HAL
open science

Convergence to steady states for solutions of a reaction-diffusion equation with mass conservation

Samira Boussaïd, Danielle Hilhorst, Thanh Nam Nguyen

To cite this version:

Samira Boussaïd, Danielle Hilhorst, Thanh Nam Nguyen. Convergence to steady states for solutions of a reaction-diffusion equation with mass conservation. 2015. hal-00817614v2

HAL Id: hal-00817614

https://hal.science/hal-00817614v2

Preprint submitted on 27 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

CONVERGENCE TO STEADY STATE FOR THE SOLUTIONS OF A NONLOCAL REACTION-DIFFUSION EQUATION

Samira Boussaïd

Département de Mathématiques, Université de Batna, Algérie
and LAMFA, Université de Picardie Jules Verne,
33 Rue Saint-Leu, 80039 Amiens Cedex 1, France

Danielle Hilhorst

CNRS, Laboratoire de Mathématique, Analyse Numérique et EDP
Université de Paris-Sud, F-91405 Orsay Cedex, France

Thanh Nam Nguyen

Laboratoire de Mathématique, Analyse Numérique et EDP
Université de Paris-Sud, F-91405 Orsay Cedex, France

(Communicated by xxxxx)

Abstract. We consider a nonlocal reaction-diffusion equation with mass conservation, which was originally proposed by Rubinstein and Sternberg as a model for phase separation in a binary mixture. We study the large time behavior of the solution and show that it converges to a stationary solution as t tends to infinity. We also evaluate the rate of convergence. In some special case, we show that the limit solution is constant.

1. **Introduction.** We consider the nonlocal initial value problem

$$(P)\begin{cases} u_t = \Delta u + f(u) - \fint_\Omega f(u) & \text{in } \Omega\times\mathbb{R}^+,\\ \partial_\nu u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+,\\ u(x,0)=u_0(x) & x\in\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N (N\geq 1)$ is a connected bounded open set with smooth boundary $\partial\Omega$; ∂_ν is the outer normal derivative to $\partial\Omega$ and

$$\fint_\Omega f(u) := \frac{1}{|\Omega|}\int_\Omega f(u(x))\,dx.$$

This model is mass conserved, namely

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \text{for all } t>0,$$

and it possesses a free energy functional which coincides with the usual Allen-Cahn functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx,$$

2010 *Mathematics Subject Classification.* Primary: 35K57, 35K20; Secondary: 35B40.

Key words and phrases. Gradient flow, Łojasiewicz inequality, stabilization of solutions, mass-conserved Allen-Cahn equation, infinite dimensional dynamical systems.

where $F(u) := \int_0^u f(s)ds$.

Problem (P) was introduced by Rubinstein and Sternberg [30] as a model for phase separation in a binary mixture. Although this problem is a nonlocal problem, we can prove the existence of an invariant set. The main result of this paper concerns the large time behavior. We show that the solution converges to a stationary solution as t tends to infinity and evaluate the rate of convergence. In some special case, we show that the limit solution is constant.

In the general case, the main tool to study the large time behavior is a Łojasiewicz inequality that was first proposed by Lojasiewicz himself [25], [27]. He showed that all bounded solutions of gradient systems in $\mathbb{R}^N$, which are systems of ordinary differential equations, converge to a stationary solution. This idea was subsequently developed in infinite-dimensional spaces for proving the convergence to steady state for bounded solutions of several local equations such as reaction-diffusion equations, wave equations and degenerate parabolic equations [11], [12], [13], [14], [20], [21] [31], [32]; let us also mention some results on nonlocal phase-field models [10], [24], and the book on gradient inequalities by Huang [18].

In the case where f is supposed to be nonincreasing on an interval containing the range of the initial function, instead of applying Łojasiewicz inequality, we show that the ω-limit set only contains a unique element. This follows from the monotony of f and the mass conservation property. This result is related to [4, Theorem 3.9, page 88] where the author studies the asymptotic behavior of solutions for parabolic equations with a monotone operator. In our case, although the operator is not monotone because of the nonlocal term, we overcome this difficulty by using the mass conservation property.

In [30], the authors consider Problem (P) with f of bistable type; a typical example is $f(s) = s - s^3$. In this paper, we assume that the function f is of the form

$$f(s) = \sum_{i=0}^{n} a_i s^i, \quad \text{where } n \geq 1 \text{ is an odd number, } a_n < 0. \tag{1.1}$$

Constants $\mathbf{s_1, s_2}$: We suppose that $s_1 < s_2$ are two constants such that

$$f(s_2) < f(s) < f(s_1) \quad \text{for all } s \in (s_1, s_2). \tag{1.2}$$

Note that we can choose s_1, s_2 such that s_1 is negative with large absolute value and s_2 is arbitrarily large.

Assumption on initial data: We will make the following hypotheses on the initial data:

$(\mathbf{H_0})$: $u_0 \in L^2(\Omega)$ and $s_1 \leq u_0(x) \leq s_2$ for a.e $x \in \Omega$.

This paper is organized as follows: in Section 2, a result on existence, uniqueness and boundedness of solutions is presented. Section 3 is devoted to prove a version of the Łojasiewicz inequality. In Section 4, we apply the Łojasiewicz inequality to prove that as $t \to +\infty$, $u(t)$ converges to a stationary solution, which we precisely compute in the case of one space dimension. The convergence rate is established in Section 5.

2. Existence, uniqueness and boundedness of solutions.

We first prove some properties of solutions of Problem (P). To begin with, we set

$$Q_T := \Omega \times (0, T), \quad T > 0.$$

Lemma 2.1 (Mass conservation)**.** *Let u be a solution of Problem (P) such that*

$$u \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \text{ and } u_t \in L^2(0,T;(H^1(\Omega))^*).$$

Then

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)\,dx \text{ for all } t > 0. \tag{2.1}$$

Proof. We take the duality product of the equation for u by 1 to obtain

$$\frac{d}{dt}\int_\Omega u + \int_\Omega \nabla u\,\nabla 1 = \int_\Omega f(u)1 - \fint_\Omega f(u)\int_\Omega 1.$$

Therefore

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = 0,$$

which implies (2.1). □

Proposition 1 (Invariant set)**.** *Let $T > 0$. Assume that $u \in C^{2,1}(\overline{\Omega} \times (0,T]) \cap C(\overline{Q}_T)$ is a solution of Problem (P) and that*

$$s_1 < u_0(x) < s_2 \quad \text{for all } x \in \Omega.$$

Then

$$s_1 < u(x,t) < s_2 \quad \text{for all } x \in \overline{\Omega}, 0 < t \le T.$$

Proof. For the purpose of contradiction, we suppose that there exists a first time $t_0 > 0$ such that $u(x_0,t_0) = s_1$ or $u(x_0,t_0) = s_2$ for some $x_0 \in \overline{\Omega}$. Without loss of generality, assume that $u(x_0,t_0) = s_2$. By the continuity of u and the definition of t_0, we have

$$s_1 \le u(x,t_0) \le s_2 \text{ for all } x \in \overline{\Omega}, \text{ and } u(x,t) < s_2 \text{ for all } x \in \overline{\Omega} \text{ and } 0 \le t < t_0. \tag{2.2}$$

