_{t_{k+1}}^{k,N} \gamma_{t_{k+1}}^{k,N} \Delta u (t_{k+1}, X_{t_{k+1}}^{N}) \right] + R_6(s),
\end{equation}
with $|R_6(s)| \leq Ch$.

3.6.7. The term $\phi_7(s) = \frac{1}{2} \beta_{s}^{k,N} \left| \frac{k,N}{s} \right|^2 \Delta u (s, X_{s}^{N})$. Using Lemma 3.10 and equation (3.17), Itô’s formula give us for $v$ of class $C^{1,2}$
\begin{align*}
d \left[ \beta_{r}^{k,N} \left| \gamma_{r}^{k,N} \right|^2 \right] (r, X_{r}^{N}) &= \left\{ \beta_{r}^{k,N} \left| \gamma_{r}^{k,N} \right|^2 \frac{\partial}{\partial r} v + \beta_{r}^{k,N} \gamma_{r}^{k,N} \eta_{r}^{k,N} \Delta v + \frac{1}{2} \beta_{r}^{k,N} \gamma_{r}^{k,N} \Delta v + 2 \beta_{r}^{k,N} \gamma_{r}^{k,N} \eta_{r}^{k,N} \Delta v + 2 \beta_{r}^{k,N} \gamma_{r}^{k,N} \eta_{r}^{k,N} \Delta v \right\} (r, X_{r}^{N}) dr + \left\{ \gamma_{r}^{k,N} \left| \beta_{r}^{k,N} \right|^2 \partial v + 2 \beta_{r}^{k,N} \gamma_{r}^{k,N} \eta_{r}^{k,N} \Delta v + 2 \beta_{r}^{k,N} \gamma_{r}^{k,N} \eta_{r}^{k,N} \Delta v \right\} (r, X_{r}^{N}) dW_r.
\end{align*}
(3.20)

Using this equation with $v = \frac{1}{2} \Delta u$, integrating between $s$ and $t_{k+1}$ and taking expectation we have
\begin{equation}
- \frac{1}{2} E \left[ \beta_{s}^{k,N} \left| \gamma_{s}^{k,N} \right|^2 \Delta u (s, X_{s}^{N}) \right] = - \frac{1}{2} E \left[ \beta_{t_{k+1}}^{k,N} \left| \gamma_{t_{k+1}}^{k,N} \right|^2 \Delta u (t_{k+1}, X_{t_{k+1}}^{N}) \right] + R_7(s),
\end{equation}
with $|R_7(s)| \leq Ch$.

3.7. Upper estimate of $J_k^{N}(t)$. We upper estimate the error $\tilde{\phi}_1(s) - \phi_1(t_k)$ where $\tilde{\phi}_1$ is one of the nine terms in the right hand side of (3.12)

3.7.1. The term $\tilde{\phi}_1(s) = \left| \gamma_{s}^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u (s, X_{s}^{N})$. Using (3.17) with $v = \frac{\partial}{\partial t} \Delta u$, integrating between $t_k$ and $s$, taking expectation and using the fact that $\gamma_{t_k}^{k,N} = \sigma (X_{t_k}^{N})$ we have
\begin{equation}
E \left[ \left| \gamma_{s}^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u (s, X_{s}^{N}) \right] = E \left[ \sigma (X_{t_k}^{N}) \right] \frac{\partial}{\partial t} \Delta u (t_k, X_{t_k}^{N}) + \tilde{R}_1(s),
\end{equation}
with
\begin{equation}
\tilde{R}_1(s) = E \int_{t_k}^{s} \left[ \left| \gamma_{r}^{k,N} \right|^2 \frac{\partial^2}{\partial r^2} \Delta u + 2 \frac{\sigma}{\sigma} \left| \gamma_{r}^{k,N} \right|^2 \frac{\partial}{\partial r} \Delta u \right] (r, X_{r}^{N}) dr
\end{equation}
\begin{equation}
+ \left\{ \beta_{r}^{k,N} + 2 \frac{\sigma}{\sigma} \right\} \left| \gamma_{r}^{k,N} \right|^2 \frac{\partial}{\partial r} \Delta u + \frac{1}{2} \left| \gamma_{r}^{k,N} \right|^4 \frac{\partial}{\partial r} \Delta u \right\} (r, X_{r}^{N}) dr.
\end{equation}
Corollary 3.15 implies that $|\tilde{R}_1(s)| \leq Ch$. 

