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Convergence to steady states for solutions of a reaction-diffusion equation with mass conservation

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Abstract. We consider a reaction-diffusion equation with mass conservation, which was originally proposed by Rubinstein and Sternberg as a model for phase separation in a binary mixture. We study the large time behavior of the solution and show that it converges to a stationary solution as t tends to infinity. We also evaluate the rate of this convergence and compute precisely the limit stationary solution in one dimension.

Keywords: Gradient flow, Lojasiewicz inequality, Stabilization of solutions, Mass-conserved Allen-Cahn equation, Infinite dimensional dynamical systems.

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1 Introduction

We consider a reaction-diffusion equation with mass conservation

$$(P) \begin{cases} u_t = \Delta u + f(u) - \int_{\Omega} f(u) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_{\nu} u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = g_0(x) & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a smooth bounded domain, ∂_{ν} is the outer normal derivative to $\partial\Omega$ and

$$\int_{\Omega} f(u) := \frac{1}{|\Omega|} \int_{\Omega} f(u(x)) dx.$$

Problem (P) was introduced by Rubinstein and Sternberg [24] as a model for phase separation in a binary mixture. Our goal is to study of the large time behavior of the solutions of Problem (P). More precisely, we show that any solution of (P) converges to a steady state. We also evaluate the rate of this convergence and compute the limit steady state in one dimension. The main tool is a Lojasiewicz inequality that was first proposed by Lojasiewicz himself [18], [20]. He showed that any bounded solution to gradient systems in \mathbb{R}^n , (which is an ODE system), converges to a stationary point. This idea was subsequently developed for infinite-dimensional gradient systems by L. Simon, who showed a version of this inequality and applied it to prove the stabilization in the Allen-Cahn equation see [26].

This model is **mass preserving**, namely

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} g_0(x) dx \quad \text{for all } t > 0,$$

and it possesses a free energy functional which coincides with the usual Allen-Cahn functional

$$\mathcal{E}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(u) dx,$$

where $F(u) := \int_0^u f(s) ds$.

In [24], the authors consider the model in which f is bistable type, typical example $f(s) = s - s^3$. In this paper, we assume that the function f is of the following form

$$f(s) = \sum_{i=1}^n a_i s^i \quad \text{where } n \geq 3 \text{ is an odd number, } a_n < 0. \quad (1)$$

Note that there exists a constant $c_1 > 0$ satisfying

$$f'(s) \leq c_1 \text{ for all } s \in \mathbb{R}. \quad (2)$$

Definition 1.1. *Two constants $s_1 < s_2$ are said to satisfy Property (C) with respect to f , if*

$$(C) : \quad f(s_2) < f(s) < f(s_1) \quad \text{for all } s \in (s_1, s_2).$$

We have two following theorems.

Theorem 1.2. *Let $g_0 \in L^\infty(\Omega)$. Problem (P) possesses a unique solution $u \in C([0, \infty); L^2(\Omega))$ which satisfies for every $T > 0$*

$$u \in L^\infty(Q_T) \cap L^2(0, T; H^1(\Omega)) \text{ and } u_t \in L^2(0, T; (H^1(\Omega))'),$$

where $Q_T := \Omega \times (0, T)$. Moreover $u \in C^\infty(\bar{\Omega} \times (0, +\infty))$ and

$$\{u(t), t \geq 1\} \text{ is relatively compact in } C^m(\bar{\Omega}) \text{ for all } m \in \mathbb{N}.$$

Theorem 1.3. *Let $g_0 \in L^\infty(\Omega)$ and let u be the unique solution of Problem (P). Then*

$$\lim_{t \rightarrow \infty} \|u(t) - \varphi\|_{C^m(\bar{\Omega})} = 0 \text{ as } t \rightarrow \infty, \text{ for all } m \in \mathbb{N},$$

where φ is a smooth solution of the stationary problem

$$(S) \begin{cases} \Delta \varphi + f(\varphi) - \int_{\Omega} f(\varphi) & = 0 \text{ in } \Omega, \\ \partial_\nu \varphi & = 0 \text{ on } \partial\Omega. \end{cases}$$

Moreover

$$\int_{\Omega} \varphi = \int_{\Omega} g_0.$$

The proofs of Theorem 1.2 and Theorem 1.3 are based on following auxiliary problem.

Problem (\bar{P}): In order to prove the main theorems, we first study Problem (P) when f is replaced by a smooth bounded function \bar{f} . More precisely, we consider the following problem

$$(\hat{P}) \begin{cases} \bar{u}_t = \Delta \bar{u} + \bar{f}(\bar{u}) - \int_{\Omega} \bar{f}(\bar{u}) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_{\nu} \bar{u} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \bar{u}(x, 0) = g_0(x) & x \in \Omega, \end{cases}$$

and \bar{f} is supposed to be smooth and satisfy

$$|\bar{F}(s)|, |\bar{f}(s)|, |\bar{f}(s)s|, |\bar{f}'(s)| < c_2 \text{ for all } s \in \mathbb{R} \text{ and a constant } c_2 > 0. \quad (3)$$

Here, $\bar{F}(s) = \int_0^s \bar{f}(\tau) d\tau$ is a primitive of \bar{f} . The corresponding Lyapunov functional is given by

$$E(\bar{u}) := \frac{1}{2} \int_{\Omega} |\nabla \bar{u}|^2 dx - \int_{\Omega} \bar{F}(\bar{u}) dx.$$

The choice of \bar{f} will be given in the proof of Theorem 1.2 (section 2). We shall show with such \bar{f} that the solution of Problem (\bar{P}) coincides with the solution of Problem (P) and deduce the result for Problem (P).

The paper is organized as follows: in section 2 we prove Theorem 1.2. In section 3, first we give a version of Lojasiewicz inequality for the energy functional E (Theorem 3.11), then apply it to prove Theorem 1.3 and to establish rate of this convergence. Section 4 is devoted to compute the limit stationary solution in one dimension.

2 The existence and uniqueness of solution

We first prove the uniqueness of the solutions of the Problem (P) and (\bar{P}).

Lemma 2.1. *(i) For any $g_0 \in L^{\infty}(\Omega)$, Problem (P) possesses at most one solution u such that*

$$u \in L^{\infty}(Q_T) \cap L^2(0, T, H^1(\Omega)) \text{ and } u_t \in L^2(0, T; H^1(\Omega)').$$

Moreover,

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} g_0(x) dx \text{ for all } t > 0. \quad (4)$$

(ii) For any $g_0 \in L^2(\Omega)$, Problem (\bar{P}) possesses at most one solution \bar{u} such that

$$\bar{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)) \text{ and } \bar{u}_t \in L^2(0, T; H^1(\Omega)').$$

Moreover,

$$\int_{\Omega} \bar{u}(x, t) dx = \int_{\Omega} g_0(x) dx \text{ for all } t > 0. \quad (5)$$

Proof. (ii) The identity (5) follows from

$$\frac{d}{dt} \int_{\Omega} \bar{u}(x, t) dx = 0.$$

Let $g'_0 \in L^2(\Omega)$ be another initial condition and let \bar{u}' be a corresponding solution. We first set $w := \bar{u} - \bar{u}'$, multiply the difference of the equations for \bar{u} and \bar{u}' by w , then integrate over Ω to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2(t) + \int_{\Omega} |\nabla w(t)|^2 = \int_{\Omega} [\bar{f}(\bar{u}) - \bar{f}(\bar{u}')] w - \int_{\Omega} w \bar{f}'(\bar{u}) [\bar{f}(\bar{u}) - \bar{f}(\bar{u}')].$$

It follows from (3) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2(t) + \int_{\Omega} |\nabla w(t)|^2 &\leq c_2 \int_{\Omega} w^2 + \frac{c_2}{|\Omega|} \left(\int_{\Omega} |w| \right)^2 \\ &\leq c_3 \int_{\Omega} w^2. \end{aligned}$$

Using Gronwall's lemma, we have

$$\int_{\Omega} |\bar{u}(t) - \bar{u}'(t)|^2 \leq \exp(2c_3 t) \int_{\Omega} |g_0 - g'_0|^2, \quad (6)$$

which implies the uniqueness.

(i) The proof of part (i) is similar to the one of part (ii). Let $\tilde{g}_0 \in L^\infty(\Omega)$ such that $\int_{\Omega} \tilde{g}_0 = \int_{\Omega} g_0$ and let \tilde{u} be a solution corresponding with the initial condition \tilde{g}_0 . Note that

$$\int_{\Omega} u(t) - \int_{\Omega} \tilde{u}(t) = \int_{\Omega} g_0 - \int_{\Omega} \tilde{g}_0 = 0. \quad (7)$$

We use (2) and the method developed in the proof of part (ii) to obtain

$$\int_{\Omega} |u(t) - \tilde{u}(t)|^2 \leq \exp(2c_1 t) \int_{\Omega} |g_0 - \tilde{g}_0|^2. \quad (8)$$

The uniqueness then follows from this estimate. \square

Lemma 2.2. *Let $g_0 \in L^2(\Omega)$. Then for any $T > 0$ arbitrary*

(i) *Problem (\hat{P}) possesses a unique solution \bar{u} which satisfies*

$$\bar{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)); \quad \bar{u}_t \in L^2(0, T; (H^1(\Omega))'). \quad (9)$$

Moreover, $\bar{u} \in C([0, \infty); L^2(\Omega))$.

(ii) *If, in addition, $g_0 \in H^1(\Omega)$ then*

$$\bar{u} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \text{ and } \bar{u}_t \in L^2(Q_T). \quad (10)$$

So that also $\bar{u} \in C([0, \infty); H^1(\Omega))$.

The mapping

$$T(t) : g_0 \longmapsto \bar{u}(t)$$

is Lipschitz continuous on $L^2(\Omega)$ for all $t > 0$ and $T(t)_{t \geq 0}$ is a semigroup on $L^2(\Omega)$.

