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# Asymptotic Nonlinearity of Vectorial Boolean Functions

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**Abstract.** We investigate the nonlinearity of functions from  $\mathbb{F}_2^m$  to  $\mathbb{F}_2^n$ . We give asymptotic bounds for almost all these functions.

## 1 Introduction

Let  $m$  and  $n$  be two positive integers. Functions from the vectorspace  $V_m = \mathbb{F}_2^m$  to the vectorspace  $V_n = \mathbb{F}_2^n$ , where  $\mathbb{F}_2$  is the finite field with two elements, are called  $(m, n)$ -functions or more generally, vectorial Boolean functions. For a cryptographic use, such functions need to fulfill many criteria in order to ensure the robustness of the cryptosystems in which they are involved [1]. Among these criteria and one very important notion is the nonlinearity of these functions that must be as high as possible in order to resist to linear cryptanalysis. A  $(m, n)$ -function is affine if and only if it is a  $\mathbb{F}_2$ -linear map plus a constant. The nonlinearity  $\mathcal{NL}(f)$  of a  $(m, n)$ -function  $f$  equals the minimum Hamming distance between all the component functions of  $f$ , that is  $v.f$  where  $v \in V_n^* = V_n \setminus \{0\}$ , and all affine  $(m, n)$ -functions. It can be computed through the Walsh transform of these components. For a given  $v \in V_n^*$ , the Walsh transform of  $v.f$  is the Fourier transform of  $\chi_{v.f}(x) = (-1)^{(v.f)(x)}$  the  $\pm 1$ -representation of  $v.f$ . Let us denote by  $\widehat{V}_m$  the set of characters of  $V_m$ . For every  $\mu \in \widehat{V}_m$ , we have

$$\widehat{\chi_{v.f}}(\mu) = \sum_{x \in V_m} (-1)^{(v.f)(x)} \mu(x),$$

where  $\mu(x) = (-1)^{x.y}$  for some  $y$  in  $V_m$ . And the nonlinearity of  $f$  is

$$\mathcal{NL}(f) = 2^{m-1} - \frac{1}{2} \max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi_{v.f}}(\mu)|.$$

Hence a function has high nonlinearity if all of its components Walsh values have low magnitudes. The covering radius bound is valid for every  $(m, n)$ -function

$$\mathcal{NL}(f) \leq 2^{m-1} - 2^{m/2-1}, \quad (1)$$

and can be achieved with equality only if  $m$  is even and  $n \leq m/2$ . For  $n \geq m$ , we have a better bound [4], and when  $n$  is sufficiently greater than  $m$ , other bounds are given in [2]. Finding better bounds than (1) in the other cases remains an open problem. Besides that, we don't have information about the distribution of the nonlinearity of  $(m, n)$ -functions. When  $n = 1$ , the distribution was studied by [8, 3, 10, 7].

In this paper, we propose asymptotic bounds which are valid for almost all  $(m, n)$ -functions. Let  $0 < \beta < 1/4$ , when  $m$  tends to infinity and  $n \leq m$ , we show in theorem 2.1 that the nonlinearity of almost all  $(m, n)$ -functions is bounded from above by  $2^{m-1} - 2^{\frac{m-1}{2}} \sqrt{(m+n) \log 2} (1 - \beta)$ . And in theorem 3.1, we prove that for any positive real  $\beta$ , when  $(m+n)$  tends to infinity and without any order for  $m$  and  $n$ , almost all  $(m, n)$ -functions have nonlinearity greater than  $2^{m-1} - 2^{\frac{m-1}{2}} \sqrt{(m+n) \log 2} (1 + \beta)$ .

To obtain the first result, we use G. Halász method in [6] concerning random trigonometric polynomials. This work inspired F. Rodier [10] to prove that almost all  $m$ -variable Boolean functions ( $(m, 1)$ -functions) have nonlinearities in the neighbourhood of  $2^{m-1} - 2^{m/2-1} \sqrt{2m \log 2}$ . We use the same scheme of proof, however, it was necessary to take more precise approximations in the case of  $(m, n)$ -functions. As for the second result, it is a generalization of F. Rodier's result [9] on Boolean functions inspired by the work of R. Salem and A. Zygmund [11] on trigonometric series.