Since $\partial_\nu u = 0$, we deduce from Hopf's maximum principle that $x_0 \in \Omega$. Therefore the function $u(\cdot,t_0)$ attains its maximum at $x_0 \in \Omega$, which implies that $\Delta u(x_0,t_0) \le 0$. By (2.2), we have

$$u_t(x_0,t_0) = \lim_{\Delta t \to 0^+} \frac{u(x_0,t_0-\Delta t) - u(x_0,t_0)}{-\Delta t} \ge 0,$$

which we substitute in Problem (P) to obtain $\fint_\Omega (f(s_2) - f(u(x,t_0)))\,dx \ge 0$. Since $s_1 \le u(x,t_0) \le s_2$ for all $x \in \Omega$, it follows from (1.2) that $f(s_2) \le f(u(x,t_0))$ for all $x \in \Omega$ so that $f(s_2) = f(u(x,t_0))$. Using (1.2) again, we obtain $u(x,t_0) = s_2$ for all x in Ω. As a consequence, we have

$$\int_\Omega u(x,t_0)\,dx = s_2|\Omega| > \int_\Omega u_0(x)\,dx,$$

which contradicts the integral preserving property in Lemma 2.1. □

Theorem 2.2. *Assume that Hypothesis $(\mathbf{H_0})$ holds. Then Problem (P) possesses a unique solution $u \in C([0,\infty);L^2(\Omega))$ which satisfies for every $T > 0$,*

$$u \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)) \text{ and } u_t \in L^2(0,T;(H^1(\Omega))^*).$$

Moreover,

$$\begin{aligned} &u \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega} \times [\varepsilon,\infty)) \text{ for all } \alpha \in (0,1), \varepsilon > 0, \\ &s_1 \le u(x,t) \le s_2 \quad \text{for all } x \in \overline{\Omega}, t > 0, \end{aligned} \tag{2.3}$$

and

$$\{u(t) : t \geq 1\} \text{ is relatively compact in } C^1(\overline{\Omega}). \tag{2.4}$$

In order to prove Theorem 2.2, we need some technical lemmas.

Lemma 2.3. *Let $u_0 \in L^2(\Omega), g \in L^p(Q_T)$ for some $p \in (1,\infty)$ and let u be the solution of the time evolution problem*

$$\begin{cases} u_t - \Delta u = g & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Then for each $0 < \varepsilon < T$, there exists a positive constant $C_0(\varepsilon, \Omega, T)$ such that

$$\|u\|_{W_p^{2,1}(Q_\varepsilon^T)} \leq C_0(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^p(Q_T)}),$$

where $Q_\varepsilon^T = \Omega \times (\varepsilon, T)$.

Remark 1. If $T = 1$, then the constant C_0 depends only on ε and Ω.

Lemma 2.4. *One has the following embedding*

$$W_p^{2,1}(Q_T) \subset C^{\lambda,\lambda/2}(\overline{Q}_T) \text{ with } \lambda = 2 - \frac{N+2}{p} \text{ if } p > \frac{N+2}{2} \text{ and } p \neq N+2.$$

Lemma 2.3 and Lemma 2.4 follow from [23, chapter 4, section 3 and chapter 2, section 3] which are stated in [5, page 206].

Proof of Theorem 2.2. One can prove in a standard way (see also Proposition 1) the existence and uniqueness of the solution of Problem (P) as well as (2.3). Next we prove (2.4). Let $\alpha \in (0,1)$, $p := \dfrac{N+2}{1-\alpha}$. Since $s_1 \leq u(t) \leq s_2$ for all $t \geq 0$, it follows that

$$\begin{aligned} \left\| f(u) - \fint_\Omega f(\ u) \right\|_{L^p(Q_0^1)} &\leq |\Omega|^{1/p} \left\| f(u) - \fint_\Omega f(\ u) \right\|_{L^\infty(Q_0^1)} \\ &\leq 2|\Omega|^{1/p} \sup_{s_1 \leq s \leq s_2} |f(s)|. \end{aligned}$$

We apply Lemma 2.3 and the embedding in Lemma 2.4 on domain Q_0^1 to obtain

$$\begin{aligned} \|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q}_\varepsilon^1)} &\leq C\left(\|u_0\|_{L^2(\Omega)} + \left\| f(u) - \fint_\Omega f(\ u) \right\|_{L^p(Q_0^1)} \right) \\ &\leq C\left(|\Omega|^{1/2} \|u_0\|_{L^\infty(\Omega)} + 2|\Omega|^{1/p} \sup_{s_1 \leq s \leq s_2} |f(s)| \right) \\ &\leq C\left(|\Omega|^{1/2}(|s_1| + |s_2|) + 2|\Omega|^{1/p} \sup_{s_1 \leq s \leq s_2} |f(s)| \right). \end{aligned}$$

Similarly, we apply Lemma 2.3 and the embedding in Lemma 2.4 on the domains Q_k^{k+1} and $Q_{k+1/2}^{k+3/2}$ to obtain

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q}_{k+\varepsilon}^{k+1})} \leq C\left(|\Omega|^{1/2}(|s_1| + |s_2|) + 2|\Omega|^{1/p} \sup_{s_1 \leq s \leq s_2} |f(s)| \right).$$

and a similar one on the domain $Q_{k+1/2}^{k+3/2}$. Finally, we deduce from the fact that k can be chosen arbitrarily large that

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q}_\varepsilon^\infty)} \leq C\left(|\Omega|^{1/2}(|s_1| + |s_2|) + 2|\Omega|^{1/p} \sup_{s_1 \leq s \leq s_2} |f(s)| \right),$$

which yields (2.4). □

As in the proof of Theorem 2.2, we apply Lemma 2.3 to deduce that $u \in W_2^{2,1}(Q_\varepsilon^T)$. This implies the following result

Corollary 1. *The solution u of Problem (P) satisfies*

$$u \in L^2(\varepsilon, T; H^2(\Omega)) \text{ and } u_t \in L^2(\varepsilon, T; L^2(\Omega)) \text{ for all } \varepsilon \in (0, T).$$

As a consequence, $u \in C((0, T]; H^1(\Omega))$.

3. **A version of Łojasiewicz inequality.** The main result of this section is the Łojasiewicz inequality stated in Theorem 3.7 below. More precisely, we prove a version of Łojasiewicz inequality for the functional E which coincides with the functional $\mathcal{E}$ on the solution orbits. We set

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega \bar{F}(u)dx,$$

where $\bar{F} \in C_c^\infty(\mathbb{R})$ is such that

$$\bar{F}(s) = \begin{cases} F(s) & \text{if } s \in [s_1 - 1, s_2 + 1] \\ \\ 0 & \text{if } s \in (-\infty, s_1 - 2) \cup (s_2 + 2, +\infty). \end{cases}$$

Then

$$E(u(t)) = \mathcal{E}(u(t)) \text{ for all } t > 0.$$

We define $\bar{f} = \bar{F}'$, then

$$\bar{f}(s) = f(s) \text{ for all } s \in [s_1 - 1, s_2 + 1].$$

This section is organized as follows: In Section 3.1, as a preparation for the proof of Theorem 3.7, we prove the differentiability of E and compute its derivative. The definition and some equivalent conditions of a critical point are given. The Łojasiewicz inequality is proved in Section 3.2.

3.1. **Some preparations.** We define the spaces

$$H = \{u \in L^2(\Omega) : \int_\Omega u(x)dx = 0\}, \text{ equipped with the norm } \|\cdot\|_H := \|\cdot\|_{L^2(\Omega)},$$

$$V = \Big\{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\Big\}, \text{ equipped with the norm } \|\cdot\|_V := \|\cdot\|_{H^1(\Omega)}.$$

Let V^* be the dual space of V. We identify H with its dual to obtain:

$$V \hookrightarrow H \hookrightarrow V^*,$$

where the embeddings $V \hookrightarrow H$, $H \hookrightarrow V^*$ are continuous, dense and compact (see e.g. [22, p. 677]). We use $\langle \cdot, \cdot \rangle$ to denote the duality product between V^* and V. We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from a Banach space X to a second Banach space Y, and we write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

We also define the spaces

$$\mathscr{L}^p(\Omega) := \{u \in L^p(\Omega) : \int_\Omega u(x)\, dx = 0\}, \tag{3.1}$$

equipped with the norm $\|\cdot\|_{\mathscr{L}^p(\Omega)} := \|\cdot\|_{L^p(\Omega)}$ and

$$X_p := \{u \in W^{2,p}(\Omega) : \partial_\nu u = 0, \int_\Omega u(x)dx = 0\}, \tag{3.2}$$

equipped with the norm $\|\cdot\|_{X_p} := \|\cdot\|_{W^{2,p}(\Omega)}$. Throughout the sequel, we denote by $C \geq 0$ a generic constant which may vary from line to line. We start with the following result.

