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Spectral analysis of hypoelliptic random walks

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Abstract

We study the spectral theory of a reversible Markov chain associated to a hypoelliptic random walk on a manifold M . This random walk depends on a parameter $h \in]0, h_0]$ which is roughly the size of each step of the walk. We prove uniform bounds with respect to h on the rate of convergence to equilibrium, and the convergence when $h \rightarrow 0$ to the associated hypoelliptic diffusion.

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1 Introduction and Results

The purpose of this paper is to study the spectral theory of a reversible Markov chain associated to a hypoelliptic random walk on a manifold M . This random walk will depend on a parameter $h \in]0, h_0]$ which is roughly the size of each step of the walk. We are in particular interested, as in [DLM11] and [DLM11], to get uniform bounds with respect to h , on the rate of convergence to equilibrium. The main tool in our approach is to compare the random walk on M with a natural random walk on a nilpotent Lie group. This idea was used by Rotschild-Stein [RS76] to prove sharp hypoelliptic estimates for some differential operators.

Let M be a smooth, connected, compact manifold of dimension m , equipped with a smooth volume form $d\mu$ such that $\int_M d\mu = 1$. We denote by μ the associated probability on M . Let $\mathcal{X} = \{X_1, \dots, X_p\}$ be a collection of smooth vector fields on M . Denote \mathcal{G} the Lie algebra generated by \mathcal{X} . In all the paper we assume that the X_k are divergence free with respect to $d\mu$

$$\forall k = 1, \dots, p, \quad \int_M X_k(f) d\mu = 0, \quad \forall f \in C^\infty(M) \quad (1.1)$$

and that they satisfy the Hörmander condition

$$\forall x \in M, \quad \mathcal{G}_x = T_x M. \quad (1.2)$$

Let $\mathfrak{r} \in \mathbb{N}$ be the smallest integer such that for any $x \in M$, \mathcal{G}_x is generated by commutators of length at most \mathfrak{r} . For $k = 1, \dots, p$ and $x_0 \in M$, denote $\mathbb{R} \ni t \mapsto e^{tX_k} x_0$ the integral curve of X_k starting from x_0 at $t = 0$.

Let $h \in]0, h_0]$ be a small parameter. Let us consider the following simple random walk $x_0, x_1, \dots, x_n, \dots$ on M , starting at $x_0 \in M$: at step n , choose $j \in \{1, \dots, p\}$ at random and $t \in [-h, h]$ at random (uniform), and set $x_{n+1} = e^{tX_j} x_n$.

Due to the condition $\text{div}(X_j) = 0$, this random walk is reversible for the probability μ on M . It is easy to compute the Markov operator T_h associated to this random walk: for any bounded and measurable function $f : M \rightarrow \mathbb{R}$ define

$$T_{k,h} f(x) = \frac{1}{2h} \int_{-h}^h f(e^{tX_k} x) dt \quad (1.3)$$

Since the vector fields X_k are divergence free, for any f, g , we have

$$\int_M T_{k,h} f(x) g(x) d\mu = \int_M f(x) T_{k,h} g(x) d\mu.$$

and the Markov operator associated to our random walk is

$$T_h f(x) = \frac{1}{p} \sum_{k=1}^p T_{k,h} f(x) \quad (1.4)$$

One has $T_h(1) = 1$, $\|T_h\|_{L^\infty \rightarrow L^\infty} = 1$, and T_h can be uniquely extended as a bounded self-adjoint operator on $L^2 = L^2(M, d\mu)$ such that $\|T_h\|_{L^2 \rightarrow L^2} = 1$. In the following, we will denote $t_h(x, dy)$ the distribution kernel of T_h , and t_h^n the kernel of T_h^n . Then, by construction, the probability for the walk starting at x_0 to be in a Borel set A after n step is equal to

$$P(x_n \in A) = \int_A t_h^n(x_0, dy)$$

The goal of this paper is to study the spectral theory of the operator T_h and the convergence of $t_h^n(x_0, dy)$ towards μ as n tends to infinity. Since T_h is Markov and self adjoint, its spectrum is a subset of $[-1, 1]$. We shall denote by $g(h)$ the spectral gap of the operator T_h . It is defined as the best constant such that the following inequality holds true for all $u \in L^2$

$$\|u\|_{L^2}^2 - \langle u, 1 \rangle_{L^2}^2 \leq \frac{1}{g(h)} \langle u - T_h u, u \rangle_{L^2} \quad (1.5)$$

The existence of a non zero spectral gap means that 1 is a simple eigenvalue of T_h , and the distance between 1 and the rest of the spectrum is equal to $g(h)$. Our first result is the following

Theorem 1.1 *There exists $h_0 > 0$, $\delta_1, \delta_2 > 0$, $A > 0$, and constants $C_i > 0$ such that for any $h \in]0, h_0]$, the following holds true.*

i) The spectrum of T_h is a subset of $[-1 + \delta_1, 1]$, 1 is a simple eigenvalue of T_h , and $\text{Spec}(T_h) \cap [1 - \delta_2, 1]$ is discrete. Moreover, for any $0 \leq \lambda \leq \delta_2 h^{-2}$, the number of eigenvalues of T_h in $[1 - h^2 \lambda, 1]$ (with multiplicity) is bounded by $C_1(1 + \lambda)^A$.

ii) The spectral gap satisfies

$$C_2 h^2 \leq g(h) \leq C_3 h^2 \quad (1.6)$$

and the following estimate holds true for all integer n

$$\sup_{x \in \Omega} \|t_h^n(x, dy) - \mu\|_{TV} \leq C_4 e^{-ng(h)} \quad (1.7)$$

Here, for two probabilities on M , $\|\nu - \mu\|_{TV} = \sup_A |\nu(A) - \mu(A)|$ where the sup is over all Borel sets A , is the total variation distance between ν and μ .

We describe now the spectrum of T_h near 1. Let $\mathcal{H}^1(\mathcal{X})$ be the Hilbert space

$$\mathcal{H}^1(\mathcal{X}) = \{u \in L^2(M), \forall j = 1, \dots, p, X_j u \in L^2(M)\}$$

Let ν be the best constant such that the following inequality holds true for all $u \in \mathcal{H}^1(\mathcal{X})$

$$\|u\|_{L^2}^2 - \langle u, 1 \rangle_{L^2}^2 \leq \frac{\mathcal{E}(u)}{\nu}, \quad (1.8)$$

where

$$\mathcal{E}(u) = \frac{1}{6} \int_M \sum_{k=1}^p |X_k u|^2 d\mu \quad (1.9)$$

By the hypoelliptic theorem of Hörmander (see [Hör85], Vol 3), one has $\mathcal{H}^1(\mathcal{X}) \subset H^s(M)$, for some $s > 0$. On the other hand, standard Taylor expansion in formula (1.3) show that for any fixed smooth function $g \in C^\infty(M)$, one has the following convergence in the space $C^\infty(M)$

$$\lim_{h \rightarrow 0} \frac{1 - T_h}{h^2} g = L(g), \quad (1.10)$$

where the operator $L = -\frac{1}{6p} \sum_k X_k^2$ is the positive Laplacian associated to the Dirichlet form $\mathcal{E}(u)$. It has a compact resolvent and spectrum $\nu_0 = 0 < \nu_1 = \nu < \nu_2 < \dots$. Let m_j be the multiplicity of ν_j . One has $m_0 = 1$ since $\text{Ker}(L)$ is spanned by the constant function 1 thanks to the Chow theorem ([Cho39]). In fact, for any $x, y \in M$ there exists a continuous curve connecting x to y which is a finite union of pieces of trajectory of one of the fields X_j .

Theorem 1.2 *One has*

$$\lim_{h \rightarrow 0} h^{-2} g(h) = \nu \quad (1.11)$$

Moreover, for any $R > 0$ and $\varepsilon > 0$ such that the intervals $[\nu_j - \varepsilon, \nu_j + \varepsilon]$ are disjoint for $\nu_j \leq R$, there exists $h_1 > 0$ such that for all $h \in]0, h_1[$

$$\text{Spec}\left(\frac{1 - T_h}{h^2}\right) \cap]0, R] \subset \cup_{j \geq 1} [\nu_j - \varepsilon, \nu_j + \varepsilon] \quad (1.12)$$

and the number of eigenvalues of $\frac{1 - T_h}{h^2}$ with multiplicities, in the interval $[\nu_j - \varepsilon, \nu_j + \varepsilon]$, is equal to m_j .

The paper is organized as follows:

In section 2, we recall some basic facts on nilpotent Lie groups, and we recall the Goodman version (see [Goo78]) of one of the main results of the Rotschild-Stein paper.

In section 3, the main result is the proposition 3.1 which gives a lower bound on a suitable power T_h^P of T_h . This in particular allows to get a first crude but fundamental bound on the L^∞ norms of eigenfunctions of T_h associated to eigenvalues close to 1.

Section 4 is devoted to the study of the Dirichlet form associated to our random walk. The fundamental result of this section is proposition 4.1. It allows to separate clearly the spectral theory of T_h in low and high frequencies with respect to the parameter h . Many tools in this section are essentially an adaptation to the semi-classical setting of the ideas contained in the Rotschild-Stein paper [RS76].

Section 5 is devoted to the proof of theorems 1.1 and 1.2. With propositions 3.1 and 4.1 in hands, the proof follows the strategy of [DLM11] and [DLM12], but with some differences and simplifications. This section contains also a paragraph on the Fourier analysis associated to T_h that will be useful in 6.

Section 6 is devoted to the proof of the convergence when $h \rightarrow 0$ of our Markov chain to the hypoelliptic diffusion on the manifold M associated to the generator $L = \frac{-1}{6p} \sum_k X_k^2$. This is probably a well known result for specialists, but we have not succeed to find a precise reference. Since this convergence follows as a simple byproduct of our estimates, we decide to include it in the paper.

Finally, the appendix contains two lemmas. Lemma 7.1 shows how to deduce from proposition 4.1 a Weyl type estimate on the eigenvalues of T_h in a neighborhood of 1. Lemma 7.2 is an elementary cohomological lemma on the Schwartz space of the nilpotent Lie algebra \mathcal{N} .

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2 The lifted operator to a nilpotent Lie algebra

We will use the notation $\mathbb{N}_q = \{1, \dots, q\}$. For any family of vector fields Z_1, \dots, Z_p and any multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_p^k$ denote $|\alpha| = k$ the length of α and let

$$Z^\alpha = H_\alpha(Z_1, \dots, Z_p) = [Z_{\alpha_1}, [Z_{\alpha_2}, \dots [Z_{\alpha_{k-1}}, Z_{\alpha_k}] \dots]] \quad (2.1)$$

Let $\mathcal{Y}_1, \dots, \mathcal{Y}_p$ be a system of generators of the free lie algebra with p generators \mathcal{F} and let \mathcal{A}^∞ be a set of multi-indexes such that $(Y^\alpha)_{\alpha \in \mathcal{A}^\infty}$ is a basis of \mathcal{F} .

Introduce \mathcal{N} the free up to step \mathfrak{r} nilpotent Lie algebra generated by p elements Y_1, \dots, Y_p , and let N be the corresponding simply connected Lie group. We have the decomposition

$$\mathcal{N} = \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_\mathfrak{r} \quad (2.2)$$

where \mathcal{N}_1 is generated by Y_1, \dots, Y_p and \mathcal{N}_j is spanned by the commutators $Y^\alpha = H_\alpha(Y_1, \dots, Y_p)$ with $|\alpha| = j$ for $2 \leq j \leq \mathfrak{r}$. Let $\mathcal{A} = \{\alpha \in \mathcal{A}^\infty, |\alpha| \leq \mathfrak{r}\}$ and $\mathcal{A}_r = \{\alpha \in \mathcal{A}, |\alpha| = r\}$. The family $(Y^\alpha)_{\alpha \in \mathcal{A}}$ is a basis for \mathcal{N} and for any $r \in \mathbb{N}_\mathfrak{r}$, $\{Y^\alpha, \alpha \in \mathcal{A}_r\}$ is a basis of \mathcal{N}_r . We denote by $D = \#\mathcal{A}$ the dimension of \mathcal{N} . The action of \mathbb{R}_+ on \mathcal{N} is given by

$$t.(v_1, v_2, \dots, v_r) = (tv_1, t^2v_2, \dots, t^\mathfrak{r}v_\mathfrak{r})$$

An homogeneous norm $|v|$ which is smooth in $\mathcal{N} \setminus o_{\mathcal{N}}$ is given by

$$|v| = \left(\sum_j |v_j|^{(2\mathfrak{r})/j} \right)^{1/(2\mathfrak{r})}$$

where $|v_j|$ is an euclidian norm on \mathcal{N}_j , and

$$Q = \sum_j j \dim(\mathcal{N}_j)$$

is the quasi homogeneous dimension of \mathcal{N} . We will identify the Lie algebra \mathcal{N} with the Lie group N by the exponential map, i.e the product law $a.b$ on \mathcal{N} is given by $\exp(a.b) = \exp(a)\exp(b)$. In particular, one has with this identification $a^{-1} = -a$ for all $a \in \mathcal{N}$. To avoid notational confusion, we will use sometime the notation $e = o_{\mathcal{N}}$, so that $a.e = e.a = a$ for all $a \in \mathcal{N}$. For $Y \in T_e\mathcal{N} \simeq \mathcal{N}$, we denote by \tilde{Y} the left invariant vector field on \mathcal{N} such that $\tilde{Y}(o_{\mathcal{N}}) = Y$, i.e

$$\tilde{Y}(f)(x) = \frac{d}{ds}(f(x.sY))|_{s=0}$$

The right invariant vector field on \mathcal{N} such that $Z(o_{\mathcal{N}}) = Y$ is defined by

$$Z(f)(x) = \frac{d}{ds}(f(sY.x))|_{s=0}$$

Here, sY is the usual product of the vector $Y \in \mathcal{N}$ by the scalar $s \in \mathbb{R}$. For $a \in \mathcal{N}$, let τ_a be the diffeomorphism of \mathcal{N} defined by $\tau_a(u) = a.u$. One has

$$\tilde{Y}(a) = d\tau_a(e)(Y)$$

Example 2.1 *The standard 3d-Heisenberg group is $\mathcal{N} = \mathbb{R}_{x,y}^2 \oplus \mathbb{R}_t \simeq \mathbb{R}^3$, with the product law*

$$(x, y, t).(x', y', t') = (x + x', y + y', t + t' + xy' - yx')$$

and the left invariant vector fields associated respectively to the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are in that case

$$\tilde{Y}_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial t}, \quad \tilde{Y}_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial t}, \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{1}{2}[\tilde{Y}_1, \tilde{Y}_2]$$

In general, for $x = (x_1, \dots, x_\tau)$ and $y = (y_1, \dots, y_\tau)$, $x_j, y_j \in \mathcal{N}_j$, the product law is given by

$$\begin{aligned} (x_1, \dots, x_\tau) \cdot (y_1, \dots, y_\tau) &= (z_1, \dots, z_\tau) \\ z_j &= x_j + y_j + P_j(x_{<j}, y_{<j}) \end{aligned} \quad (2.3)$$

with the notation $x_{<j} = (x_1, \dots, x_{j-1})$, and where P_j is a polynomial of degree j with respect to the homogeneity on \mathcal{N} , i.e

$$P_j((t.x)_{<j}, (t.y)_{<j}) = t^j P_j(x_{<j}, y_{<j})$$

which is compatible with the identity $t.(x.y) = (t.x).(t.y)$.

Let $\lambda : \mathcal{N} \rightarrow \mathcal{G}$ be the unique linear map such that for any $\alpha \in \mathcal{A}$, $\lambda(Y^\alpha) = X^\alpha$. Then λ is a Lie homomorphism ‘‘up to step τ ’’:

$$\lambda([Y^\alpha, Y^\beta]) = [X^\alpha, X^\beta] \quad (2.4)$$

for any multi-indexes α, β such that $|\alpha| + |\beta| \leq \tau$.

Let $x_0 \in M$. There exists a subset $\mathcal{A}_{x_0} \subset \mathcal{A}$ such that $(X^\alpha(x))_{\alpha \in \mathcal{A}_{x_0}}$ is a basis of $T_x M$ for any x close to x_0 . Therefore, there exists a neighborhood Ω_0 of the origin $o_{\mathcal{N}}$ in \mathcal{N} and a neighborhood V_0 of x_0 in M such that the map Λ

$$\Lambda : u = \sum_{\alpha \in \mathcal{A}} u_\alpha Y^\alpha \in \Omega_0 \mapsto e^{\lambda(u)} x_0 = e^{\sum_{\alpha \in \mathcal{A}} u_\alpha X^\alpha} x_0$$

is a submersion from Ω_0 onto V_0 , and the map $W_{x_0} : \mathcal{C}^\infty(V_0) \rightarrow \mathcal{C}^\infty(\Omega_0)$ defined by $W_{x_0} f(u) = f(e^{\lambda(u)} x_0)$ is injective. Since Λ is a submersion, there exists a system of coordinates $\theta : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathcal{N}$ defined near $o_{\mathcal{N}}$, where $m + n = D$, such that $\Lambda \theta : \mathbb{R}^m \rightarrow M$ is a system of coordinates near x_0 and in these coordinates one has $\Lambda(x, y) = x$. We thus may assume that in these coordinates one has $\Omega_0 = V_0 \times U_0$ where U_0 is a neighborhood of $0 \in \mathbb{R}^n$.

Example 2.2 Take for example the two vectors fields in \mathbb{R}^2 , $X_1 = \partial_x$, $X_2 = x\partial_y$ so that $[X_1, X_2] = \partial_y$. Then on take for \mathcal{N} the 3d-Heisenberg group, and the map λ is given by, with $T = 2\partial_t = [Y_1, Y_2]$

$$\lambda(u_1 Y_1 + u_2 Y_2 + u_3 T) = u_1 X_1 + u_2 X_2 + u_3 [X_1, X_2] = u_1 \partial_x + (u_3 + u_2 x) \partial_y$$

Thus we get

$$e^{\lambda(u)}(x, y) = (x + u_1, y + u_3 + u_2 x + \frac{1}{2} u_1 u_2) \quad (2.5)$$

Let $I_h = \{|u_1| < h, |u_2| < h, |u_3| < h^2\}$. One has $\text{Vol}(I_{\epsilon, h}) = 8h^4$. Observe on this example that the set $\tilde{B}_{h, (x, y)} = \{e^{\lambda(u)}(x, y), u \in I_h\}$, when (x, y) is fixed and h small, has volume of order: h^2 when $x \neq 0$, and h^3 when $x = 0$.

Let us now recall the notion of order of a vector field used in [RS76] and [Goo78]. Denote $\{\delta_t\}_{t>0}$ the one parameter group of dilating automorphisms on \mathcal{N} :

$$\delta_t Y^\alpha = t^{|\alpha|} Y^\alpha.$$

Let Ω be a compact neighborhood of $o_{\mathcal{N}}$ in \mathcal{N} . For any $m \in \mathbb{N}$, let

$$C_m^\infty = \{u \in C^\infty(\Omega, \mathbb{R}), f(u) = \mathcal{O}(|u|^m)\}.$$

We have the filtration $C^\infty(\Omega) = C_0^\infty \supseteq C_1^\infty \supseteq \dots$, and $C_m^\infty \cdot C_n^\infty \subseteq C_{m+n}^\infty$. Let $T : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$. We say that T is of order less than k at 0, if $T(C_m^\infty) \subseteq C_{m-k}^\infty$ for all integers $m \geq 0$. If ∂_α denotes the differentiation in the direction Y^α , then a vector field $T = \sum_\alpha \varphi_\alpha \partial_\alpha$ is of order $\leq k$ iff $\varphi_\alpha \in C_{|\alpha|-k}^\infty$ for all α , with the convention $C_m^\infty = C_0^\infty$ for $m \leq 0$.