We begin by proving the lower bound of  $\max_{v \in V_n^*; \mu \in \widehat{V}_m} |\widehat{\chi_{v,f}}(\mu)|$  which is more difficult.

## 2 The lower bound

Let  $u(x)$ , that will be completely constructed in section 2.3, be a function on  $\mathbb{R}$  satisfying

$$0 \leq u(x) \leq 1 \quad \forall x \in \mathbb{R}, \quad u(x) = \begin{cases} 0 & \text{for } |x| \leq M \\ 1 & \text{for } |x| \geq M + \Delta, \end{cases}$$

where  $M = 2^{\frac{m+1}{2}} \sqrt{(m+n) \log 2} (1-\beta)$  with  $0 < \beta < 1/4$  and  $\Delta = \sqrt{\frac{2^m}{\log 2^m}}$ . We consider the random variable  $\eta$  on the space of  $(m, n)$ -functions

$$\eta(f) = \int_{V_n^*} \int_{\widehat{V}_m} u(\widehat{\chi}_{v,f}(\mu)) d\mu dv,$$

where  $d\mu$  (resp.  $dv$ ) is a uniform measure over  $\widehat{V}_m$  (resp.  $V_n^*$ ) of total mass 1.  $\eta(f) = 0$  is equivalent to  $\max_{v \in V_n^*; \mu \in \widehat{V}_m} |\widehat{\chi}_{v,f}(\mu)| \leq M$ . When  $n \leq m$ , we shall prove by applying Chebyshev's inequality that this occurs with probability tending to 0 for large enough  $m$ .

The function  $u(x)$  is the real Fourier transform of a measure  $U$  on  $\mathbb{R}$

$$u(x) = \int_{\mathbb{R}} \exp(-2\pi itx) dU(t).$$

Hence

$$\eta(f) = \int_{V_n^*} \int_{\widehat{V}_m} \int_{\mathbb{R}} \exp(-2\pi it \widehat{\chi}_{v,f}(\mu)) dU(t) d\mu dv.$$

Before evaluating the first and second moment of  $\eta$ , some estimations are necessary but we chose to give the proof later. The following proposition is given in [6] and [10] but we repeat it for the reader's convenience.

**Proposition 2.1.** *When  $m$  tends to infinity, we have the following estimations:*

$$\int_{\mathbb{R}} |dU(t)| = O(m), \tag{2}$$

$$\int_{\mathbb{R}} |t|^p |dU(t)| = O\left(\frac{m}{2^m}\right)^{p/2} \quad \text{for } 1 \leq p \leq 32, \tag{3}$$

$$\left| \int_{\mathbb{R}} \exp(-2^{m+1} \pi^2 t^2) t^p dU(t) \right| = O\left(2^{-m\frac{p}{2} - (m+n)(1-2\beta)} m^{p/2-1/2}\right) \quad \text{for } 0 \leq p \leq 24. \tag{4}$$

*Proof.* See section 2.3.

## 2.1 Expectation of $\eta$

**Lemma 2.1.**

$$E(\eta) = \int_{\mathbb{R}} \exp(-2^{m+1} \pi^2 t^2) dU(t) + O\left(2^{-2m-n+2\beta(m+n)} m^{3/2}\right) + O\left(2^{-2m} m^4\right). \tag{5}$$

*Proof.* We have

$$E(\eta) = \int_{V_n^*} \int_{\widehat{V}_m} \int_{\mathbb{R}} E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu))) dU(t) d\mu dv.$$

The random variables  $\chi_{v.f}(x)\mu(x)$  are independent in  $x$  and take values  $+1$  and  $-1$  with probability  $1/2$ . Thus

$$\begin{aligned} E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu))) &= E\left(\prod_{x \in V_m} \exp(-2\pi it \chi_{v.f}(x)\mu(x))\right) \\ &= \prod_{x \in V_m} E(\exp(-2\pi it \chi_{v.f}(x)\mu(x))) \\ &= \cos^{2^m}(2\pi t) \\ &= \exp\left(-2^{m+1}\pi^2 t^2 - \frac{4}{3}\pi^4 2^m t^4\right) + O(2^m t^6) \end{aligned} \tag{6}$$