3.7.2. The term $\tilde{\phi}_2(s) = \beta_{k,N} \left| \gamma_{k,N}^Z \partial^3 u(s, X^N_s) \right|^2$. For an $F_s$-measurable random variable $Z$, we have $E \left( Z \tilde{\phi}_2(s) \right) = E \left( Z \tilde{\phi}(X^N_{t_k+1}) \right)$. Using (3.20) with $v = \partial^3 u$, integrating between $t_k$ and $s$ and taking expectation we have

$$E \left[ \beta_{k,N} \left| \gamma_{k,N}^Z \partial^3 u(s, X^N_s) \right|^2 \right] = E \left[ b \left( X^N_{t_k+1} \right) \left( \sigma \left( X^N_{t_k} \right) \right)^2 \partial^3 u(t_k, X^N_k) + \tilde{R}_2(s) \right],$$

where

$$\tilde{R}_2(s) := E \int_{t_k}^{s} \left\{ 2 \gamma_{k,N} \partial_{t} \eta_{r,N}^k \eta_{r,N}^k + \beta_{k,N} \bigg| \eta_{r,N}^k \bigg|^2 \bigg| r - t_k \bigg|^2 + 2 \gamma_{k,N} \eta_{r,N}^k z_{r,N}^k (r - t_k) \right\} \partial^3 u + \beta_{k,N} \bigg| \gamma_{k,N}^Z \bigg|^2 \left( \partial_t \gamma_{k,N} \right)^4 u + \frac{1}{2} \beta_{\sigma}^k \bigg| \gamma_{r,N}^Z \bigg|^4 \left( \partial_t \gamma_{r,N} \right)^4 u \bigg| (r, X^N_r) \bigg) \, dr.$$

Corollary 3.15 implies that $|\tilde{R}_2(s)| \leq Ch.$

3.7.3. The term $\tilde{\phi}_3(s) = \frac{1}{2} \bigg| \gamma_{k,N}^k \bigg|^4 \partial^4 u(s, X^N_s)$. Using Lemma 3.10 and Itô’s formula we deduce

$$d \bigg| \gamma_{r,N}^k \bigg|^4 = \bigg| \gamma_{r,N}^k \bigg|^2 \left\{ 4 \partial_t \gamma_{r,N} \gamma_{r,N}^k + 6 \bigg| \eta_{r,N}^k \bigg|^2 \bigg| r - t_k \bigg|^2 \right\} dr + 4 \bigg( \gamma_{r,N}^k \bigg)^3 \eta_{r,N}^k (r - t_k) dW_r.$$

Using this equation and (1.13) with $v = \frac{1}{2} \partial^4 u$ and applying Itô formula, we have

$$\frac{1}{2} E \left[ \bigg| \gamma_{r,N}^k \bigg|^4 \partial^4 u(s, X^N_s) \right] = \frac{1}{2} E \left[ \left( \sigma \left( X^N_{t_k} \right) \right)^4 \partial^4 u(t_k, X^N_k) \right] + \tilde{R}_3(s),$$

where $\tilde{R}_3(s) \leq Ch$ by Corollary 3.15.

3.7.4. The term $\tilde{\phi}_4(s) = 2 \gamma_{k,N} \gamma_{k,N}^Z \Delta u(s, X^N_s)$. Using (3.19) with $v = 2 \Delta u$ we have

$$E \left[ 2 \gamma_{k,N} \gamma_{k,N}^Z \Delta u(s, X^N_s) \right] = E \left[ 2 \gamma_{k,N} \gamma_{k,N}^Z \Delta u(t_k, X^N_k) \right] + \tilde{R}_4(s),$$

and Corollary 3.15 implies $|\tilde{R}_4(s)| \leq Ch.$

3.7.5. The term $\tilde{\phi}_5(s) := \tilde{R}_5(s) := |s - t_k|^2 \bigg| \eta_{s,N}^k \bigg|^2 \partial^3 u(s, X^N_s) + 2 (s - t_k) \bigg| \gamma_{s,N}^k \bigg|^2 \eta_{s,N}^k \partial^3 u(s, X^N_s).$

Using Corollary 3.15, we have $|\tilde{R}_5(s)| \leq Ch.$

3.7.6. The term $\tilde{\phi}_6(s) = \frac{\partial}{\partial t} (\sigma^2 \Delta u)(s, X^N_s)$. Using (3.13) with $v = \frac{\partial}{\partial t} (\sigma^2 \Delta u)$, integrating between $t_k$ and $s$ and taking expectation, we have

$$- E \left[ \frac{\partial}{\partial t} (\sigma^2 \Delta u)(s, X^N_s) \right] = - E \left[ \frac{\partial}{\partial t} (\sigma^2 \Delta u)(t_k, X^N_k) \right] + \tilde{R}_6(s),$$

with $|\tilde{R}_6(s)| \leq Ch$ by Corollary 3.15.