The following result is the Goodman version of one of the results of the article [RS76] by L.Rothschild and E.Stein.

Theorem 2.3 *Decreasing Ω_0 if necessary, there exists C^∞ vector fields Z_1, \dots, Z_p on Ω_0 such that for any $\alpha \in \mathcal{A}$ we have*

- i) $Z^\alpha W_{x_0} = W_{x_0} X^\alpha$
- ii) $Z^\alpha = \tilde{Y}^\alpha + R_\alpha$, where R_α is a vector field of order $\leq |\alpha| - 1$ at 0.
- iii) $Z^\alpha = H_\alpha(Z_1, \dots, Z_p)$ for all $\alpha \in \mathcal{A}$.

Observe that in the previous coordinate system (x, y) on Ω_0 , one can write for $\alpha \in \mathcal{A}$

$$X^\alpha = \sum_j a_{\alpha,j}(x) \frac{\partial}{\partial x_j}, \quad Z^\alpha = \sum_j a_{\alpha,j}(x) \frac{\partial}{\partial x_j} + \sum_l b_{\alpha,l}(x, y) \frac{\partial}{\partial y_l} \quad (2.6)$$

As an obvious consequence of this theorem, we have the following, with $W = W_{x_0}$, and $\tilde{\lambda}(u) = \sum_{\alpha \in \mathcal{A}} u_\alpha Z^\alpha$.

Proposition 2.4 *Let $f \in C^0(V_0)$ and let $\omega_0 \subset\subset \Omega_0$ be a neighborhood of $o_{\mathcal{N}}$. Then, there exists $r_0 > 0$ such that for all $|u| \leq r_0$, and $v \in \omega_0$, we have*

$$(Wf)(e^{\tilde{\lambda}(u)}v) = W(f_u)(v) \quad (2.7)$$

where the function f_u is defined near x_0 by $f_u(x) = f(e^{\lambda(u)}x)$.

Using this proposition, we can easily compute the action of W on the operator T_h acting on functions with support close to x_0 . We get immediatly

$$WT_h = \tilde{T}_h W, \quad \tilde{T}_h = \frac{1}{p} \sum_{k=1}^p \tilde{T}_{k,h} \quad (2.8)$$

where for $u \in \mathcal{N}$ small.

$$\tilde{T}_{k,h} g(u) = \frac{1}{2h} \int_{-h}^h g(e^{tZ_k} u) dt \quad (2.9)$$

Using the notation $T^\alpha = T_{\alpha_k,h} \dots T_{\alpha_1,h}$ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_k)$ we get for any $u \in \mathcal{N}$ close to $o_{\mathcal{N}}$ such that $\Lambda(u) = x$

$$T^\alpha f(x) = W(T^\alpha f)(u) = \frac{1}{(2h)^k} \int_{[-h,h]^k} (Wf)(e^{t_1 Z_{\alpha_1}} \dots e^{t_k Z_{\alpha_k}} u) dt_1 \dots dt_k \quad (2.10)$$

3 Rough bounds on eigenfunctions

Let us recall from section 2, that for $u = \sum_{\alpha \in \mathcal{A}} u_\alpha Y^\alpha \in \mathcal{N}$, the vector field $\lambda(u)$ on M is defined by $\lambda(u) = \sum_{\alpha \in \mathcal{A}} u_\alpha X^\alpha$. Let $\epsilon > 0$ and $I_{\epsilon,h}$ be the neighborhood of $o_{\mathcal{N}}$ in \mathcal{N} defined by

$$I_{\epsilon,h} = \left\{ u = \sum_{\alpha \in \mathcal{A}} u_\alpha Y^\alpha, \quad u_\alpha \in]-\epsilon h^{|\alpha|}, \epsilon h^{|\alpha|}[\right\}$$

For any $x \in M$ we define a positive measure $S_h^\epsilon(x, dy)$ on M by the formula

$$\forall f \in C^0(M), \quad \int f(y) S_h^\epsilon(x, dy) = h^{-Q} \int_{u \in I_{\epsilon, h}} f(e^{\lambda(u)} x) du \quad (3.1)$$

where $du = \Pi_\alpha du_\alpha$ is the left and right invariant Haar measure on \mathcal{N} . Let us introduce the numerical sequence $(b_n)_{n \in \mathbb{N}^*}$ defined by $b_1 = 1$ and $b_{n+1} = 2b_n + 2$, so that for all $n \in \mathbb{N}^*$, we have $b_n = 3 \cdot 2^{n-1} - 2$. For all $r = 1, \dots, \mathfrak{r}$ denote $a_r = \sharp \mathcal{A}_r = \dim \mathcal{N}_r$, and let $P = \sum_{r=1}^{\mathfrak{r}} a_r b_r$.

Proposition 3.1 *There exists $\epsilon > 0$, $c > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $x \in M$*

$$t_h^P(x, dy) = \rho_h(x, dy) + c S_h^\epsilon(x, dy) \quad (3.2)$$

where $\rho_h(x, dy)$ is a non-negative Borel measure on M for all $x \in M$.

Remark 3.2 *As in [DLM11], one can deduce from proposition 3.1 that the inequality (3.2) holds true for $t_h^N(x, dy)$ as soon as $N \geq P$, eventually with different constants $\epsilon > 0$, $c > 0$ and $h_0 > 0$ depending on N .*

Before proving this proposition, let us give two simple but fundamental corollaries. Like in [DLM11], these two corollaries will play a key role in the proof of theorems 1.1 and 1.2. Here, we use the same notation for a bounded measurable family in x of non negative Borel measure $k(x, dy)$ and the corresponding operator $f \mapsto K(f)(x) = \int f(y) k(x, dy)$ acting on L^∞ .

Corollary 3.3 *There exists $h_0 > 0$ and $\gamma < 1$ such that for all $h \in]0, h_0]$ and all $x \in M$*

$$\|\rho_h(x, dy)\|_{L^\infty \rightarrow L^\infty} \leq \gamma < 1 \quad (3.3)$$

Proof. By definition, the non-negative measure ρ_h is given by $\rho_h(x, dy) = t_h^P(x, dy) - c S_h^\epsilon(x, dy)$. Therefore

$$\left| \int_M f(x) d\rho_h(x, dy) \right| \leq \|f\|_{L^\infty} \int_M d\rho_h(x, dy) \leq \|f\|_{L^\infty} (1 - c \inf_{x \in M} \int_M S_h^\epsilon(x, dy)) \quad (3.4)$$

since $t_h^P(x, dy)$ is a Markov kernel. From (3.1), one has $\int_M S_h^\epsilon(x, dy) = h^{-Q} \text{meas}(I_{\epsilon, h}) = (2\epsilon)^D$. Combined with (3.4), this implies the result. \square

Corollary 3.4 *Let $a \in]\gamma^{\frac{1}{P}}, 1]$ be fixed. There exists $C = C_a > 0$ such that for any $\lambda \in [a, 1]$ and any $f \in L^2(M, d\mu)$ we have*

$$T_h f = \lambda f \implies \|f\|_{L^\infty} \leq C h^{-\frac{Q}{2}} \|f\|_{L^2} \quad (3.5)$$

Proof. Suppose $T_h f = \lambda f$, then $T_h^P f = \lambda^P f$. Hence, $S_h^\epsilon f = \lambda^P f - \rho_h(f)$ and then

$$\|S_h^\epsilon f\|_{L^\infty} \geq \lambda^P \|f\|_{L^\infty} - \gamma \|f\|_{L^\infty} \geq c_a \|f\|_{L^\infty} \quad (3.6)$$

with $c_a = a^P - \gamma$. On the other hand, since $u \mapsto e^{\lambda(u)} x$ is a submersion from a neighborhood of $o_{\mathcal{N}} \in \mathcal{N}$ onto a neighborhood of $x \in M$, we get by Cauchy-Schwarz inequality

$$|S_h^\epsilon f(x)| \leq h^{-Q} \text{meas}(I_{\epsilon, h})^{1/2} \left(\int_{u \in I_{\epsilon, h}} |f(e^{\lambda(u)} x)|^2 du \right)^{1/2} \leq C h^{-Q/2} \|f\|_{L^2(M)} \quad (3.7)$$

Putting together (3.6) and (3.7), we obtain the announced result. \square

Let us now prove Proposition 3.1. We have to show that there exists $c, \epsilon > 0$ independent of h small, such that for any non negative continuous function f on M , one has $T_h^P f(x) \geq c S_h^\epsilon f(x)$. Since M is compact and the operator T_h moves supports of functions at distance at most h , we can assume without loss of generality that f is supported near some point $x_0 \in M$ where we can apply the results of section 2. Recall $\tilde{\lambda}(u) = \sum_{\alpha \in \mathcal{A}} u_\alpha Z^\alpha$. From proposition 2.4 one has $f(e^{\lambda(w)}x) = W(f)(e^{\tilde{\lambda}(w)}w)$ for any w close to $o_{\mathcal{N}}$ such that $\Lambda(w) = x$. Using also (2.8), we are thus reduced to prove that there exists $c, \epsilon > 0$ independent of h small such that for any non negative continuous function g on \mathcal{N} supported near $o_{\mathcal{N}}$, one has

$$\tilde{T}_h^P g(w) \geq ch^{-Q} \int_{u \in I_{\epsilon, h}} g(e^{\tilde{\lambda}(u)}w) du \quad (3.8)$$

For any non-commutative sequence (A_k) of operators, we denote $\prod_{k=1}^K A_k = A_K \dots A_1$ (i.e A_1 is the first operator acting). Endowing \mathcal{A}_r with the lexicographical order, we can write $\mathcal{A}_r = \{\alpha_1 < \dots < \alpha_{a_r}\}$ and for any non-commutative sequence (B_α) indexed by \mathcal{A} , we define $\prod_{\alpha \in \mathcal{A}_r} B_\alpha = \prod_{j=1}^{a_r} B_{\alpha_j}$ and $\prod_{\alpha \in \mathcal{A}} B_\alpha = \prod_{r=1}^r \prod_{\alpha \in \mathcal{A}_r} B_\alpha$.

Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_p^k$, and $t = (t_1, \dots, t_k) \in \mathbb{R}^k$ close to 0. One defines by induction on $|\alpha|$ a smooth diffeomorphism $\phi_\alpha(t)$ of \mathcal{N} near $o_{\mathcal{N}}$, with $\phi_\alpha(0) = Id$, by the following formulas: If $|\alpha| = 1$ and $\alpha = j \in \{1, \dots, p\}$, set $\phi_\alpha(t)(w) = e^{tZ_j}w$. If $|\alpha| = k \geq 2$, set $\alpha = (j, \beta)$, with $\beta \in \mathbb{N}_p^{k-1}$ and $t = (t_1, t')$ with $t' \in \mathbb{R}^{k-1}$ and set

$$\phi_\alpha(t) = \phi_\beta^{-1}(t') e^{-t_1 Z_j} \phi_\beta(t') e^{t_1 Z_j} \quad (3.9)$$

Observe that $\phi_\alpha(t) = Id$ if one of the t_j is equal to 0. The map $(t, w) \mapsto \phi_\alpha(t)(w)$ is smooth, and one has in local coordinates on \mathcal{N} and for t close to 0

$$\phi_\alpha(t)(w) = w + (\prod_{1 \leq l \leq |\alpha|} t_l) Z^\alpha(w) + r_\alpha(t, w) \quad (3.10)$$

with $r_\alpha(t, w) \in (\prod_{1 \leq l \leq |\alpha|} t_l) O(|t|)$. From (3.9), one gets easily by induction on k the following lemma.

Lemma 3.5 *For $2 \leq k \leq \mathfrak{r}$, there exists maps:*

$$\epsilon_k : \{1, \dots, b_k\} \rightarrow \{\pm 1\}, \quad \ell_k : \{1, \dots, b_k\} \rightarrow \{1, \dots, k\}, \quad j_k : \{1, \dots, b_k\} \rightarrow \{1, \dots, p\}$$

such that $\epsilon_k(1) = 1$, $\epsilon_k(b_k/2) = -1$, $\ell_k(1) = 1$, $\ell_k(b_k/2) = 1$, $\# \ell_k^{-1}(j) = 2^j$ for $j \leq k-1$, $\# \ell_k^{-1}(k) = 2^{k-1}$, $j_k(m) = \alpha_{\ell_k(m)}$, and such that for all $t = (t_1, \dots, t_k)$ one has

$$\phi_\alpha(t) = \prod_{m=1}^{b_k} e^{\epsilon_k(m) t_{\ell_k(m)} Z_{j_k(m)}} \quad (3.11)$$

Since g is non negative, one has

$$\tilde{T}_h^P g(w) \geq \frac{1}{p^P} \prod_{\alpha \in \mathcal{A}} \prod_{k=1}^{b_{|\alpha|}} T_{j_{|\alpha|}(k), h} g(w) \quad (3.12)$$

Therefore, we are reduced to prove that there exists $\epsilon, c > 0$ independent of h small and w near $o_{\mathcal{N}}$ such that the following inequality holds true.

$$h^{-P} \int_{[-h, h]^P} g\left(\prod_{\alpha \in \mathcal{A}} \prod_{k=1}^{b_{|\alpha|}} e^{t_{|\alpha|, k} Z_{j_{|\alpha|}(k)}} w\right) dt \geq ch^{-Q} \int_{z \in I_{\epsilon, h}} g(e^{\tilde{\lambda}(z)}w) dz \quad (3.13)$$

Let $\Phi_w : \mathbb{R}^P \rightarrow \mathcal{N}$ be the smooth map defined for $s = (s_{\alpha,k})_{\alpha \in \mathcal{A}, k=1, \dots, b_{|\alpha|}} \in \mathbb{R}^P$ by the formula

$$\Phi_w(s) = \left(\prod_{r=1}^r \prod_{\alpha \in \mathcal{A}_r} \prod_{k=1}^{b_r} e^{s_{\alpha,k} Z_{j_{|\alpha|}(k)}} \right) w \quad (3.14)$$

Since $(Z^\beta(w))_{\beta \in \mathcal{A}}$ is a basis of $T_w \mathcal{N}$, $u = (u_\beta)_{\beta \in \mathcal{A}} \mapsto e^{\sum_{\beta \in \mathcal{A}} u_\beta Z^\beta} w$ is a local coordinate system centered at $w \in \mathcal{N}$, and therefore, there exists smooth functions $U_{\beta,w}(s)$ such that

$$\Phi_w(s) = e^{\sum_{\beta \in \mathcal{A}} U_{\beta,w}(s) Z^\beta} w \quad (3.15)$$

Moreover, it follows easily from the Campbell-Hausdorff formula, that one has $U_{\beta,w}(s) \in O(|s|^{|\beta|})$ near $s = 0$. Let now $\kappa : \mathbb{R}^Q \rightarrow \mathbb{R}^P$ the map defined by

$$(t_{\alpha,l})_{\alpha \in \mathcal{A}, l \in \mathbb{N}_{|\alpha|}} \mapsto (\epsilon_\alpha(k) t_{\alpha, \ell_{|\alpha|}(k)})_{\alpha \in \mathcal{A}, k=1, \dots, b_{|\alpha|}} \quad (3.16)$$

Then, from lemma 3.5 we have the following identity for any $t = (t_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^Q$.

$$\Phi_w \circ \kappa(t) = \prod_{\alpha \in \mathcal{A}} \phi_\alpha(t_\alpha) w \quad (3.17)$$

From (3.10) and Campbell-Hausdorff formula, one gets

$$\begin{aligned} \prod_{\alpha \in \mathcal{A}} \phi_\alpha(t_\alpha) w &= e^{\sum_{\beta \in \mathcal{A}} f_\beta(t) Z^\beta} w \\ f_\beta(t) &= \prod_{1 \leq l \leq |\beta|} t_{\beta,l} + g_\beta((t_\gamma)_{|\gamma| < |\beta|}) + r_\beta(t) \end{aligned} \quad (3.18)$$

with g_β an homogeneous polynomial of degree $|\beta|$ depending only on $(t_\gamma)_{|\gamma| < |\beta|}$ and $r_\beta(t) \in O(|t|^{|\beta|+1})$. Let $\delta \in]\frac{1}{2}, 1[$ and define $\xi = (\xi_{\alpha,k})_{\alpha \in \mathcal{A}, k \in \mathbb{N}_{|\alpha|}} \in \mathbb{R}^Q$ by $\xi_{\alpha,1} = 0$ and $\xi_{\alpha,k} = \delta h$ for $k = 2, \dots, |\alpha|$. Let $\zeta : \mathbb{R}^D \rightarrow \mathbb{R}^Q$ be the map defined by the formula

$$\begin{aligned} s &= (s_\alpha)_{\alpha \in \mathcal{A}} \mapsto (\zeta_{\alpha,k}(s))_{\alpha \in \mathcal{A}, k \in \mathbb{N}_{|\alpha|}} \\ \zeta_{\alpha,1}(s) &= s_\alpha, \quad \text{and} \quad \zeta_{\alpha,k}(s) = 0 \quad \forall k \geq 2 \end{aligned} \quad (3.19)$$

and let $\sigma : \mathbb{R}^{P-D} \rightarrow \mathbb{R}^P$ be the map defined by the formula

$$\begin{aligned} v &= (v_{\alpha,k})_{\alpha \in \mathcal{A}, k=2, \dots, b_{|\alpha|}} \mapsto (\sigma_{\alpha,k}(v))_{\alpha \in \mathcal{A}, k=1, \dots, b_{|\alpha|}} \\ \sigma_{\alpha,1}(v) &= 0, \quad \text{and} \quad \sigma_{\alpha,k}(v) = v_{\alpha,k} \quad \forall k \neq 1 \end{aligned} \quad (3.20)$$

Set $\hat{\kappa}_\xi(u, v) = \kappa(\zeta(u) + \xi) + \sigma(v)$, and let $\Psi_w : \mathbb{R}^D \times \mathbb{R}^{P-D} \rightarrow \mathcal{N}$ be defined by

$$\Psi_w(u, v) = \Phi_w(\hat{\kappa}_\xi(u, v)) \quad (3.21)$$

Then, it follows from (3.15) that there exists smooth maps $\hat{\varphi}_{\alpha,w}(u, v)$ such that

$$\Psi_w(u, v) = e^{\sum_{\alpha \in \mathcal{A}} \hat{\varphi}_{\alpha,w}(u, v) Z^\alpha} w \quad (3.22)$$