Remark 2.3. Consider the operator $-\Delta$ with homogeneous Neumann boundary conditions. Denote by

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_i \leq \dots$$

the eigenvalues of the Laplacian operator, and $w_i, i = 1, \dots$ the corresponding unit eigenfunctions. Remark that $w_1 = \frac{1}{|\Omega|^{1/2}}, \int_{\Omega} w_i = 0$ for $i \geq 2$ and that the w_i 's are smooth functions up to boundary. They constitute an orthonormal basis of $L^2(\Omega)$ and also an orthogonal basis of $H^1(\Omega)$.

Proof of Lemma 2.2

We look for an approximate solution of the form

$$\bar{u}_m(t) = \sum_{i=1}^m g_{mi}(t) w_i$$

satisfying

$$\int_{\Omega} \bar{u}_{mt} w_j + \int_{\Omega} \nabla \bar{u}_m \nabla w_j = \int_{\Omega} \bar{f}(\bar{u}_m) w_j - \int_{\Omega} \bar{f}'(\bar{u}_m) \int_{\Omega} w_j \quad (11)$$

for $j = 1, \dots, m$ and

$$\bar{u}_m(0) = \bar{u}_{m0} := \sum_{i=1}^m g_{mi}^0 w_i \rightarrow g_0 \text{ in } L^2(\Omega) \text{ as } m \rightarrow \infty. \quad (12)$$

Note that $\int_{\Omega} w_j = 0$ for $2 \leq j \leq m$, we deduce that the equations (11) form a nonlinear differential system for the functions g_{m1}, \dots, g_{mm} :

$$g'_{m1}(t) = 0, \quad (13)$$

$$g'_{mj} - \lambda_j g_{mj} = \int_{\Omega} \bar{f}'\left(\sum_{i=1}^m g_{mi}(t) w_i\right) w_j \quad \text{for } 2 \leq j \leq m. \quad (14)$$

The condition (12) is equivalent to the m scalar initial conditions

$$g_{m1}(0) = g_{m1}^0, \quad (15)$$

$$g_{mj}(0) = g_{mj}^0 \text{ for } 2 \leq j \leq m. \quad (16)$$

It follows from (13) and (15) that

$$g_{m1}(t) = g_{m1}^0 \text{ for all } t \geq 0. \quad (17)$$

Substituting (17) in (14), we obtain a nonlinear differential system of $m - 1$ variables. Then the nonlinear differential system (14) with the initial condition (16) has a maximal solution defined on some interval $(0, T_m)$. In fact $T_m = \infty$ because of the following *a priori* estimates.

***A priori* estimates for the proof of Lemma 2.2(i):** First, by (12) and (17), we have

$$\int_{\Omega} \bar{u}_m(t) dx = \int_{\Omega} \bar{u}_{m0}(x) dx \rightarrow \int_{\Omega} g_0(x) dx \text{ as } m \rightarrow \infty \quad (18)$$

for all $0 \leq t < T_m$. Consequently, there exists a constant c_4 independent of m such that

$$\left| \int_{\Omega} \bar{u}_m(t) dx \right| \leq c_4 \text{ for all } 0 \leq t < T_m. \quad (19)$$

Multiplying (11) by g_{mj} and summing on $j = 1, \dots, m$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\bar{u}_m|^2 + \int_{\Omega} |\nabla \bar{u}_m|^2 = \int_{\Omega} \bar{f}(\bar{u}_m) \bar{u}_m - \int_{\Omega} \bar{f}'(\bar{u}_m) \int_{\Omega} \bar{u}_m.$$

We use (3) and (19) to deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\bar{u}_m|^2 + \int_{\Omega} |\nabla \bar{u}_m|^2 \leq c_2(|\Omega| + c_4).$$

Therefore we integrate this inequality from 0 to t with t arbitrary, and then take $t = T$ to deduce that

$$\|\bar{u}_m\|_{L^\infty(0,T;L^2(\Omega))}, \|\bar{u}_m\|_{L^2(0,T;H^1(\Omega))} \leq K_1(\|g_0\|_{L^2(\Omega)} + T), \quad (20)$$

where K_1 is a constant independent of m .

Now we will give an estimate for \bar{u}_{mt} . Fix any $\eta \in H^1(\Omega)$, with $\|\eta\|_{H^1(\Omega)} \leq 1$, and write $\eta = \eta_1 + \eta_2$ where $\eta_1 \in \text{span}\{w_i\}_{i=1}^m$ and $\int_{\Omega} \eta_2 w_i = 0$ for all $i = 1, \dots, m$. Note that $\eta_2 \in H^1(\Omega)$ and that

$$\int_{\Omega} \nabla \eta_2 \nabla w_i = -\lambda_i \int_{\Omega} \eta_2 w_i = 0 \text{ for all } 1 \leq i \leq m$$

so that η_2 is also orthogonal to $\text{span}\{w_i\}_{i=1}^m$ with respect to the scalar product in $H^1(\Omega)$. In particular, η_2 is orthogonal to η_1 in $H^1(\Omega)$, hence

$$\|\eta_1\|_{H^1(\Omega)} \leq \|\eta\|_{H^1(\Omega)} \leq 1.$$

We deduce from (11) that

$$\int_{\Omega} \bar{u}_{mt} \eta = \int_{\Omega} \bar{u}_{mt} \eta_1 = - \int_{\Omega} \nabla \bar{u}_m \nabla \eta_1 + \int_{\Omega} \bar{f}(\bar{u}_m) \eta_1 - \int_{\Omega} \bar{f}'(\bar{u}_m) \int_{\Omega} \eta_1.$$

Therefore,

$$\begin{aligned} \left| \int_{\Omega} \bar{u}_{mt} \eta \right| &\leq \left(\sum_{i=1}^N \left\| \frac{\partial \bar{u}_m}{\partial x_i} \right\|_{L^2(\Omega)} \right) \left(\sum_{i=1}^N \left\| \frac{\partial \eta_1}{\partial x_i} \right\|_{L^2(\Omega)} \right) + 2c_2 \|\eta_1\|_{L^1(\Omega)} \\ &\leq \|\nabla \bar{u}_m\|_{L^2(\Omega)} \|\nabla \eta_1\|_{L^2(\Omega)} + c_5 \|\eta_1\|_{L^2(\Omega)} \\ &\leq \|\nabla \bar{u}_m\|_{L^2(\Omega)} + c_5, \end{aligned}$$

where c_5 is a constant independent of m . Since $\eta \in H^1(\Omega)$ is an arbitrary function such that $\|\eta\|_{H^1(\Omega)} \leq 1$, it follows that

$$\|\bar{u}_{mt}\|_{(H^1(\Omega))'} \leq \|\nabla \bar{u}_m\|_{L^2(\Omega)} + c_5.$$

This together with (20) implies that there exists a positive constant K_2 independent of m such that

$$\|\bar{u}_{mt}\|_{L^2(0,T;(H^1(\Omega))')} \leq K_2. \quad (21)$$

The estimates (20) and (21) will be necessary for the proof (i).

A priori estimates for the proof of Lemma 2.2(ii): We suppose that $\bar{u}_{0m} = \sum_{i=1}^m g_{mi}^0 w_i \rightarrow g_0$ in $H^1(\Omega)$. First, note that

$$\int_{\Omega} \Delta \bar{u}_m dx = 0, \text{ and } \int_{\Omega} \bar{u}_{mt} dx = 0.$$

Multiplying Equation (11) by $-\lambda_j g_{mj}$, summing on $j = 1, \dots, m$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \bar{u}_m|^2 + \int_{\Omega} |\Delta \bar{u}_m|^2 &= - \int_{\Omega} \bar{f}(\bar{u}_m) \Delta \bar{u}_m + \int_{\Omega} \bar{f}(\bar{u}_m) \int_{\Omega} \Delta \bar{u}_m \\ &= \int_{\Omega} \bar{f}'(\bar{u}_m) |\nabla \bar{u}_m|^2 \\ &\leq c_2 \int_{\Omega} |\nabla \bar{u}_m|^2. \end{aligned}$$

Integrating this inequality from 0 to T ($T > 0$ is arbitrary), we deduce that there exists a constant K_3 independent of m such that

$$\|\bar{u}_m\|_{L^\infty(0,T;H^1(\Omega))}, \|\bar{u}_m\|_{L^2(0,T;H^2(\Omega))} \leq K_3. \quad (22)$$

Finally, we multiply (11) by $g'_{mj}(t)$ and sum on $j = 1, \dots, m$ to obtain

$$\int_{\Omega} \bar{u}_{mt}^2 + \int_{\Omega} \nabla \bar{u}_m \nabla \bar{u}_{mt} = \int_{\Omega} \bar{f}(\bar{u}_m) \bar{u}_{mt} - \int_{\Omega} \bar{f}(\bar{u}_m) \int_{\Omega} \bar{u}_{mt} = \frac{d}{dt} \int_{\Omega} \bar{F}(\bar{u}_m).$$

So that

$$\int_0^T \int_{\Omega} \bar{u}_{mt}^2 + \frac{1}{2} \int_{\Omega} |\nabla \bar{u}_m(T)|^2 = \int_{\Omega} \bar{F}(\bar{u}_m(T)) - \int_{\Omega} \bar{F}(\bar{u}_m(0)) + \frac{1}{2} \int_{\Omega} |\nabla \bar{u}_m(0)|^2,$$

which implies that

$$\|\bar{u}_{mt}\|_{L^2(Q_T)} \leq K_4, \quad (23)$$

where K_4 is independent of m .

Passing to the limit: It follows from (20) and (21) that there exists a subsequence of \bar{u}_m , still denoted by \bar{u}_m , such that

$$\begin{aligned} \bar{u}_m &\rightharpoonup \bar{u} \text{ weakly in } L^2(0, T; H^1(\Omega)), \text{ weak-star in } L^\infty(0, T; L^2(\Omega)), \\ \bar{u}_m &\rightarrow \bar{u} \text{ strongly in } L^2(0, T; L^2(\Omega)), \\ \bar{u}_{mt} &\rightharpoonup \bar{u}_t \text{ weakly in } L^2(0, T; (H^1(\Omega))'). \end{aligned}$$

Moreover, since \bar{f} is Lipschitz continuous on \mathbb{R} ,

$$\bar{f}(\bar{u}_m) \rightarrow \bar{f}(\bar{u}) \text{ strongly in } L^2(0, T; L^2(\Omega)).$$

Passing to the limit in (11) we deduce that

$$\bar{u}_t = \Delta \bar{u} + \bar{f}(\bar{u}) - \int_{\Omega} \bar{f}(\bar{u}) \text{ in } L^2(0, T; (H^1(\Omega))').$$

Since $\bar{u} \in L^2(0, T; H^1(\Omega))$ and since $\bar{u}_t \in L^2(0, T; (H^1(\Omega))')$, one can show that $\bar{u} \in C([0, T]; L^2(\Omega))$. Moreover $\bar{u}(0, x) = g_0(x)$ by classical arguments.