Lemma 3.1. *Let $u, h \in L^1(\Omega)$, $p \in [1, \infty)$ be arbitrary and let g be a continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$|g(s)|, |g'(s)| \leq C \text{ for all } s \in \mathbb{R}. \tag{3.3}$$

Then

$$\int_0^1 g(u + \tau h) d\tau \to g(u) \quad \text{in } L^p(\Omega) \text{ as } \|h\|_{L^1(\Omega)} \to 0.$$

Proof. By Jensen's inequality and (3.3),

$$\begin{aligned}\left|\int_0^1 (g(u+\tau h) - g(u)) d\tau\right|^p &\leq \int_0^1 |g(u+\tau h) d\tau - g(u)|^p d\tau \\ &\leq C \int_0^1 |g(u+\tau h) d\tau - g(u)| d\tau \\ &\leq C|h|.\end{aligned}$$

Thus

$$\left(\int_\Omega \left|\int_0^1 (g(u+\tau h) - g(u)) d\tau\right|^p\right)^{\frac{1}{p}} \leq C \left(\int_\Omega |h|\right)^{\frac{1}{p}}.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *The functional E is twice continuously Fréchet differentiable on V. We denote by E' and L the first and second derivatives of E, respectively. Then*

(i) The first derivative

$$E' : V \longrightarrow V^* \text{ is given by}$$

$$\langle E'(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h - \int_\Omega \bar{f}(u) h \quad \text{for all } u, h \in V. \tag{3.4}$$

(ii) The second derivative

$$L : V \longrightarrow \mathcal{L}(V, V^*) \text{ is given by}$$

$$\langle L(u)h, k\rangle_{V^*,V} = \int_\Omega \nabla h \nabla k - \int_\Omega \bar{f}'(u) hk \quad \text{for all } u, h, k \in V. \tag{3.5}$$

As a consequence,

$$\langle L(u)h, k\rangle_{V^*,V} = \langle h, L(u)k\rangle_{V,V^*}. \tag{3.6}$$

Proof. We write E as the difference of E_1 and E_2, where

$$E_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx \text{ and } E_2(u) = \int_\Omega \bar{F}(u) \, dx. \tag{3.7}$$

Obviously, E_1 is twice continuously Fréchet differentiable. Its derivatives are easily identified in the formulas (3.4) and (3.5). We now compute the first and second derivative of E_2.

(i) By Taylor formula,

$$\bar{F}(u+h) - \bar{F}(u) = h \int_0^1 \bar{f}(u + \tau h) d\tau =: h\zeta \text{ for all } u, h \in V,$$

where

$$\zeta(x) := \int_0^1 \bar{f}(u(x) + \tau h(x))d\tau.$$

It follows that

$$\begin{aligned}\left|E_2(u+h) - E_2(u) - \int_\Omega \bar{f}(u)h\,dx\right| &\le \int_\Omega |\zeta - \bar{f}(u)|\,|h|\,dx \\ &\le C\|\zeta - \bar{f}(u)\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)} \\ &\le C\|\zeta - \bar{f}(u)\|_{L^2(\Omega)}\|h\|_V.\end{aligned}$$

We deduce from Lemma 3.1 that

$$\zeta = \int_0^1 \bar{f}(u + \tau h)d\tau \to \bar{f}(u) \text{ in } L^2(\Omega) \text{ as } \|h\|_V \to 0.$$

Therefore

$$\left|E_2(u+h) - E_2(u) - \int_\Omega \bar{f}(u)h\,dx\right| = o(\|h\|_V) \text{ as } \|h\|_V \to 0.$$

This implies that the first derivative E_2' exists and

$$\langle E_2'(u), h\rangle_{V^*,V} = \int_\Omega \bar{f}(u)h\,dx.$$

(ii) The Fréchet differentiability of E_2' is shown in a similar way. Choose $p \in (2, +\infty)$ such that V is continuously embedded in $L^p(\Omega)$. Let T be the linear mapping from V to V^* given by

$$\langle T\,h, k\rangle_{V^*,V} = \int_\Omega \bar{f}'(u)h\,k\,dx.$$

We will use below a generalized Hölder inequality based on the identity

$$\frac{1}{p} + \frac{1}{p} + \frac{p-2}{p} = 1.$$

For every $u, h, k \in V$ and for

$$\eta(x) := \int_0^1 \bar{f}'(u(x) + \tau h(x))d\tau$$

we have

$$\begin{aligned}&\left|\langle E_2'(u+h) - E_2'(u) - Th, k\rangle_{V^*,V}\right| \\ &\qquad \le \int_\Omega |\eta - \bar{f}'(u)|\,|h|\,|k|\,dx \\ &\qquad \le \|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_{L^p(\Omega)}\|k\|_{L^p(\Omega)} \\ &\qquad \le C\|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V\|k\|_V. \qquad (3.8)\end{aligned}$$

Consequently, we have

$$\|E_2'(u+h) - E_2'(u) - T\,h\|_{V^*} \le C\|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V. \qquad (3.9)$$

Since $1 < p/(p-2) < +\infty$, we deduce from Lemma 3.1 that

$$\|\eta - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)} \to 0 \quad \text{as } \|h\|_V \to 0,$$

which together with (3.9) follows that

$$\|E_2'(u+h) - E_2'(u) - Th\|_{V^*} = o(\|h\|_V).$$

Therefore,

$$\langle E_2''(u)h, k\rangle_{V^*,V} = \int_\Omega \bar{f}'(u)h\,k \text{ for all } u, h, k \in V.$$

We also note that

$$\begin{aligned}|\langle (E_2''(u) - E_2''(v))h, k\rangle_{V^*,V}| &\le \int_\Omega |\bar{f}'(u) - \bar{f}'(v)|\,|h|\,|k|\,dx \\ &\le C\|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V\|k\|_V.\end{aligned}$$

Hence

$$\|E_2''(u) - E_2''(v)\|_{\mathcal{L}(V,V^*)} \le C\|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/p-2}(\Omega)},$$

which implies the continuity of E_2''. Finally, (3.6) is an immediate consequence of (3.5). □

We define a continuous bilinear form from $V \times V \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega \nabla u \nabla v\,dx.$$

The following lemma is an immediate consequence of the Lax-Milgram theorem (cf. [3, Corollary 5.8]). We omit its proof.

Lemma 3.3. *There exists an isomorphism A from V onto V^* such that*

$$a(u, v) = \langle Au, v\rangle_{V^*,V} \text{ for all } u, v \in V. \tag{3.10}$$

Corollary 2. *The first and second derivatives of E can be represented in V^* as:*

$$E'(u) = Au - \bar{f}(u) + \fint_\Omega \bar{f}(u), \tag{3.11}$$

$$L(u)h = Ah - \bar{f}'(u)h + \fint_\Omega \bar{f}'(u)h, \tag{3.12}$$

for all $u, h \in V$.