From (3.17) one has

$$\Psi_w(u, 0) = \Phi_w(\kappa(\zeta(u) + \xi)) = \prod_{\alpha \in \mathcal{A}} \phi_\alpha(u_\alpha, \delta h, \dots, \delta h) w$$

and therefore from (3.18) we get, since $\hat{\kappa}_\xi(u, v)$ is linear in ξ, u, v

$$\hat{\varphi}_{\alpha,w}(u, v) = u_\alpha (\delta h)^{|\alpha|-1} + g_{\alpha,w}((u_\gamma)_{|\gamma| < |\alpha|}, \delta h) + p_{\alpha,w}(u, \delta h, v) + q_{\alpha,w}(u, \delta h, v) \quad (3.23)$$

where $g_{\alpha,w}(u, s)$ is a homogenous polynomial of degree $|\alpha|$ depending only on u_γ for $|\gamma| < |\alpha|$, $p_{\alpha,w}(u, s, v)$ is a homogenous polynomial of degree $|\alpha|$ in (u, s, v) such that $p_{\alpha,w}(u, s, 0) = 0$, and $q_{\alpha,w}(u, s, v) \in O((u, s, v)^{1+|\alpha|})$ near $(u, s, v) = (0, 0, 0)$. Moreover, from $\phi_\alpha(0, \delta h, \dots, \delta h) = Id$, one get $g_{\alpha,w}(0, s) = 0$ and also $q_{\alpha,w}(0, s, 0) = 0$. Observe that w is just a smooth parameter in the above constructions. Thus, we will remove the dependance in w in what follows. Define now

$$\begin{aligned} \mathfrak{Q} : \mathbb{R}^P &= \mathbb{R}^D \times \mathbb{R}^{P-D} \longrightarrow \mathbb{R}^P \\ (u, v) &= ((u_\alpha)_{\alpha \in \mathcal{A}}, (v_{\alpha,k})_{\alpha \in \mathcal{A}, k=2, \dots, b_{|\alpha|}}) \mapsto ((\hat{\varphi}_\alpha(u, v))_{\alpha \in \mathcal{A}}, v) \end{aligned} \quad (3.24)$$

and for $\eta, \epsilon > 0$ let

$$\Delta_{\epsilon, \eta} = \{(u, v) = ((u_\alpha)_{\alpha \in \mathcal{A}}, (v_{\alpha,k})_{\alpha \in \mathcal{A}, k=2, \dots, b_{|\alpha|}}) \in \mathbb{R}^P, |u_\alpha| < \epsilon h, \text{ and } |v_{\alpha,k}| < \eta h \text{ for all } \alpha, k\}$$

Lemma 3.6 *Let $\delta \in]\frac{1}{2}, 1[$ be fixed. There exists $0 < \eta \ll \epsilon < 1/2$ and $h_0 > 0$ such that the restriction $\mathfrak{Q}_{\epsilon, \eta}$ of \mathfrak{Q} to $\Delta_{\epsilon, \eta}$ enjoys the following:*

1. *there exists $U_{\epsilon, \eta}$ open neighbourhood of $0 \in \mathbb{R}^P$ such that $\mathfrak{Q}_{\epsilon, \eta} : \Delta_{\epsilon, \eta} \rightarrow U_{\epsilon, \eta}$ is a C^∞ diffeomorphism.*

2. *there exists some constant $C > 0$ such that for all $h \in]0, h_0]$ and all $(u, v) \in \Delta_{\epsilon, \eta}$*

$$h^{Q-D}/C \leq \mathbf{J}\mathfrak{Q}_{\epsilon, \eta}(u, v) := |\det(D_{(u,v)}\mathfrak{Q}_{\epsilon, \eta})| \leq Ch^{Q-D}$$

3. *there exists $M \geq 1$ such that for all $h \in]0, h_0]$ the set $U_{\epsilon, \eta}$ contains $I_{\epsilon/M, h} \times]-\eta h, \eta h[^{P-D}$ where $I_{\epsilon/M, h} = \prod_{\alpha \in \mathcal{A}}]-\epsilon h^{|\alpha|}/M, \epsilon h^{|\alpha|}/M[$.*

Proof. The proof is just a scaling argument. Set $u_\alpha = h\tilde{u}_\alpha$, $v_{\alpha,k} = h\tilde{v}_{\alpha,k}$ and $\hat{\varphi}_\alpha = h^{|\alpha|}z_\alpha$. Then the map \mathfrak{Q} becomes after scaling $\tilde{\mathfrak{Q}} : (\tilde{u}, \tilde{v}) \mapsto (z, \tilde{v})$, and from (3.23) one has

$$z_\alpha = \tilde{u}_\alpha \delta^{|\alpha|-1} + g_\alpha((\tilde{u}_\gamma)_{|\gamma| < |\alpha|}, \delta) + p_\alpha(\tilde{u}, \delta, \tilde{v}) + h\tilde{q}_\alpha(\tilde{u}, \delta, \tilde{v}, h)$$

$p_\alpha(\tilde{u}, \delta, 0) = 0$, $\tilde{q}_\alpha(\tilde{u}, \delta, \tilde{v}, h)$ is smooth and vanish at order $|\alpha| + 1$ at 0 as a function of $(\tilde{u}, \delta, \tilde{v})$ and $g_\alpha(0, \delta) = 0$, $\tilde{q}_\alpha(0, \delta, 0, h) = 0$. From the triangular structure above, it is obvious that $\tilde{\mathfrak{Q}}$ is a smooth diffeomorphism at $0 \in \mathbb{R}^P$, such that $\tilde{\mathfrak{Q}}(0) = 0$. Thus, for $\eta \ll \epsilon$, $h \leq h_0$ small and $M \gg 1$, we get the inclusion $\{|z_\alpha| < \epsilon/M, |\tilde{v}_{\alpha,k}| < \eta\} \subset \tilde{\mathfrak{Q}}(\{|\tilde{u}_\alpha| < \epsilon, |\tilde{v}_{\alpha,k}| < \eta\})$. One has by construction $|\det(D_{(u,v)}\mathfrak{Q})| = h^{Q-D}|\det(D_{(\tilde{u}, \tilde{v})}\tilde{\mathfrak{Q}})|$. The proof of lemma 3.6 is complete. \square

It is now easy to verify that (3.13) holds true. One has $\det D_{(u,v)}\hat{\kappa}_\xi = 1$ for all $(u, v) \in \mathbb{R}^P$ and for $\frac{1}{2} < \delta < 1$, and $0 < \eta \ll \epsilon < 1/2$ there exists some numbers $-1 < \alpha_i < \beta_i < 1$, $i = 1, \dots, P-D$ depending only on ϵ, η, δ and such that $\hat{\kappa}_\xi(\Delta_{\epsilon, \eta})$ is contained in the set $\hat{\Delta}_{\epsilon, \eta} = \{(t, s), t \in [-\epsilon h, \epsilon h]^D, s \in \prod_{i=1}^{P-D} [\alpha_i h, \beta_i h]\}$. Using again the positivity of g and the change of variable $\hat{\kappa}$, we obtain, with a constant $c >$ changing from line to line

$$\begin{aligned} h^{-P} \int_{[-h, h]^P} g(\Phi(t)) dt &\geq h^{-P} \int_{\hat{\Delta}_{\epsilon, \eta}} g(\Phi(t)) dt \geq h^{-P} \int_{\hat{\kappa}_\xi(\Delta_{\epsilon, \eta})} g(\Phi(t)) dt \\ &\geq ch^{-P} \int_{\Delta_{\epsilon, \eta}} g(\Phi \circ \hat{\kappa}_\xi(u, v)) dudv = ch^{-P} \int_{\Delta_{\epsilon, \eta}} g(\Psi(u, v)) dudv \end{aligned} \quad (3.25)$$

Thanks to Lemma 3.6, we can use the change of variable $\mathfrak{Q}_{\epsilon, \eta}$ to get

$$\begin{aligned} h^{-P} \int_{\Delta_{\epsilon, \eta}} g(\Psi(u, v)) dudv &\geq ch^{D-P-Q} \int_{U_{\epsilon, \eta}} g(e^{\sum_{\alpha \in \mathcal{A}} z_\alpha Z^\alpha} w) dz dv \\ &\geq ch^{-Q} \int_{I_{\epsilon', \eta}} g(e^{\sum_{\alpha \in \mathcal{A}} z_\alpha Z^\alpha} w) dz = ch^{-Q} \int_{z \in I_{\epsilon', h}} g(e^{\tilde{\lambda}(z)} w) dz \end{aligned} \quad (3.26)$$

whith $\epsilon' = \epsilon/M$ and M is given by Lemma 3.6. The proof of proposition 3.1 is complete.

4 Dirichlet form

Let \mathcal{E}_h be the rescaled Dirichlet form associated to the Markov kernel T_h

$$0 \leq \mathcal{E}_h(u) = \left(\frac{1 - T_h}{h^2} u | u \right)_{L^2}, \quad u \in L^2(M, d\mu) \quad (4.1)$$

The main result of this section is the following proposition.

Proposition 4.1 *Under the hypoelliptic hypothesis (1.2), there exists $C, h_0 > 0$ such that the following holds true for all $h \in]0, h_0]$: for all $u \in L^2(M, d\mu)$ such that*

$$\|u\|_{L^2}^2 + \mathcal{E}_h(u) \leq 1 \quad (4.2)$$

there exists $v_h \in \mathcal{H}^1(\mathcal{X})$ and $w_h \in L^2$ such that

$$u = v_h + w_h, \quad \forall j, \|X_j v_h\|_{L^2} \leq C, \quad \|w_h\|_{L^2} \leq Ch \quad (4.3)$$

This proposition is easy to prove when the vector fields X_j span the tangent bundle at each point, by elementary Fourier analysis. Under the hypoelliptic hypothesis, the proof is more involved, and will be done in several steps.

Step 1: Localization and reduction to the nilpotent Lie algebra.

Let us first verify that for all $\varphi \in C^\infty(M)$, there exists C_φ independent of $h \in]0, 1]$ such that

$$\mathcal{E}_h(\varphi u) \leq C_\varphi (\|u\|_{L^2}^2 + \mathcal{E}_h(u)) \quad (4.4)$$

One has $1 - T_h = \frac{1}{p} \sum_{k=1}^p (1 - T_{k,h})$ and

$$2((1 - T_{k,h})u|u) = \int_M \frac{1}{2h} \int_{-h}^h |u(x) - u(e^{tX_k}x)|^2 dt d\mu(x)$$

Since $\sup_{x \in M} |\varphi(x) - \varphi(e^{tX_k}x)| \leq C|t|$, this implies for some constant C_φ and all k

$$((1 - T_{k,h})\varphi u | \varphi u) \leq C_\varphi ((1 - T_{k,h})u | u) + h^2 \|u\|_{L^2}^2$$

and therefore, (4.4) holds true. Thus, in the proof of proposition 4.1, we may assume that $u \in L^2(M, d\mu)$ is supported in a small neighborhood of a given point $x_0 \in M$ where theorem 2.3 applies. More precisely, with the notations of section 2, we may assume in the coordinate system $\Lambda\theta$ centered at $x_0 \simeq 0$ that u is supported in the closed ball $B_r^m = \{x \in \mathbb{R}^m, |x| \leq r\} \subset V_0$. Let $\chi(y) \in C_0^\infty(U_0)$ with support in $B_r^n \subset U_0$, such that $\int \chi(y) dy = 1$. Set $g(x, y) = \chi(y)u(x)$. One has $g(x, y) = \chi(y)W_{x_0}(u)(x, y)$. By hypothesis, one has

$$\|u\|_{L^2}^2 + \mathcal{E}_h(u) \leq 1$$

This implies for all k

$$2((1 - T_{k,h})u|u) = \int_M \frac{1}{2h} \int_{-h}^h |u(x) - u(e^{tX_k}x)|^2 dt d\mu(x) \leq ph^2$$

Thus, for any compact $K \subset U_0$, there exist C_K such that for all k and $h \in]0, h_0]$, one has

$$\int_{V_0 \times K} \frac{1}{2h} \int_{-h}^h |u(x) - u(e^{tX_k}x)|^2 dt dx dy \leq C_K h^2 \quad (4.5)$$

Here, h_0 is small enough so that $e^{tX_k}x$ remains in V_0 for $|t| \leq h_0$ and $x \in B_r$. Let $\phi(x, y) = \chi(y)$. One has $\sup_{x,y} |\phi(x, y) - \phi(e^{tZ_k}(x, y))| \leq C|t|$ and $\|g\|_{L^2} \leq C$. Thus, decreasing h_0 , we get from (4.5) that there exists a constant C independent of k and $h \in]0, h_0]$ such that

$$\int_{V_0 \times U_0} \frac{1}{2h} \int_{-h}^h |g(x, y) - g(e^{tZ_k}(x, y))|^2 dt dx dy \leq Ch^2 \quad (4.6)$$

Therefore, there exists C_0 independent $h \in]0, h_0]$ such that one has

$$\|g\|_{L^2(\mathcal{N})}^2 + \sum_{j=1}^p h^{-2} \int_{V_0 \times U_0} \frac{1}{2h} \int_{-h}^h |g(x, y) - g(e^{tZ_k}(x, y))|^2 dt dx dy \leq C_0 \quad (4.7)$$

Lemma 4.2 *There exists $C_1, h_0 > 0$ such that for all $h \in]0, h_0]$, any g with support in $B_r^m \times B_r^n$, such that (4.7) holds true can be written on the form*

$$g = f_h + l_h, \quad \sum_{k=1}^p \|Z_k f_h\|_{L^2(V_0 \times U_0)} \leq C_1, \quad \|l_h\|_{L^2(V_0 \times U_0)} \leq C_1 h$$

Let us assume that lemma 4.2 holds true. Then one can write $g = \chi(y)u(x) = f_h + l_h$. Let $\psi(x, y) \in C_0^\infty(V_0 \times U_0)$ equal to 1 near $B_r^m \times B_r^n$. Set

$$v_h = \int \psi(x, y) f_h(x, y) dy, \quad w_h = \int \psi(x, y) l_h(x, y) dy$$

One has $v_h + w_h = \int \psi(x, y) \chi(y) u(x) dy = \int \chi(y) u(x) dy = u(x)$ and $\|w_h\|_{L^2} \leq Ch$. Moreover, we get from (2.6)

$$X_k(v_h) = \int (Z_k - \sum_l b_{k,l}(x, y) \frac{\partial}{\partial y_l}) \psi(x, y) f_h(x, y) dy$$

Since $f_h, Z_k(f_h) \in O_{L^2}(1)$ and $\int b \frac{\partial}{\partial y_l} (\psi f_h) dy = - \int \frac{\partial}{\partial y_l} (b) \psi f_h dy \in O_{L^2}(1)$, we get that (4.3) holds true. We are thus reduced to prove lemma 4.2.

For any given k , the vector field Z_k is not singular; thus, decreasing V_0, U_0 if necessary, there exists coordinates $(z_1, \dots, z_D) = (z_1, z')$ such that $Z_k = \frac{\partial}{\partial z_1}$. Using Fourier transform in z_1 , we get that if g satisfies (4.7), one has

$$2 \int (1 - \frac{\sin h \zeta_1}{h \zeta_1}) |\hat{g}(\zeta_1, z')|^2 d\zeta_1 dz' = \int \frac{1}{2h} \int_{-h}^h |1 - e^{it\zeta_1}|^2 dt |\hat{g}(\zeta_1, z')|^2 d\zeta_1 dz' \leq C'_0 h^2 \quad (4.8)$$

Let $a > 0$ small. There exists $c > 0$ such that $(1 - \frac{\sin h \zeta_1}{h \zeta_1}) \geq ch^2 \zeta_1^2$ for $h|\zeta_1| \leq a$ and $(1 - \frac{\sin h \zeta_1}{h \zeta_1}) \geq c$ for $h|\zeta_1| > a$. Since $g(z_1, z') = \frac{1}{2\pi} \int_{h|\zeta_1| \leq a} e^{iz_1 \zeta_1} \hat{g}(\zeta_1, z') d\zeta_1 + \frac{1}{2\pi} \int_{h|\zeta_1| > a} e^{iz_1 \zeta_1} \hat{g}(\zeta_1, z') d\zeta_1 = v_{h,k} + w_{h,k}$, we get from (4.8) that g satisfies for some C_0 independent of $h \in]0, h_0]$

$$\begin{aligned} \|g\|_{L^2(\mathcal{N})} &\leq C_0, \quad \text{support}(g) \subset V_0 \times U_0 \\ \forall k, \quad g &= v_{h,k} + w_{h,k} \\ \|Z_k v_{h,k}\|_{L^2(\mathcal{N})} &\leq C_0, \quad \|w_{h,k}\|_{L^2(\mathcal{N})} \leq C_0 h \end{aligned} \quad (4.9)$$

and we want to prove that the decomposition $g = v_{h,k} + w_{h,k}$ may be chosen independent of k , i.e there exists $C > 0$ independent of h such that

$$\begin{aligned} g &= v_h + w_h \\ \forall k, \quad \|Z_k v_h\|_{L^2(\mathcal{N})} &\leq C \\ \|w_h\|_{L^2(\mathcal{N})} &\leq Ch \end{aligned} \quad (4.10)$$

In order to prove the implication (4.9) \Rightarrow (4.10) we will construct operators $\Phi, C_j, B_{k,j}, R_l$, depending on h , acting on L^2 functions with support in a small neighborhood of $o_{\mathcal{N}}$ in \mathcal{N} , with values in $L^2(\mathcal{N})$, such that $\Phi, C_j, B_{k,j}, R_l, C_j h Z_j, B_{k,j} h Z_k$ are uniformly in h bounded on L^2 and

$$\begin{aligned} 1 - \Phi &= \sum_{j=1}^p C_j h Z_j + h R_0 \\ Z_j \Phi &= \sum_{k=1}^p B_{k,j} Z_k + R_j \end{aligned} \tag{4.11}$$

and then we set

$$v_h = \Phi(g), \quad w_h = (1 - \Phi)(g)$$

With this decomposition of g , we get

$$\begin{aligned} w_h &= \sum_{j=1}^p C_j h Z_j (v_{h,j} + w_{h,j}) + h R_0(g) \in O_{L^2}(h) \\ Z_k(v_h) &= \sum_{j=1}^p B_{j,k} Z_j (v_{h,j} + h \frac{1}{h} w_{h,j}) + R_k(g) \in O_{L^2}(1) \end{aligned}$$

We are thus reduced to prove the existence of the operators $\Phi, C_j, B_{k,j}, R_l$, with the suitable bounds on L^2 , and such that (4.11) holds true. This is a problem on the Lie algebra \mathcal{N} with vector fields Z_j given by the Rothschild-Stein Goodman theorem 2.3. We will first do this construction in the special case where the vector fields Z_j are equal to the left invariant vector fields \tilde{Y}_j on \mathcal{N} . In that special case, we will have $R_l = 0$ in formula (4.11). We will conclude in the general case by a suitable h-pseudodifferential calculus.

Step 2: The case of left invariant vector fields on \mathcal{N} .

Let $f * u$ be the convolution on \mathcal{N}

$$f * u(x) = \int_{\mathcal{N}} f(x.y^{-1})u(y)dy = \int_{\mathcal{N}} f(z)u(z^{-1}.x)dz$$

Here, dy is the left (and right) invariant Haar measure on \mathcal{N} , which is simply equal to the Lebesgue measure $dy_1 \dots dy_t$ in the coordinates used in formula (2.3). Then for $u \in L^1(\mathcal{N})$, the map $f \mapsto f * u$ is bounded on $L^q(\mathcal{N})$ by $\|u\|_{L^1}$ for any $q \in [1, \infty]$. The vector fields \tilde{Y}_j are divergence free for the Haar measure dy .