for  $|t| \leq \frac{1}{3\pi}$ , by applying on (6)

$$\log \cos y = -\frac{y^2}{2} - \frac{y^4}{12} + O(y^6) \quad \text{for } |y| \leq 1$$

and

$$\exp(-a) = \exp(-b) + O(b-a) \quad \text{for } a, b \geq 0. \tag{7}$$

For  $|t| > \frac{1}{3\pi}$ , we use the trivial bound 1 for the integrand. This gives

$$\begin{aligned} &\int_{\mathbb{R}} E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu))) dU(t) \\ &= \int_{-\frac{1}{3\pi}}^{\frac{1}{3\pi}} \exp\left(-2^{m+1}\pi^2 t^2 - \frac{4}{3}\pi^4 2^m t^4\right) dU(t) + O\left(2^m \int_{\mathbb{R}} t^6 |dU(t)|\right) \\ &\quad + O\left(\int_{|t| \geq \frac{1}{3\pi}} |dU(t)|\right). \end{aligned}$$

We extend the first integral over the real line making the same error as the third term, that can be included in the second one. This yields

$$\int_{\mathbb{R}} \exp\left(-2^{m+1}\pi^2 t^2 - \frac{4}{3}\pi^4 2^m t^4\right) dU(t) + O\left(2^m \int_{\mathbb{R}} t^6 |dU(t)|\right).$$

By (3), the remainder equals  $O(2^{-2m} m^3)$ . As for the main term, we use

$$\exp(-a) = 1 - a + O(a^2) \quad \text{for } a > 0,$$

in addition to (3) and (4) as follows

$$\begin{aligned}
& \int_{\mathbb{R}} \exp\left(-2^{m+1}\pi^2 t^2 - \frac{4}{3}\pi^4 2^m t^4\right) dU(t) \\
&= \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) \left(1 - \frac{4}{3}\pi^4 2^m t^4 + O(2^{2m} t^8)\right) dU(t) \\
&= \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) + O(2^{-2m-n+2\beta(m+n)} m^{3/2}) + O\left(2^{2m} \int_{\mathbb{R}} t^8 |dU(t)|\right) \\
&= \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) + O(2^{-2m-n+2\beta(m+n)} m^{3/2}) + O(2^{-2m} m^4).
\end{aligned}$$

The proof is complete recalling that the total mass over  $V_n^*$  and  $\widehat{V}_m$  is 1.  $\square$

## 2.2 The second moment

$\eta^2(f)$  consists of three sums

$$\begin{aligned}
\eta^2(f) &= \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ (v,\mu)=(v',\mu')}} u(\widehat{X}_{v.f}(\mu)) u(\widehat{X}_{v'.f}(\mu')) d\mu dv d\mu' dv' \\
&\quad + \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ v=v' \\ \mu \neq \mu'}} u(\widehat{X}_{v.f}(\mu)) u(\widehat{X}_{v'.f}(\mu')) d\mu dv d\mu' dv' \\
&\quad + \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ v \neq v'}} u(\widehat{X}_{v.f}(\mu)) u(\widehat{X}_{v'.f}(\mu')) d\mu dv d\mu' dv',
\end{aligned}$$

which we denote respectively by  $\eta_1^2(f)$ ,  $\eta_2^2(f)$  et  $\eta_3^2(f)$ .

**Lemma 2.2.**

$$E(\eta_1^2) \leq \frac{1}{2^m(2^n - 1)} E(\eta).$$

*Proof.*

$$\begin{aligned}
\eta_1^2(f) &= \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ (v,\mu)=(v',\mu')}} u(\widehat{X}_{v.f}(\mu)) u(\widehat{X}_{v'.f}(\mu')) d\mu dv d\mu' dv' \\
&= \frac{1}{2^m(2^n - 1)} \int_{V_n^* \times \widehat{V}_m} u^2(\widehat{X}_{v.f}(\mu)) d\mu dv \\
&\leq \frac{1}{2^m(2^n - 1)} \int_{V_n^* \times \widehat{V}_m} u(\widehat{X}_{v.f}(\mu)) d\mu dv = \frac{1}{2^m(2^n - 1)} \eta(f)
\end{aligned}$$