Proof. Since $\bar{f}$ is bounded, $\bar{f}(u) - \fint_\Omega \bar{f}(u) \in H \hookrightarrow V^*$. Therefore

$$Au - \bar{f}(u) + \fint_\Omega \bar{f}(u) \in V^*.$$

We also note that

$$\int_\Omega \left(\fint_\Omega \bar{f}(u)\right)h = \fint_\Omega \bar{f}(u) \int_\Omega h = 0 \text{ for all } h \in V,$$

thus

$$\langle Au - \bar{f}(u) + \fint_\Omega \bar{f}(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h - \int_\Omega \bar{f}(u)h.$$

This together with (3.4) implies that

$$E'(u) = Au - \bar{f}(u) + \fint_\Omega \bar{f}(u).$$

Identity (3.12) may be proved in a similar way. □

Lemma 3.4. *Let $\mathcal{L}^p(\Omega), X_p$ be the Banach spaces as in (3.1) and (3.2). Assume that $p \geq 2$. Then, for any $g \in \mathscr{L}^p(\Omega)$, there exists a unique solution $u \in X_p$ of the equation*

$$Au = g \text{ in } V^*.$$

Moreover,

$$\langle Aw, v\rangle = \langle -\Delta w, v\rangle \text{ for all } w \in X_p, v \in V. \tag{3.13}$$

Proof. It follows from Lemma 3.3 that the equation

$$Au = g \text{ in } V^* \tag{3.14}$$

has a unique solution $u \in V$ so that it is enough to prove that $u \in X_p$. For this purpose, we consider the elliptic problem

$$\begin{cases} -\Delta \tilde{u} = g & \text{in } \Omega, \\ \partial_\nu \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $g \in H$, we apply the Fredholm alternative to deduce that this problem possesses a unique solution $\tilde{u} \in V$. Note that $g \in \mathscr{L}^p(\Omega)$, so that we deduce from [2] that $\tilde{u} \in W^{2,p}(\Omega)$ so that also $\tilde{u} \in X_p$. On the other hand, for all $v \in V$, we have

$$\langle A\tilde{u}, v\rangle_{V^*,V} = a(\tilde{u}, v) = \int_\Omega \nabla\tilde{u}\nabla v\, dx = \langle -\Delta\tilde{u}, v\rangle_{V^*,V} = \langle g, v\rangle_{V^*,V}.$$

Therefore, $\tilde{u}$ coincides with the unique solution of equation (3.14). In other words,

$$u = \tilde{u} \in X_p.$$

Moreover, for all $w \in X_p, v \in V$,

$$\langle -\Delta w, v\rangle_{V^*,V} = \int_\Omega \nabla w \nabla v\, dx = \langle Aw, v\rangle_{V^*,V}.$$

Then (3.13) follows. □

Definition 3.5. We say that $\varphi \in V$ is a critical point of E if

$$E'(\varphi) = 0 \quad \text{in} \quad V^*.$$

Lemma 3.6. *For every $\varphi \in V$, the following assertions are equivalent:*

(i) φ is a critical point of E,
(ii) $\varphi \in X_2$ and φ satisfies the equations

$$(S) \begin{cases} -\Delta\varphi = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) & \text{in } \Omega, \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. $(i) \Rightarrow (ii)$ Assume that $\varphi \in V$ is a critical point of E. We deduce from (3.11) that

$$A(\varphi) = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) \quad \text{in } V^*.$$

Since $\bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) \in H$, it follows from Lemma 3.4 that $\varphi \in X_2$ satisfies the equations

$$\begin{cases} -\Delta\varphi = \bar{f}(\varphi) - \fint_\Omega \bar{f}(\varphi) & \text{in } \Omega, \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

$(ii) \Rightarrow (i)$ It follows from (3.11) that

$$E'(\varphi) = A\varphi - \bar{f}(\varphi) + \fint_\Omega \bar{f}(\varphi),$$

which together with (3.13) implies that

$$E'(\varphi) = -\Delta\varphi - \bar{f}(\varphi) + \fint_\Omega \bar{f}(\varphi) = 0,$$

where the last identity follows from the fact that φ is a solution of Problem (S). Thus φ is a critical point of E.

□

3.2. **Łojasiewicz inequality.**

Theorem 3.7 (Łojasiewicz inequality)**.** *Let $\varphi \in V$ be a critical point of the functional E such that $s_1 \le \varphi \le s_2$. Then there exist constants $\theta \in (0, \frac{1}{2}]$ and $C, \sigma > 0$ such that*

$$|E(u) - E(\varphi)|^{1-\theta} \le C\|E'(u)\|_{V^*}, \tag{3.15}$$

for all $\|u - \varphi\|_V \le \sigma$. In this case, we say that E satisfies the Łojasiewicz inequality in φ. The number θ will be called the Lojasiewicz exponent.

We check below that all hypotheses in [8, Corollary 3.11] are satisfied so that the result of Theorem 3.7 will follow from [8, Corollary 3.11]. We need the following result.

Lemma 3.8. *Let φ be a critical point of E. Then, $L(\varphi)$ is a Fredholm operator from V to V^* of index 0 i.e. $\operatorname{Rg} L(\varphi)$ is closed in V^* and*

$$\dim\ker L(\varphi) = \operatorname{codim}(\operatorname{Rg} L(\varphi)) < +\infty,$$

where $\operatorname{codim}\operatorname{Rg} L(\varphi) := \dim(V^*/\operatorname{Rg} L(\varphi))$. As a consequence, V^* is the direct sum of $\operatorname{Rg} L(\varphi)$ and $\ker L(\varphi)$.

Proof. We first prove that the linear operator

$$\begin{aligned} T : V &\longrightarrow V^* \\ h &\longmapsto -\bar{f}'(\varphi)h + \fint_\Omega \bar{f}'(\varphi)h \end{aligned}$$

is compact. Indeed, note that we have for all $h \in V$

$$\begin{aligned} \|Th\|_H &\le \|\bar{f}'(\varphi)h\|_{L^2(\Omega)} + \left\| \fint_\Omega \bar{f}'(\varphi)h \right\|_{L^2(\Omega)} \\ &\le C\|h\|_{L^2(\Omega)} \le C\|h\|_V. \end{aligned}$$

Therefore T is continuous from V to H, which together with the compactness of the embedding $H \hookrightarrow V^*$ implies that T is compact from V to V^*.

Next, since A is an isomorphism from V onto V^*, it is also a Fredholm operator of index

$$\operatorname{ind} A := \dim\ker A - \operatorname{codim}\operatorname{Rg} A = 0.$$

It follows that $L(\varphi) = A + T$, as a sum of a Fredholm operator and a compact operator, is also a Fredholm operator with the same index (cf. [3, p. 168]). This completes the proof of Lemma 3.8. □

Before proving Theorem 3.7, we recall the definition of an analytic map on a neighborhood of a point (cf. [33, Definition 8.8, p. 362]). A map T from a Banach space X into a Banach space Y is called analytic on a neighborhood of $z \in X$ if there exists $\varepsilon > 0$ such that for all $h \in X$, $\|h\|_X \leq \varepsilon$,

$$T(z+h) - T(z) = \sum_{i \geq 1} T_i(z)[h, \dots, h] \text{ in } Y,$$

where $T_i(z)$ is a symmetric i-linear form on X with values in Y and

$$\sum_{i \geq 1} \|T_i(z)\|_{\mathcal{L}_i(X,Y)} \|h\|_X^i < \infty.$$

Here, $\mathcal{L}_i(X, Y)$ is the space of bounded i-linear operators from X^i to Y.