If f is a function on \mathcal{N} , and $a \in \mathcal{N}$, let $\tau_a(f)$ be the function defined by $\tau_a(f)(x) = f(a^{-1}.x)$. One has for any $a \in \mathcal{N}$ and $Y \in T_e \mathcal{N} \simeq \mathcal{N}$, $\tau_a \tilde{Y} = \tilde{Y} \tau_a$ and the following formula holds true:

$$\begin{aligned} \tau_a(f) &= \delta_a * f \\ \tilde{Y} f &= f * \tilde{Y} \delta_e \end{aligned} \tag{4.12}$$

Let us denote by \mathcal{T}_h the scaling operator $\mathcal{T}_h(f)(x) = h^{-Q} f(h^{-1}.x)$. One has $h.(x^{-1}) = (h.x)^{-1}$ and $\mathcal{T}_h(f * g) = \mathcal{T}_h(f) * \mathcal{T}_h(g)$. The action of \mathcal{T}_h on the space $\mathcal{D}'(\mathcal{N})$ of distributions on \mathcal{N} , compatible with the action on functions, is given by $\langle \mathcal{T}_h(T), \phi \rangle = \langle T, x \mapsto \phi(h.x) \rangle$. Thus one has $\mathcal{T}_h \delta_e = \delta_e$ and $\mathcal{T}_h(\tilde{Y}_j(\delta_e)) = h \tilde{Y}_j(\delta_e)$ for $j \in \{1, \dots, p\}$.

Let $\mathcal{S}(\mathcal{N})$ be the Schwartz space on \mathcal{N} , and $\varphi \in \mathcal{S}(\mathcal{N})$, with $\int_{\mathcal{N}} \varphi(x) dx = 1$. For $h \in]0, 1]$, let Φ_h be the operator defined by

$$\Phi_h(f) = f * \varphi_h, \quad \varphi_h(x) = h^{-Q} \varphi(h^{-1} \cdot x) = \mathcal{T}_h(\varphi) \quad (4.13)$$

Since the Jacobian of the transformation $x \mapsto h \cdot x$ is equal to h^Q , one has $\|\varphi_h\|_{L^1} = \|\varphi\|_{L^1}$ for all $h \in]0, 1]$, and therefore the operator Φ_h is uniformly bounded on L^2 .

If we define the operators $B_{k,j,h}$ by $B_{k,j,h}(f) = f * \mathcal{T}_h(\varphi_{k,j})$, with $\varphi_{k,j} \in \mathcal{S}(\mathcal{N})$, the equation

$$\tilde{Y}_j \Phi_h = \sum_{k=1}^p B_{k,j,h} \tilde{Y}_k$$

is equivalent to find the $\varphi_{k,j} \in \mathcal{S}(\mathcal{N})$ such that

$$\tilde{Y}_j \varphi = \sum_{k=1}^p \tilde{Y}_k \delta_e * \varphi_{k,j} \quad (4.14)$$

One has $\int_{\mathcal{N}} \tilde{Y}_j(\varphi)(x) dx = 0$, and since $f \mapsto \tilde{Y}_k \delta_e * f$ is the right invariant vector field \mathcal{Z}_k on \mathcal{N} such that $\mathcal{Z}_k(o_{\mathcal{N}}) = Y_k$, the equation (4.14) is solvable thanks to lemma 7.2 of the appendix. Moreover, the operators Φ_h , $B_{k,j,h}$ and $B_{k,j,h} \tilde{Y}_k$ are uniformly in $h \in]0, 1]$ bounded on L^2 . (one has $B_{k,j,h}(\tilde{Y}_k(f)) = f * \mathcal{T}_h(\tilde{Y}_k(\delta_e) * \varphi_{k,j})$ and $\tilde{Y}_k(\delta_e) * \varphi_{k,j} \in \mathcal{S}(\mathcal{N})$).

Let now $c_j \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$ be Schwartz for $|x| \geq 1$, and quasi homogeneous of degree $-Q + 1$ near $o_{\mathcal{N}}$ (i.e $c_j(t \cdot x) = t^{-Q+1} c_j(x)$ for $0 < |x| \leq 1$ and $t > 0$ small). Let $C_{j,h}$ be the operators defined by $C_{j,h}(f) = f * \mathcal{T}_h(c_j)$. Then the equation $1 - \Phi_h = \sum_j C_{j,h} \tilde{Y}_j$ is equivalent to solve

$$\delta_e - \varphi = \sum_j \tilde{Y}_j \delta_e * c_j \quad (4.15)$$

In order to solve (4.15), we denote by $E \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$ the (unique) fundamental solution, quasi homogeneous of degree $-Q + 2$ on \mathcal{N} of the hypoelliptic equation (for the existence of E , we refer to [Fol75], theorem (2.1), p.172)

$$\delta_e = \sum_{j=1}^p \mathcal{Z}_j^2(E), \quad \mathcal{Z}_j(f) = \tilde{Y}_j \delta_e * f$$

Let $\psi \in C_0^\infty(\mathcal{N})$ with $\psi(x) = 1$ near $e = o_{\mathcal{N}}$. We will choose c_j of the form

$$c_j = \psi \mathcal{Z}_j(E) - d_j, \quad d_j \in \mathcal{S}(\mathcal{N}) \quad (4.16)$$

Then the equation (4.15) is equivalent to solve

$$\varphi + \sum_{j=1}^p [\mathcal{Z}_j, \psi] \mathcal{Z}_j(E) = \varphi_0 = \sum_{j=1}^p \mathcal{Z}_j(d_j) \quad (4.17)$$

One has $\varphi_0 \in \mathcal{S}(\mathcal{N})$ and $\int_{\mathcal{N}} \varphi_0(x) dx = 0$ since $\int_{\mathcal{N}} \varphi(x) dx = 1$ and $\int_{\mathcal{N}} \sum_{j=1}^p [\mathcal{Z}_j, \psi] \mathcal{Z}_j(E) dx = -\int_{\mathcal{N}} \sum_{j=1}^p \psi \mathcal{Z}_j^2(E) dx = -1$. Thus, the equation (4.14) is solvable thanks to lemma 7.2. Moreover, since $c_j \in L^1(\mathcal{N})$, the operators $C_{j,h}$ are uniformly in h bounded on L^2 . It remains to verify that the operators $C_{j,h} \tilde{Y}_j$ are uniformly in h bounded on L^2 . One has $C_{j,h} \tilde{Y}_j(f) = f * \mathcal{T}_h(\mathcal{Z}_j(c_j))$. Since $\|\mathcal{T}_h(f)\|_{L^2} = h^{-Q/2} \|f\|_{L^2}$ it is equivalent to prove that the operator $g \mapsto g * \mathcal{Z}_j(c_j)$ is bounded on L^2 . By construction one has $\mathcal{Z}_j(c_j) = \psi \mathcal{Z}_j^2(E) + l_j$, $l_j \in \mathcal{S}(\mathcal{N})$.

With the terminology of [Fol75], the distribution $Z_j^2(E)$ is homogeneous of degree 0 (i.e quasi homogeneous of degree $-Q$), thus of the form $Z_j^2(E) = a_j \delta_e + f_j$ where $f_j \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$, quasi homogeneous of degree $-Q$ and such that $\int_{b < |u| < b'} f_j(u) du = 0$. Thus by [Fol75], proposition 1.9, p.167, the operator $g \mapsto g * Z_j(c_j)$ is bounded on L^2 .

Step 3: A suitable h -pseudodifferential calculus on \mathcal{N} .

Let Z^α be the smooth vector fields defined in a neighborhood Ω of $o_{\mathcal{N}}$ in \mathcal{N} given by the Goodman theorem 2.3. In this last step, we will finally construct the operators such that (4.11) holds true. We first recall the construction of the map $\Theta(a, b)$ which play a crucial role in the construction of a parametrix for hypoelliptic operators in [RS76]. Let us recall that $(Y_\alpha^\alpha = H_\alpha(Y_1, \dots, Y_p) \in T_e \mathcal{N}, \alpha \in \mathcal{A})$ is a basis of $T_e \mathcal{N}$. For $a \in \mathcal{N}$ close to e and $u = \sum_{\alpha \in \mathcal{A}} u_\alpha Y^\alpha \in T_e \mathcal{N}$ close to 0, let $\Lambda(u) = \sum_{\alpha \in \mathcal{A}} u_\alpha Z^\alpha$ and

$$\Phi(a, u) = e^{\Lambda(u)} a \quad (4.18)$$

Clearly, $(a, u) \mapsto (a, \Phi(a, u))$ is a diffeomorphism of a neighborhood of $(e, 0)$ in $\mathcal{N} \times T_e \mathcal{N}$ onto a neighborhood of (e, e) in $\mathcal{N} \times \mathcal{N}$, and $\Phi(a, 0) = a$. We denote by $\Theta(a, b)$ the map defined in a neighborhood of (e, e) in $\mathcal{N} \times \mathcal{N}$ into a neighborhood of $o_{\mathcal{N}}$ in $\mathcal{N} \simeq T_e \mathcal{N}$ by

$$\Phi(a, \Theta(a, b)) = b \quad (4.19)$$

For $b = \Phi(a, u)$, one has $\Phi(b, -u) = e^{\Lambda(-u)}(e^{\Lambda(u)} a) = e^{-\Lambda(u)}(e^{\Lambda(u)} a) = a$. Thus one has the symmetry relation

$$\Theta(a, b) = -\Theta(b, a) = \Theta(b, a)^{-1} \quad (4.20)$$

Observe that in the special case $Z_j = \tilde{Y}_j$, $\Lambda(u)$ is equal to the left invariant vector field on \mathcal{N} such that $\Lambda(u)(o_{\mathcal{N}}) = u$, i.e $\Lambda(u) = \tilde{u}$ and $\Phi(a, u) = e^{\tilde{u}} a = a.u$, and this implies in that case

$$\Theta(a, b) = a^{-1}.b \quad (4.21)$$

Let $\varphi \in \mathcal{S}(\mathcal{N})$, with $\int_{\mathcal{N}} \varphi(x) dx = 1$. By step 2, there exists functions $\varphi_{k,j} \in \mathcal{S}(\mathcal{N})$, and $c_j \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$, Schwartz for $|x| \geq 1$, quasi homogeneous of degree $-Q + 1$ near $o_{\mathcal{N}}$, such that the following holds true.

$$\begin{aligned} \tilde{Y}_j(\varphi) &= \sum_{k=1}^p Z_k(\varphi_{k,j}) \\ \delta_e - \varphi &= \sum_j Z_j(c_j) \end{aligned} \quad (4.22)$$

Let $\omega_0 \subset\subset \omega_1$ be small neighborhoods of $o_{\mathcal{N}}$ such that $\Theta(y, x)$ is well defined for $(y, x) \in \omega_0 \times \omega_1$, and $\chi \in C_0^\infty(\omega_1)$ be equal to 1 in a neighborhood of $\bar{\omega}_0$. We define the operators Φ_h , $B_{k,j,h}$ and $C_{j,h}$ for $1 \leq j, k \leq p$ by the formulas

$$\begin{aligned} \Phi_h(f)(x) &= \chi(x) h^{-Q} \int_{\mathcal{N}} \varphi(h^{-1}.\Theta(y, x)) f(y) dy \\ B_{k,j,h}(f)(x) &= \chi(x) h^{-Q} \int_{\mathcal{N}} \varphi_{k,j}(h^{-1}.\Theta(y, x)) f(y) dy \\ C_{j,h}(f)(x) &= \chi(x) h^{-Q} \int_{\mathcal{N}} c_j(h^{-1}.\Theta(y, x)) f(y) dy \end{aligned} \quad (4.23)$$

All these operators are of the form

$$A_h(f)(x) = h^{-Q} \int_{\mathcal{N}} g(x, h^{-1} \cdot \Theta(y, x)) f(y) dy \quad (4.24)$$

where the function $g(x, \cdot)$ is smooth in x , with compact support ω_1 , and takes values in $L^1(\mathcal{N})$, i.e. $\sup_{x \in \omega_1} \|\partial_x^\beta g(x, \cdot)\|_{L^1(\mathcal{N})} < \infty$ for all β . The function $A_h(f)$ is well defined for $f \in L^\infty(\mathcal{N})$ such that $\text{support}(f) \subset \omega_0$. We have introduced the cutoff $\chi(x)$ just to have $A_h(f)(x)$ defined for all $x \in \mathcal{N}$, and one has $A_h(f)(x) = 0$ for all $x \notin \omega_1$.

Lemma 4.3 *Let $g(x, \cdot)$ be smooth in x with compact support in ω_1 , with values in $L^1(\mathcal{N})$. Then the operator A_h defined by (4.24) is uniformly in $h \in]0, 1]$ bounded from $L^q(\omega_0)$ into $L^q(\mathcal{N})$ for all $q \in [1, \infty]$.*

Proof. The proof is standard. By interpolation, it is sufficient to treat the two cases $q = \infty$ and $q = 1$. In the case $q = \infty$, the jacobian of the change of coordinates $y \mapsto u = \Theta(y, x)$ is bounded by C for all $x \in \omega_1, y \in \omega_0$. Thus we get

$$|A_h(f)(x)| \leq C \|f\|_{L^\infty(\omega_0)} h^{-Q} \int_{\mathcal{N}} |g(x, h^{-1} \cdot u)| du = C \|f\|_{L^\infty(\omega_0)} \|g(x, \cdot)\|_{L^1}$$

Since $x \mapsto g(x, \cdot)$ is smooth in x with values in $L^1(\mathcal{N})$, one has $C_\infty = \sup_{x \in \omega_1} \|g(x, \cdot)\|_{L^1} < \infty$. Thus we get $\|A_h(f)\|_{L^\infty} \leq C C_\infty \|f\|_{L^\infty(\omega_0)}$.

For $q = 1$, we first extend g as a smooth L -periodic function of $x \in \mathcal{N}$, with L large enough, $g(x, u) = \sum_{k \in \mathbb{Z}^D} g_k(u) e^{2i\pi k \cdot x/L}$, the equality being valid for $x \in \omega_1$. Observe that $\|g_k\|_{L^1(\mathcal{N})}$ is rapidly decreasing in k . Then one has

$$A_h(f)(x) = \sum_k A_{h,k}(f)(x) e^{ik \cdot x/L}, \quad A_{h,k}(f)(x) = h^{-Q} \int_{\mathcal{N}} g_k(h^{-1} \cdot \Theta(y, x)) f(y) dy$$

The jacobian of the change of coordinates $(x, y) \mapsto (u = \Theta(y, x), y)$ is bounded by C for all $(x, y) \in \omega_1 \times \omega_0$, and one has

$$\int_{\omega_1} |A_{h,k}(f)(x)| dx \leq C h^{-Q} \int_{\mathcal{N}} \int_{\omega_0} |g_k(h^{-1} \cdot u)| |f(y)| dy du = C \|f\|_{L^1} \|g_k\|_{L^1}$$

Thus we get $\sup_{h \in]0, 1]} \|A_{h,k}\|_{L^1} = d_k$ with d_k rapidly decreasing in k , and this implies $\sup_{h \in]0, 1]} \|A_h\|_{L^1} \leq \sum_k d_k < \infty$. The proof of lemma 4.3 is complete. \square

Observe that in the special case $Z_j = \tilde{Y}_j$, using (4.21), we get that the operators $\Phi_h, B_{k,j,h}, C_{j,h}$ defined by the formula (4.23) are precisely equal, up to the factor $\chi(x)$, to the operators we have constructed in step 2.

In the general case, it remains to show the following:

i) The operators $R_{l,h}$ defined by

$$\begin{aligned} R_{0,h} &= h^{-1} \left(1 - \Phi_h - \sum_{j=1}^p C_{j,h} Z_j \right) \\ R_{j,h} &= Z_j \Phi_h - \sum_{k=1}^p B_{k,j,h} Z_k, \quad 1 \leq j \leq p \end{aligned} \quad (4.25)$$

are uniformly bounded in $h \in]0, 1]$ on L^2 .

ii) The operators $C_{j,h}hZ_j$ and $B_{k,j,h}hZ_k, k > 0$ are uniformly bounded in $h \in]0, 1]$ on L^2 .

For the verification of i) and ii), we just follow the natural strategy which is developed in [RS76]. If f is a function defined near $a \in \mathcal{N}$, let $\Phi_a(f)$ be the function defined near 0 in $\mathcal{N} \simeq T_e\mathcal{N}$ by $\Phi_a(f)(u) = f(\Phi(a, u))$. The following fundamental lemma is proven in [RS76] (theorem 5) and also in [Goo78] (section 5, "Estimation of the error").

Lemma 4.4 *For all $j \in \{1, \dots, p\}$, and $a \in \mathcal{N}$ near e , the vector field $V_{j,a}$ defined near 0 in \mathcal{N}*

$$V_{j,a}(g) = \Phi_a(Z_j(\Phi_a^{-1}g)) - \tilde{Y}_j(g) \quad (4.26)$$

is of order ≤ 0 at 0. If we introduce the system of coordinates $(u_\alpha) = (u_{l,k})$ with $l(\alpha) = |\alpha|$ and $1 \leq k \leq a_l = \dim(\mathcal{N}_l)$, we thus have

$$V_{j,a} = \sum_{l=1}^r \sum_{k=1}^{a_l} v_{j,l,k}(a, u) \frac{\partial}{\partial u_{l,k}} \quad (4.27)$$

where the functions $v_{j,l,k}(a, u)$ are smooth and satisfy $v_{j,l,k}(a, u) \in O(|u|^l)$.

Let us denote by $A_h[g]$ an operator of the form (4.24). Recall that $g(x, u)$ is smooth in x with compact support in ω_1 , with values in $L^1(\mathcal{N})$. More precisely, we have two cases to consider: a) g is Schwartz in u , and b) g is smooth in u in $\mathcal{N} \setminus \{o_{\mathcal{N}}\}$, Schwartz for $|u| \geq 1$, and quasi homogeneous of degree $-Q+1$ near $o_{\mathcal{N}}$. We have to compute the kernel of the operators $Z_j A_h[g]$ and $A_h[g] Z_j$.