If $g_0 \in H^1(\Omega)$, it follows from (22) and (23) that \bar{u} satisfies (10). By [10, Theorem 4, p. 288] it follows $\bar{u} \in C([0, \infty); H^1(\Omega))$.

The uniqueness is proved in Lemma 2.1. The Lipschitz continuity of $T(t)$ on $L^2(\Omega)$ follows from (6). \square

The following corollary follows directly from Lemma 2.2.

Corollary 2.4. *For any $g_0 \in L^2(\Omega)$ and $\delta > 0$, we have*

$$\bar{u} \in C([\delta, \infty); H^1(\Omega)).$$

We now prove more regularity properties of the solution. For this, we recall some technical lemmas which will be used in what follows. Denote by $Q_a^b = \Omega \times (a, b)$ for $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$.

Lemma 2.5. *Let $u_0 \in L^2(\Omega)$, $g \in L^p(Q_T)$ for some $p \in (1, \infty)$ and let u be the solution of the time evolution problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & x \in \Omega. \end{cases}$$

Then

(i) *for each $0 < \delta < 1$, there exists a positive constant C such that*

$$\|u\|_{W_p^{2,1}(Q_\delta^T)} \leq C(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^p(Q_T)}).$$

(ii) *If we suppose that $u_0 \in W^{2,p}(\Omega)$ then we obtain the estimate*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C(\|u_0\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(Q_T)}).$$

Lemma 2.6. *One has the following embedding*

$$W_p^{2,1}(Q_T) \subset C^{\lambda, \lambda/2}(\bar{Q}_T) \text{ with } \lambda = 2 - \frac{N+2}{p} \text{ if } p > \frac{N+2}{2} \text{ and } p \neq N+2.$$

Lemma 2.5 and Lemma 2.6 follow from [17, chapter 4, section 3 and chapter 2, section 3] which are stated in [4, p. 206].

Proposition 2.7. *Let $\bar{u} \in C^{2,1}(\bar{\Omega} \times (0, T]) \cap C(\bar{\Omega} \times [0, T])$ be the solution of Problem (\bar{P}) with the initial condition g_0 . Assume that $s_1 < g_0 < s_2$ then*

$$s_1 < \bar{u}(x, t) < s_2$$

for $x \in \bar{\Omega}$ and $0 < t \leq T$.

Proof. For the purpose of contradiction, we suppose that there exists a first time $0 < t_0 \leq T$ such that $\bar{u}(x_0, t_0) = s_1$ or $\bar{u}(x_0, t_0) = s_2$ for some $x_0 \in \bar{\Omega}$. Without loss of generality, assume that $\bar{u}(x_0, t_0) = s_2$. By the continuity of \bar{u} and the definition of t_0 , we have

$$s_1 \leq \bar{u}(x, t_0) \leq s_2 \text{ for all } x \in \bar{\Omega}, \text{ and } \bar{u}(x, t) < s_2 \text{ for all } x \in \bar{\Omega} \text{ and } 0 \leq t < t_0. \quad (24)$$

Since $\partial_\nu \bar{u} = 0$, we deduce from Hopf's maximum principle that $x_0 \in \Omega$. Therefore the function $\bar{u}(\cdot, t_0)$ attains its maximum at $x_0 \in \Omega$, which implies that $\Delta \bar{u}(x_0, t_0) \leq 0$. By (24), we have

$$\bar{u}_t(x_0, t_0) = \lim_{\Delta t \rightarrow 0^+} \frac{\bar{u}(x_0, t_0 - \Delta t) - \bar{u}(x_0, t_0)}{-\Delta t} \geq 0,$$

which we substitute in Problem (\bar{P}) to obtain $\int_{\Omega} (\bar{f}(s_2) - \bar{f}(\bar{u}(x, t_0))) dx \geq 0$. Since $s_1 \leq \bar{u}(x, t_0) \leq s_2$ for all $x \in \Omega$, it follows that $\bar{f}(s_2) \leq \bar{f}(\bar{u}(x, t_0))$ for all $x \in \Omega$ so that $\bar{f}(s_2) = \bar{f}(\bar{u}(x, t_0))$ and hence $\bar{u}(x, t_0) = s_2$ for all x in Ω . But the inequality $\bar{u}(x, \frac{t_0}{2}) < s_2 = \bar{u}(x, t_0)$ for all $x \in \Omega$ implies that

$$\int_{\Omega} \bar{u}(x, \frac{t_0}{2}) dx < \int_{\Omega} \bar{u}(x, t_0) dx,$$

which contradicts the mass preserving property. \square

Lemma 2.8. *Let $0 < \delta < \frac{1}{2}$, $\alpha \in (0, 1)$ be arbitrary. Let s_1, s_2 be two constants which satisfy Property (\mathcal{C}) with respect to \bar{f} . We assume that $s_1 < g_0 < s_2$ then*

$$(i) \quad s_1 \leq \bar{u}(x, t) \leq s_2 \text{ for all } x \in \bar{\Omega}, t > 0,$$

$$(ii) \quad \|\bar{u}\|_{C^{2m+1+\alpha, m+\frac{1+\alpha}{2}}(Q_\delta^\infty)} \leq C(m, \delta, s_1, s_2, \Omega) \text{ for any } m \in \mathbb{N}.$$

Proof. (i) The main idea of the proof is to approximate the initial function g_0 by a sequence of smooth functions and to first obtain uniform a priori estimates for the corresponding solutions of Problem (\bar{P}) . To begin with, we choose a sequence $g_{0n} \in C^\infty(\bar{\Omega})$ such that $s_1 < g_{0n} < s_2$ converges to g_0 in $L^2(\Omega)$ as $n \rightarrow \infty$ (cf. [1, Lemma 2.18, p. 29]). We denote by \bar{u}_n the corresponding solutions of Problem (\bar{P}) . Since $|\bar{f}| \leq c_2$, we deduce from Lemma 2.5(ii) that $\bar{u}_n \in W_p^{2,1}(Q_T)$ for all $p > 1$, which by the Sobolev embedding in Lemma 2.6 implies that $\bar{u}_n \in C^{1+\alpha, (1+\alpha)/2}(\bar{Q}_T)$ for all $\alpha \in (0, 1)$. Applying a standard bootstrap argument (cf. [17, Theorem 10.1, p. 351]), we deduce that $\bar{u}_n \in C^{3+\alpha, 1+\frac{1+\alpha}{2}}(\bar{Q}_T)$, which by Proposition 2.7 is such that $s_1 < \bar{u}_n < s_2$.

In view of Lemma 2.5(i) we also deduce that $\bar{u} \in W_p^{2,1}(Q_\delta^T)$ for all $\delta \in (0, \frac{1}{2})$ and $p > 1$, which in turn implies that $\bar{u} \in C(\bar{\Omega} \times (0, T])$. Since by (6),

$$\int_{\Omega} |\bar{u}(t) - \bar{u}_n(t)|^2 \leq \exp(2c_3 t) \int_{\Omega} |g_0 - g_{0n}|^2$$

it follows that

$$\bar{u}_n(t) \rightarrow \bar{u}(t) \text{ in } L^2(\Omega) \text{ for all } t > 0.$$

Thus $s_1 \leq \bar{u} \leq s_2$ in $\bar{\Omega} \times (0, T]$. Finally, since T is arbitrary,

$$s_1 \leq \bar{u}(x, t) \leq s_2 \text{ for all } x \in \bar{\Omega}, t > 0.$$

(ii) Reasoning in the same way as for the function \bar{u}_n on the domains Q_k^{k+1} and $Q_{k+1/2}^{k+3/2}$, we deduce that

$$\|\bar{u}\|_{C^{1+\alpha, (1+\alpha)/2}(Q_{k+\delta}^{k+1})} \leq C \left(\|\bar{u}(k)\|_{L^2(\Omega)} + \left\| \bar{f}(\bar{u}) - \int_{\Omega} \bar{f}(\bar{u}) \right\|_{L^p(Q_k^{k+1})} \right) \leq C(\delta, s_1, s_2, \Omega)$$

and a similar one on the domain $Q_{k+1/2}^{k+3/2}$. Therefore, we have

$$\|\bar{u}\|_{C^{1+\alpha, (1+\alpha)/2}(Q_{\delta}^{\infty})} \leq C(\delta, s_1, s_2, \Omega).$$

By standard bootstrap arguments [17, Theorem 10.1, p. 351], Lemma 2.8(ii) holds. \square

Corollary 2.9. *Let s_1, s_2 be two constants which satisfy Property (C) with respect to \bar{f} . Assume that $s_1 < g_0 < s_2$ then $\{\bar{u}(t) : t \geq 1\}$ is relatively compact in $C^m(\bar{\Omega})$ for all $m \in \mathbb{N}$.*

In particular, $\{\bar{u}(t) : t \geq 1\}$ is relatively compact in $H^1(\Omega)$.

Proof of Theorem 1.2

Construction function \bar{f} : first, choose two constants s_1, s_2 (such that $|s_1|, |s_2|$ large enough) which satisfy Property (C) and satisfy

$$s_1 < g_0(x) < s_2 \text{ for almost } x \in \Omega.$$

It is easy to see that such s_1, s_2 exist. Now, let $\bar{f} \in C_c^\infty(\mathbb{R})$ be given by

$$\bar{f}(s) = \begin{cases} 0 & \text{if } s \leq s_1 - 2, \\ f(s) & \text{if } s \in [s_1 - 1, s_2 + 1], \\ 0 & \text{if } s \geq s_2 + 2. \end{cases}$$

Obviously, this function satisfy (3).