Proof of Theorem 3.7. In order prove Theorem 3.7, we apply [8, Corollary 3.11] for

$$X := X_p, \quad Y := \mathscr{L}^p(\Omega),$$

where $p > N$. In this case, there holds the embedding $W^{2,p}(\Omega) \subset C^{1,\lambda}(\overline{\Omega})$ with $\lambda = 1 - \dfrac{n}{p}$. Note that

$$E'(u) = -\Delta u - \bar{f}(u) + \fint_\Omega \bar{f}(u) \in \mathscr{L}^p(\Omega),$$

for all $u \in X_p$. In view of Lemma 3.8, it is sufficient to prove that E' is analytic in a neighborhood of φ. Indeed, let ε be small enough such that for all $h \in X_p$ with $\|h\|_{X_p} \leq \varepsilon$, we have

$$\|h\|_{C(\overline{\Omega})} \leq C\|h\|_{X_p} < 1.$$

Since

$$\bar{f}(s) = f(s) = \sum_{i=0}^n a_i s^i \quad \text{for all } s \in (s_1 - 1, s_2 + 1),$$

we perform a Taylor's expansion to deduce for all $h \in X_p$ with $\|h\|_{X_p} \leq \varepsilon$ that

$$\bar{f}(\varphi(x) + h(x)) - \bar{f}(\varphi(x)) = \sum_{i=1}^n \frac{\bar{f}^{(i)}(\varphi(x))}{i!} h^i(x).$$

It follows that

$$\begin{aligned} E'(\varphi + h) - E'(\varphi) &= -\Delta h + \sum_{i=1}^n \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i - \sum_{i=1}^n \fint_\Omega \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i \, dx \\ &= \sum_{i=1}^n T_i[h, \dots, h], \end{aligned}$$

where

$$T_1[h] := -\Delta h + \bar{f}'(\varphi)h - \fint_\Omega \bar{f}'(\varphi)h$$

and

$$T_i[h, \dots, h] := \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i - \fint_\Omega \frac{\bar{f}^{(i)}(\varphi)}{i!} h^i \quad \text{for all } 1 < i \leq n.$$

We now prove that $T_i \in \mathcal{L}_i(X_p, \mathscr{L}^p(\Omega))$. For all $h_1, \dots, h_i \in X_p$, and $1 < i \le n$, we have

$$\begin{aligned}\Big\|T_i[h_1,..,h_i]\Big\|_{\mathscr{L}^p(\Omega)} &\le C\Big\|T_i[h_1,..,h_i]\Big\|_{L^\infty(\Omega)}\\ &\le C\Big\|\frac{\bar f^{(i)}(\varphi)}{i!}h_1\dots h_i\Big\|_{L^\infty(\Omega)} + C\Big\|\fint_\Omega \frac{\bar f^{(i)}(\varphi)}{i!}h_1\dots h_i\Big\|_{L^\infty(\Omega)}\\ &\le C\prod_{j=1}^i \|h_j\|_{L^\infty(\Omega)} \le C\prod_{j=1}^i \|h_j\|_{X_p},\end{aligned}$$

which implies that $T_i \in \mathcal{L}_i(X_p, \mathscr{L}^p(\Omega))$ for all $1 < i \le n$. In the case $i = 1$, since $-\Delta$ is linear, continuous from X_p to $\mathscr{L}^p(\Omega)$, we easily deduce that $T_1 \in \mathcal{L}(X_p, \mathscr{L}^p(\Omega))$. Therefore E' is analytic on a neighborhood of φ. This completes the proof of Theorem 3.7. □

4. **Large time behavior.**

Theorem 4.1. *Let* $(\mathbf{H_0})$ *hold and let* u *be the unique solution of Problem* (P)*. Then there exists a function* φ *such that*

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{C^1(\overline{\Omega})} = 0 \text{ as } t \to \infty.$$

Moreover, $s_1 \le \varphi \le s_2$,

$$\int_\Omega \varphi = \int_\Omega u_0,$$

and φ *is a solution of the stationary problem*

$$(S)\begin{cases}\Delta\varphi = -f(\varphi) + \fint_\Omega f(\varphi) & \text{in } \Omega,\\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega.\end{cases}$$

This section is devoted to the proof of Theorem 4.1 by applying the Łojasiewicz inequality. In some case we also compute the limit stationary solution (see Theorem 4.5 below).

Lemma 4.2. *Suppose that* $(\mathbf{H_0})$ *is satisfied and let* u *be the solution of Problem* (P)*. Then*

(i) For all $0 < s \le t < \infty$,

$$E(u(s)) = E(u(t)) + \int_s^t \int_\Omega |u_t|^2\,dx. \tag{4.1}$$

(ii) Further, $E(u(\cdot))$ *is continuous, nonincreasing on* $(0, +\infty)$*, and there exists* e *such that*

$$\lim_{t\to\infty} E(u(t)) = e.$$

Proof. (i) In view of Corollary 1, for $t > 0$ we have

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \int_\Omega \big(-\Delta u - \bar f(u)\big)u_t\\ &= \int_\Omega \big(-\Delta u - \bar f(u) + \fint_\Omega \bar f(u)\big)u_t\\ &= -\int_\Omega u_t^2(x,t)\,dx \le 0.\end{aligned}$$

As a consequence, for all $0 < s \leq t < \infty$

$$E(u(s)) = E(u(t)) + \int_s^t \int_\Omega |u_t|^2 \, dx.$$

(ii) We recall that the function $\bar{F}$ is bounded on $\mathbb{R}$. Therefore the function $t \mapsto E(u(t))$, which is nonincreasing and bounded from below, converges to a limit as $t \to \infty$. □

Definition 4.3. We define the ω-limit set of u_0 by

$$\omega(u_0) := \{\varphi \in H^1(\Omega) : \exists t_n \to \infty, u(t_n) \to \varphi \text{ in } H^1(\Omega) \text{ as } n \to \infty\}.$$

Lemma 4.4. *Suppose that* $(\mathbf{H_0})$ *is satisfied and let* u *be the solution of Problem* (P)*. Then*

(i) $\omega(u_0)$ *is a non-empty, compact set of* $H^1(\Omega)$*.*

(ii) For all $\varphi \in \omega(u_0)$

$$E(\varphi) = e,$$

where e *is defined as in Lemma 4.2(ii).*

(iii) Let $\varphi \in \omega(u_0)$ *then* $s_1 \leq \varphi \leq s_2$ *and is a stationary solution of Problem* (P)*, which implies that*

$$\begin{cases} -\Delta\varphi = f(\varphi) - \fint_\Omega f(\varphi) & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

(iv) $d(u(t), \omega(u_0)) \to 0$ *as* $t \to \infty$*, where*

$$d(u(t), \omega(u_0)) := \inf_{\varphi \in \omega(u_0)} \|u(t) - \varphi\|_{H^1(\Omega)}.$$

Proof. (i) This is an immediate consequence of the relative compactness of solution orbits in $H^1(\Omega)$ which is a consequence of Theorem 2.2.

(ii) Let $\varphi \in \omega(u_0)$ and let $\{u(t_n)\}$ be such that

$$u(t_n) \to \varphi \text{ in } H^1(\Omega) \text{ as } n \to +\infty.$$

We deduce from the continuity of E on $H^1(\Omega)$ that

$$E(\varphi) = \lim_{n\to\infty} E(u(t_n) = e,$$

where e is as in Lemma 4.2.

(iii) Since

$$s_1 \leq u(x,t) \leq s_2 \quad \text{for all } x \in \overline{\Omega}, t > 0,$$

It follows that

$$s_1 \leq \varphi(x) \leq s_2 \text{ for all } x \in \overline{\Omega}. \tag{4.2}$$

Next, we prove that φ is a stationary solution. We denote here by $u(t; w)$ the solution of Problem (P) corresponding to initial function w. Let $\{t_n\}$ be such that

$$u(t_n; u_0) \to \varphi \text{ in } H^1(\Omega) \text{ as } n \to \infty.$$

This implies in particular that

$$u(t_n; u_0) \to \varphi \text{ in } L^2(\Omega) \text{ as } n \to \infty.$$

It follows that for all $t \geq 0$,

$$u(t; u(t_n; u_0)) \to u(t; \varphi) \text{ in } L^2(\Omega) \text{ as } n \to \infty.$$

In other words,

$$u(t+t_n;u_0) \to u(t;\varphi) \text{ in } L^2(\Omega) \text{ as } n \to \infty.$$

Since $\{u(\tau;u_0) : \tau \geq 1\}$ is relatively compact in $H^1(\Omega)$, so that

$$u(t_n+t;u_0) \to u(t;\varphi) \text{ in } H^1(\Omega) \text{ as } n \to \infty.$$

It follows that $u(t;\varphi) \in \omega(u_0)$. This together with (ii) implies that for all $t \geq s \geq 0$.

$$E(u(t;\varphi)) = E(u(s;\varphi)) = e.$$

In view of Lemma 4.2 , we have $t \geq s > 0$.

$$0 = E(u(t;\varphi)) - E(u(s;\varphi)) = -\int_s^t \int_\Omega |u_t(\varphi)|^2 \, dxdt.$$

As a consequence, for all $t > 0$, $u_t(t;\varphi) = 0$. In other words, $u(t;\varphi)$ the solution of Problem (P) with the initial function φ is independent of time. Therefore φ is a stationary solution of Problem (P), which implies that

$$\begin{cases} -\Delta\varphi = f(\varphi) - \fint_\Omega f(\varphi) & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

The identity

$$\int_\Omega \varphi = \int_\Omega u_0,$$

follows from the mass conservation property.