3.7.7. The term $\tilde{\phi}_7(s) = \beta_{s,N} \partial (\sigma^2 \Delta u)(s, X^N_s)$. Using (3.14) with $v = \partial (\sigma^2 \Delta u)$, integrating between $t_k$ and $s$, taking expectation we have

$$- E \left[ \beta_{s,N} \partial (\sigma^2 \Delta u)(s, X^N_s) \right] = - E \left[ b \left( X^N_k \right) \partial (\sigma^2 \Delta u)(t_k, X^N_k) \right] + \tilde{R}_7(s),$$

with $|\tilde{R}_7(s)| \leq Ch$ by Corollary 3.15.
3.7.8. The term \( \tilde{\phi}_{8}(s) = \frac{1}{2} |\gamma_{t_{k},N}^{k,k,N}|^{2} \Delta \left( \sigma^{2} \Delta u \right)(s, X_{s}^{N}) \). Using (3.17) with \( v = \frac{1}{2} \Delta (\sigma^{2} \Delta u) \), integrating between \( t_{k} \) and \( s \), and finally taking expectation we have

\[
-\frac{1}{2} E \left[ |\gamma_{t_{k}}^{k,k,N}(s_{k},N)}|^{2} \Delta \left( \sigma^{2} \Delta u \right)(s, X_{s}^{N}) \right] = -\frac{1}{2} E \left[ |\sigma(X_{t_{k}}^{N})|^{2} \Delta \left( \sigma^{2} \Delta u \right)(t_{k}, X_{t_{k}}^{N}) \right] + \tilde{R}_{8}(s),
\]

with \( |\tilde{R}_{8}(s)| \leq Ch \) by Corollary 3.15.

3.8. Proof Theorem 2.1 (i). The identity (3.18) and the upper estimate in section 3.6 and 3.7 imply that

\[
Ef \left( X_{t}^{N} \right) - Ef \left( X_{t} \right) = \sum_{k=0}^{N-1} E \int_{t_{k}}^{t_{k+1}} \left\{ \int_{t}^{t_{k+1}} \partial_{t} u \left( t, X_{t_{k+1}}^{N} \right) ds + \frac{1}{2} \int_{t}^{t_{k}} \partial_{S} u \left( t_{k+1} \right) ds \right\} dt + R
\]

where

\[
R := \sum_{k=0}^{N-1} \int_{t_{k}}^{t_{k+1}} \int_{t_{k}}^{t_{k+1}} R_{j}(s) d\tau ds + \sum_{k=0}^{N-1} \int_{t_{k}}^{t_{k+1}} \int_{t_{k}}^{t_{k+1}} \tilde{R}_{j}(s) d\tau ds.
\]

Hence \( |R| \leq C h^{2} \). Note that \( \beta_{t_{k+1}}^{k,N} = b \left( X_{t_{k+1}}^{N} \right) \). Using (3.3) and (3.5) we deduce that

\[
z_{t_{k+1}}^{k,N} = \sigma \left( X_{t_{k+1}}^{N} \right) \left( S_{b} b' \right) \left( X_{t_{k+1}}^{N} \right) \quad \text{and} \quad \gamma_{t_{k+1}}^{k,N} = \sigma \left( X_{t_{k+1}}^{N} \right) \left[ 1 + h(S_{b} b') \right] \left( X_{t_{k+1}}^{N} \right).
\]

Therefore, we deduce that

\[
i_{k}^{N} \left( t_{k+1} \right) = \frac{\partial}{\partial t} \left( b \partial u \right) \left( t_{k+1}, X_{t_{k+1}}^{N} \right) + \left[ b \partial (b \partial u) \right] \left( t_{k+1}, X_{t_{k+1}}^{N} \right) + \frac{1}{2} \sigma^{2} \left( X_{t_{k+1}}^{N} \right) \left[ S_{b}^{2} \partial \Delta \left( b \partial u \right) \right] \left( t_{k+1}, X_{t_{k+1}}^{N} \right) - \frac{1}{2} \sigma^{2} \left( X_{t_{k+1}}^{N} \right) \left[ S_{b}^{2} \partial \Delta u \right] \left( t_{k+1}, X_{t_{k+1}}^{N} \right) - \frac{1}{2} \sigma^{2} \left( X_{t_{k+1}}^{N} \right) \left[ b S_{b}^{2} \partial u \right] \left( t_{k+1}, X_{t_{k+1}}^{N} \right).
\]