Note that s_1, s_2 satisfy Property (C) with respect to \bar{f} . According to Lemma 2.7, we have $\bar{u}(t) \in [s_1, s_2]$. On the other hand, $\bar{f}(s) = f(s)$ for all $s \in [s_1, s_2]$, so that $\bar{f}(\bar{u}) = f(\bar{u})$. It follows that \bar{u} coincides with the unique solution of Problem (P). Now, the assertions of the theorem follow from Lemma 2.2, Lemma 2.8(ii) and Corollary 2.9. \square

Remark 2.10. We can extend all results in this section to f being a smooth function in \mathbb{R} . We summarize two important results.

- (i) Theorem 1.2 holds provided that f is smooth.
- (ii) **A comparison result:** let f be a smooth function and let $s_1 < s_2$ be two constants which satisfy Property (C) with respect to f . Assume that $s_1 < g_0 < s_2$ then $s_1 \leq u(x, t) \leq s_2$.

3 Large time behavior

In this section, we first give some modifications for Problem (P), which bright out a slight view for this problem. Lojasiewicz inequality is proven in section 3.2, then we apply the inequality to prove the convergence of the solution to a steady state. A result of velocity of the convergence is also established in this section.

3.1 Modification setting

In the following, by mass conservation property, we may without loss of generality assume that u satisfies the condition

$$\int_{\Omega} u(x, t) dx = 0 \quad \text{for all } t \geq 0.$$

In fact, it suffices to replace the solution u by $u - m_0$ with $m_0 := \frac{1}{|\Omega|} \int_{\Omega} g_0(x) dx$ to note that $u - m_0$ satisfies Problem (P) with f replaced by $f_1(s) := f(s + m_0)$. The question, whether u converges as $t \rightarrow \infty$ is not affected by this normalization.

Next, we fix two constants s_1, s_2 (such that $|s_1|, |s_2|$ large enough) which satisfy Property (C) and satisfy

$$s_1 < g_0(x) < s_2 \quad \text{for almost } x \in \Omega.$$

Function \bar{f} is given as in the proof of Theorem 1.2. More precisely, $\bar{f} \in C_c^\infty(\mathbb{R})$ and

$$\bar{f}(s) = \begin{cases} 0 & \text{if } s \leq s_1 - 2, \\ f(s) & \text{if } s \in [s_1 - 1, s_2 + 1], \\ 0 & \text{if } s \geq s_2 + 2. \end{cases}$$

By using the fact that the solution of Problem (\bar{P}) coincides with the unique solution of Problem (P) , we shall study the large time behavior for Problem (\bar{P}) and deduce the result for Problem (P) .

3.2 A version of Lojasiewicz inequality

We define the spaces

$H = \{u \in L^2(\Omega), \int_{\Omega} u(x)dx = 0\}$, equipped with the norm $\|\cdot\|_H := \|\cdot\|_{L^2(\Omega)}$,

$V = \left\{u \in H^1(\Omega) : \int_{\Omega} u(x)dx = 0\right\}$, equipped with the norm $\|\cdot\|_V := \|\cdot\|_{H^1(\Omega)}$.

Let V^* be the dual space of V and \cdot . We identify H with its dual to obtain:

$$V \hookrightarrow H \hookrightarrow V^*,$$

where the embeddings $V \hookrightarrow H$, $H \hookrightarrow V^*$ are continuous, dense and compact (see e.g. [15, p. 677]). We use $\langle \cdot, \cdot \rangle$ to denote the scalar product in H and the scalar product for the duality V^*, V . We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from a Banach space X to a second Banach space Y , and we write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

Throughout the following, we denote by $C \geq 0$ a generic constant which may vary from line to line. We start with the following result.

Lemma 3.1. *Let $1 \leq p < \infty$ and let g be a continuous, bounded function on \mathbb{R} . We define an operator $B : H^1(\Omega) \rightarrow L^p(\Omega)$ by $B(u)(x) := \bar{f}(u(x))$. Then B is continuous from $H^1(\Omega)$ to $L^p(\Omega)$.*

Proof. It follows from [16, Lemma 16.1, p. 60] that B is continuous from $L^2(\Omega)$ to $L^p(\Omega)$. The result holds because of the continuous embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. \square

Lemma 3.2. *The functional E is twice continuously Fréchet differentiable on V . We denote by E', L be the first and second derivative of E , respectively. Then*

(i) *The first derivative*

$$E' : V \longrightarrow V^* \text{ is given by}$$

$$\langle E'(u), h \rangle_{V^*, V} = \int_{\Omega} \nabla u \nabla h - \int_{\Omega} \bar{f}(u)h \quad \text{for all } u, h \in V. \quad (25)$$

(ii) *The second derivative*

$L : V \longrightarrow \mathcal{L}(V, V^*)$ is given by

$$\langle L(u)h, k \rangle_{V^*, V} = \int_{\Omega} \nabla h \nabla k - \int_{\Omega} \bar{f}'(u) h k \quad \text{for all } u, h, k \in V. \quad (26)$$

Proof. We write E as the difference of E_1 and E_2 , where

$$E_1(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx \quad \text{and} \quad E_2(u) = \int_{\Omega} \bar{F}(u) dx. \quad (27)$$

Obviously, E_1 is twice continuously Fréchet differentiable. Its derivatives are easily identified in the formula (25) and (26). We now prove the differentiability of E_2 .

By Taylor's formula, there exists $\theta(x) \in (0, 1)$ such that

$$\bar{F}(u + h) - \bar{F}(u) = \bar{f}(u + \theta h)h \quad \text{for all } u, h \in V.$$

It follows that

$$\begin{aligned} & \left| E_2(u + h) - E_2(u) - \int_{\Omega} \bar{f}(u)h dx \right| \\ & \leq \int_{\Omega} |\bar{f}(u + \theta h) - \bar{f}(u)| |h| dx \leq C \|\bar{f}(u + \theta h) - \bar{f}(u)\|_{L^2(\Omega)} \|h\|_V. \end{aligned}$$

Note that $u + \theta h$ tends to u in $H^1(\Omega)$ as $h \rightarrow 0$ in V ; it follows from Lemma 3.1 that $\|\bar{f}(u + \theta h) - \bar{f}(u)\|_{L^2(\Omega)}$ tends to 0 as $h \rightarrow 0$ in V . Thus

$$\left| E_2(u + h) - E_2(u) - \int_{\Omega} \bar{f}(u)h dx \right| = o(\|h\|_V) \quad \text{as } h \rightarrow 0.$$

This implies that the first derivative E_2' exists and

$$\langle E_2'(u), h \rangle_{V^*, V} = \int_{\Omega} \bar{f}(u)h dx.$$

The Fréchet differentiability of E_2' is shown in a similar way. Choose $p > 2$ such that V is continuously embedded in $L^p(\Omega)$. Let T be a linear mapping from V to V^* given by

$$\langle T h, k \rangle_{V^*, V} = \int_{\Omega} \bar{f}'(u)h k dx.$$

We will use below a generalized Holder inequality based on the identity

$$\frac{1}{p} + \frac{1}{p} + \frac{p-2}{p} = 1.$$

For every $u, h, k \in V$, there exist $\eta(x) \in (0, 1)$ such that

$$\begin{aligned} & \left| \langle E_2'(u+h) - E_2'(u) - Th, k \rangle_{V^*, V} \right| \\ & \leq \int_{\Omega} |\bar{f}'(u+\eta h) - \bar{f}'(u)| |h| |k| dx \\ & \leq \|\bar{f}'(u+\eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)} \|h\|_{L^p(\Omega)} \|k\|_{L^p(\Omega)} \\ & \leq C \|\bar{f}'(u+\eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)} \|h\|_V \|k\|_V, \end{aligned} \quad (28)$$

It follows from (28) that

$$\|E_2'(u+h) - E_2'(u) - Th\|_{V^*} \leq C \|\bar{f}'(u+\eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)} \|h\|_V.$$

Since $p/(p-2) < +\infty$ and since \bar{f}' is bounded, $\|\bar{f}'(u+\eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}$ tends to 0 as $h \rightarrow 0$. Thus

$$\|E_2'(u+h) - E_2'(u) - Th\|_{V^*} = o(\|h\|_V)$$

which implies that

$$\langle E_2''(u)h, k \rangle_{V^*, V} = \int_{\Omega} \bar{f}'(u) h k \text{ for all } u, h, k \in V.$$

On the other hand,

$$\begin{aligned} |\langle (E_2''(u) - E_2''(v))h, k \rangle_{V^*, V}| & \leq \int_{\Omega} |\bar{f}'(u) - \bar{f}'(v)| |h| |k| dx \\ & \leq C \|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/(p-2)}(\Omega)} \|h\|_V \|k\|_V, \end{aligned}$$

so that

$$\|E_2''(u) - E_2''(v)\|_{\mathcal{L}(V, V^*)} \leq C \|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/(p-2)}(\Omega)}.$$

This estimate implies the continuity of E_2'' . □

We define a continuous bilinear form from $V \times V \rightarrow \mathbb{R}$ by

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx.$$

Lemma 3.3. *There exists an isomorphism A from V onto V^* such that*

$$a(u, v) = \langle Au, v \rangle_{V^*, V} \text{ for all } u, v \in V. \quad (29)$$

Proof. For each $u \in V$ we define the functional T_u on V by $T_u(v) = a(u, v)$. By continuity of the bilinear form a , $|T_u(v)| \leq C\|u\|_V\|v\|_V$. Thus, $T_u \in V^*$ and

$$\|T_u\|_{V^*} \leq C\|u\|_V. \quad (30)$$

We define the operator $A : V \rightarrow V^*$ by $A(u) = T_u$. It follows from (30) that A is continuous.

Next, we prove that A is injective. Let $u \in V$ such that $Au = 0$. Then $a(u, v) = 0$ for all $v \in V$. In particular, $a(u, u) = 0$. Recall that with Poincaré-Wirtinger inequality for all $w \in V$,

$$\int_{\Omega} w^2 dx = \int_{\Omega} (w - \fint_{\Omega} w)^2 dx \leq C(\Omega) \int_{\Omega} |\nabla w|^2 dx, \quad (31)$$

we deduce that $\int_{\Omega} u^2 dx = 0$, hence $u = 0$.