We first compute the kernel of $Z_j A_h(g)$. For any fixed y , perform the change of coordinates $x = \Phi_y(u)$ so that $\Theta(y, x) = u$. Denote Z_j^x the vector field Z_j acting on the variable x . Using lemma 4.4, we get

$$\begin{aligned} Z_j(A_h[g](f))(x) &= h^{-Q} \int_{\mathcal{N}} Z_j^x(g(x, h^{-1} \cdot \Theta(y, x))) f(y) dy = \\ &h^{-Q} \int_{\mathcal{N}} h^{-1} (\tilde{Y}_j^u g)(x, h^{-1} \cdot \Theta(y, x)) f(y) dy \\ &+ h^{-Q} \int_{\mathcal{N}} (Z_j^x g)(x, h^{-1} \cdot \Theta(y, x)) f(y) dy \\ &+ \sum_{l=1}^r \sum_{k=1}^{a_l} h^{-Q} \int_{\mathcal{N}} v_{j,l,k}(y, \Theta(y, x)) h^{-l} \frac{\partial g}{\partial u_{l,k}}(x, h^{-1} \cdot \Theta(y, x)) f(y) dy \end{aligned} \quad (4.28)$$

By lemma 4.3, the second term in (4.28) is uniformly bounded in $h \in]0, 1]$, from $L^2(\omega_0)$ into $L^2(\mathcal{N})$. The same holds true for the third term. To see this point, following the proof of lemma 4.3, first write $v_{j,l,k}(y, u) = \sum_n v_{j,l,k,n}(u) e^{2i\pi n \cdot y/L}$, with $v_{j,l,k,n}(u)$ rapidly decreasing in n and $O(|u|^l)$ near $u = o_{\mathcal{N}}$. We are then reduce to show that an operator of the form

$$R_h(f) = h^{-Q} \int_{\mathcal{N}} h^{-l} G(\Theta(y, x)) \frac{\partial g}{\partial u_{l,k}}(x, h^{-1} \cdot \Theta(y, x)) f(y) dy$$

with $G(u)$ smooth and $G(u) \in O(|u|^l)$, is uniformly bounded in $h \in]0, 1]$ from $L^2(\omega_0)$ into $L^2(\mathcal{N})$ by a constant which depends linearly on a finite number of derivatives of G . Clearly, there exists such a constant C such that $h^{-l} |G(\Theta(y, x))| \leq C |h^{-1} \cdot \Theta(y, x)|^l$. Thus the result follows from the proof of lemma 4.3, since $|u|^l \frac{\partial g}{\partial u_{l,k}}(x, u)$ is L^1 in u in both case a) and b) (the vector field $|u|^l \frac{\partial}{\partial u_{l,k}}$

is of order 0).

If we denote by R_h any operator uniformly bounded on L^2 , we have thus proven

$$Z_j A_h[g] = h^{-1} A_h[\tilde{Y}_j^u g] + R_h \quad (4.29)$$

Let us now compute the kernel of $A_h[g]Z_j$. The basic observation is the following identity (recall $u^{-1} = -u$ and $\mathcal{Z}_j(f) = \tilde{Y}_j(\delta_e) * f$ is the right invariant vector field such that $\mathcal{Z}_j(0) = Y_j$)

$$-\tilde{Y}_j(f(-u)) = \mathcal{Z}_j(f)(-u) \quad (4.30)$$

Let l_j be the smooth function such that ${}^t Z_j = -Z_j + l_j$. For any given x perform the change of coordinates $y = \Phi_x(u)$. By (4.20), one has $\Theta(y, x) = -\Theta(x, y) = -u$. We thus get from lemma 4.4 and (4.30) the following formula:

$$\begin{aligned} A_h[g](Z_j(f))(x) &= h^{-Q} \int_{\mathcal{N}} g(x, h^{-1} \cdot \Theta(y, x)) Z_j(f)(y) dy \\ &= h^{-Q} \int_{\mathcal{N}} (-Z_j^y + l_j(y))(g(x, h^{-1} \cdot \Theta(y, x))) f(y) dy \\ &= h^{-Q} \int_{\mathcal{N}} h^{-1} (\mathcal{Z}_j^u g)(x, h^{-1} \cdot \Theta(y, x)) f(y) dy \\ &+ h^{-Q} \int_{\mathcal{N}} g(x, h^{-1} \cdot \Theta(y, x)) l_j(y) f(y) dy \\ &+ \sum_{l=1}^{\tau} \sum_{k=1}^{a_l} h^{-Q} \int_{\mathcal{N}} v_{j,l,k}(x, -\Theta(y, x)) h^{-l} \frac{\partial g}{\partial u_{l,k}}(x, h^{-1} \cdot \Theta(y, x)) f(y) dy \end{aligned} \quad (4.31)$$

As above, this gives the identity, with R_h uniformly bounded on L^2

$$A_h[g]Z_j = h^{-1} A_h[\mathcal{Z}_j^u g] + R_h \quad (4.32)$$

Observe that formulas (4.22), (4.29) and (4.32) imply that (4.25) holds true. Moreover, from (4.32) and lemma 4.3, the operators $B_{k,j,h} h Z_k, k > 0$ are uniformly bounded in $h \in]0, 1]$ on L^2 . In order to get from (4.32) the same uniform bounds for the operators $C_{j,h} h Z_j$, we just observe that in the case where $g(x, u)$ is quasi homogeneous in u of degree $-Q + 1$ near $o_{\mathcal{N}}$, one has $\mathcal{Z}_j^u g(x, u) = C_j(x) \delta_e + f_j(x, u)$ with $\int_{b < |u| < b'} f_j(x, u) du = 0$ and we conclude as in the end of step 2 by the proposition 1.9 of [Fol75].

The proof of proposition 4.1 is complete.

5 Proof of theorems 1.1 and 1.2

This section is devoted to the proof of theorems 1.1 and 1.2. Let \mathcal{B}_h be the bilinear form associated to the rescaled Dirichlet form \mathcal{E}_h

$$\mathcal{B}_h(f, g) = \left(\frac{1 - T_h}{h^2} f | g \right)_{L^2}, \quad f, g \in L^2(M, d\mu) \quad (5.1)$$

Proposition 5.1 *Let $f \in \mathcal{H}^1(\mathcal{X})$. Let $(r_h, \gamma_h) \in \mathcal{H}^1(\mathcal{X}) \times L^2$ such that r_h converge weakly (when $h \rightarrow 0$) in $\mathcal{H}^1(\mathcal{X})$ to $r \in \mathcal{H}^1(\mathcal{X})$, and $\sup_h \|\gamma_h\|_{L^2} < \infty$. Then*

$$\lim_{h \rightarrow 0} \mathcal{B}_h(f, r_h + h\gamma_h) = \frac{1}{6p} \sum_{k=1}^p (X_k f | X_k r)_{L^2} \quad (5.2)$$

Proof. Write $r_h = r + r'_h$. The weak limit of r'_h in $\mathcal{H}^1(\mathcal{X})$ is 0. Since $\mathcal{B}_h(f, r_h) = \mathcal{B}_h(f, r) + \mathcal{B}_h(f, r'_h)$, we have to prove the two assertions:

$$\lim_{h \rightarrow 0} \mathcal{B}_h(f, r) = \frac{1}{6p} \sum_{k=1}^p (X_k f | X_k r)_{L^2}, \quad \forall f, r \in \mathcal{H}^1(\mathcal{X}) \quad (5.3)$$

and under the hypothesis that the weak limit of r_h in $\mathcal{H}^1(\mathcal{X})$ is 0

$$\lim_{h \rightarrow 0} \left(\frac{1 - T_{k,h}}{h^2} f | r_h + h\gamma_h \right)_{L^2} = 0, \quad \forall k \in \{1, \dots, p\} \quad (5.4)$$

In order to verify (5.4), since M is compact, we may assume that f is supported in a small neighborhood of a point $x_0 \in M$ where the Goodman theorem 2.3 applies. With the notations of section 2, we may thus assume in the coordinate system $\Lambda\theta$ centered at $x_0 \simeq 0$ that f, r_h, γ_h are supported in the closed ball $B_r^m = \{x \in \mathbb{R}^m, |x| \leq r\} \subset V_0$. Let $\chi(y) \in C_0^\infty(U_0)$ with support in $B_r^n \subset U_0$, such that $\int \chi(y) dy = 1$ and write $d\mu(x) = \rho(x) dx$ with ρ smooth. For $u, v \in L^2(M)$ supported in B_r^m , one has

$$(u|v)_{L^2} = \int_{V_0} u(x) \overline{v(x)} d\mu(x) = \int_{V_0 \times U_0} u(x) \overline{\rho(x)\chi(y)v(x)} dx dy$$

Set $\tilde{f}(x, y) = W_{x_0}(f)(x, y) = f(x)$, $\tilde{r}_h(x, y) = \rho(x)\chi(y)r_h(x)$, $\tilde{\gamma}_h(x, y) = \rho(y)\chi(y)\gamma_h(x)$. We get from (2.8)

$$\left(\frac{1 - T_{k,h}}{h^2} f | r_h + h\gamma_h \right)_{L^2} = \int_{V_0 \times U_0} \left(\frac{1 - \tilde{T}_{k,h}}{h^2} \tilde{f} \right) \overline{\tilde{r}_h + h\tilde{\gamma}_h} dx dy \quad (5.5)$$

Observe that $\tilde{\gamma}_h$ is bounded in $L^2(V_0 \times U_0)$. Since the injection $\mathcal{H}^1(\mathcal{X}) \subset L^2(M)$ is compact, r_h converge strongly to 0 in L^2 , and therefore \tilde{r}_h converge strongly to 0 in $L^2(V_0 \times U_0)$. Moreover, $Z_k(\tilde{r}_h)$ converge weakly to 0 in $L^2(V_0 \times U_0)$. Finally, since $\tilde{T}_{k,h}$ increase the support of at most $\simeq h$, we may replace \tilde{f} by $F = \theta(y)\tilde{f}$ with $\theta \in C_0^\infty$ equal to 1 near the support of χ . Then F is compactly supported in $V_0 \times U_0$ and satisfies $F \in L^2$ and $Z_k F \in L^2$. Since the vector field Z_k is not singular, decreasing V_0, U_0 if necessary, there exists coordinates $(z_1, \dots, z_D) = (z_1, z')$ such that $Z_k = \frac{\partial}{\partial z_1}$. One has $dx dy = q(z) dz$ with $q > 0$ smooth. Set $q\tilde{r}_h = R_h, q\tilde{\gamma}_h = Q_h$. Using Fourier transform in z_1 , it remains to show

$$\begin{aligned} \lim_{h \rightarrow 0} I_h = 0, \quad I_h &= h^{-2} \int \left(1 - \frac{\sin(h\xi_1)}{h\xi_1}\right) \hat{F}(\xi_1, z') \overline{\hat{R}_h(\xi_1, z')} d\xi_1 dz' \\ \lim_{h \rightarrow 0} J_h = 0, \quad J_h &= h^{-1} \int \left(1 - \frac{\sin(h\xi_1)}{h\xi_1}\right) \hat{F}(\xi_1, z') \overline{\hat{Q}_h(\xi_1, z')} d\xi_1 dz' \end{aligned} \quad (5.6)$$

Recall that Q_h is bounded in L^2 , R_h converge strongly to zero in L^2 , $\partial_{z_1} R_h$ converge weakly to zero in L^2 and $F, \partial_{z_1} F \in L^2$. We write the first integral in (5.6) on the form

$$I_h = \int \psi(h\xi_1) \xi_1 \hat{F}(\xi_1, z') \overline{\xi_1 \hat{R}_h(\xi_1, z')} d\xi_1 dz'$$

with $\psi(x) = x^{-2}(1 - \frac{\sin(x)}{x})$. One has $\psi \in C^\infty(\mathbb{R})$ and $|\psi(x)| \leq C \frac{1}{1+x^2}$. Then we write $I_h = I_{1,h} + I_{2,h}$ with $I_{1,h}$ defined by the integral over $|\xi_1| \leq M$ and $I_{2,h}$ defined by the integral over $|\xi_1| > M$. Since $\xi_1 \hat{R}_h(\xi_1, z')$ is bounded in L^2 , and $\psi \in L^\infty$ we get by Cauchy-Schwarz

$$|I_{2,h}| \leq C \left(\int_{|\xi_1| > M} |\xi_1 \hat{F}(\xi_1, z')|^2 d\xi_1 dz' \right)^{1/2} \rightarrow 0 \quad \text{when } M \rightarrow \infty$$

On the other hand, one has $\psi(x) = \psi(0) + \tau(x)$ with $\psi(0) = 1/6$ and $\sup_{x \in \mathbb{R}} \tau(x)/x \leq C_0$. Thus we get

$$I_{1,h} = \frac{1}{6} \int_{|\xi_1| \leq M} \xi_1 \hat{F}(\xi_1, z') \overline{\xi_1 \hat{R}_h(\xi_1, z')} d\xi_1 dz' + \int_{|\xi_1| \leq M} \tau(h\xi_1) \xi_1 \hat{F}(\xi_1, z') \overline{\xi_1 \hat{R}_h(\xi_1, z')} d\xi_1 dz' \quad (5.7)$$

For any fixed M , the first term in (5.7) goes to 0 when $h \rightarrow 0$ since $\xi_1 \hat{R}_h(\xi_1, z')$ converge weakly to 0 in L^2 and $\xi_1 \hat{F}(\xi_1, z') \in L^2$. Since $\xi_1 \hat{R}_h(\xi_1, z')$ is bounded in L^2 by say A , by Cauchy-Schwarz, the second term is bounded by $C_0 h M A \|\partial_{z_1} F\|_{L^2}$. Thus one has $\lim_{h \rightarrow 0} I_h = 0$.

We proceed exactly in the same way to prove $\lim_{h \rightarrow 0} J_h = 0$: one has with $x\psi = \phi$

$$J_h = \int \phi(h\xi_1) \xi_1 \hat{F}(\xi_1, z') \overline{\hat{Q}_h(\xi_1, z')} d\xi_1 dz'$$

and we use the fact that $\phi \in L^\infty$, $\hat{Q}_h(\xi_1, z')$ is bounded in L^2 , $\phi(0) = 0$ and $\phi(x)/x \in L^\infty(\mathbb{R})$.

Let us now verify (5.3). From (1.10) this is obvious if f is smooth and $r \in \mathcal{H}^1(\mathcal{X})$. Standard smoothing arguments show that $C^\infty(M)$ is dense in $\mathcal{H}^1(\mathcal{X})$. Let now $f \in \mathcal{H}^1(\mathcal{X})$ and choose $f_h \in C^\infty(M)$ converging strongly to f in $\mathcal{H}^1(\mathcal{X})$. Then $\lim_{h \rightarrow 0} (X_k f_h | X_k r)_{L^2} = (X_k f | X_k r)_{L^2}$ and from (5.4) one has also $\lim_{h \rightarrow 0} \mathcal{B}_h(f_h, r) = \lim_{h \rightarrow 0} \mathcal{B}_h(r, f_h) = \mathcal{B}_h(f, r)$.

The proof of proposition 5.1 is complete. \square

5.1 Proof of theorem 1.1.

Let $|\Delta_h|$ be the rescaled (non negative) Laplacien associated to the Markov kernel T_h :

$$|\Delta_h| = \frac{1 - T_h}{h^2} \quad (5.8)$$

From proposition 4.1 and lemma 7.1, there exists $h_0 > 0$ and $C_4, C_5 > 0$ independent of $h \in]0, h_0]$, such that $\text{Spec}(|\Delta_h|) \cap [0, \lambda]$ is discrete for all $\lambda \leq C_4 h^{-2}$ and one has the Weyl type estimate

$$\#(\text{Spec}(|\Delta_h|) \cap [0, \lambda]) \leq C_5 < \lambda >^{dim(M)/2s}, \quad \forall \lambda \leq C_4 h^{-2}. \quad (5.9)$$

In particular, since $T_h(1) = 1$, 1 is an isolated eigenvalue of T_h . Let us verify that 1 is a simple eigenvalue of T_h . Let $f \in L^2 = L^2(M, d\mu)$ such that $T_h(f) = f$. One has for any $g \in L^2$

$$((1 - T_h)g|g)_{L^2} = \frac{1}{2} \int \int |g(x) - g(y)|^2 t_h(x, dy) d\mu(x) \quad (5.10)$$

Thus we get for all $k \in \{1, \dots, p\}$

$$\int_M \int_{-h}^h |f(x) - f(e^{tX_k} x)|^2 dt d\mu(x) = 0$$

This gives $f(x) - f(e^{tX_k} x) = 0$ for almost all $(x, h) \in M \times]-h, h[$. Therefore, one has $X_k f = 0$ in $\mathcal{D}'(M)$ for all k , and this implies $f = Cte$ thanks to Hörmander and Chow theorems. One can also give a more direct argument: one has $T_h^P(f) = f$, and therefore if one use 5.10 with the Markov kernel T_h^P and proposition 3.1, we get

$$\int_M \int_{u \in I_{\epsilon, h}} |f(x) - f(e^{\lambda(u)} x)|^2 du d\mu(x) = 0$$

Since $u \mapsto e^{\lambda(u)} x$ is a submersion, this implies $f(x) - f(y) = 0$ for almost all (x, y) in a neighborhood of the diagonal in $M \times M$, and therefore $f = Cte$.

Let us now verify that there exists $\delta_1 > 0$ such that for all $h \in]0, h_0]$, the spectrum of T_h is a subset of $[-1 + \delta_1, 1]$. It is sufficient to prove that the same holds true for an odd power T_h^{2N+1} of T_h . We are thus reduce to show that there exists $h_0, C_0 > 0$ such that the following inequality holds true for all $h \in]0, h_0]$ and all $f \in L^2(\Omega)$:

$$(f + T_h^{2N+1}f|f)_{L^2} = \frac{1}{2} \int_{M \times M} t_h^{2N+1}(x, dy) |f(x) + f(y)|^2 d\mu(x) \geq C_0 \|f\|_{L^2}^2. \quad (5.11)$$

Take N large enough such that proposition 3.1 applies for T_h^{2N+1} , i.e $t_h^{2N+1}(x, dy) \geq cS_h^\epsilon(x, dy)$. Then we are reduce to show that there exists C independent of h such that

$$\int_{M \times M} S_h^\epsilon(x, dy) |f(x) + f(y)|^2 d\mu(x) \geq C \|f\|_{L^2}^2. \quad (5.12)$$

From the definition (3.1) of S_h^ϵ , we get

$$\int_{M \times M} S_h^\epsilon(x, dy) |f(x) + f(y)|^2 d\mu(x) = \int_M h^{-Q} \int_{u \in I_{\epsilon, h}} |f(x) + f(e^{\lambda(u)}x)|^2 dud\mu(x) = B$$

Define A by the formula

$$A = \int_M h^{-2Q} \int_{u \in I_{\epsilon/2, h}} \int_{v \in I_{\epsilon/2, h}} |f(e^{\lambda(v)}y) + f(e^{\lambda(u)}y)|^2 dudvd\mu(y)$$

Since $\lambda(v)$ is divergence free as a linear combination with constant coefficients of commutators of the vector fields X_k , the change of variables $e^{\lambda(v)}y = x$ gives

$$A = \int_M h^{-2Q} \int_{u \in I_{\epsilon/2, h}} \int_{v \in I_{\epsilon/2, h}} |f(x) + f(e^{\lambda(u-v)}x)|^2 dudvd\mu(x)$$

Therefore, one has for some constant $c_\epsilon > 0$ independent of h , $B \geq c_\epsilon A$. Clearly, one has

$$\int_M \operatorname{Re} \left(\int_{u \in I_{\epsilon/2, h}} \int_{v \in I_{\epsilon/2, h}} f(e^{\lambda(v)}y) \overline{f}(e^{\lambda(u)}y) dudv \right) d\mu(y) \geq 0$$

and this implies, still using the change of variables $e^{\lambda(v)}y = x$

$$\begin{aligned} A &\geq 2 \int_M h^{-2Q} \int_{u \in I_{\epsilon/2, h}} \int_{v \in I_{\epsilon/2, h}} |f(e^{\lambda(v)}y)|^2 dudvd\mu(y) \\ &= 2\epsilon^D \int_M h^{-Q} \int_{v \in I_{\epsilon/2, h}} |f(e^{\lambda(v)}y)|^2 dvd\mu(y) = 2\epsilon^{2D} \int_M |f(x)|^2 d\mu(x) \end{aligned} \quad (5.13)$$

From (5.13) and $B \geq c_\epsilon A$, we get that (5.12) holds true.