noting that  $0 \leq u(x) \leq 1$ ,  $\forall x \in \mathbb{R}$ .  $\square$

For  $E(\eta_2^2)$  (resp.  $E(\eta_3^2)$ ), we use the representation of  $u$  as a Fourier transform

$$\begin{aligned} E(\eta_2^2) &= E\left(\int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ v=v' \\ \mu \neq \mu'}} u(\widehat{\chi}_{v.f}(\mu)) u(\widehat{\chi}_{v.f}(\mu')) d\mu dv d\mu' dv'\right) \\ &= \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ v=v' \\ \mu \neq \mu'}} \int_{\mathbb{R}^2} E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu) - 2\pi it' \widehat{\chi}_{v.f}(\mu'))) dU(t) dU(t') d\mu dv d\mu' dv'. \end{aligned}$$

We evaluate the integrand in the following lemma.

**Lemma 2.3.** *Given  $v \in V_n^*$  and  $\mu, \mu' \in \widehat{V}_m$  such that  $\mu \neq \mu'$ . For  $t$  and  $t'$  of absolute value smaller than  $\frac{1}{3\pi}$ , we have*

$$E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu) - 2\pi it' \widehat{\chi}_{v.f}(\mu'))) = \exp\left(-2^m \sum_{i=1}^4 \sum_{j=0}^i c_{i,j} t^{2i-2j} t'^{2j}\right) + 2^m O(|t| + |t'|)^{10},$$

where  $c_{i,j}$  are positive reals.

*Proof.* The random variables  $\chi_{v.f}(x)$  ( $t\mu(x) + t'\mu'(x)$ ) are independent in  $x$ , and take values  $(t\mu(x) + t'\mu'(x))$  and  $-(t\mu(x) + t'\mu'(x))$  with probability  $1/2$ . Thus

$$\begin{aligned} &E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu) - 2\pi it' \widehat{\chi}_{v.f}(\mu'))) \\ &= E\left(\prod_{x \in V_m} \exp(-2\pi i \chi_{v.f}(x) (t\mu(x) + t'\mu'(x)))\right) \\ &= \prod_{x \in V_m} E(\exp(-2\pi i \chi_{v.f}(x) (t\mu(x) + t'\mu'(x)))) \\ &= \prod_{x \in V_m} \cos(2\pi(t\mu(x) + t'\mu'(x))). \end{aligned}$$

Since  $\mu \neq \mu'$ , they agree (resp. disagree)  $2^{m-1}$  times

$$\begin{aligned} &\prod_{x \in V_m} \cos(2\pi(t\mu(x) + t'\mu'(x))) \\ &= \cos^{2^{m-1}}(2\pi(t+t')) \cos^{2^{m-1}}(2\pi(t-t')) \\ &= \exp\left(-2^m \left(\sum_{i=1}^4 c_i (t+t')^{2i} + O(t+t')^{10} + \sum_{i=1}^4 c_i (t-t')^{2i} + O(t-t')^{10}\right)\right), \end{aligned}$$

for  $|t| \leq \frac{1}{3\pi}$ ,  $|t'| \leq \frac{1}{3\pi}$ , by applying

$$\log \cos y = -\frac{y^2}{2} - \frac{y^4}{12} - \frac{y^6}{45} - \frac{17y^8}{2520} + O(y^{10}) \quad \text{for } |y| \leq 1.$$

Simplifying and using (7) give the result.  $\square$

**Lemma 2.4.**

$$E(\eta_2^2) = \frac{1}{2^n - 1} \left( \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) \right)^2 + O(2^{-3m-3n+4\beta(m+n)} m) + O(2^{-4m-n} m^9).$$

*Proof.* Using the previous lemma together with the trivial bound 1 for the integrand outside the square  $|t| \geq \frac{1}{3\pi}$ ,  $|t'| \geq \frac{1}{3\pi}$  give

$$\begin{aligned} & \int_{\mathbb{R}^2} E(\exp(-2\pi it \widehat{\chi_{v,f}}(\mu) - 2\pi it' \widehat{\chi_{v,f}}(\mu'))) dU(t) dU(t') \\ &= \int_{-\frac{1}{3\pi}}^{\frac{1}{3\pi}} \int_{-\frac{1}{3\pi}}^{\frac{1}{3\pi}} \exp\left(-2^m \sum_{i=1}^4 \sum_{j=0}^i c_{i,j} t^{2i-2j} t'^{2j}\right) dU(t) dU(t') \\ &+ O\left(2^m \int_{\mathbb{R}^2} (|t| + |t'|)^{10} |dU(t)| |dU(t')|\right) \\ &+ O\left(\int_{|t| \geq \frac{1}{3\pi}} \int_{\mathbb{R}} |dU(t)| |dU(t')|\right). \end{aligned} \quad (8)$$