We now claim that A is surjective. First, note that a is coercive because of Poincaré-Wirtinger inequality (31). It follows from Lax-Milgram theorem that for all $T \in V^*$ there exists a unique element $u \in V$ such that

$$a(u, v) = \langle T, v \rangle_{V^*, V} \text{ for all } v \in V.$$

This follows $\langle T, v \rangle_{V^*, V} = \langle T_u, v \rangle_{V^*, V}$, hence $T = T_u = Au$ so that A is surjective. By a consequence of the open mapping theorem [3, Corollary 2.7, p. 35] theorem, we conclude that A is an isomorphism from V onto V^* . \square

Corollary 3.4. *The first and second derivatives of E can be represented in V^* as:*

$$E'(u) = Au - \bar{f}(u) + \fint_{\Omega} \bar{f}(u), \quad (32)$$

$$L(u)h = Ah - \bar{f}'(u)h + \fint_{\Omega} \bar{f}'(u)h, \quad (33)$$

for all $u, h \in V$.

Proof. Since \bar{f} is bounded, $\bar{f}(u) - \fint_{\Omega} \bar{f}(u) \in H \hookrightarrow V^*$. Therefore,

$$Au - \bar{f}(u) + \fint_{\Omega} \bar{f}(u) \in V^*.$$

Since

$$\int_{\Omega} \left(\int_{\Omega} \bar{f}(u) \right) h = \int_{\Omega} \bar{f}(u) \int_{\Omega} h = 0 \text{ for all } h \in V,$$

it follows that

$$\langle Au - \bar{f}(u) + \int_{\Omega} \bar{f}(u), h \rangle_{V^*, V} = \int_{\Omega} \nabla u \nabla h - \int_{\Omega} \bar{f}(u) h.$$

This together with (25) implies that

$$E'(u) = Au - \bar{f}(u) + \int_{\Omega} \bar{f}(u).$$

Identity (33) may be proved in a similar way. \square

We define

$$\mathcal{L}^p(\Omega) := \{u \in L^p(\Omega), \int_{\Omega} u(x) dx = 0\},$$

equipped with the norm $\|\cdot\|_{\mathcal{L}^p(\Omega)} := \|\cdot\|_{L^p(\Omega)}$ and

$$X_p := \{u \in W^{2,p}(\Omega), \partial_{\nu} u = 0, \int_{\Omega} u(x) dx = 0\}.$$

Lemma 3.5. *Let $p \geq 2$, then for any $g \in \mathcal{L}^p(\Omega)$, there exists a unique solution $u \in X_p$ of the equation*

$$Au = g \text{ in } V^*. \quad (34)$$

Moreover, $A = -\Delta$ on X_p .

Proof. It follows from Lemma 3.3 that Equation (34) has a unique solution $u \in V$. We now claim that $u \in X_p$. Consider the elliptic problem

$$\begin{cases} -\Delta \tilde{u} = g & \text{in } \Omega, \\ \partial_{\nu} \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

First, since $g \in H$, it follows from the Fredholm alternative that this problem possesses a unique solution $\tilde{u} \in V$. Next, since $g \in \mathcal{L}^p(\Omega)$, we deduce from [2] that $\tilde{u} \in W^{2,p}(\Omega)$ so that also $\tilde{u} \in X_p$. In fact, \tilde{u} satisfies Equation (34) since

$$\langle g, v \rangle_{V^*, V} = \langle -\Delta \tilde{u}, v \rangle_{V^*, V} = \int_{\Omega} \nabla \tilde{u} \nabla v dx = a(\tilde{u}, v) = \langle A\tilde{u}, v \rangle_{V^*, V}$$

for all $v \in V$. By the uniqueness of the solution of Equation (34), $u = \tilde{u} \in X_p$. On the other hand, for all $w \in X_p, v \in V$

$$\langle -\Delta w, v \rangle_{V^*, V} = \int_{\Omega} \nabla w \nabla v \, dx = \langle Aw, v \rangle_{V^*, V},$$

so that $A = -\Delta$ on X_p . □

Definition 3.6. We say that $\varphi \in V$ is a critical point of E if $E'(\varphi) = 0$.

Lemma 3.7. For every $\varphi \in V$, the following assertions are equivalent:

- (i) φ is a critical point of E ,
- (ii) $\varphi \in X_2$ and φ satisfies the equations

$$-\Delta \varphi - \bar{f}(\varphi) + \int_{\Omega} \bar{f}(\varphi) = 0 \text{ in } \Omega, \quad (35)$$

$$\partial_{\nu} \varphi = 0 \quad \text{on } \partial\Omega. \quad (36)$$

Moreover, φ is $C^{\infty}(\bar{\Omega})$.

Proof. (ii) \Rightarrow (i). It follows directly from Lemma 3.5 and the formula (32).
(i) \Rightarrow (ii). Assume that $\varphi \in V$ is a critical point of E . We deduce from (32) that

$$A(\varphi) = \bar{f}(\varphi) - \int_{\Omega} \bar{f}(\varphi) \quad \text{in } V^*.$$

Since $\bar{f}(\varphi) - \int_{\Omega} \bar{f}(\varphi) \in H$, then $A(\varphi) \in H$. It follows from Lemma 3.5 that $\varphi \in X_2$ and $A = -\Delta$. Therefore φ satisfies (35).

Finally, we deduce that $\varphi \in C^{\infty}(\bar{\Omega})$ from the boundedness of $\bar{f}(\varphi) - \int_{\Omega} \bar{f}(\varphi)$, Sobolev embedding theorem and a standard bootstrap argument. □

Lemma 3.8. Let φ be a critical point of E . Then operator $L(\varphi)$ is Fredholm from V to V^* . Moreover,

- (i) $\ker L(\varphi)$ is finite-dimensional and contained in $C^{\infty}(\bar{\Omega})$.
- (ii) $\langle u, v \rangle_{V, V^*} = 0$ for all $u \in \ker L(\varphi)$ and $v \in \text{Rg } L(\varphi)$,

(iii) V^* is the topological direct sum of $\ker L(\varphi) \subset V \hookrightarrow V^*$ and $\operatorname{Rg} L(\varphi)$,

(iv) if $g \in \mathcal{L}^p(\Omega) \cap \operatorname{Rg} L(\varphi)$ for $p \geq 2$ and $u \in V$ solves the equation

$$L(\varphi)u = g \text{ in } V^*$$

then $u \in X_p$. Consequently,

$$\operatorname{Rg} (L(\varphi)|_{X_p}) = \operatorname{Rg} L(\varphi) \cap \mathcal{L}^p(\Omega).$$

Proof. We first prove that the linear operator

$$\begin{aligned} T : V &\longrightarrow V^* \\ h &\longmapsto -\bar{f}'(\varphi)h + \int_{\Omega} \bar{f}'(\varphi)h. \end{aligned}$$

is compact. Indeed, it follows from the compact embedding $H \hookrightarrow V^*$ and the following estimate

$$\begin{aligned} \|Th\|_H &\leq \|\bar{f}'(\varphi)h\|_{L^2(\Omega)} + \left\| \int_{\Omega} \bar{f}'(\varphi)h \right\|_{L^2(\Omega)} \\ &\leq C(\|h\|_{L^2(\Omega)} + \|h\|_{L^1(\Omega)}) \\ &\leq C\|h\|_V. \end{aligned}$$

Recall that since A is an isomorphism from V onto V^* , it is also a Fredholm operator of index

$$\operatorname{ind} A := \dim \ker A - \operatorname{codim} \operatorname{Rg} A = 0.$$

It follows that $L(\varphi) = A + T$, as a sum of a Fredholm operator and a compact operator, is also a Fredholm operator with the same index [3, p. 168]. Therefore,

$$\operatorname{Rg} L(\varphi) \text{ is closed in } V^* \text{ and } \dim \ker L(\varphi) = \operatorname{codim} \operatorname{Rg} L(\varphi) < \infty. \quad (37)$$

(i) Using similar arguments as the proof in Lemma 3.7, we deduce that if $h \in \ker L(\varphi)$ then $h \in X_2$ and satisfies the equation:

$$\begin{aligned} -\Delta h - \bar{f}'(\varphi)h + \int_{\Omega} \bar{f}'(\varphi)h &= 0 \text{ in } \Omega, \\ \partial_{\nu} h &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Note that $\bar{f}'(\varphi) \in C^\infty(\bar{\Omega})$; we deduce that $h \in C^\infty(\bar{\Omega})$ from a Sobolev embedding theorem and a bootstrap argument.

(ii) We may identify the linear operator $L(\varphi)$ with a bilinear symmetric form on $V \times V$ (e.g see [29, Section 10.5.3 p. 82]). Thus, for every $u \in \ker L(\varphi), v = L(\varphi)w, w \in V$,

$$\langle u, v \rangle_{V, V^*} = \langle u, L(\varphi)w \rangle_{V, V^*} = \langle L(\varphi)u, w \rangle_{V^*, V} = 0,$$

which implies (ii).

(iii) Using part (ii), we deduce that for every $u \in \ker L(\varphi) \cap \text{Rg } L(\varphi)$, $\langle u, u \rangle_{V^*, V} = 0$, hence $u = 0$. It follows that $\ker L(\varphi) \cap \text{Rg } L(\varphi) = \{0\}$. On the other hand, $\dim \ker L(\varphi) = \text{codim } \text{Rg } L(\varphi)$ so that V^* is the algebraic direct sum of $\ker L(\varphi)$ and $\text{Rg } L(\varphi)$.

Since $\ker L(\varphi)$ is finite-dimensional, it is closed in V^* . It follows from (37) that $\text{Rg } L(\varphi)$ is closed in V^* , thus V^* is the topological direct sum of $\ker L(\varphi)$ and $\text{Rg } L(\varphi)$.