Lemma 5.2 *There exists $C_2, C_3 > 0$ such that the spectral gap of T_h satisfies*

$$C_2 h^2 \leq g(h) \leq C_3 h^2 \quad (5.14)$$

Proof. The right inequality in (5.14) is an obvious consequence of the min-max principle since for any $f \in C^\infty(M)$ one has $\lim_{h \rightarrow 0} \frac{1 - T_h}{h^2} f = L(f)$. From (5.9), we get that for any $a \in]0, 1]$, $m_a = \#(\operatorname{Spec}(T_h) \cap [1 - ah^2, 1])$ is bounded by a constant independent of h small, and we have to verify that there exists $h_0 > 0$ and $a > 0$ independent of $h \in]0, h_0]$ such that $m_a = 0$. If this is not true, there exists two sequences $\epsilon_n, h_n \rightarrow 0$ and a sequence $f_n \in L^2$, with $\|f_n\|_{L^2} = 1$ and $(f_n|1)_{L^2} = \int_M f_n d\mu = 0$ such that

$$T_{h_n} f_n = (1 - h_n^2 \epsilon_n) f_n$$

This implies $\mathcal{E}_{h_n}(f_n) = \epsilon_n$. Using proposition 4.1, we get $f_n = v_n + h_n\gamma_n$ with $\sup_n \|\gamma_n\|_{L^2} < \infty$ and $\|v_n\|_{\mathcal{H}^1(\mathcal{X})} \leq C$. The hypoelliptic theorem of Hörmander implies the existence of $s > 0$ such that one has $\mathcal{H}^1(\mathcal{X}) \subset H^s(M)$, hence the injection $\mathcal{H}^1(\mathcal{X}) \subset L^2(M)$ is compact. As a direct byproduct, we get (up to extraction of a subsequence) that the sequence f_n converge strongly in L^2 to some $f \in \mathcal{H}^1(\mathcal{X})$, and v_n converge weakly in $\mathcal{H}^1(\mathcal{X})$ to f . Set $v_n = f + r_n$. Then r_n converge weakly to 0 in $\mathcal{H}^1(\mathcal{X})$, $f_n = f + r_n + h_n\gamma_n$, and one has

$$\mathcal{E}_{h_n}(f_n) = \mathcal{E}_{h_n}(f) + 2Re(\mathcal{B}_{h_n}(f, r_n + h_n\gamma_n)) + \mathcal{E}_{h_n}(r_n + h_n\gamma_n)$$

Since one has $\mathcal{E}_h(\cdot) \geq 0$, proposition 5.1 implies

$$\frac{1}{6p} \sum_{k=1}^p \|X_k f\|_{L^2}^2 = \lim_{n \rightarrow \infty} \mathcal{E}_{h_n}(f) \leq \liminf_{n \rightarrow \infty} \mathcal{E}_{h_n}(f_n) = 0 \quad (5.15)$$

and therefore $f = Cte$. But since f_n converge strongly in L^2 to f , one has $\|f\|_{L^2} = 1$ and $(f|1)_{L^2} = \int_M f d\mu = 0$. This is a contradiction. The proof of lemma 5.2 is complete \square

To conclude the proof of theorem 1.1, it remains to prove the total variation estimate (1.7). Let Π_0 be the orthogonal projector in $L^2(M, d\mu)$ onto the space of constant functions

$$\Pi_0(f)(x) = \int_M f d\mu \quad (5.16)$$

Then

$$2 \sup_{x \in M} \|t_h^n(x, dy) - \mu\|_{TV} = \|T_h^n - \Pi_0\|_{L^\infty \rightarrow L^\infty} \quad (5.17)$$

Thus, we have to prove that there exist C_0, h_0 , such that for any n and any $h \in]0, h_0]$, one has

$$\|T_h^n - \Pi_0\|_{L^\infty \rightarrow L^\infty} \leq C_0 e^{-ng(h)} \quad (5.18)$$

Observe that since $g(h) \simeq h^2$, and $\|T_h^n - \Pi_0\|_{L^\infty \rightarrow L^\infty} \leq 2$, in the proof of (5.18), we may assume $n \geq Ch^{-2}$ with C large. Let $E_{h,L}$ be the (finite dimensional) subspace of $L^2(M, d\mu)$ span by the eigenvectors $e_{j,h}$ of $|\Delta_h|$, associated with eigenvalues $\lambda_{j,h} \leq C_4 h^{-2}$, with $C_4 > 0$ small enough. Here, the subscript L means "low frequencies". Recall from (5.9) $\dim(E_{h,L}) \leq Ch^{-\dim(M)/2s}$. We will denote by J_h the set of indices

$$J_h = \{j, \lambda_{j,h} \leq C_4 h^{-2}\} \quad (5.19)$$

Lemma 5.3 *There exist $p > 2$ and C independent of $h \in]0, h_0]$ such that for all $u \in E_{h,L}$, the following inequality holds true*

$$\|u\|_{L^p(M)}^2 \leq C(\mathcal{E}_h(u) + \|u\|_{L^2}^2) \quad (5.20)$$

Proof. We denote by $C > 0$ a constant independent of h , changing from line to line. Let $u \in E_{h,L}$ such that $\mathcal{E}_h(u) + \|u\|_{L^2}^2 \leq 1$. From proposition 4.1, one has $u = v_h + w_h$ with $\|v_h\|_{\mathcal{H}^1(\mathcal{X})} \leq C$ and $\|w_h\|_{L^2} \leq Ch$. From the continuous imbedding $\mathcal{H}^1(\mathcal{X}) \subset H^s(M) \subset L^q(M)$ with $s > 0, q > 2, s = \dim(M)(1/2 - 1/q)$, we get

$$\|v_h\|_{L^q} \leq C$$

One has $u = \sum_{\lambda_{j,h} \leq C_4 h^{-2}} z_{j,h} e_{j,h}$ with $\sum_{\lambda_{j,h} \leq C_4 h^{-2}} |z_{j,h}|^2 \leq 1$. From corollary 3.4, one has for $C_4 > 0$ small enough $\|e_{j,h}\|_{L^\infty} \leq Ch^{-Q/2}$. Therefore by Cauchy-Schwarz we get

$$\|u\|_{L^\infty} \leq Ch^{-Q/2} \left(\sum_{\lambda_{j,h} \leq C_4 h^{-2}} |z_{j,h}|^2 \right)^{1/2} (\dim(E_{h,L}))^{1/2} \leq Ch^{-Q/2 - \dim(M)/4s} \quad (5.21)$$

From the proof of proposition 4.1 (see lemma 4.3), one has $\|v_h\|_{L^\infty} \leq C\|u\|_{L^\infty}$. Thus we get $\|w_h\|_{L^\infty} \leq \|u\|_{L^\infty} + \|v_h\|_{L^\infty} \leq Ch^{-Q/2-dim(M)/4s}$. Since $\|w_h\|_{L^2} \leq Ch$ we get by interpolation that there exists $q' > 2$ such that

$$\|w_h\|_{L^{q'}} \leq C$$

Then (5.20) holds true with $p = \min(q, q') > 2$. The proof of lemma 5.3 is complete. \square

We are now ready to prove (5.18), essentially following the strategy of [DLM11], but with some simplifications. We split T_h in 2 pieces, according to the spectral theory. We write $T_h - \Pi_0 = T_{h,1} + T_{h,2}$ with

$$T_{h,1}(x, y) = \sum_{\lambda_{1,h} \leq \lambda_{j,h} \leq C_4 h^{-2}} (1 - h^2 \lambda_{j,h}) e_{j,h}(x) e_{j,h}(y) \quad (5.22)$$

One has $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n$, and we will get the bound (5.18) for each of the two terms. We start by very rough bounds. From $\|e_{j,h}\|_{L^\infty} \leq Ch^{-Q/2}$, $|(1 - h^2 \lambda_{j,h})| \leq 1$, we get with $A = Q/2 + dim(M)/4s$, as in the proof of (5.21) with C independent of $n \geq 1$ and h

$$\|T_{h,1}^n\|_{L^\infty \rightarrow L^\infty} \leq \|T_{h,1}^n\|_{L^2 \rightarrow L^\infty} \leq Ch^{-A} \quad (5.23)$$

Since T_h^n is bounded by 1 on L^∞ , we get from $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n$

$$\|T_{h,2}^n\|_{L^\infty \rightarrow L^\infty} \leq Ch^{-A} \quad (5.24)$$

Let P be the integer defined at the beginning of section 3. Let M_h be the Markov operator $M_h = T_h^P$. Write $n = kP + r$ with $0 \leq r < P$. From proposition 3.1 and corollary 3.3 one has $M_h = \rho_h + R_h$ with

$$\begin{aligned} \|\rho_h\|_{L^\infty \rightarrow L^\infty} &\leq \gamma < 1, \\ \|R_h\|_{L^2 \rightarrow L^\infty} &\leq C_0 h^{-Q/2}. \end{aligned} \quad (5.25)$$

From this, we deduce that for any $k = 1, 2, \dots$, one has $M_h^k = A_{k,h} + B_{k,h}$, with $A_{1,h} = \rho_h$, $B_{1,h} = R_h$ and the recurrence relation $A_{k+1,h} = \rho_h A_{k,h}$, $B_{k+1,h} = \rho_h B_{k,h} + R_h M_h^k$. Thus one gets, since M_h^k is bounded by 1 on L^2 ,

$$\begin{aligned} \|A_{k,h}\|_{L^\infty \rightarrow L^\infty} &\leq \gamma^k, \\ \|B_{k,h}\|_{L^2 \rightarrow L^\infty} &\leq C_0 h^{-Q/2} (1 + \gamma + \dots + \gamma^k) \leq C_0 h^{-Q/2} / (1 - \gamma). \end{aligned} \quad (5.26)$$

Let $\theta = 1 - C_4 < 1$ so that $\|T_{h,2}\|_{L^2 \rightarrow L^2} \leq \theta$. Then one has

$$\|T_{h,2}^n\|_{L^\infty \rightarrow L^2} \leq \|T_{h,2}^n\|_{L^2 \rightarrow L^2} \leq \theta^n \quad (5.27)$$

For $m \geq 1$, $k \geq 1$, and $0 \leq r < P - 1$, one gets, using the fact that T_h is bounded by 1 on L^∞ and (5.24), (5.26), and (5.27)

$$\begin{aligned} \|T_{h,2}^{kP+r+m}\|_{L^\infty \rightarrow L^\infty} &= \|T_h^r M_h^k T_{h,2}^m\|_{L^\infty \rightarrow L^\infty} \leq \|M_h^k T_{h,2}^m\|_{L^\infty \rightarrow L^\infty} \\ &\leq \|A_{k,h} T_{h,2}^m\|_{L^\infty \rightarrow L^\infty} + \|B_{k,h} T_{h,2}^m\|_{L^\infty \rightarrow L^\infty} \\ &\leq Ch^{-A} \gamma^k + C_0 h^{-Q/2} \theta^m / (1 - \gamma). \end{aligned} \quad (5.28)$$

Thus we get, that there exists $C > 0$, $\mu > 0$, and a large constant $B \gg 1$ such that

$$\|T_{h,2}^n\|_{L^\infty \rightarrow L^\infty} \leq C e^{-\mu n}, \quad \forall h, \quad \forall n \geq B \log(1/h), \quad (5.29)$$

and thus the contribution of $T_{h,2}^n$ is far smaller than the bound we have to prove in (5.18). It remains to study the contribution of $T_{h,1}^n$.

From lemma 5.3, using the interpolation inequality $\|u\|_{L^2}^2 \leq \|u\|_{L^p}^{\frac{p}{p-1}} \|u\|_{L^1}^{\frac{p-2}{p-1}}$, we deduce the Nash inequality, with $1/d = 2 - 4/p > 0$

$$\|u\|_{L^2}^{2+1/d} \leq C(\mathcal{E}_h(u) + \|u\|_{L^2}^2) \|u\|_{L^1}^{1/d}, \quad \forall u \in E_{h,L} \quad (5.30)$$

For $\lambda_{j,h} \leq C_4 h^{-2}$, one has $h^2 \lambda_{j,h} \leq 1$, and thus for any $u \in E_{h,L}$, one gets $\mathcal{E}_h(u) \leq \|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2$, thus we get from 5.30

$$\|u\|_{L^2}^{2+1/d} \leq C h^{-2} (\|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2 + h^2 \|u\|_{L^2}^2) \|u\|_{L^1}^{1/d}, \quad \forall u \in E_{h,L} \quad (5.31)$$

From (5.29) and $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n$, we get that there exists C_2 such that for all h and all $n \geq B \log(1/h)$ one has $\|T_{1,h}^n\|_{L^\infty \rightarrow L^\infty} \leq C_2$ and thus since $T_{1,h}$ is self adjoint on L^2 , $\|T_{1,h}^n\|_{L^1 \rightarrow L^1} \leq C_2$. Fix $p \simeq B \log(1/h)$. Take $g \in L^2$ such that $\|g\|_{L^1} \leq 1$ and consider the sequence $c_n, n \geq 0$

$$c_n = \|T_{h,1}^{n+p} g\|_{L^2}^2 \quad (5.32)$$

Then, $0 \leq c_{n+1} \leq c_n$ and from 5.31 and $T_{h,1}^{n+p} g \in E_{h,L}$, we get

$$\begin{aligned} c_n^{1+\frac{1}{2d}} &\leq C h^{-2} (c_n - c_{n+1} + h^2 c_n) \|T_{h,1}^{n+p} g\|_{L^1}^{1/d} \\ &\leq C C_2^{1/d} h^{-2} (c_n - c_{n+1} + h^2 c_n) \end{aligned} \quad (5.33)$$

Thus there exist A which depends only on C, C_2, d , such that for all $0 \leq n \leq h^{-2}$, one has $c_n \leq (\frac{A h^{-2}}{1+n})^{2d}$ (this is the key point in the argument, for a proof of this estimate, see [DSC98]). Thus for all $0 \leq n \leq h^{-2}$ and with $p \simeq B \log(1/h)$ one has

$$\|T_{h,1}^{n+p} g\|_{L^2} \leq \left(\frac{A h^{-2}}{1+n}\right)^d \|g\|_{L^1} \quad (5.34)$$

which implies by duality since $T_{1,h}$ is selfadjoint on L^2

$$\|T_{h,1}^{n+p} g\|_{L^\infty} \leq \left(\frac{A h^{-2}}{1+n}\right)^d \|g\|_{L^2} \quad (5.35)$$

Thus there exist C_0 , such that for $N \simeq h^{-2}$, one has

$$\|T_{h,1}^{N+p} g\|_{L^\infty} \leq C_0 \|g\|_{L^2} \quad (5.36)$$

and so we get for any $m \geq 0$ and with $N \simeq h^{-2}$

$$\|T_{h,1}^{N+p+m} g\|_{L^\infty} \leq C_0 (1 - h^2 \lambda_{1,h})^m \|g\|_{L^2} \quad (5.37)$$

Thus for $n \geq h^{-2} + N + p$, since $h^2 \lambda_{1,h} = g(h)$ and $0 \leq (1-r)^m \leq e^{-mr}$ for $r \in [0, 1]$, we get

$$\|T_{h,1}^n\|_{L^\infty \rightarrow L^\infty} \leq C_0 e^{-(n-(N+p)g(h))} = C_0 e^{(N+p)g(h)} e^{-ng(h)} \leq C_0' e^{-ng(h)} \quad (5.38)$$

The proof of theorem 1.1 is complete.

5.2 Proof of theorem 1.2

The proof of Theorem 1.2 is exactly the same that the one given in [DLM12]. Let $R > 0$ be fixed. If $\nu_h \in [0, R]$ and $u_h \in L^2(M)$ satisfy $|\Delta_h|u_h = \nu_h u_h$ and $\|u_h\|_{L^2} = 1$, then, thanks to proposition 4.1, u_h can be decomposed as $u_h = v_h + w_h$ with $\|w_h\|_{L^2} = O(h)$ and v_h bounded in $\mathcal{H}^1(\mathcal{X})$. Hence (extracting a subsequence if necessary) it may be assumed that v_h weakly converges in $\mathcal{H}^1(\mathcal{X})$ to a limit v and that ν_h converges to a limit ν . Hence u_h converge strongly in L^2 to v . It now follows from proposition 5.1 that for any $f \in C^\infty(M)$,

$$\nu(f|v) = \lim_{h \rightarrow 0} (f|\nu_h u_h) = \lim_{h \rightarrow 0} (|\Delta_h|(f)|u_h) = \lim_{h \rightarrow 0} \mathcal{B}_h(f, v_h + w_h) = \frac{1}{6p} \sum_{k=1}^p (X_k f | X_k v)_{L^2} = (f|Lv) \quad (5.39)$$

Since f is arbitrary, it follows that $(L - \nu)v = 0$. By the Weyl type estimate (5.9) the number of eigenvalues $|\Delta_h|$ in the interval $[0, R]$ is uniformly bounded. Moreover, the dimension of an orthonormal basis is preserved by strong limit. So the above argument proves that for any $\epsilon > 0$ small, there exists $h_\epsilon > 0$ such that for $h \in]0, h_\epsilon]$, one has

$$\text{Spec}(|\Delta_h|) \cap [0, R] \subset \cup_j [\nu_j - \epsilon, \nu_j + \epsilon] \quad (5.40)$$

and

$$\#\text{Spec}(|\Delta_h|) \cap [\nu_j - \epsilon, \nu_j + \epsilon] \leq m_j \quad (5.41)$$

The fact that one has equality in (5.41) for ϵ small follows exactly like in the proof of theorem 2 iii) in [DLM12]: this use only proposition 5.1, the min-max principle and a compactness argument. The proof of theorem 1.2 is complete.

Remark 5.4 *Observe that the estimate (5.14) on the spectral gap is a direct consequence of theorem 1.2, and moreover observe that in the proof of theorem 1.2 we only use proposition 5.1 in the special case $f \in C^\infty(M)$, and that for $f \in C^\infty(M)$, proposition 5.1 is obvious. However, we think that the fact that proposition 5.1 holds true for any function $f \in \mathcal{H}^1(\mathcal{X})$ is interesting by itself, and since it is an easy byproduct of proposition 4.1, we decide to include it in the paper.*

5.3 Elementary Fourier Analysis

We conclude this section by collecting some basic results on the Fourier analysis theory (uniformly with respect to h) associated to the spectral decomposition of T_h . These results are consequences of the preceding estimates. We start with the following lemma which gives an honest L^∞ estimate on the eigenfunction $e_{j,h} \in E_{h,L}$. Recall $\langle x \rangle = (1 + x^2)^{1/2}$.