Similarly, \( \gamma_{t_{k}}^{k,N} = \sigma \left( X_{t_{k}}^{N} \right) \). So we have

\[
E_{j_{k}}^{N} \left( t_{k} \right) = E \sigma^{2} \frac{\partial}{\partial t} \Delta u \left( t_{k}, X_{t_{k}}^{N} \right) + E b \left( X_{t_{k}}^{N} \right) \sigma^{2} \partial^{2} u \left( t_{k}, X_{t_{k}}^{N} \right) + \frac{1}{2} E \sigma^{4} \partial^{4} u \left( t_{k}, X_{t_{k}}^{N} \right)
\]

\[
+ 2 E S_{b} b' \left( X_{t_{k+1}}^{N} \right) \sigma^{2} \Delta u \left( t_{k}, X_{t_{k}}^{N} \right) - E \frac{\partial}{\partial t} \left( \sigma^{2} \Delta u \right) \left( t_{k}, X_{t_{k}}^{N} \right)
\]

\[
- E b \left( X_{t_{k+1}}^{N} \right) \partial \left( \sigma^{2} \Delta u \right) \left( t_{k}, X_{t_{k}}^{N} \right) - \frac{1}{2} E \sigma^{2} \Delta \left( \sigma^{2} \Delta u \right) \left( t_{k}, X_{t_{k}}^{N} \right) .
\]

Notice that \( b \) and \( \sigma \) do not depend upon \( t \); hence after simplification we have

\[
i_{k}^{N} \left( t_{k+1} \right) = b \partial (b \partial u) \left( t_{k+1}, X_{t_{k+1}}^{N} \right) - b^{2} \Delta u \left( t_{k+1}, X_{t_{k+1}}^{N} \right) + \frac{1}{2} \sigma^{2} \left( X_{t_{k+1}}^{N} \right) \left( S_{b}^{2} b' \partial u \right) \left( t_{k+1}, X_{t_{k+1}}^{N} \right),
\]

\[
E_{j_{k}}^{N} \left( t_{k} \right) = E b \left( X_{t_{k+1}}^{N} \right) \sigma^{2} \partial^{2} u \left( t_{k}, X_{t_{k+1}}^{N} \right) + \frac{1}{2} E \sigma^{4} \partial^{4} u \left( t_{k}, X_{t_{k+1}}^{N} \right) - E b \left( X_{t_{k+1}}^{N} \right) \partial \left( \sigma^{2} \Delta u \right) \left( t_{k}, X_{t_{k+1}}^{N} \right)
\]

\[
- \frac{1}{2} E \sigma^{2} \Delta \left( \sigma^{2} \Delta u \right) \left( t_{k}, X_{t_{k+1}}^{N} \right) + 2 E b' S_{b} \left( X_{t_{k+1}}^{N} \right) \sigma^{2} \Delta u \left( t_{k}, X_{t_{k+1}}^{N} \right) .
\]

Corollary 3.15 implies the existence of a constant \( C \), such that for all \( k = 0, \ldots, N - 1 \):

\[
|E \left( i_{k}^{N} \left( t_{k+1} \right) + j_{k}^{N} \left( t_{k} \right) \right) | \leq C.
\]

Using this bound with (3.21) proves the first part of Theorem 2.1.
Lemma 3.18. Let \( v \) and \( w \) in \( C_b^{\infty, \infty}([0, T] \times \mathbb{R}) \). Then there exists a constant \( C \) independent of \( h \) such that
\[
\left| \sum_{k=0}^{N-1} h \int_{t_k}^{t_{k+1}} v \left( t, X^N_{t_k} \right) \right| \leq C h.
\]

Proof. Using (3.13) multiply by \( w(t_k, X^N_{t_k}) \) and taking expected value, we deduce for \( v \in C^{1,2} \),
\[
Ev \left( t_{k+1}, X^N_{t_{k+1}} \right) = Evw \left( t_k, X^N_{t_k} \right) + Ak,
\]
where
\[
Ak := Ew \left( t_k, X^N_{t_k} \right) \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta^k N \partial v + \frac{1}{2} |\gamma^k t|^2 \Delta v \right\} (t, X^N_t) dt.
\]
This yields
\[
\sum_{k=0}^{N-1} h \int_{t_k}^{t_{k+1}} v \left( t, X^N_{t_k} \right) = Ev \left( t_{k+1}, X^N_{t_{k+1}} \right) - Evw \left( t_k, X^N_{t_k} \right) = \sum_{k=0}^{N-1} \left( hA_k + hB_k + C_k \right).
\]