(iv) Since $g \in \text{Rg } L(\varphi)$, there exists $u \in V$ satisfying

$$Au = \bar{f}'(\varphi)u - \int_{\Omega} \bar{f}'(\varphi)u + g.$$

We write

$$Au = \bar{f}'(\varphi)u - \int_{\Omega} \bar{f}'(\varphi)u + g \in H,$$

thus $u \in X_2$ and $A = -\Delta$. We have

$$-\Delta u - \bar{f}'(\varphi)u = -\int_{\Omega} \bar{f}'(\varphi)u + g \in L^p(\Omega),$$

note that $\bar{f}'(\varphi) \in C^\infty(\bar{\Omega})$ and use elliptic regularity theory to deduce that $u \in X_p$. From this we obtain (iv). \square

We equip the norm $\|\cdot\|_{X_p} := \|\cdot\|_{W^{2,p}(\Omega)}$ on X_p .

Lemma 3.9. *Assume that $\varphi \in X_p, p > \max\{N, 2\}$ then we can consider E' as a mapping from X_p to $\mathcal{L}^p(\Omega)$ and it may be represented of the following form*

$$E'(\varphi) = -\Delta\varphi - \bar{f}(\varphi) + \int_{\Omega} \bar{f}(\varphi).$$

If, in addition $s_1 \leq \varphi \leq s_2$ then there exists a neighborhood $U(\varphi)$ of φ in X_p such that

$$E' \Big|_{U(\varphi)} \mapsto \mathcal{L}^p(\Omega)$$

is analytic.

Remark 3.10. We consider here the standard definition of analyticity (see [28, Definition 8.8, p. 362]):

A mapping T from a Banach space X into a Banach space Y is called analytic on a neighborhood of $z \in X$ if it may be represented as

$$T(z+h) - T(z) = \sum_{k \geq 1} T_k(z)[h, \dots, h] \text{ in } Y,$$

for any $h \in X$, $\|h\|_X \leq \varepsilon$, ε small enough,

where $T_k(z)$ is a symmetric k -linear form on X with values in Y and

$$\sum_{k \geq 1} \|T_k(z)\|_{\mathcal{L}_k(X,Y)} \|h\|^k < \infty.$$

Here, $\mathcal{L}_k(X, Y)$ is the space of bounded k -linear operators $X^k \rightarrow Y$.

Proof. If $\varphi \in X_p$, then we can see easily that

$$E'(\varphi) = -\Delta\varphi - \bar{f}(\varphi) + \int_{\Omega} \bar{f}(\varphi),$$

and obviously, $E'(\varphi) \in \mathcal{L}^p(\Omega)$.

Since \bar{f} is a polynomial on $(s_1 - 1, s_2 + 1)$, using Taylor's expansion we have

$$\bar{f}(\varphi(x) + h(x)) - \bar{f}(\varphi(x)) = \sum_{k=1}^n \frac{\bar{f}^{(k)}(\varphi(x))}{k!} h^k(x),$$

for all $h \in X_p$ such that $\|h\|_{C(\bar{\Omega})} \leq C\|h\|_{X_p} < 1$. It follows that

$$\begin{aligned} E'(\varphi + h) - E'(\varphi) &= -\Delta h + \sum_{k=1}^n \frac{\bar{f}^{(k)}(\varphi)}{k!} h^k - \sum_{k=1}^n \int_{\Omega} \frac{\bar{f}^{(k)}(\varphi)}{k!} h^k dx \\ &= \sum_{k=1}^n T_k[h, \dots, h], \end{aligned}$$

where

$$T_1[h] = -\Delta h + \bar{f}'(\varphi)h - \int_{\Omega} \bar{f}'(\varphi)h \quad \text{and}$$

$$T_k[h, \dots, h] = \frac{\bar{f}^{(k)}(\varphi)}{k!}h^k - \int_{\Omega} \frac{\bar{f}^{(k)}(\varphi)}{k!}h^k \quad \text{for all } 1 < k \leq n.$$

Note that for all $1 < k \leq n$ and for all $h_1, \dots, h_k \in X_p$

$$\begin{aligned} \left| T_k[h_1, \dots, h_k] \right| &\leq \left| \frac{\bar{f}^{(k)}(\varphi)}{k!}h_1 \dots h_k \right| + \left| \int_{\Omega} \frac{\bar{f}^{(k)}(\varphi)}{k!}h_1 \dots h_k \right| \\ &\leq C \prod_{i=1}^k \|h_i\|_{L^\infty(\Omega)} \leq C \prod_{i=1}^k \|h_i\|_{X_p}, \end{aligned}$$

so that

$$\left\| T_k[h_1, \dots, h_k] \right\|_{\mathcal{L}^p(\Omega)} \leq C \prod_{i=1}^k \|h_i\|_{X_p}.$$

This estimate implies that $T_k \in \mathcal{L}_k(X_p, \mathcal{L}^p(\Omega))$ for all $1 < k \leq n$. It is easy to see that $T_1 \in \mathcal{L}(X_p, \mathcal{L}^p(\Omega))$. It follows that E' is analytic in a neighborhood of φ . \square

Theorem 3.11. (*Lojasiewicz inequality*). *Let $\varphi \in V$ be a critical point of the functional E such that $s_1 \leq \varphi \leq s_2$. Then there exist constants $\theta \in (0, \frac{1}{2}]$ and $C, \sigma > 0$ such that*

$$|E(u) - E(\varphi)|^{1-\theta} \leq C \|E'(u)\|_{V^*}$$

for all $\|u - \varphi\|_V \leq \sigma$.

In order to study Lojasiewicz inequality of a functional $E \in C^2(V)$ near a critical point φ , Chill gives the following conditions:

(C₁) The kernel $V_0 := \ker L(\varphi)$ is a complemented subspace of V , that means there exists a projection $P \in \mathcal{L}(V)$ such that $V_0 = \text{Rg } P$. We denote by $P' \in \mathcal{L}(V^*)$ the adjoint projection of P .

(C₂) There exists a Banach space W with the following properties

- (i) W is continuously embedded in V^* and invariant under P' (i.e. $P'(W) \subset W$),

- (ii) $E' \in C^1(V, W)$,
 - (iii) $\text{Rg } L(\varphi) = \ker P' \cap W$.
- (C₃) There exists Banach spaces $X \subset V$ and $Y \subset W$ such that
- (i) the spaces X and Y are invariant under projection P and P' , respectively,
 - (ii) the restriction of the derivative E' in X is analytic in a neighborhood of φ with value in Y ,
 - (iii) $\ker L(\varphi)$ is contained in X and finite dimensional,
 - (iv) $\text{Rg } L(\varphi)|_X = \ker P' \cap Y$.

It follows from [8, Corollary 3.11] that if the conditions (C₁), (C₂), (C₃) hold then there exist $\sigma > 0, \theta \in (0, \frac{1}{2}]$, $C \geq 0$ such that

$$|E(u) - E(\varphi)|^{1-\theta} \leq C \|E'(u)\|_W \text{ for all } \|u - \varphi\|_V \leq \sigma.$$

We say that the functional E satisfies the Lojasiewicz inequality near φ and the constant θ will be called the Lojasiewicz exponent.

Our task is to verify the conditions (C₁), (C₂) (C₃). We need the following lemma

Lemma 3.12. *The conditions (C₁), (C₂) hold with $W = V^*$ and P the projection onto $\ker L(\varphi)$ along $\text{Rg } L(\varphi) \cap V$. In this case, the adjoint projection P' is an extension of P to V^* .*

Remark 3.13. Before giving the proof of Lemma 3.12, we recall a useful property in linear algebra. Let T be a linear mapping from a vector space to X into itself. Assume that $T^2 = T$, then

$$X = \ker T \oplus \text{Rg } T.$$

Proof. Because of Lemma 3.8(iii), there exists a projection $Q \in \mathcal{L}(V^*)$ onto $\ker L(\varphi)$ along $\text{Rg } L(\varphi)$. Since V and H are invariant under the projection Q , V and H can be represented as

$$V = \ker L(\varphi) \oplus (\text{Rg } L(\varphi) \cap V) \text{ and } H = \ker L(\varphi) \oplus (\text{Rg } L(\varphi) \cap H) \quad (38)$$

where the sums are algebraic direct sums. Moreover, the identities (38) are also topological direct sums since their components are closed in the corresponding spaces.

Let P, P_H be the restriction of Q to V, H , respectively. Then, $P \in \mathcal{L}(V), P_H \in \mathcal{L}(H)$. We shall prove that $P' = Q$.

By Lemma 3.8(ii), $\ker L(\varphi)$ is orthogonal to $\text{Rg } L(\varphi) \cap H$ with respect to the scalar product in H . Thus, P_H is an orthogonal projection in H . As a consequence P_H is an auto-adjoint operator of H .

Let $P' \in \mathcal{L}(V^*)$ be the adjoint projection of P . For all $u \in V, v \in H$,

$$\langle u, P'v \rangle_{V, V^*} = \langle Pu, v \rangle_{V, V^*} = \langle P_H u, v \rangle_{H, H} = \langle u, P_H v \rangle_{H, H} = \langle u, Qv \rangle_{V, V^*}$$

It follows that $P' = Q$ on H . On the other hand, H is dense in V^* we deduce that $P' = Q$. Put $W = V^*$ then it is easy to see that the conditions (\mathbf{C}_1) and (\mathbf{C}_2) hold with the above projections P and P' . \square

Proof of Theorem 3.11

It remain to verify the condition (\mathbf{C}_3) . Let $p > \max\{N, 2\}$, choose $X = X_p$ and $Y = \mathcal{L}^p(\Omega)$.

- (i) It follows from Lemma 3.8(i) that $\ker L(\varphi) \subset C^\infty(\bar{\Omega})$, thus $P(V), P(H), P'(V), P'(H)$ are contained in $\ker L(\varphi) \subset C^\infty(\bar{\Omega})$, which is a subspace of $X_p, \mathcal{L}^p(\Omega)$.
- (ii) Condition \mathbf{C}_3 (ii) follows from Lemma 3.9(i).
- (iii) Condition \mathbf{C}_3 (iii) follows from Lemma 3.8(i).
- (iv) Condition \mathbf{C}_3 (iv) follows from Lemma 3.8(iv).