Lemma 5.5 *There exists C independent of h such that for any eigenfunction $e_{j,h} \in E_{h,L}$, $\|e_{j,h}\|_{L^2} = 1$, associated to the eigenvalue $1 - h^2 \lambda_{j,h}$ of T_h the following inequality holds true*

$$\|e_{j,h}\|_{L^\infty} \leq C \langle \lambda_{j,h} \rangle^d \quad (5.42)$$

Proof. This is a byproduct of the preceding estimate (5.35). Apply this inequality to $g = e_{j,h}$. This gives

$$(1 - h^2 \lambda_{j,h})^{n+p} \|e_{j,h}\|_{L^\infty} \leq \left(\frac{Ah^{-2}}{1+n} \right)^d \quad (5.43)$$

Thus we get with $n \simeq h^{-2} \langle \lambda_{j,h} \rangle^{-1}$

$$\|e_{j,h}\|_{L^\infty} \leq \left(\frac{Ah^{-2}}{h^{-2} \langle \lambda_{j,h} \rangle^{-1} + 1} \right)^d (1 - h^2 \lambda_{j,h})^{-h^{-2} \langle \lambda_{j,h} \rangle^{-1} - B \log(1/h)} \leq C \langle \lambda_{j,h} \rangle^d \quad (5.44)$$

The proof of lemma 5.5 is complete. \square

Let $h_0 > 0$ be a small given real number. We will use the following notations. If X is a Banach space, we denote by X_h the space $L^\infty([0, h_0], X)$, i.e the space of functions $h \mapsto x_h$ from $h \in]0, h_0]$ into X such that $\sup_{h \in]0, h_0]} \|x_h\|_X < \infty$. For $a \geq 0$, the notation $x_h \in O_X(h^a)$ means that there exists C independent of h such that $\|x_h\|_X \leq Ch^a$, and $x_h \in O_X(h^\infty)$ means $x_h \in O_X(h^a)$ for all a . We denote $C_h^\infty = \bigcap_{k \geq 0} C_h^k(M)$.

Let $\Pi_{h,L}$ be the L^2 -orthogonal projection on $E_{h,L}$, and denote $\Pi_{h,2} = Id - \Pi_{h,L}$. Let $(e_{j,h})_{j \in J_h}$ be an orthonormal basis of $E_{h,L}$ with $T_h(e_{j,h}) = (1 - h^2 \lambda_{j,h})e_{j,h}$. For $f \in L^2$ we denote by $c_{j,h}(f) = (f|e_{j,h})$ the corresponding Fourier coefficient of f . Recall that J_h is defined in (5.19).

Proposition 5.6 *Let $f_h \in C_h^\infty$. For all integer N , the following holds true.*

$$|\Delta_h|^N f_h \in C_h^\infty \quad \text{and} \quad \exists C_N, \quad \sup_{h \in]0, h_0]} \sum_{j \in J_h} \lambda_{j,h}^N |c_{j,h}(f_h)|^2 \leq C_N \quad (5.45)$$

Moreover, one has the following estimates

$$\Pi_{h,L}(f_h) \in O_{L^\infty(M)}(1) \quad (5.46)$$

$$\Pi_{h,2}(f_h) \in O_{L^\infty(M)}(h^N) \quad (5.47)$$

Proof. Let X be a vector field on M , and $f \in C^\infty(M)$. The smooth function $F(t, x) = f(e^{tX} x)$ satisfy the transport equation

$$\partial_t F = X(f), \quad F(0, x) = f(x)$$

Thus, one has by Taylor expansion at $t = 0$, and for any integer N

$$F(t, x) = \sum_{n \leq N} \frac{t^n}{n!} X^n(f)(x) + t^{N+1} r_N(t, x)$$

with $r_N(t, x)$ smooth. From the definition of T_h , we thus get

$$T_h f(x) = \sum_{n \text{ even} \leq N} \frac{h^n}{(n+1)!} \left(\frac{1}{p} \sum_{k=1}^p X_k^n(f)(x) \right) + h^{N+1} \tilde{r}_N(h, x)$$

with $\tilde{r}_N(h, x) \in C_h^\infty$. This implies for $f_h \in C_h^\infty$

$$|\Delta_h| f_h = L(f_h) + h^2 g_h, \quad g_h \in C_h^\infty$$

Therefore, one has $|\Delta_h| f_h \in C_h^\infty$, hence by induction $|\Delta_h|^N f_h \in C_h^\infty$ for all N . The second assertion of (5.45) follows from $\sup_{h \in]0, h_0]} \|g_h\|_{L^2} < \infty$ for any $g_h \in C_h^\infty$ and the fact

$$\sum_{j \in J_h} \lambda_{j,h}^N |c_{j,h}(f_h)|^2 = \|\Pi_{h,L} |\Delta_h|^N f_h\|_{L^2}^2 \leq \| |\Delta_h|^N f_h \|_{L^2}^2$$

For the proof of (5.46), we just write

$$\Pi_{h,L}(f_h) = \sum_{j \in J_h} c_{j,h}(f_h) e_{j,h}$$

and we use the estimate (5.42) of lemma 5.5 to get the bound

$$\begin{aligned} \|\Pi_{h,L}(f_h)\|_{L^\infty} &\leq C \sum_{j \in J_h} |c_{j,h}(f_h)| < \lambda_{j,h} >^d \\ &\leq C \left(\sum_{j \in J_h} |c_{j,h}(f_h)|^2 < \lambda_{j,h} >^{2d+2N} \right)^{1/2} \left(\sum_{j \in J_h} < \lambda_{j,h} >^{-2N} \right)^{1/2} \end{aligned}$$

From the Weyl type estimate (5.9), there exists N and C independent of h such that

$$\left(\sum_{j \in J_h} < \lambda_{j,h} >^{-2N} \right)^{1/2} \leq C$$

and therefore (5.46) follows from (5.45). It remains to prove the estimate (5.47). We first prove the weaker estimate

$$\Pi_{h,2}(f_h) \in O_{L^2(M)}(h^N) \quad (5.48)$$

Observe that $\Pi_{h,2}(f_h)$ satisfies for all $N \geq 1$ the equation

$$h^{2N} \Pi_{h,2}(|\Delta_h|^N f_h) = (h^2 |\Delta_h|)^N \Pi_{h,2}(f_h) = (Id - T_h \Pi_{h,2})^N \Pi_{h,2}(f_h) \quad (5.49)$$

By (5.27), the operator $Id - T_h \Pi_{h,2} = Id - T_{h,2}$ is invertible on L^2 with inverse bounded by $(1 - \theta)^{-1}$. Since $|\Delta_h|^N f_h \in C_h^\infty$ we get from (5.49) $\Pi_{h,2}(f_h) \in O_{L^2}(h^{2N})$.

Set $g_h = \Pi_{h,2}(f_h)$. One has $|\Delta_h|^N f_h = \Pi_{h,L}(|\Delta_h|^N f_h) + |\Delta_h|^N g_h$. From (5.45) and (5.46), one has $\Pi_{h,L}(|\Delta_h|^N f_h) \in O_{L^\infty}(1)$. Thus we get $|\Delta_h|^N g_h \in O_{L^\infty}(1)$ for any N . Let $M_h = T_h^P$, and $|\tilde{\Delta}_h| = (Id + T_h + \dots + T_h^{P-1})|\Delta_h|$. Then g_h satisfies the equation

$$h^2 |\tilde{\Delta}_h| g_h = g_h - M_h g_h \quad (5.50)$$

As in (5.25), write $M_h = \rho_h + R_h$. Since T_h is bounded by 1 on L^∞ , one gets

$$g_h - \rho_h g_h = h^2 r_h + R_h g_h, \quad r_h = |\tilde{\Delta}_h| g_h \in O_{L^\infty}(1). \quad (5.51)$$

By the second line of (5.25) and (5.48) one has $R_h g_h \in O_{L^\infty}(h^\infty)$, and by the first line of (5.25), the operator $Id - \rho_h$ is invertible on L^∞ with inverse bounded by $(1 - \gamma)^{-1}$. Thus we get from (5.51) $g_h \in O_{L^\infty}(h^2)$. Since $|\tilde{\Delta}_h| g_h = \Pi_{h,2}(|\tilde{\Delta}_h| f_h)$ and $|\tilde{\Delta}_h| f_h \in C_h^\infty$, the same estimates shows $|\tilde{\Delta}_h| g_h = r_h \in O_{L^\infty}(h^2)$. Then (5.51) implies $g_h \in O_{L^\infty}(h^4)$. By induction, we get $g_h \in O_{L^\infty}(h^{2N})$ for all N . The proof of proposition 5.6 is complete. \square

Let $F_k = Ker(L - \nu_k)$. Recall $m_k = dim(F_k)$ is the multiplicity of the eigenvalue ν_k of L . Let us denote by \mathcal{J}_k the set of indices j such that for h small, $\lambda_{j,h}$ is close to ν_k , and $F_{h,k} = span(e_{j,h}, j \in \mathcal{J}_k)$. By theorem 1.2 and his proof, the set \mathcal{J}_k is independent of $h \in]0, h_k]$ for h_k small, and one has $\sharp(\mathcal{J}_k) = dim(F_{h,k}) = k$ for $h \in]0, h_k]$. Let Π_{F_k} and $\Pi_{F_{h,k}}$ the L^2 -orthogonal projectors on F_k and $F_{h,k}$.

Lemma 5.7 *For all $f \in F_k$ one has*

$$\lim_{h \rightarrow 0} \|f - \Pi_{F_{h,k}}(f)\|_{L^\infty} = 0 \quad (5.52)$$

Proof. For $f \in F_k$, and h small, one has

$$f - \Pi_{F_{h,k}}(f) = \sum_{j \in J_h \setminus \mathcal{J}_k} c_{j,h}(f) e_{j,h} + \Pi_{h,2}(f) \quad (5.53)$$

One has $f \in C_h^\infty$, and thus by (5.47), we get

$$\Pi_{h,2}(f) \in O_{L^\infty}(h^\infty) \quad (5.54)$$

Since $f \in F_k$, for any given $j \in J_h \setminus \mathcal{J}_k$, one has $\lim_{h \rightarrow 0} c_{j,h}(f) = \lim_{h \rightarrow 0} (f|_{e_{j,h}})_{L^2} = 0$. Therefore, it remains to prove

$$\lim_{N \rightarrow \infty} \sup_{h \in]0, h_0]} \sum_{j \in J_h, j \geq N} |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} = 0 \quad (5.55)$$

Let $N \gg \nu_k$. From (5.42), Cauchy-Schwarz inequality, (5.45), and the Weyl type estimate (5.9), there exist N_0 and a constant $C(f)$ independent of h such that one has the estimate

$$\begin{aligned} \sum_{j \in J_h, j \geq N} |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} &\leq C \sum_{j \in J_h, j \geq N} |c_{j,h}(f)| < \lambda_{j,h} >^d \\ &\leq C \left(\sum_{j \in J_h} |c_{j,h}(f)|^2 < \lambda_{j,h} >^{2d+2N_0} \right)^{1/2} \left(\sum_{j \in J_h, j \geq N} < \lambda_{j,h} >^{-2N_0} \right)^{1/2} \\ &\leq C(f) \sup_{h \in]0, h_0]} \left(\sum_{j \in J_h, j \geq N} < \lambda_{j,h} >^{-2N_0} \right)^{1/2} \rightarrow 0 \quad (N \rightarrow \infty) \end{aligned} \quad (5.56)$$

In fact, since by (5.9) one has $\#\{j, \lambda_{j,h} \leq m\} \leq C_5 < m >^{dim(M)/2s}$, one can choose $N_0 = 1 + dim(M)/4s$. Then one has

$$\sup_{h \in]0, h_0]} \sum_{j \in J_h, j \geq N} < \lambda_{j,h} >^{-2N_0} \leq C_5 \sum_{m \geq m(N)} < m >^{-2N_0} < m + 1 >^{dim(M)/2s}$$

with $m(N)$ the bigger integer such that $\lambda_{N,h} \geq m(N)$ for any $h \in]0, h_0]$. Observe that (5.9) implies $\lim_{N \rightarrow \infty} m(N) = \infty$. The proof of lemma 5.7 is complete. \square

6 The hypoelliptic diffusion

We refer to the paper of J.-M. Bismut [Bis81] and references therein for a construction of the hypoelliptic diffusion associated to the generator L .

For a given $x_0 \in M$, let $X_{x_0} = \{\omega \in C^0([0, \infty[, M), \omega(0) = x_0\}$ be the set of continuous paths from $[0, \infty[$ to M , starting at x_0 , equipped with the topology of uniform convergence on compact subsets of $[0, \infty[$, and let \mathcal{B} be the Borel σ -field generated by the open sets in X_{x_0} . We denote by W_{x_0} the Wiener measure on X_{x_0} associated to the hypoelliptic diffusion with generator L . Let $p_t(x, y)d\mu(y)$ be the heat kernel, i.e the kernel of the self-adjoint operator e^{-tL} , $t \geq 0$. Then W_{x_0} is the unique probability on (X_{x_0}, \mathcal{B}) , such that for any $0 < t_1 < t_2 < \dots < t_k$ and any Borel sets A_1, \dots, A_k in M , one has

$$\begin{aligned} W_{x_0}(\omega(t_1) \in A_1, \omega(t_2) \in A_2, \dots, \omega(t_k) \in A_k) = \\ \int_{A_1 \times A_2 \times \dots \times A_k} p_{t_k - t_{k-1}}(x_k, x_{k-1}) \dots p_{t_2 - t_1}(x_2, x_1) p_{t_1}(x_1, x_0) d\mu(x_1) d\mu(x_2) \dots d\mu(x_k) \end{aligned} \quad (6.1)$$

Let us first introduce some notations. Let $Y = \{1, \dots, p\} \times [-1, 1]$ and let ρ be the uniform probability on Y . For any function $g(k, s)$ on Y , one has

$$\int_Y g d\rho = \frac{1}{2p} \sum_{k=1}^p \int_{-1}^{+1} g(k, s) ds \quad (6.2)$$

We denote by $Y^{\mathbb{N}}$ the infinite product space $Y^{\mathbb{N}} = \{\underline{y} = (y_1, y_2, \dots, y_n, \dots), y_j \in Y\}$. Equipped with the product topology, it is a compact metrisable space, and we denote by $\rho^{\mathbb{N}}$ the product probability on $Y^{\mathbb{N}}$. Let $M^{\mathbb{N}}$ be the infinite product space $M^{\mathbb{N}} = \{\underline{x} = (x_1, x_2, \dots, x_n, \dots), x_j \in M\}$. Equipped with the product topology, $M^{\mathbb{N}}$ is a compact metrisable space. For $h \in]0, 1]$, and $x_0 \in M$, let $\pi_{x_0, h}$ be the continuous map from $Y^{\mathbb{N}}$ into $M^{\mathbb{N}}$ defined by

$$\pi_{x_0, h}((k_j, s_j)_{j \geq 1}) = (x_j)_{j \geq 1}, \quad x_j = e^{s_j h X_{k_j}} \dots e^{s_2 h X_{k_2}} e^{s_1 h X_{k_1}} x_0 \quad (6.3)$$

We will use the notation $X_{h, x_0}^n = (\pi_{x_0, h})_n$. This means that X_{h, x_0}^n is the position after n step of the random walk starting at x_0 . Let $\mathcal{P}_{x_0, h}$ be the probability on $M^{\mathbb{N}}$ defined by $\mathcal{P}_{x_0, h} = (\pi_{x_0, h})_*(\rho^{\mathbb{N}})$. Then by construction, one has for all Borel sets A_1, \dots, A_k in M

$$\begin{aligned} \mathcal{P}_{x_0, h}(x_1 \in A_1, x_2 \in A_2, \dots, x_k \in A_k) \\ = \int_{A_1 \times A_2 \times \dots \times A_k} t_h(x_{k-1}, dx_k) \dots t_h(x_1, dx_2) t_h(x_0, dx_1) \end{aligned} \quad (6.4)$$

Let $j_{x_0, h}$ be the map from $Y^{\mathbb{N}}$ into X_{x_0} defined by, with $\underline{y} = ((k_j, s_j)_{j \geq 1})$

$$\begin{aligned} j_{x_0, h}(\underline{y}) = \omega \iff \forall j \geq 0, \forall t \in [0, h^2], \quad \omega(jh^2 + t) = e^{\frac{t}{h^2} h s_j X_{k_j}} x_j \\ \text{with } x_j = (\pi_{x_0, h}(\underline{y}))_j \text{ if } j \geq 1 \end{aligned} \quad (6.5)$$

Let $P_{x_0, h}$ be the probability on X_{x_0} defined as the image of $\rho^{\mathbb{N}}$ by the continuous map $j_{x_0, h}$. Our aim is to prove the following theorem of weak convergence of $P_{x_0, h}$ to the Wiener measure W_{x_0} when $h \rightarrow 0$.

Theorem 6.1 *For any bounded continuous function $\omega \mapsto f(\omega)$ on X_{x_0} , one has*

$$\lim_{h \rightarrow 0} \int f dP_{x_0, h} = \int f dW_{x_0} \quad (6.6)$$

Observe that the proof below shows that our study of the Markov kernel T_h on M is also a way to prove the existence of the Wiener measure W_{x_0} associated to the hypoelliptic diffusion. Let g be a Riemannian distance on M and let d_g the associated distance. We start by proving that the family of probability $P_{x_0, h}$ is tight, hence compact by the Prohorov theorem.

Proposition 6.2 *For any $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that the following holds true for any $T > 0$.*

$$\lim_{\delta \rightarrow 0} \left(\sup_{h \in]0, h_\varepsilon]} P_{x_0, h} \left(\max_{|s-t| \leq \delta, 0 \leq s, t \leq T} d_g(\omega(s), \omega(t)) > \varepsilon \right) \right) = 0 \quad (6.7)$$

Proof. We start with the following lemma.