We extend integration in the first term over  $\mathbb{R}^2$  making the same error as the third term, that is smaller than the second one.

Noting that  $c_{1,0} = c_{1,1} = 2\pi^2$ , and applying

$$\exp(-a) = 1 - a + \frac{a^2}{2} - \frac{a^3}{6} + O(a^4), \quad \text{for } a > 0,$$

the first term then becomes

$$\begin{aligned} & \int_{\mathbb{R}^2} \exp(-2^{m+1}\pi^2 t^2) \exp(-2^{m+1}\pi^2 t'^2) \left( 1 - 2^m \sum_{i=2}^4 \sum_{j=0}^i c_{i,j} t^{2i-2j} t'^{2j} + 2^{2m} \sum_{i=4}^8 \sum_{j=0}^i l_{i,j} t^{2i-2j} t'^{2j} \right. \\ & \left. - 2^{3m} \sum_{i=6}^{12} \sum_{j=0}^i p_{i,j} t^{2i-2j} t'^{2j} + 2^{4m} O\left(\sum_{i=8}^{16} \sum_{j=0}^i r_{i,j} t^{2i-2j} t'^{2j}\right) \right) dU(t) dU(t') \end{aligned}$$

and by (4), we get

$$\begin{aligned} & \left( \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) \right)^2 - 2^m \sum_{i=2}^4 \sum_{j=0}^i O(2^{-mi-2(m+n)(1-2\beta)} m^{i-1}) \\ & + 2^{2m} \sum_{i=4}^8 \sum_{j=0}^i O(2^{-mi-2(m+n)(1-2\beta)} m^{i-1}) - 2^{3m} \sum_{i=6}^{12} \sum_{j=0}^i O(2^{-mi-2(m+n)(1-2\beta)} m^{i-1}) \\ & + 2^{4m} O\left(\sum_{i=8}^{16} \sum_{j=0}^i r_{i,j} \int_{\mathbb{R}} t^{2i-2j} |dU(t)| \int_{\mathbb{R}} t'^{2j} |dU(t')|\right). \end{aligned} \quad (9)$$



Terms in (9) with  $i = j$  or  $j = 0$  are equal  $2^{4m}O(m)O\left(\frac{m}{2^m}\right)^8$  by (2) and (3). The other terms are equal  $2^{4m}O\left(\frac{m}{2^m}\right)^8$  by (3). This gives

$$\left(\int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t)\right)^2 + O(2^{-3m-2n+4\beta(m+n)}m) + O(2^{-4m}m^9).$$

As for (8), it can be estimated just like (9) using (2) and (3), yielding  $O(2^{-4m}m^6)$ . We end the calculations by integrating over the other variables.  $\square$

**Lemma 2.5.**

$$E(\eta_3^2) = \left(1 - \frac{1}{2^n - 1}\right)E^2(\eta).$$

*Proof.* We have

$$E(\eta_3^2) = \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ v \neq v'}} \int_{\mathbb{R}^2} E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu) - 2\pi it' \widehat{\chi}_{v'.f}(\mu'))) dU(t)dU(t')d\mu dv d\mu' dv'.$$

Since  $v \neq v'$ , the random variables  $\widehat{\chi}_{v.f}(\mu)$  and  $\widehat{\chi}_{v'.f}(\mu')$  are independent. Thus

$$\begin{aligned} & E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu) - 2\pi it' \widehat{\chi}_{v'.f}(\mu'))) \\ &= E(\exp(-2\pi it \widehat{\chi}_{v.f}(\mu))) E(\exp(-2\pi it' \widehat{\chi}_{v'.f}(\mu'))) \\ &= \cos^{2m}(2\pi t) \cos^{2m}(2\pi t'), \end{aligned}$$

as calculated previously in (6). And,

$$\begin{aligned} E(\eta_3^2) &= \int_{\substack{V_n^{*2} \times \widehat{V}_m^2 \\ v \neq v'}} d\mu dv d\mu' dv' \left(\int_{\mathbb{R}} \cos^{2m}(2\pi t) dU(t)\right)^2 \\ &= \left(1 - \frac{1}{2^n - 1}\right)E^2(\eta). \end{aligned}$$