These prove theorem. \square

3.3 Application of Lojasiewicz inequality

Lemma 3.14. *Let \bar{u} be the solution of Problem (\bar{P}) corresponding with the initial condition g_0 . Then*

- (i) *For all $0 < s \leq t < \infty$,*

$$E(\bar{u}(s)) = E(\bar{u}(t)) + \int_s^t \int_{\Omega} |\bar{u}_t|^2 dx. \quad (39)$$

Moreover if $g_0 \in V$ we can take $s = 0$ in (39).

(ii) Further, $e := \lim_{t \rightarrow \infty} E(\bar{u}(t))$ exists.

Proof. (i) By Lemma 2.8, \bar{u} is a smooth function on $\bar{\Omega} \times (0, \infty)$ so that

$$\begin{aligned} \frac{d}{dt} E(\bar{u}(t)) &= \int_{\Omega} (-\Delta \bar{u} - f(\bar{u})) \bar{u}_t \\ &= \int_{\Omega} (-\Delta \bar{u} - f(\bar{u}) + \int_{\Omega} f(\bar{u})) \bar{u}_t \\ &= - \int_{\Omega} \bar{u}_t^2(x, t) dx \leq 0. \end{aligned}$$

As a consequence, for all $0 < s \leq t < \infty$

$$E(\bar{u}(s)) = E(\bar{u}(t)) + \int_s^t \int_{\Omega} |\bar{u}_t|^2 dx.$$

If $g_0 \in V$, using the continuity of E in V , noting that $\bar{u}_t \in L^2(\Omega \times (0, T))$ we deduce that (39) also holds for $s = 0$.

(ii) We recall that the function \bar{F} is bounded on \mathbb{R} . Therefore the function $t \rightarrow E(\bar{u}(t))$, which is nonincreasing and bounded from below, converges to a limit as $t \rightarrow \infty$. \square

We denote by $S(t)$ the semigroup on H that corresponding with problem (\bar{P}) and define the ω -limit set of g_0 by

$$\omega(g_0) := \{\varphi \in V : \text{there exists } \{t_n\} \text{ such that } S(t_n)v_0 \rightarrow \varphi \text{ in } V \text{ as } t_n \rightarrow \infty\}.$$

We have the following result

Lemma 3.15. (i) $\omega(g_0)$ is non-empty, compact of V . Furthermore, $\omega(g_0)$ is positive invariant under $S(t)$, i.e., $S(t)(\omega(g_0)) \subset \omega(g_0)$.

(ii) The functional E is constant on $\omega(g_0)$. If $\varphi \in \omega(g_0)$ then $s_1 \leq \varphi \leq s_2$ and φ is a critical points of E , i.e., $E'(\varphi) = 0$.

(iii) $d(S(t)v_0, \omega(g_0)) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. (i) Since $\{\bar{u}(t), t \geq 1\}$ is compact in V , we can easily show that $\omega(g_0)$ is non-empty, compact of V . Next, note that if $\psi \in \omega(g_0)$, then there exists a sequence $t_n \rightarrow \infty$ such that $\psi = \lim_{n \rightarrow \infty} S(t_n)g_0$. For all $t \geq 0$, we have

$$S(t)\psi = \lim_{n \rightarrow \infty} S(t + t_n)g_0 \in \omega(g_0).$$

This shows that $\omega(g_0)$ is positive invariant.

(ii) First, we prove that E is constant on $\omega(g_0)$. Let $e = \lim_{t \rightarrow \infty} E(S(t)g_0)$ as in Lemma 3.14. For any $\varphi \in \omega(g_0)$, we have $\varphi = \lim_{n \rightarrow \infty} S(t_n)g_0$ for some sequence $t_n \rightarrow \infty$. Since E is continuous in V ,

$$E(\varphi) = \lim_{n \rightarrow \infty} E(S(t_n)g_0) = e.$$

i.e., E is constant on $\omega(g_0)$.

Note that $S(t_n)v_0 \rightarrow \varphi$ in also $L^2(\Omega)$ so that we can extract a subsequence of $S(t_n)v_0$ which converges almost everywhere on Ω . On the other hand $s_1 \leq S(t_n)v_0 \leq s_2$ for all $n \geq 0$, therefore

$$s_1 \leq \varphi \leq s_2.$$

We now prove that φ is a critical of E . Since $\omega(g_0)$ is positive invariance, $E(S(t)\varphi) = E(\varphi)$ for all $t \geq 0$. It follows from Lemma 3.14 that

$$\int_0^t \int_{\Omega} |\varphi_t|^2 dx dt = 0 \text{ for all } t \geq 0,$$

so that $\varphi_t = 0$ for all $t \geq 0$. Hence, φ satisfies the equation

$$\begin{aligned} -\Delta\varphi - \bar{f}(\varphi) + \int_{\Omega} \bar{f}(\varphi) &= 0 \text{ in } \Omega, \\ \partial_{\nu}\varphi &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Therefore, φ is a critical point of the functional E by Lemma 3.7.

(iii) Assume by contradiction that there exists a sequence $t_n \rightarrow \infty$ and $\varepsilon > 0$ such that $d(S(t_n)g_0, \omega(g_0)) \geq \varepsilon$. By compactness, there exists a subsequence $t_{n_k} \rightarrow \infty$ such that $S(t_{n_k})g_0 \rightarrow w \in \omega(g_0)$. Therefore, $d(S(t_{n_k})g_0, \omega(g_0)) = 0$ as $k \rightarrow \infty$, which is absurd. \square

Proof of Theorem 1.3

We will first apply Lojasiewicz inequality for Problem \bar{P} . Recall that since E is constant on $\omega(g_0)$, as in Lemma 3.15 we can write

$$e = E(v) \text{ for all } v \in \omega(g_0). \quad (40)$$

It follows from Theorem 3.11 that E satisfies the Lojasiewicz inequality in the neighborhood of every $\varphi \in \omega(g_0)$; in other words, we have that for every $\varphi \in \omega(g_0)$ there exist constants $\theta \in (0, \frac{1}{2}]$, $C \geq 0$ and $\delta > 0$ such that

$$|E(v) - E(\varphi)|^{1-\theta} \leq C \|E'(v)\|_{V^*} \text{ whenever } \|v - \varphi\|_V \leq \delta. \quad (41)$$

Since the functional E is continuous on V , we may choose δ small enough so that

$$|E(v) - E(\varphi)| < 1 \text{ whenever } \|v - \varphi\|_V \leq \delta. \quad (42)$$

It follows from the compactness of $\omega(g_0)$ in V that there exists a neighborhood \mathcal{U} of $\omega(g_0)$ composed of finitely many balls $B_j, j = 1, \dots, J$, with center φ_j and radius δ_j . In each of the ball B_j , inequality (42) and the Lojasiewicz inequality (41) hold for some constants θ_j and C_j . We define $\bar{\theta} = \min \{\theta_j, j = 1, \dots, J\}$ and $\bar{C} = \max \{C_j, j = 1, \dots, J\}$ to deduce from (40), (41) and (42) that

$$|E(v) - e|^{1-\bar{\theta}} \leq \bar{C} \|E'(v)\|_{V^*} \text{ for } v \in \mathcal{U}.$$

Using Lemma 3.15(iii), there exists $t_0 \geq 0$ such that $\bar{u}(t) \in \mathcal{U}$ for all $t \geq t_0$. Hence, for every $t \geq t_0$, there holds

$$\begin{aligned} -\frac{d}{dt} |E(\bar{u}(t)) - e|^{\bar{\theta}} &= \bar{\theta} |E(\bar{u}(t)) - e|^{\bar{\theta}-1} \left(-\frac{dE}{dt}(\bar{u}(t)) \right) \\ &\geq \frac{\bar{\theta}}{\bar{C}} \frac{\|\bar{u}_t\|_{L^2(\Omega)}^2}{\|E'(\bar{u}(t))\|_{V^*}}. \end{aligned} \quad (43)$$

Note that for all $t \geq t_0$, $\bar{u}(t) \in C^\infty(\bar{\Omega})$, so that $E'(\bar{u}(t)) \in H$ and it can be written of the form

$$E'(\bar{u}(t)) = -\Delta \bar{u} - \bar{f}(\bar{u}) + \int_{\Omega} \bar{f}'(\bar{u}) = -\bar{u}_t.$$

Applying continuous embedding $H \hookrightarrow V^*$, we have

$$\|E'(\bar{u}(t))\|_{V^*} \leq \bar{C} \|E'(\bar{u}(t))\|_{L^2(\Omega)} = \bar{C} \|\bar{u}_t\|_{L^2(\Omega)} \text{ for all } t \geq t_0, \quad (44)$$

where \bar{C} is a positive constant. Combining (43) and (44) we obtain

$$-\frac{d}{dt} |E(\bar{u}(t)) - e|^{\bar{\theta}} \geq C_0 \|\bar{u}_t\|_{L^2(\Omega)}.$$

Here, $C_0 = \frac{\bar{\theta}}{\bar{C}\bar{C}}$. Thus

$$\|\bar{u}(t_1) - \bar{u}(t_2)\|_{L^2} \leq \int_{t_1}^{t_2} \|\bar{u}_t\|_{L^2} \leq \frac{1}{C_0} (|E(\bar{u}(t_1)) - e|^{\bar{\theta}} - |E(\bar{u}(t_2)) - e|^{\bar{\theta}})$$

for all $t_0 \leq t_1 \leq t_2$. Therefore, $\|\bar{u}(t_1) - \bar{u}(t_2)\|_{L^2(\Omega)}$ tends to zero as $t_1 \rightarrow \infty$ so that $\{\bar{u}(t)\}$ is a Cauchy sequence in H . Consequently, there exists $\varphi \in H$ such that $\lim_{t \rightarrow \infty} \bar{u}(t) = \varphi$ exists in H . It follows from Corollary 2.9(ii) that $\varphi \in C^\infty(\bar{\Omega})$ and that $\bar{u}(t) \rightarrow \varphi$ in $C^m(\bar{\Omega})$ for all $m \in \mathbb{N}$. We conclude that

$$\lim_{t \rightarrow \infty} \|u(t) - \varphi\|_{C^m(\bar{\Omega})} = 0,$$

and φ is a solution of the stationary problem (S). Moreover, by conserved mass property

$$\int_{\Omega} \varphi \, dx = \int_{\Omega} g_0 \, dx,$$

which completes the proof of Theorem 1.3. \square

3.4 Rate of the convergence

In this section, we estimate the rate of the convergence of $u(t)$ to φ . The proof is based on Lojasiewicz inequality. We consider two cases, when Lojasiewicz exponent $\theta = \frac{1}{2}$ and $\theta \in (0, \frac{1}{2})$.