(iv) For the purpose of contradiction, we assume that there exists a sequence $t_n \to \infty$ and $\varepsilon_0 > 0$ such that

$$d(u(t_n), \omega(u_0)) \geq \varepsilon_0 \text{ for all } n > 0. \tag{4.3}$$

Note that there exists a subsequence $t_{n_k} \to \infty$ and $w \in H^1(\Omega)$ such that

$$u(t_{n_k}) \to w \in \omega(u_0) \text{ in } H^1(\Omega) \text{ as } k \to \infty,$$

Therefore, $d(u(t_{n_k}), \omega(u_0)) = 0$ as $k \to \infty$, which is in contradiction with (4.3). □

Proof of Theorem 4.1. We will first prove Theorem 4.1 in the case

$$\int_\Omega u_0(x) = 0.$$

By the mass conservation property, we have

$$\int_\Omega u(x,t) = 0.$$

As a consequence, $u(t) \in V$ for all $t > 0$. Recall from Lemma 4.4(ii) that

$$E\big|_{\omega(u_0)} = e. \tag{4.4}$$

It follows from Lemma 4.4(iii) and Lemma 3.6 that for all $\varphi \in \omega(u_0)$,

$$s_1 \leq \varphi \leq s_2$$

and that φ is a critical point of E. We apply Theorem 3.7 to deduce that E satisfies the Łojasiewicz inequality in the neighborhood of every $\varphi \in \omega(u_0)$. In other words, for every $\varphi \in \omega(u_0)$ there exist constants $\theta \in (0, \frac{1}{2}], C \geq 0$ and $\delta > 0$ such that

$$|E(v) - E(\varphi)|^{1-\theta} \leq C\|E'(v)\|_{V^*} \text{ whenever } \|v - \varphi\|_V \leq \delta. \tag{4.5}$$

Since E is continuous on V, we may choose δ small enough so that

$$|E(v) - E(\varphi)| < 1 \text{ whenever } \|v - \varphi\|_V \leq \delta. \tag{4.6}$$

It follows from the compactness of $\omega(u_0)$ in V that there exists a neighborhood $\mathcal{U}$ of $\omega(u_0)$ composed of finitely many balls $B_j, j = 1, ..., J$, with center φ_j and radius δ_j. In each of the ball B_j, inequality (4.6) and the Łojasiewicz inequality (4.5) hold for some constants θ_j and C_j. We define $\bar{\theta} = \min\{\theta_j, j = 1, ..., J\}$ and $\bar{C} = \max\{C_j, j = 1, ..., J\}$ to deduce from (4.4), (4.5) and (4.6) that

$$|E(v) - e|^{1-\bar{\theta}} \leq \bar{C}\|E'(v)\|_{V^*} \text{ for } v \in \mathcal{U}.$$

It follows from Lemma 4.4(iv) that there exists $t_0 \geq 0$ such that $u(t) \in \mathcal{U}$ for all $t \geq t_0$. Hence, for every $t \geq t_0$, there holds

$$\begin{aligned} -\frac{d}{dt}|E(u(t)) - e|^{\bar{\theta}} &= \bar{\theta}|E(u(t)) - e|^{\bar{\theta}-1}\Big(-\frac{dE}{dt}(u(t))\Big) \\ &\geq \frac{\bar{\theta}}{\bar{C}}\frac{\|u_t\|^2_{L^2(\Omega)}}{\|E'(u(t))\|_{V^*}}, \end{aligned} \tag{4.7}$$

where we have also used (4.1). Note that for all $t \geq t_0$, $E'(u(t)) \in H$ and it can be written of the form

$$E'(u(t)) = -\Delta u - \bar{f}(u) + \fint_\Omega \bar{f}(u) = -u_t.$$

Applying the continuous embedding $H \hookrightarrow V^*$, we have

$$\|E'(u(t))\|_{V^*} \leq \hat{C}\|E'(u(t))\|_{L^2(\Omega)} = \hat{C}\|u_t\|_{L^2(\Omega)} \text{ for all } t \geq t_0, \tag{4.8}$$

where $\hat{C}$ is a positive constant. Combining (4.7) and (4.8) we obtain

$$-\frac{d}{dt}|E(u(t)) - e|^{\bar{\theta}} \geq \widetilde{C}\|u_t\|_{L^2(\Omega)}.$$

Here $\widetilde{C} = \dfrac{\bar{\theta}}{\bar{C}\hat{C}}$. Thus

$$\|u(t_1) - u(t_2)\|_{L^2(\Omega)} \leq \int_{t_1}^{t_2} \|u_t\|_{L^2(\Omega)} \leq \frac{1}{\widetilde{C}}(|E(u(t_1)) - e|^{\bar{\theta}} - |E(u(t_2)) - e|^{\bar{\theta}})$$

for all $t_0 \leq t_1 \leq t_2$. Therefore $\|u(t_1) - u(t_2)\|_{L^2(\Omega)}$ tends to zero as $t_1 \to \infty$ so that $\{u(t)\}$ is a Cauchy sequence in H. As a consequence, there exists $\varphi \in H$ such that $\lim_{t\to\infty} u(t) = \varphi$ exists in H, hence by the relative compactness of solution orbits in $C^1(\overline{\Omega})$ we have

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{C^1(\overline{\Omega})} = 0.$$

In the general case, when

$$\int_\Omega u_0(x)\,dx \neq 0,$$

instead of considering Problem (P), we consider the Problem $(\widehat{P})$:

$$(\widehat{P})\begin{cases}\widehat{u}_t = \Delta\widehat{u} + \hat{f}(\widehat{u}) - \fint_\Omega \hat{f}(\widehat{u}) & \text{in } \Omega\times\mathbb{R}^+,\\ \partial_\nu\widehat{u} = 0 & \text{on } \partial\Omega\times\mathbb{R}^+,\\ \widehat{u}(x,0) = u_0(x) - m_0, & x\in\Omega.\end{cases}$$

where $m_0 := \fint_\Omega u_0$, and $\hat{f}(s) := \bar{f}(s+m_0)$. We note that

$$s_1 - m_0 \le \hat{u}(x,0) \le s_2 - m_0, \qquad \int_\Omega \hat{u}(x,0) = 0.$$

Moreover,

$$u = \widehat{u} + m_0$$

and $\hat{f}$ is analytic on $(s_1 - 1 - m_0, s_2 + 1 - m_0)$. Repeating the above arguments for Problem $(\widehat{P})$, we deduce that there exists a smooth stationary solution ψ of Problem $(\widehat{P})$ such that

$$\lim_{t\to\infty} \|\widehat{u}(t) - \psi\|_{C^1(\overline{\Omega})} = 0.$$

It follows that for $\varphi := \psi + m_0$, we have

$$\lim_{t\to\infty} \|u(t) - \varphi\|_{C^1(\overline{\Omega})} = 0.$$

The proof of Theorem 4.1 is complete. □

Theorem 4.5. *We suppose that the hypothesis (H_0) is satisfied. We assume further that*

$$f'(s) \le 0 \text{ for all } s\in[s_1,s_2].$$

Then

$$u(t) \to \fint_\Omega u_0 \text{ in } C^1(\overline{\Omega}) \text{ as } t\to\infty.$$

Proof. Let $\varphi\in\omega(u_0)$; it is sufficient to show that

$$\varphi(x) \equiv \fint_\Omega u_0 =: m_0. \tag{4.9}$$

First we note that φ satisfies $\fint_\Omega \varphi = m_0$ and

$$(S)\begin{cases}-\Delta\varphi = f(\varphi) - \fint_\Omega f(\varphi) & \text{in } \Omega,\\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega.\end{cases}$$

Then we multiply the partial differential equation in (S) by φ and integrate over Ω to obtain

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2 &= \int_\Omega f(\varphi)\varphi - \frac{1}{|\Omega|}\int_\Omega f(\varphi)\int_\Omega \varphi\\ &= \int_\Omega f(\varphi)(\varphi - m_0)\\ &= \int_\Omega (f(\varphi) - f(m_0))(\varphi - m_0) \le 0.\end{aligned}$$

Thus by Poincaré inequality

$$\int_\Omega |\varphi - m_0|^2 \le 0,$$

which yields (4.9).