$\square$

**Lemma 2.6.**

$$\frac{1}{E(\eta)} = O\left(2^{(m+n)(1-\beta)^2} \sqrt{m}\right). \quad (10)$$

*Proof.* We have

$$E(\eta) = \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) + O(2^{-2m-n+2\beta(m+n)}m^{3/2}) + O(2^{-2m}m^4).$$

The Fourier transform of  $\exp(-2^{m+1}\pi^2 t^2)$  is  $\frac{1}{\sqrt{2^{m+1}\pi}} \exp\left(-\frac{x^2}{2^{m+1}}\right)$ . Hence, by Plancherel's theorem, and the left-hand inequality of (13), we have

$$\begin{aligned}
\int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) &= \frac{1}{\sqrt{2^{m+1}\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2^{m+1}}\right) u(x) dx \\
&\geq \frac{1}{\sqrt{2^{m+1}\pi}} \int_{|x| \geq M+\Delta} \exp\left(-\frac{x^2}{2^{m+1}}\right) dx \\
&= \frac{1}{\sqrt{\pi}} \int_{|y| \geq \frac{M+\Delta}{\sqrt{2^{m+1}}}} \exp(-y^2) dy \\
&\geq \sqrt{\frac{2^{m+1}}{\pi}} \frac{\exp\left(-\frac{(M+\Delta)^2}{2^{m+1}}\right)}{M+\Delta} \left(1 - \frac{2^m}{(M+\Delta)^2}\right) \\
&\geq C_1 \sqrt{2^{m+1}} \frac{\exp\left(-\frac{M^2}{2^{m+1}}\right)}{M+\Delta} \\
&\geq C_2 2^{-(m+n)(1-\beta)^2} m^{-1/2}.
\end{aligned}$$

Adding the fact that

$$O\left(2^{-2m-n+2\beta(m+n)} m^{3/2}\right) + O\left(2^{-2m} m^4\right) = o\left(2^{-(m+n)(1-\beta)^2} m^{-1/2}\right),$$

proves the result.  $\square$

**Theorem 2.1.** *Let  $0 < \beta < \frac{1}{4}$  and  $\gamma$  any positive real. When  $m$  tends to infinity and  $n \leq m$ , we have*

$$P\left(\max_{\substack{v \in V_n^* \\ \mu \in \hat{V}_m}} |\widehat{\chi_{v,f}}(\mu)| \leq 2^{\frac{m+1}{2}} \sqrt{(m+n) \log 2} (1-\beta)\right) = P(\eta = 0) = O(m^{-\gamma}).$$

*Proof.* When  $\eta = 0$ ,  $\eta$  deviates from its expectation by  $E(\eta)$ , and by Tchebitcheff's inequality

$$P(\eta = 0) \leq P(|\eta - E(\eta)| \geq E(\eta)) \leq \frac{E(\eta^2) - E^2(\eta)}{E^2(\eta)}.$$

We have

$$\begin{aligned}
E(\eta^2) - E^2(\eta) &\leq \frac{E(\eta)}{2^m(2^n - 1)} + \frac{1}{2^n - 1} \left( \left( \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) \right)^2 - E^2(\eta) \right) \\
&\quad + O\left(2^{-3m-3n+4\beta(m+n)} m\right) + O\left(2^{-4m-n} m^9\right),
\end{aligned}$$

and by (5), we get

$$E(\eta^2) - E^2(\eta) \leq \frac{E(\eta)}{2^m(2^n - 1)} + \frac{1}{2^n - 1} \left( \left( \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) dU(t) \right) O(2^{-2m-n+2\beta(m+n)} m^{3/2}) \right. \\ \left. + O(2^{-4m} m^8) + E(\eta) O(2^{-2m} m^4) \right) + O(2^{-3m-3n+4\beta(m+n)} m) + O(2^{-4m-n} m^9)$$