Using the Cauchy-Schwarz inequality, the fact that \( \frac{\partial}{\partial t} v, \partial v \) and \( \Delta v \) have polynomial growth so that Corollary 3.15 can be applied, we deduce
\[
|A_k|^2 \leq E \left[ |w \left( t_k, X^N_{t_k} \right)|^2 \right] E \left[ \left| \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta^k N \partial v + \frac{1}{2} |\gamma^k t|^2 \Delta v \right\} (t, X^N_t) dt \right|^2 \right]
\]
\[
\leq C h \int_{t_k}^{t_{k+1}} \left| \left\{ \frac{\partial}{\partial t} v + \beta^k N \partial v + \frac{1}{2} |\gamma^k t|^2 \Delta v \right\} (t, X^N_t) \right|^2 dt,
\]
\[
\leq C h^2.
\]
Hence, \|A_k\| \leq C h which implies \( h \sum_{0 \leq k \leq N-1} |A_k| \leq C h \).
Since \((vw)(t_k,.) \) is in \( C_b^{\infty} \), we use Theorem 2.1 (i), changing \( T \) by \( t_k \), which yields \|B_k\| \leq C h and then \( h \sum_{0 \leq k \leq N-1} |B_k| \leq C h \). Finally, Itô’s formula implies
\[
C_k = E \int_{t_k}^{t_{k+1}} \left\{ (vw)(t_k, X_k) - (vw)(t, X_t) \right\} dt
\]
\[
= E \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \left\{ \frac{\partial}{\partial t} (vw) + b \partial (vw) + \frac{1}{2} \sigma^2 \Delta (vw) \right\} (s, X_s) ds dt.
\]
Once more the polynomial growth imposed on \( v, w \) and their partial derivatives implies that \|C_k\| \leq C h^2 and then \( \sum_{0 \leq k \leq N-1} |C_k| \leq C h \). This concludes the proof. \( \square \)

Now we introduce the function \( \psi_{ih} : [0, T] \times \mathbb{R} \to \mathbb{R} \) defined by
\[
\psi_{ih}(t, x) := \frac{1}{2} b \partial (b \partial u)(t, x) - \frac{1}{2} b^2 \Delta u(t, x) + \frac{1}{4} \sigma^2 \partial^2_S h b^2 \partial u(t, x) + \frac{1}{4} b \sigma^2 \partial \Delta u(t, x)
\]
\[
+ \frac{1}{8} \sigma^4 \partial^4 u(t, x) + \frac{1}{2} b' S \partial \sigma^2 \Delta u(t, x) - \frac{1}{4} b \partial (\sigma^2 \Delta u)(t, x) - \frac{1}{8} \sigma^2 \Delta (\sigma^2 \Delta u)(t, x) \tag{3.24}
\]
Using the expression of $i_k^{N}$ in (3.22) (resp. $j_k^{N}$ in (3.23)) and the previous lemma, we deduce
\[
\left| \frac{1}{2} h \sum_{k=0}^{N-1} E i_k^{N} (t_{k+1}) + \frac{1}{4} h \sum_{k=0}^{N_1} E j_k^{N} (t_k) - \int_0^T E \psi_{ih} (t, X_t) dt \right| \leq Ch. \tag{3.25}
\]
Using the definitions of $\psi_i$ and $\psi_{ih}$ given in (2.2) and (3.24) respectively, we have
\[
\psi_{ih} (t, x) - \psi_i (t, x) = \frac{1}{4} \left\{ \sigma^2 S_h^2 b'' \partial u + b \sigma^2 \partial^3 u + 2 b S_h \sigma^2 \Delta u - \sigma^2 \Delta (b \partial u) \right\} (t, x)
\begin{align*}
&= \frac{1}{4} \sigma^2 (S_h^2 - 1) b'' \partial u(t, x) + \frac{1}{2} b (S_h - 1) \sigma^2 \Delta u(t, x).
\end{align*}
\]
Since $(S_h - 1)(x) = \frac{b''(x)}{b''(0)}(x)$ and $|S_h(x) + 1| \leq C$ for $h \in (0, h^*)$, we have $|(S_h - 1)(x)| + |(S_h^2 - 1)(x)| \leq Ch$, where as usually $C$ does not depend on $N$ and $h$. This yields
\[
\left| \int_0^T E \{ \psi_{ih} (t, X_t) - \psi_i (t, X_t) \} dt \right| \leq Ch.
\]
This last equation with (3.21) and (3.25) concludes the proof.

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References


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