□

5. **Convergence rate.** In this section, we evaluate the rate of the convergence of the solution to the stationary solution. The proof is based once more on the Łojasiewicz inequality. We consider two cases: the Lojasiewicz exponent $\theta = \frac{1}{2}$ and $\theta \in (0, \frac{1}{2})$. These cases were studied by Haraux and Jendoubi [14] and Haraux, Jendoubi and Kavian [15].

5.1. **The case $\theta = \frac{1}{2}$.** We will apply the following result.

Lemma 5.1 (see [14], Lemma 2.2)**.** *Let $t_0 \ge 0$ be arbitrary. Assume that there exist positive constants γ and a such that*

$$\int_t^{+\infty} \|u_t\|^2_{L^2(\Omega)}\, ds \le a \exp(-\gamma t) \quad \text{for all } t \ge t_0.$$

Then for all $\tau \ge t \ge t_0$,

$$\|u(t) - u(\tau)\|_{L^2(\Omega)} \le \sqrt{a} b \exp(-\frac{\gamma t}{2}),$$

where $b := \dfrac{\exp(\frac{\gamma}{2})}{\exp(\frac{\gamma}{2}) - 1}$.

Theorem 5.2. *Let $(\mathbf{H_0})$ hold. Assume further that Theorem 3.7 holds for $\theta = \frac{1}{2}$; then there exist positive constants K, δ such that*

$$\|u(t) - \varphi\|_{L^\infty(\Omega)} \le K \exp(-\delta t) \quad \text{for all } t \ge 0.$$

Proof. As in the proof of Theorem 4.1, it is sufficient to prove this result for the function u with the assumption that

$$\int_\Omega u_0 = 0.$$

We have

$$\frac{d}{dt}(E(u) - E(\varphi)) = \langle E'(u), u_t\rangle = -\langle E'(u), E'(u)\rangle = -\|E'(u)\|^2_H. \tag{5.1}$$

Note that

$$u(t) \to \varphi \text{ in } V \text{ as } t \to \infty,$$

we deduce that for σ as in Theorem 3.7 there exists $T_0 > 0$ such that for all $t \ge T_0$

$$\|u(t) - \varphi\|_V \le \sigma.$$

Therefore, by Theorem 3.7, we have for all $t \ge T_0$

$$(E(u(t)) - E(\varphi))^{\frac{1}{2}} = |E(u(t)) - E(\varphi)|^{\frac{1}{2}} \le C\|E'(u(t))\|_{V^*}.$$

By using the continuous embedding $H \hookrightarrow V^*$, we obtain

$$(E(u(t)) - E(\varphi))^{\frac{1}{2}} \le C_1\|E'(u(t))\|_H,$$

which implies that

$$(E(u(t)) - E(\varphi)) \le C_1^2\|E'(u(t))\|^2_H,$$

or equivalently,

$$-\|E'(u(t))\|_H^2 \leq -\frac{1}{C_1^2}(E(u(t)) - E(\varphi)).$$

This together with (5.1) implies that

$$\frac{d}{dt}(E(u(t)) - E(\varphi)) \leq -C_2(E(u(t)) - E(\varphi)) \text{ for all } t \geq T_0, \tag{5.2}$$

where $C_2 := 1/C_1^2$. We also note that

$$y(t) := \Big(E(u(T_0)) - E(\varphi)\Big) \exp(-C_2(t - T_0))$$

is the unique solution of the differential equation

$$\begin{cases} \dfrac{d}{dt}y(t) = -C_2 y \text{ for } t \geq T_0, \\ \\ y(T_0) = E(u(T_0)) - E(\varphi). \end{cases}$$

Therefore, by [16, Theorem 6.1, page 31] and the differential inequality (5.2), we deduce that for all $t \geq T_0$

$$E(u(t)) - E(\varphi) \leq \Big(E(u(T_0)) - E(\varphi)\Big) \exp(-C_2(t - T_0)).$$

In view of (4.1), this implies that for all $t \geq T_0$

$$\int_t^\infty \|u_t(s)\|_H^2 \, ds \leq \Big(E(u(T_0) - E(\varphi)\Big) \exp(-C_2(t - T_0)).$$

Setting $a := \Big(E(u(T_0) - E(\varphi)\Big) \exp(C_2 T_0) > 0$, we obtain the inequality

$$\int_t^\infty \|u_t(s)\|_{L^2(\Omega)}^2 = \int_t^\infty \|u_t(s)\|_H^2 \, ds \leq a \exp(-C_2 t) \text{ for all } t \geq T_0.$$

We deduce from Lemma 5.1 that

$$\|u(t) - \varphi\|_{L^2(\Omega)} \leq \sqrt{a} \frac{\exp(\frac{C_2}{2})}{\exp(\frac{C_2}{2}) - 1} \exp(-\frac{C_2 t}{2}) \text{ for all } t \geq T_0. \tag{5.3}$$

Note that for a function $w \in C^1(\overline{\Omega})$, we can apply the Gagliardo-Nirenberg inequality (e.g. see [3, page 314])

$$\|w\|_{L^\infty(\Omega)} \leq C\|w\|_{L^2(\Omega)}^{1-\beta} \|w\|_{W^{1,r}(\Omega)}^{\beta},$$

where

$$\beta = \frac{1}{2}(\frac{1}{2} + \frac{1}{N} - \frac{1}{r}), r > N;$$

to obtain

$$\|w\|_{L^\infty(\Omega)} \leq C\|w\|_{L^2(\Omega)}^{1-\beta} \|w\|_{C^1(\overline{\Omega})}^{\beta}. \tag{5.4}$$

Thus the conclusion of Theorem 5.2 follows from (5.3), (5.4) and (2.4). □

5.2. **The case $\theta \in (0, \frac{1}{2})$.** We will apply the following lemma.