When divided by  $E^2(\eta)$ , we can check using (10) and (4) that every term is smaller than  $O(m^{-\gamma})$ .  $\square$

### 2.3 Proof of proposition 2.1

Before giving the proof, we first complete the construction of  $u$ . Let us fix a 34 times continuously differentiable function  $\alpha$  on  $[0, 1]$ , which takes 0 at 0, 1 at 1, takes values between 0 and 1, and with vanishing derivatives up to the 18th order at 0 and 1. By choosing  $u(x)$  to be equal  $\alpha\left(\frac{|x|-M}{\Delta}\right)$  for  $M \leq |x| \leq M + \Delta$ ,  $u(x)$  is then a 34 times differentiable function on  $\mathbb{R}$  with  $|u^r(x)| \leq \frac{\text{constant}}{\Delta^r}$ , for  $r = 0, 1, \dots, 34$ .

*Proof.* The measure  $U$ , having  $u$  as its Fourier transform, can be written as the sum of the Dirac measure at the origin and

$$g(t) = \int_{\mathbb{R}} \exp(-2\pi itx)(u(x) - 1)dx = \int_{-M-\Delta}^{M+\Delta} \exp(-2\pi itx)(u(x) - 1)dx.$$

We have

$$|g(t)| \leq 2(M + \Delta) = O(M). \quad (11)$$

And integration by parts gives

$$|t^r g(t)| \leq \int_{-M-\Delta}^{M+\Delta} |u^{(r)}(x)| dx = O\left(\frac{1}{\Delta^{r-1}}\right) \quad \text{for } r = 1, \dots, 34. \quad (12)$$

To prove (2), we use (11) for  $|t| \leq \frac{1}{\Delta}$  and (12) with  $r = 2$  for  $|t| \geq \frac{1}{\Delta}$

$$\int_{\mathbb{R}} |dU(t)| = 1 + \int_{\mathbb{R}} |g(t)| dt = O\left(\frac{M}{\Delta}\right) = O(m).$$

To prove (3), we use (12) with  $r = p$  for  $|t| \leq \frac{1}{\Delta}$  and with  $r = p + 2$  for  $|t| \geq \frac{1}{\Delta}$

$$\int_{\mathbb{R}} |t^p| |dU(t)| = \int_{\mathbb{R}} |t^p| |g(t)| dt = O\left(\frac{1}{\Delta^p}\right) = O\left(\frac{m}{2^m}\right)^{p/2} \quad \text{for } p = 1, \dots, 32.$$

To prove (4), we use the Plancherel's theorem. The Fourier transform of  $t^p U$  is  $\frac{i^p}{(2\pi)^p} u^{(p)}(x)$  and that of  $\exp(-2^{m+1}\pi^2 t^2)$  is  $\frac{1}{\sqrt{2^{m+1}\pi}} \exp\left(-\frac{x^2}{2^{m+1}}\right)$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) t^p dU(t) \right| &= \frac{1}{\sqrt{2^{m+1}\pi} (2\pi)^p} \left| \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2^{m+1}}\right) u^{(p)}(x) dx \right| \\ &= O\left(\frac{1}{\Delta^p \sqrt{2^m}}\right) \int_{|x| \geq M} \exp\left(-\frac{x^2}{2^{m+1}}\right) dx. \end{aligned}$$

To evaluate the integral of the exponential, we have [5]

$$\left(1 - \frac{1}{2y^2}\right) \frac{\exp(-y^2)}{-2y} < \int_{-\infty}^y \exp(-x^2) dx < \frac{\exp(-y^2)}{-2y}, \quad (13)$$

for every  $y < 0$ . Using the right-hand inequality of (13), we get

$$\left| \int_{\mathbb{R}} \exp(-2^{m+1}\pi^2 t^2) t^p dU(t) \right| = O\left(2^{(-m\frac{p}{2} - (m+n)(1-\beta)^2)} m^{p/2-1/2}\right). \quad (14)$$