3.4.1 When Lojasiewicz exponent $\theta \in (0, \frac{1}{2})$

We need the following lemma.

Lemma 3.16 (see [13], Lemma 3.3). *Assume that for all $t \geq t_0$, some $\alpha > 0$ and a constant $K > 0$*

$$\int_t^\infty \|u_t\|_H^2 \leq K t^{-2\alpha-1}.$$

Then, we have

$$\|u(t) - u(\tau)\|_H \leq \frac{\sqrt{K}}{1 - 2^{-\alpha}} t^{-\alpha} \quad \text{for all } \tau \geq t \geq t_0.$$

Consequently,

$$\|u(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1 - 2^{-\alpha}} t^{-\alpha} \quad \text{for all } t \geq t_0.$$

Proof.

\square

Theorem 3.17. *Assume that Theorem 3.11 holds for $\theta \in (0, \frac{1}{2})$, then for $\alpha := \frac{\theta}{1-2\theta} > 0$ and a constant $K > 0$, we have*

$$\|u(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } t > 0.$$

Proof. By the modification on section 3.1, we only need to prove this theorem for function \bar{u} . Since $\bar{u}(x, t)$ is smooth for all $t > 0$, we have

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) = \langle E'(\bar{u}), \bar{u}_t \rangle = -\langle E'(\bar{u}), E'(\bar{u}) \rangle = -\|E'(\bar{u})\|_H^2. \quad (45)$$

Since $\bar{u}(t)$ tends to φ as $t \rightarrow \infty$. Therefore there exists $T_0 > 0$ such that for all $t \geq T_0$

$$\|\bar{u}(t) - \varphi\| \leq \sigma \quad (\sigma \text{ in Theorem 3.11}).$$

It follows that for all $t \geq T_0$

$$C\|E'(\bar{u})\|_{V^*} \geq |E(\bar{u}) - E(\varphi)|^{1-\theta}.$$

Therefore, by using the continuous embedding $H \hookrightarrow V^*$, we obtain

$$C_1\|E'(\bar{u})\|_H \geq |E(\bar{u}) - E(\varphi)|^{1-\theta} = (E(\bar{u}) - E(\varphi))^{1-\theta}. \quad (46)$$

Combining (47) and (48), we get

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) \leq -C_2(E(\bar{u}) - E(\varphi))^{2(1-\theta)} \quad \text{for all } t \geq T_0,$$

where $C_2 := 1/C_1^2$.

$$\text{Note that } y(t) := \left((E(\bar{u}(T_0)) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t-T_0) \right)^{-1/(1-2\theta)}$$

is the unique solution of the differential equation

$$\begin{cases} \frac{d}{dt}y(t) = -C_2y^{2(1-\theta)} \text{ for } t \geq T_0, \\ y(T_0) = E(\bar{u}(T_0) - E(\varphi)). \end{cases}$$

We use a differential inequality in [22, Lemma 2.7, p.53] to deduce for all $t \geq T_0$ that

$$\begin{aligned}
E(\bar{u}(t)) - E(\varphi) &\leq \left((E(\bar{u}(T_0)) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t-T_0) \right)^{-1/(1-2\theta)} \\
&= \left((E(\bar{u}(T_0)) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)t \right)^{-1/(1-2\theta)} \\
&= \left((E(\bar{u}(T_0)) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)\frac{t}{2} + C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \\
&\leq \left(C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \quad \text{for all } t \geq \bar{T}_0, \text{ with some } \bar{T}_0 > T_0 \text{ large enough.}
\end{aligned}$$

It follows that for all $t \geq \bar{T}_0$

$$\int_t^\infty \|\bar{u}_t(s)\|^2 ds \leq Kt^{-2\alpha-1}$$

Here, $K := \left(\frac{C_2(1-2\theta)}{2} \right)^{-1/(1-2\theta)}$ and $\alpha := \frac{\theta}{1-2\theta} > 0$. Now, according to Lemma 3.16, we obtain

$$\|\bar{u}(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } t \geq \bar{T}_0.$$

□

3.4.2 When Lojasiewicz exponent $\theta = \frac{1}{2}$

Lemma 3.18 (see [12], Lemma 2.2). *Assume that there exists two constants $\gamma > 0$ and $a > 0$ such that for all $t \in [0, T]$,*

$$\int_t^{+\infty} \|u_t\|_H^2 ds \leq a \exp(-\gamma t).$$

Then for all $\tau \geq t \geq 0$, we have

$$\|u(t) - u(\tau)\|_H \leq \sqrt{ab} \exp\left(-\frac{\gamma t}{2}\right).$$

Theorem 3.19. *Assume that Theorem 3.11 holds for $\theta \in (0, \frac{1}{2})$, then for constants $K, \delta > 0$, we have*

$$\|u(t) - \varphi\|_H \leq K \exp(-\delta t) \quad \text{for all } t > 0.$$

Proof. By the modification on section 3.1, we only need to prove this theorem for function \bar{u} . Since $\bar{u}(x, t)$ is smooth for all $t > 0$, we have

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) = \langle E'(\bar{u}), \bar{u}_t \rangle = -\langle E'(\bar{u}), E'(\bar{u}) \rangle = -\|E'(\bar{u})\|_H^2. \quad (47)$$

Since $\bar{u}(t)$ tends to φ as $t \rightarrow \infty$. Therefore there exists $T_0 > 0$ such that for all $t \geq T_0$

$$\|\bar{u}(t) - \varphi\| \leq \sigma \quad (\sigma \text{ in Theorem 3.11}).$$

It follows that for all $t \geq T_0$

$$C\|E'(\bar{u})\|_{V^*} \geq |E(\bar{u}) - E(\varphi)|^{\frac{1}{2}}.$$

Therefore, by using the continuous embedding $H \hookrightarrow V^*$, we obtain

$$C_1\|E'(\bar{u})\|_H \geq |E(\bar{u}) - E(\varphi)|^{\frac{1}{2}} = (E(\bar{u}) - E(\varphi))^{\frac{1}{2}}. \quad (48)$$

Combining (47) and (48), we get

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) \leq -C_2(E(\bar{u}) - E(\varphi)) \quad \text{for all } t \geq T_0,$$

where $C_2 := 1/C_1^2$.

Note that $y(t) := \left(E(\bar{u}(T_0) - E(\varphi)) \right) \exp(-C_2(t - T_0))$ is the unique solution of the differential equation

$$\begin{cases} \frac{d}{dt}y(t) = -C_2y \text{ for } t \geq T_0, \\ y(T_0) = E(\bar{u}(T_0) - E(\varphi)). \end{cases}$$

We use a differential inequality in [22, Lemma 2.7, p.53] to deduce for all $t \geq T_0$ that

$$E(\bar{u}(t)) - E(\varphi) \leq \left(E(\bar{u}(T_0) - E(\varphi)) \right) \exp(-C_2(t - T_0)).$$

In particular for all $t \geq T_0$,

$$\int_t^\infty \|\bar{u}_t(s)\|_H^2 ds \leq \left(E(\bar{u}(T_0)) - E(\varphi) \right) \exp(-C_2(t - T_0)).$$

Now, Using Lemma 3.18, we deduce the result of the theorem. \square

4 Stationary solution in one dimension

We consider Problem (P) in one dimension.

$$(P) \begin{cases} u_t = u_{xx} + f(u) - \int_{\Omega} f(u) & \text{in } \Omega \times \mathbb{R}^+, \\ u_x(x, t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = g_0(x) & x \in \Omega. \end{cases}$$

Theorem 4.1. *Set $N = 1$. Let $a < b$ be such that satisfy Property (C) with respect to f . We also assume that $f'(s) \leq 0$ for all $s \in [a, b]$. Assume that $a < g_0(x) < b$ for almost $x \in \Omega$, then*

$$\varphi = \int_{\Omega} g_0.$$

Proof. First, note that we have

$$a \leq u(x, t) \leq b \text{ for all } x \in \Omega, t \geq 0.$$

It follows that $a \leq \varphi \leq b$. Set $w := \varphi_x$, differentiating the equations of φ in stationary problem (S) with respect to x , we obtain

$$(P_1) \begin{cases} w_{xx} + f'(\varphi)w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $c(x) := f'(\varphi) \leq 0$, we apply maximum principle to deduce that $w \equiv 0$ is the unique solution of Problem (P₁). Consequently, φ is constant. This together with mass conservation property implies that

$$\varphi = \int_{\Omega} g_0.$$

\square

5 Appendix

Lemma 5.1. *Let T be a linear mapping from V to V such that $T^2 = T$. then*

(i)

$$V = \ker T \oplus \operatorname{Rg} T.$$

(ii) *Let W be a subspace of V . Assume that W is invariant i.e. $T(W) \subset W$. Then*

$$W = (\ker T \cap W) \oplus (\operatorname{Rg} T \cap W).$$

Proof. (ii) First, note that we have

$$T|_W : W \longrightarrow W$$

satisfies $(T|_W)^2 = T|_W$. It follows that

$$W = \ker (T|_W) \oplus \operatorname{Rg} (T|_W).$$

Since

$$\begin{aligned} \ker (T|_W) &= \{y \in W : T(x) = 0\} = \{x \in W \text{ and } x \in \ker T\} \\ &= \ker T \cap W, \end{aligned}$$

and since

$$\operatorname{Rg} (T|_W) = T(W) \subset \operatorname{Rg} T \cap W,$$

then

$$W = \ker (T|_W) \oplus \operatorname{Rg} (T|_W) \subset (\ker T \cap W) \oplus (\operatorname{Rg} T \cap W).$$

But,

$$W \supset (\ker T \cap W) \oplus (\operatorname{Rg} T \cap W).$$

It follows that

$$W = (\ker T \cap W) \oplus (\operatorname{Rg} T \cap W).$$

□

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