Lemma 5.3 (see [15], Lemma 3.3)**.** *Let $t_0 > 0$ be arbitrary. Assume that there exist two positive constants α and K such that*

$$\int_t^\infty \|u_t\|^2_{L^2(\Omega)} \le K t^{-2\alpha-1} \quad \text{for all } t \ge t_0.$$

Then

$$\|u(t) - u(\tau)\|_{L^2(\Omega)} \le \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } \tau \ge t \ge t_0.$$

Theorem 5.4. *Let $(\mathbf{H_0})$ hold. Assume further that Theorem 3.7 holds for $\theta \in (0, \frac{1}{2})$ and set $\alpha := \dfrac{\theta}{1-2\theta} > 0$. Then there exists a positive constant M such that*

$$\|u(t) - \varphi\|_{L^\infty(\Omega)} \le M t^{-\alpha} \quad \text{for all } t > 0.$$

Proof. As in the proof of Theorem 4.1, it is sufficient to prove this result for the function u in the case

$$\int_\Omega u_0 = 0.$$

We have

$$\frac{d}{dt}(E(u) - E(\varphi)) = \langle E'(u), u_t \rangle = -\langle E'(u), E'(u) \rangle = -\|E'(u)\|^2_H. \tag{5.5}$$

Note that

$$u(t) \to \varphi \text{ in } V \text{ as } t \to \infty,$$

we deduce that for σ as in Theorem 3.7 there exists $T_0 > 0$ such that for all $t \ge T_0$

$$\|u(t) - \varphi\|_V \le \sigma.$$

Therefore, by Theorem 3.7, we have for all $t \ge T_0$

$$|E(u(t)) - E(\varphi)|^{1-\theta} \le C\|E'(u(t))\|_{V^*}.$$

By applying the continuous embedding $H \hookrightarrow V^*$, we obtain

$$(E(u(t)) - E(\varphi))^{1-\theta} = |E(u(t)) - E(\varphi)|^{1-\theta} \le C_1 \|E'(u(t))\|_H,$$

which implies that

$$(E(u(t)) - E(\varphi))^{2(1-\theta)} \le C_1^2 \|E'(u(t))\|^2_H,$$

or equivalently,

$$-\|E'(u)\|^2_H \le -\frac{1}{C_1^2}(E(u) - E(\varphi))^{2(1-\theta)}.$$

This together with (5.5) implies that

$$\frac{d}{dt}(E(u) - E(\varphi)) \le -C_2(E(u) - E(\varphi))^{2(1-\theta)} \text{ for all } t \ge T_0, \tag{5.6}$$

where $C_2 := 1/C_1^2$. We also note that

$$y(t) := \left((E(u(T_0) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t - T_0) \right)^{-1/(1-2\theta)}$$

is the unique solution of the differential equation

$$\begin{cases} \dfrac{d}{dt}y(t) = -C_2 y^{2(1-\theta)} \text{ for } t \geq T_0, \\ \\ y(T_0) = E(u(T_0) - E(\varphi)). \end{cases}$$

Therefore, by [16, Theorem 6.1, page 31] and the differential inequality (5.6), we deduce that

$$\begin{aligned} E(u(t)) - E(\varphi) &\leq \left((E(u(T_0) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t-T_0) \right)^{-1/(1-2\theta)} \\ &= \left((E(u(T_0) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)t \right)^{-1/(1-2\theta)} \\ &= \left((E(u(T_0) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)\frac{t}{2} + C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \\ &\leq \left(C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \quad \text{for all } t \geq 2T_0. \end{aligned}$$

It follows that for all $t \geq 2T_0$

$$\int_t^\infty \|u_t(s)\|^2 \leq \left(C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)}.$$

We set $K := \left(\dfrac{C_2(1-2\theta)}{2} \right)^{-1/(1-2\theta)}$ and $\alpha := \dfrac{\theta}{1-2\theta} > 0$, then

$$\int_t^\infty \|u_t(s)\|^2 \, ds \leq K t^{-2\alpha-1},$$

which by Lemma 5.3 implies that

$$\|u(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } t \geq 2T_0.$$

This together with (5.4) and (2.4) completes the proof of Theorem 5.4. □

Acknowledgement. The authors would like to thank Alain Haraux for some helpful discussions.

REFERENCES

[1] R. A. Adams, J. H. F. Fournier, *Sobolev spaces*, 2nd edition, Academic Press, 2003.

[2] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., **12**, (1959), 623–727.

[3] H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Spinger, 2010.

[4] H. Brézis, *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North Holland, 1973

[5] D. Brochet, D. Hilhorst, X. Chen, *Finite-dimensional exponential attractor for the phase field model*, Appl. Anal., **49** (1993), 197–212.

[6] T. Cazenave, A. Haraux *An introduction to semilinear evolution equations*, Oxford University Press, USA, 1998.

[7] R. Chill, E. Fašangová, J. Prüss, *Convergence to steady states of solutions of the Cahn-Hilliard and Caginalp equations with boundary conditions*, Math. Nachr. **279** (2006), 1448–1462.

[8] R. Chill, *On the Lojasiewicz-Simon gradient inequality*, J. Funct. Anal. **201** (2003), 572-601.

[9] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 1998.

[10] E. Feireisl, F. Issard-Roch, and H. Petzeltová, *A non-smooth version of the Łojasiewicz-Simon theorem with applications to non-local phase-field systems*, J. Differential Equations. **199** (2004), no 1, 1–21.

[11] E. Feireisl and F. Simondon. *Convergence for semilinear degenerate parabolic equations in several space dimensions*, J. Dynam. Differential Equations. **12**(3), 2000, 647–673.

[12] A. Haraux, M. A. Jendoubi, *Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity*, Calc. Var. Partial Differential Equations. **9** (1999), 95–124.

[13] A. Haraux, M. A. Jendoubi, *Convergence of global and bounded solutions of the wave equation with linear dissipation and analytic nonlinearity*, J. Differential Equations. **144** (1998), 302–312.

[14] A. Haraux, M. A. Jendoubi, *On the convergence of global and bounded solutions of some evolution equations*, J. Evol. Equ. **7** (2007), 449–470.

[15] A. Haraux, M. A. Jendoubi, O. Kavian, *Rate of decay to equilibrium in some semilinear parabolic equations*, J. Evol. Equ. **3** (2003), 463–484.

[16] J. K. Hale, *Ordinary differential equations*, first edition, Wiley, New York, 1969.

[17] D. Henry, *Geometric theory of semilinear parabolic equations*, Springer, Berlin 1981.

[18] S. Huang, *Gradient inequalities. With application to asymptotic behavior and stability of gradient-like*, Mathematical Surveys and Monographs, **126**. American Mathematical Society, Providence, RI, (2006).

[19] S. Huang, P. Takáč, *Convergence in gradient-like systems which are asymptotically autonomous and analytic*, Nonlinear Anal. Ser. A: Theory Methods. **46**, (2001), 675–698.

[20] M. A. Jendoubi, *A simple unified approach to some convergence theorems of L. Simon*, J. Funct. Anal. **153** (1998), 187–202.

[21] M. A. Jendoubi, *Convergence of global and bounded solutions of the wave equation with linear dissipation and analytic nonlinearity*, J. Differential Equations **144** (1998), 302–312.

[22] S. Huang, P. Takáč, *Convergence in gradient-like systems which are asymptotically autonomous and analytic*, Nonlinear Anal. Ser. A: Theory Methods. **46**, (2001), 675–698.

[23] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Uraltseva, *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs, vol **23**, Providence R I. American Mathematical Society (1968).

[24] S. Londen, H. Petzeltová, *Convergence of solutions of a non-local phase-field system*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), no. 3, 653–670.

[25] S. Łojasiewicz, *Ensembles semi-analytiques*, I.H.E.S., Bures-sur-Yvette, 1965.

[26] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques reéls, Colloques Internationaux du C.N.R.S*, **117**, Les Equations aux Derivées Partielles (1963), 87–89.

[27] S. Łojasiewicz, *Sur la géométrie semi- et sous-analytique*, Ann. Inst. Fourier (Grenoble), **43**, (1993), 1575–1595.

[28] M. Marion, *Attractors for reaction-diffusion equations: existence and estimate of their dimension*, Appl. Anal. **25** (1987), 10–147.

[29] J. C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge University Press, 2001.

[30] J. Rubinstein and P. Sternberg, *Nonlocal reaction-diffusion equations and nucleation*, IMA J. of Appl. Math. **48** (1992), 249–264.

[31] P. Rybka, K.-H. Hoffmann, *Convergence of solutions to Cahn-Hilliard equation*, Commun. PDE. **24**, (1999), 1055–1077.

[32] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. **118**, (1983), 525–571.

[33] E. Zeidler, *Nonlinear functional analysis and its applications*, Vol I, Spinger-Verlag, 1998.

E-mail address: samira.boussaid@u-picardie.fr
E-mail address: danielle.hilhorst@math.u-psud.fr
E-mail address: thanh-nam.nguyen@math.u-psud.fr