□

### 3 The upper bound

**Lemma 3.1.** *Let  $\lambda$  be a real number,  $v \in V_n^*$  and  $\mu \in \widehat{V}_m$ . Then, for  $f$  running in the space of  $(m, n)$ -functions*

$$E(\exp(\lambda \widehat{\chi}_{v,f}(\mu))) \leq \exp(2^{m-1} \lambda^2).$$

*Proof.* The random variables  $\chi_{v,f}(x)\mu(x)$  are independent in  $x$  and take values  $+1$  and  $-1$  with probability  $1/2$ . Thus

$$\begin{aligned} E(\exp(\lambda \widehat{\chi}_{v,f}(\mu))) &= E\left(\prod_{x \in V_m} \exp(\lambda \chi_{v,f}(x)\mu(x))\right) \\ &= \prod_{x \in V_m} E(\exp(\lambda \chi_{v,f}(x)\mu(x))) \\ &= \prod_{x \in V_m} \cosh \lambda. \end{aligned}$$

And

$$\cosh \lambda \leq \exp \frac{\lambda^2}{2}.$$

□

**Theorem 3.1.** *Let  $m$  and  $n$  be any positive integers and  $\beta$  any positif real. Then*

$$P\left(\max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)| \geq 2^{\frac{m+1}{2}} \sqrt{(m+n) \log 2} (1+\beta)\right) \leq 2^{-(m+n)(2\beta+\beta^2)+1}.$$

*Proof.* There exists  $(v_0, \mu_0)$  in  $V_n^* \times \widehat{V}_m$  such that  $\max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)| = |\widehat{\chi}_{v_0,f}(\mu_0)|$ .

Let  $\lambda$  be a positive real, we have

$$\begin{aligned} \exp\left(\lambda \max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)|\right) &\leq \exp(\lambda \widehat{\chi}_{v_0,f}(\mu_0)) + \exp(-\lambda \widehat{\chi}_{v_0,f}(\mu_0)) \\ &\leq 2^{m+n} \int_{V_n^*} \int_{\widehat{V}_m} (\exp(\lambda \widehat{\chi}_{v,f}(\mu)) + \exp(-\lambda \widehat{\chi}_{v,f}(\mu))) d\mu dv. \end{aligned}$$

When  $f$  ranges over the space of  $(m, n)$ -functions

$$E\left(\exp\left(\lambda \max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)|\right)\right) \leq 2^{m+n} \int_{V_n^*} \int_{\widehat{V}_m} E(\exp(\lambda \widehat{\chi}_{v,f}(\mu)) + \exp(-\lambda \widehat{\chi}_{v,f}(\mu))) d\mu dv.$$

Using lemma 3.1 and recalling that the total mass over  $V_n^*$  and  $\widehat{V}_m$  is 1, we have

$$\begin{aligned} E\left(\exp\left(\lambda \max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)|\right)\right) &\leq 2^{m+n+1} \exp(2^{m-1}\lambda^2) \\ &= 2^{-(m+n)(2\beta+\beta^2)+1} \exp(2^{m-1}\lambda^2 + (m+n)(1+\beta)^2 \log 2). \end{aligned}$$

Thus,

$$E\left(\exp\left(\lambda \max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)| - 2^{m-1}\lambda^2 - (m+n)(1+\beta)^2 \log 2\right)\right) \leq 2^{-(m+n)(2\beta+\beta^2)+1}.$$

Consequently,

$$P\left(\exp\left(\lambda \max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)| - 2^{m-1}\lambda^2 - (m+n)(1+\beta)^2 \log 2\right) \geq 1\right) \leq 2^{-(m+n)(2\beta+\beta^2)+1}.$$

And finally,

$$P\left(\max_{\substack{v \in V_n^* \\ \mu \in \widehat{V}_m}} |\widehat{\chi}_{v,f}(\mu)| \geq 2^{m-1}\lambda + \frac{(m+n)(1+\beta)^2 \log 2}{\lambda}\right) \leq 2^{-(m+n)(2\beta+\beta^2)+1}.$$

The best bound is obtained when  $\lambda = 2^{\frac{1-m}{2}} \sqrt{(m+n) \log 2} (1+\beta)$ , which gives the result.  $\square$

When  $(m + n)$  tends to infinity, we obtain then a lower bound of the nonlinearity of almost all  $(m, n)$ -functions.

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