Lemma 6.3 *Let $f \in C^\infty(M)$. There exists C such that for all $h \in]0, h_0]$, one has*

$$\forall \delta \in [0, 1], \quad \sup_{nh^2 \leq \delta} \|T_h^n(f) - f - nh^2 |\Delta_h| f\|_{L^\infty} \leq C \delta^2 \quad (6.8)$$

Proof. We may assume $\delta > 0$ and $n \geq 1$. Then $nh^2 \leq \delta$ implies $h \leq \sqrt{\delta}$. With the notation of section 5, one has

$$\begin{aligned} T_h^n(f) - f - nh^2 |\Delta_h| f &= \sum_{j \in J_h} c_{j, h}(f) \left((1 - h^2 \lambda_{j, h})^n - 1 - nh^2 \lambda_{j, h} \right) e_{j, h} + R(n, h) \\ R(n, h) &= T_h^n \Pi_{h, 2}(f) - \Pi_{h, 2}(f + nh^2 |\Delta_h| f) \end{aligned} \quad (6.9)$$

One has $|\Delta_h|f \in C_h^\infty$ by (5.45), T_h is bounded by 1 on L^∞ , and $nh^2 \leq \delta \leq 1$. Thus from (5.47) we get

$$\sup_{nh^2 \leq \delta} \|R(n, h)\|_{L^\infty} \in O(h^\infty) \subset O(\delta^\infty) \quad (6.10)$$

For all $j \in J_h$ one has $h^2\lambda_{j,h} \in [0, 1]$ and for all $x \in [0, 1]$

$$|(1-x)^n - 1 - nx| \leq \frac{n(n-1)}{2}x^2$$

Therefore we get

$$\left\| \sum_{j \in J_h} c_{j,h}(f) \left((1 - h^2\lambda_{j,h})^n - 1 - nh^2\lambda_{j,h} \right) e_{j,h} \right\|_{L^\infty} \leq \frac{n^2 h^4}{2} \sum_{j \in J_h} \lambda_{j,h}^2 |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} \quad (6.11)$$

By the Weyl type estimate (5.9), (5.42) and (5.45), there exists a constant C such that

$$\sup_{h \in]0, h_0]} \sum_{j \in J_h} \lambda_{j,h}^2 |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} \leq C$$

Therefore (6.8) is consequence of (6.10) and (6.11). The proof of lemma 6.3 is complete. \square

The proof of proposition 6.2 is now standard and proceeds as follows. Let $\varepsilon_0 > 0$ small with respect to the injectivity radius of the Riemannian manifold (M, g) , and let $\varepsilon \in]0, \varepsilon_0]$ be fixed. One has

$$\rho^{\mathbb{N}}(d_g(X_{h,x_0}^n, x_0) > \varepsilon) = \int_{d_g(y,x_0) > \varepsilon} t_h^n(x_0, dy) = T_h^n(1_{d_g(y,x_0) > \varepsilon})(x_0) \quad (6.12)$$

Let $\varphi(r) \in C^\infty([0, \infty[)$ be a nondecreasing function equal to 0 for $r \leq 3/4$ and equal to 1 for $r \geq 1$. Set

$$\varphi_{x_0, \varepsilon}(x) = \varphi\left(\frac{d_g(x, x_0)}{\varepsilon}\right) \quad (6.13)$$

Then $\varphi_{x_0, \varepsilon}$ is a smooth function, and from $1_{d_g(y,x_0) > \varepsilon} \leq \varphi_{x_0, \varepsilon} \leq 1$, we get since T_h is Markovian,

$$0 \leq T_h^n(1_{d_g(y,x_0) > \varepsilon}) \leq T_h^n(\varphi_{x_0, \varepsilon}) \quad (6.14)$$

Since T_h moves the support at distance $\leq ch$, one has $\varphi_{x_0, \varepsilon}(x_0) + nh^2(|\Delta_h|\varphi_{x_0, \varepsilon})(x_0) = 0$ for $ch \leq \varepsilon/2$. From lemma 6.3, we thus get that there exists $h_\varepsilon > 0$ and C_ε such that

$$\sup_{h \in]0, h_\varepsilon]} \sup_{nh^2 \leq \delta} T_h^n(\varphi_{x_0, \varepsilon})(x_0) \leq C_\varepsilon \delta^2 \quad (6.15)$$

Since M is compact, it is clear from the proof of lemma 6.3 that we may assume C_ε independent of $x_0 \in M$. From (6.12), (6.14) and (6.15) we get

$$\sup_{x_0 \in M} \sup_{h \in]0, h_\varepsilon]} \sup_{nh^2 \leq \delta} \rho^{\mathbb{N}}(d_g(X_{h,x_0}^n, x_0) > \varepsilon) \leq C_\varepsilon \delta^2 \quad (6.16)$$

Let $T > 0$ be given. One has for $h \in]0, h_\varepsilon]$ the following inequalities.

$$\begin{aligned} & \rho^{\mathbb{N}}(\exists j < l \leq h^{-2}T, (l-j)h^2 \leq \delta, d_g(X_{h,x_0}^j, X_{h,x_0}^l) > 4\varepsilon) \\ & \leq \frac{C}{\delta} \sup_{y_0 \in M} \rho^{\mathbb{N}}(\exists j < l \leq h^{-2}\delta, d_g(X_{h,y_0}^j, X_{h,y_0}^l) > 4\varepsilon) \\ & \leq \frac{C}{\delta} \sup_{y_0 \in M} \rho^{\mathbb{N}}(\exists j \leq h^{-2}\delta, d_g(X_{h,y_0}^j, y_0) > 2\varepsilon) \\ & \leq \frac{2C}{\delta} \sup_{z_0 \in M, nh^2 \leq \delta} \rho^{\mathbb{N}}(d_g(X_{z_0}^n, z_0) > \varepsilon) \end{aligned} \quad (6.17)$$

$$\text{(by (6.16))} \leq 2C C_\varepsilon \delta$$

In fact, for the first inequality in (6.17), we just use the fact that the interval $[0, T]$ is a union of $\simeq C/\delta$ intervals of length $\delta/2$. The second inequality is obvious since the event $\{\exists j < l \leq h^{-2}\delta, d_g(X_{h,y_0}^j, X_{h,y_0}^l) > 4\varepsilon\}$ is a subset of $\{\exists j \leq h^{-2}\delta, d_g(X_{h,y_0}^j, y_0) > 2\varepsilon\}$. For the third one, we use the fact that the event $A = \{\exists j \leq h^{-2}\delta, d_g(X_{h,y_0}^j, y_0) > 2\varepsilon\}$ is contained in $B \cup_{j < k} (C_j \cap D_j)$ with $B = \{d_g(X_{h,y_0}^k, y_0) > \varepsilon\}$ (k is the greatest integer $\leq \delta h^{-2}$), $C_j = \{d_g(X_{h,y_0}^j, X_{h,y_0}^k) > \varepsilon\}$, $D_j = \{d_g(X_{h,y_0}^j, y_0) > 2\varepsilon \text{ and } d_g(X_{h,y_0}^l, y_0) \leq 2\varepsilon \text{ for } l < j\}$, and the fact that C_j and D_j are independent and the D_j are disjoint.

Since $P_{x_0, h} = (j_{x_0, h})_*(\rho^{\mathbb{N}})$, (6.7) follows easily from (6.17) and the definition (6.5) of the map $j_{x_0, h}$. The proof of proposition 6.2 is complete. \square

With the result of proposition 6.2, the proof of theorem 6.1 follows now the classical proof of weak convergence of a sequence of random walks in the Euclidian space \mathbb{R}^d to the Brownian motion on \mathbb{R}^d , for which we refer to ([KS88], chapter 2.4). We have to prove that any weak limit P_{x_0} of a sequence P_{x_0, h_k} , $h_k \rightarrow 0$, is equal to the Wiener measure W_{x_0} . We denote by $\omega_h(t)$ the map from $Y^{\mathbb{N}}$ into M defined by $\omega_h(t)(y) = j_{x_0, h}(y)(t)$. By theorem 4.15 of [KS88] it is sufficient to show that for any $m \geq 1$, any $0 < t_1 < \dots < t_m$, and any continuous function $f(x_1, \dots, x_m)$ defined on the space M^m , one has

$$\lim_{h \rightarrow 0} \int_{Y^{\mathbb{N}}} f(\omega_h(t_1), \dots, \omega_h(t_m)) d\rho^{\mathbb{N}} = \int f(x_1, \dots, x_m) p_{t_m - t_{m-1}}(x_m, x_{m-1}) \dots p_{t_2 - t_1}(x_2, x_1) p_{t_1}(x_1, x_0) d\mu(x_1) d\mu(x_2) \dots d\mu(x_m) \quad (6.18)$$

As in [KS88], we may assume $m = 2$. For a given $t \geq 0$, let $n(t, h) \in \mathbb{N}$ be the greatest integer such that $h^2 n(t, h) \leq t$. By (6.5), one has for some $c > 0$ independent of h and $\underline{y} \in Y^{\mathbb{N}}$, $d_g(\omega_h(t), X_{h, x_0}^{n(t, h)}) \leq ch$. Since f is uniformly continuous on M^m , we are reduce to prove

$$\begin{aligned} & \lim_{h \rightarrow 0} \int f(X_{h, x_0}^{n(t_1, h)}, X_{h, x_0}^{n(t_2, h)}) d\rho^{\mathbb{N}} \\ &= \int f(x_1, x_2) p_{t_2 - t_1}(x_2, x_1) p_{t_1}(x_1, x_0) d\mu(x_1) d\mu(x_2) \end{aligned} \quad (6.19)$$

From (6.4), one has

$$\begin{aligned} & \int f(X_{h, x_0}^{n(t_1, h)}, X_{h, x_0}^{n(t_2, h)}) d\rho^{\mathbb{N}} \\ &= \int f(x_1, x_2) t_h^{n(t_2, h) - n(t_1, h)}(x_1, dx_2) t_h^{n(t_1, h)}(x_0, dx_1) \end{aligned} \quad (6.20)$$

By (6.19), (6.20), we have to show that for any continuous function $f(x_1, x_2)$ on the product space $M \times M$, one has

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{M \times M} f(x_1, x_2) t_h^{n(t_2, h) - n(t_1, h)}(x_1, dx_2) t_h^{n(t_1, h)}(x_0, dx_1) \\ &= \int_{M \times M} f(x_1, x_2) p_{t_2 - t_1}(x_2, x_1) p_{t_1}(x_1, x_0) d\mu(x_1) d\mu(x_2) \end{aligned} \quad (6.21)$$

or equivalently

$$\begin{aligned} & \lim_{h \rightarrow 0} T_h^{n(t_1, h)} \left(T_h^{n(t_2, h) - n(t_1, h)} (f(x_1, \cdot))(x_1) \right) (x_0) \\ &= e^{-t_1 L} \left(e^{-(t_2 - t_1)L} (f(x_1, \cdot))(x_1) \right) (x_0) \end{aligned} \quad (6.22)$$

Since $\|T_h^{n(t,h)}\|_{L^\infty} \leq 1$ and $\|e^{-tL}\|_{L^\infty} \leq 1$, the following "central limit" theorem will conclude the proof of theorem 6.1.

Lemma 6.4 *For all $f \in C^0(M)$, and all $t > 0$, one has*

$$\lim_{h \rightarrow 0} \|e^{-tL}(f) - T_h^{n(t,h)}(f)\|_{L^\infty} = 0 \quad (6.23)$$

Since one has $\|T_h^{n(t,h)}\|_{L^\infty} \leq 1$ and $\|e^{-tL}\|_{L^\infty} \leq 1$, it is sufficient to prove that (6.23) holds true for $f \in \mathcal{D}$, with \mathcal{D} a dense subset of the space $C^0(M)$, and therefore we may assume that $f \in F_k$ is an eigenvector of L associated to the eigenvalue ν_k . We set $n = n(t, h)$, and we use the notation of section 5. One has

$$\begin{aligned} T_h^n(f) &= \sum_{j \in \mathcal{J}_k} c_{j,h}(f)(1 - h^2 \lambda_{j,h})^n e_{j,h} + R_{t,h}(f) \\ R_{t,h}(f) &= \sum_{j \in \mathcal{J}_h \setminus \mathcal{J}_k} c_{j,h}(f)(1 - h^2 \lambda_{j,h})^n e_{j,h} + T_h^n \Pi_{h,2}(f) \end{aligned} \quad (6.24)$$

One has $|(1 - h^2 \lambda_{j,h})^n| \leq 1$ and T_h is bounded by 1 on L^∞ . By (5.54) and (5.55), we thus get

$$\lim_{h \rightarrow 0} \|R_{t,h}(f)\|_{L^\infty} = 0$$

One has $\lim_{h \rightarrow 0} (1 - h^2 \lambda_{j,h})^{n(t,h)} = e^{-t\nu_k}$ for all $j \in \mathcal{J}_k$. Moreover, one has $\#\mathcal{J}_k = m_k$ and $\sup_{h \in]0, h_0]} \sup_{j \in \mathcal{J}_k} \|e_{j,h}\|_{L^\infty} < \infty$ by lemma 5.5. Therefore lemma 5.7 and $e^{-tL}(f) = e^{-t\nu_k} f$ implies

$$\lim_{h \rightarrow 0} \left\| \sum_{j \in \mathcal{J}_k} c_{j,h}(f)(1 - h^2 \lambda_{j,h})^n e_{j,h} - e^{-tL}(f) \right\|_{L^\infty} = 0$$

The proof of lemma 6.4 is complete. \square

7 Appendix

Let $P = P(x, \partial_x)$ be an elliptic second order differential operator on M , with smooth coefficients, such that $P = P^* \geq Id$, where P^* is the formal adjoint on $L^2(M, \mu) = L^2$. Let $(e_j)_{j \geq 1}$ be an orthonormal basis of eigenfunctions of P in L^2 , and $1 \leq \nu_1 \leq \nu_2 \dots$ be the associated eigenvalues. By the classical Weyl formula, one has

$$\#\{j, \nu_j^{1/2} \leq r\} \simeq r^{\dim(X)} \quad (7.1)$$

For $s \in \mathbb{R}$ and $f = \sum_j f_j e_j$ in the Sobolev space $H^s(M)$, we set

$$\|v\|_{H^s}^2 = \sum_j \nu_j^s |f_j|^2 = (P^s f | f)_{L^2}$$

Let us recall that this H^s -norm depends on P , but an other choice for P gives an equivalent norm. The following elementary lemma is useful for us.

Lemma 7.1 *Let $s > 0$ and $A_h = A_h^* \geq 0$, $h \in]0, 1]$ a family of non negative self-adjoint bounded operators acting on $L^2(M, \mu)$. Assume that there exists a constant $C_0 > 0$ independent of h such that*

$$((Id + A_h)u | u) \leq 1 \Rightarrow \exists (v, w) \in H^s \times L^2 \text{ such that } u = v + w, \|v\|_{H^s} \leq C_0, \|w\|_{L^2} \leq C_0 h \quad (7.2)$$

Let $C_1 < \frac{1}{4C_0^2}$. There exists $C_2 > 0$ independent of h such that $\text{Spec}(A_h) \cap [0, \lambda - 1]$ is discrete for all $\lambda \leq C_1 h^{-2}$ and

$$\#(\text{Spec}(A_h) \cap [0, \lambda - 1]) \leq C_2 < \lambda >^{dim(M)/2s}, \quad \forall \lambda \leq C_1 h^{-2} \quad (7.3)$$

Here, $\#(\text{Spec}(A_h) \cap [0, r])$ is the number of eigenvalues of A_h in the interval $[0, r]$ with multiplicities, and $< \lambda > = \sqrt{1 + \lambda^2}$.

Proof. Let $B_h = Id + A_h$. Let C_h be the bounded operator on L^2 defined by

$$C_h \left(\sum_j u_j e_j \right) = \sum_j \min(h^{-1}, \nu_j^{s/2}) u_j e_j$$

For $u = v + w$ one has

$$\|C_h u\|_{L^2}^2 \leq 2\|C_h v\|_{L^2}^2 + 2\|C_h w\|_{L^2}^2 \leq 2(\|v\|_{H^s}^2 + h^{-2}\|w\|_{L^2}^2)$$

From (7.2), we get for all $u \in L^2$

$$\|C_h u\|_{L^2}^2 \leq 4C_0^2 (B_h u | u) \quad (7.4)$$

For any non negative selfadjoint bounded operator T on L^2 , set for $j \geq 1$

$$\lambda_j(T) = \min_{dim(F)=j} \left(\max_{u \in F, \|u\|_{L^2}=1} (Tu | u) \right)$$

It is well known that if $\#\{j, \lambda_j(T) \in [0, a[] < \infty$, the spectrum of T in $[0, a[$ is discrete and in that case, the $\lambda_j(T) \in [0, a[$ are the eigenvalues of T in $[0, a[$ with multiplicities. From (7.4), we get for all $j \geq 1$ the inequality

$$\lambda_j(B_h) \geq \frac{1}{4C_0^2} \lambda_j(C_h^2) \quad (7.5)$$

For all j such that $\nu_j^s < h^{-2}$, one has $\lambda_j(C_h^2) = \nu_j^s$, and therefore, for all $\lambda < h^{-2}$, we get from (7.1), $\#\{j, \lambda_j(C_h^2) \leq \lambda\} \leq C < \lambda >^{dim(M)/2s}$. Therefore, the spectrum of B_h in $[0, h^{-2}/4C_0^2[$ is discrete, and (7.3) follows from (7.5) and $\text{Spec}(A_h) = \text{Spec}(B_h) - 1$. The proof of lemma 7.1 is complete. \square

Lemma 7.2 *Let $\mathcal{N} = \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_\mathfrak{r}$ be the free up to rank \mathfrak{r} nilpotent Lie algebra with p generators. Let (Y_1, \dots, Y_p) be a basis of \mathcal{N}_1 and let $(\mathcal{Z}_1, \dots, \mathcal{Z}_p)$ be the right invariant vector fields on \mathcal{N} such that $\mathcal{Z}_j(0) = Y_j$. Let $\mathcal{S}(\mathcal{N})$ be the Schwartz space of \mathcal{N} . Let $\varphi \in \mathcal{S}(\mathcal{N})$, be such that $\int_{\mathcal{N}} \varphi dx = 0$. Then there exists $\varphi_k \in \mathcal{S}(\mathcal{N})$ such that*

$$\varphi = \sum_{k=1}^p \mathcal{Z}_k(\varphi_k) \quad (7.6)$$

Proof. Let $Y^\alpha = H_\alpha(Y_1, \dots, Y_p)$ and let \mathcal{Z}^α be the right invariant vector fields on \mathcal{N} such that $\mathcal{Z}^\alpha(0) = Y^\alpha$. Let $u_\alpha, \alpha \in \mathcal{A}$ be the coordinates on \mathcal{N} associated to the basis $(Y^\alpha, \alpha \in \mathcal{A})$ of \mathcal{N} . Let ∂_α be the derivative in the direction of u_α . Let $\varphi \in \mathcal{S}(\mathcal{N})$ such that $\int_{\mathcal{N}} \varphi dx = 0$. Using the Fourier transform in coordinates (u_α) , and $\hat{\varphi}(0) = 0$, one get easily that there exists functions $\psi_\alpha \in \mathcal{S}(\mathcal{N})$ such that

$$\varphi = \sum_{\alpha \in \mathcal{A}} \partial_\alpha(\psi_\alpha) \quad (7.7)$$

By (2.3), the vector field \mathcal{Z}^α is of the form

$$\mathcal{Z}^\alpha = \partial_\alpha + \sum_{|\beta| > |\alpha|} p_{\alpha,\beta}(u_{<|\beta|}) \partial_\beta = \partial_\alpha + \sum_{|\beta| > |\alpha|} \partial_\beta p_{\alpha,\beta}(u_{<|\beta|})$$

where the $p_{\alpha,\beta}$ are polynomials in u depending only on (u_1, \dots, u_j) with $j < |\beta|$. Therefore, there exists polynomials $q_{\alpha,\beta}$ such that

$$\partial_\alpha = \mathcal{Z}^\alpha + \sum_{|\beta| > |\alpha|} \mathcal{Z}^\beta q_{\alpha,\beta}$$

Since the Schwartz space $\mathcal{S}(\mathcal{N})$ is stable by multiplication by polynomials, we get from (7.7) that there exists $\phi_\alpha \in \mathcal{S}(\mathcal{N})$ such that

$$\varphi = \sum_{\alpha \in \mathcal{A}} \mathcal{Z}^\alpha(\phi_\alpha) \tag{7.8}$$

For $|\alpha| > 1$, there exists $j \in \{1, \dots, p\}$ and β with $|\beta| = |\alpha| - 1$ such that $\mathcal{Z}^\alpha = \mathcal{Z}_j \mathcal{Z}^\beta - \mathcal{Z}^\beta \mathcal{Z}_j$. By induction on $|\alpha|$, since the Schwartz space $\mathcal{S}(\mathcal{N})$ is stable by the vector fields \mathcal{Z}_j , this shows that for any α and $\phi \in \mathcal{S}(\mathcal{N})$, there exists $\phi_j \in \mathcal{S}(\mathcal{N})$ such that $\mathcal{Z}^\alpha(\phi) = \sum_{j=1}^p \mathcal{Z}_j(\phi_j)$. Thus (7.6) follows from (7.8). The proof of lemma 7.2 is complete. \square

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