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MODELING A HARD, THIN CURVILINEAR INTERFACE

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ABSTRACT. In this paper, some results obtained on the asymptotic behavior of hard, thin curvilinear interfaces i.e., in cases where the interphase and adherents have comparable rigidities, are presented. The case of hard interfaces is investigated in terms of cylindrical coordinates and some analytical examples are presented.

1. Introduction. In this paper, the terms “interphase” and “interface” will be used in line with the following definition: an interphase is a thin volumic zone where exchanges occur between two materials or structures; and an interface is a contact surface, as well as being a mathematical expression for an interphase. In the fields of mechanical and civil engineering, interphases are known to play a crucial role in structure assemblies. However, since they are so thin, it is difficult to account directly for them in a complete analysis (the number of degrees of freedom is liable to be very large in a computational model). The strategy used here to overcome this problem consists in performing an asymptotic analysis. In this case, the adhesive is eliminated geometrically and a simpler equivalent interface model is obtained. If a relationship is obtained between the stress vector and the jump in the displacement, we have what we call an imperfect interface; otherwise it will be a perfect interface. It has been established in previous studies that if the stiffness of the interphase is similar to that of the adherents, various mathematical approaches (matching asymptotic expansions, energy methods, Γ -convergence, etc.), can be used, which yield a perfect interface model at the first order in the expansion procedure and an imperfect interface mode at the second order. The model obtained here is non-local, due to the presence of tangential derivatives. The aim of the present study was to obtain an interface law in terms of cylindrical coordinates in the context of elasticity, by performing a micro-mechanical analysis.

This paper consists of three parts. In section 2, some elementary examples are presented. In section 3, some general mathematical results are recalled. Section 4

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is devoted to drawing up the interface law in terms of polar coordinates, in order to model the gluing along some particular surfaces.

2. Some elementary examples.

2.1. A simple one-dimensional example in one dimension: A bar subjected to traction. We take the example of the equilibrium of an elastic bar AC divided into two parts, AB with stiffness E_1 and BC with stiffness E_2 (see Fig. 1). The bar is fixed at point A , a constant load $f = a$ is imposed on BC and a given load F is imposed at point C . We have $AB = \varepsilon$; $BC = L - \varepsilon$.

By simply integrating the equilibrium equations, we have

$$\begin{aligned}
 0 \leq x \leq \varepsilon \quad u(x) &= \frac{F + a(L - \varepsilon)}{E_1}x; \\
 \sigma &= F + a(L - \varepsilon) \\
 \varepsilon \leq x \leq L \quad u(x) &= -\frac{1}{2E_2}(x - \varepsilon)^2 + \frac{F + a(L - \varepsilon)}{E_2}(x - \varepsilon) + \frac{F + a(L - \varepsilon)}{E_1}\varepsilon; \\
 \sigma &= F + a(L - x)
 \end{aligned} \tag{1}$$

where u is the displacement field, and σ is the stress field. These two fields depend on ε .

At this stage, it is therefore possible to expand the displacement field in BC and the stress into powers of ε :

$$\begin{aligned}
 u(x) &= u^0(x) + \varepsilon u^1(x) + \dots \\
 \sigma(x) &= \sigma^0(x) + \varepsilon \sigma^1(x) + \dots
 \end{aligned} \tag{2}$$

where

$$\begin{aligned}
 u^0(x) &= -\frac{1}{2E_2}x^2 + \frac{F + aL}{E_2}x + \frac{F + aL}{E_1} \\
 u^1(x) &= \frac{1}{E_2}x - \frac{F + a(L - x)}{E_2} + \frac{F + aL}{E_1} \\
 \sigma^0(x) &= -ax + F + aL \\
 \sigma^1(x) &= 0
 \end{aligned} \tag{3}$$

where the jump in the displacement between points C and A is given by

$$\begin{aligned}
 [u^0] &= 0 \\
 [u^1] &= -\frac{F + aL}{E_1} + \frac{F + aL}{E_2}.
 \end{aligned} \tag{4}$$

If we expand the second order equation (the second equation in (4)), it emerges that the jump in the displacement is not equal to zero and an imperfect interface law is obtained.

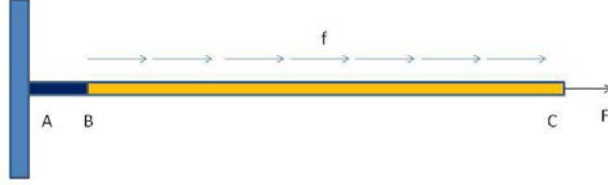


FIGURE 1. The example of a composite bar subjected to traction loading.

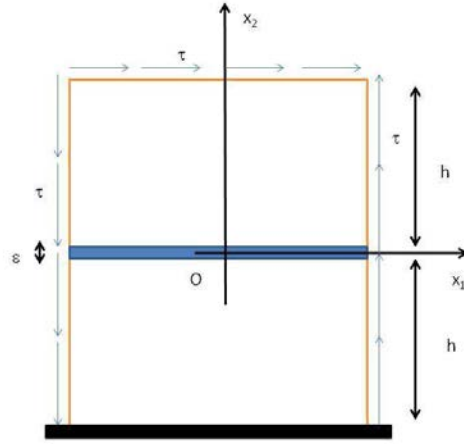


FIGURE 2. The case of a composite block.

2.2. A two-dimensional example: A composite block subjected to a shear load. In this section, we recall an example previously presented by the authors in [15]: that of the equilibrium of an elastic composite structure comprising two adherents, with shear modulus μ and height h , cemented together by a central adhesive with shear modulus $\bar{\mu}$ and thickness ε (see Fig. 2). The adhesion between the adhesive and the adherents is perfect. The lower base of the block is kept fixed. The upper base and the lateral surfaces are subjected to a given constant tangential load τ . Using standard notations, the displacement field is given by

$$\begin{aligned}
 u_1 &= \frac{\tau}{\mu}(x_2 + h + \varepsilon/2) \text{ in the lower adherent} \\
 u_1 &= \frac{\tau}{\mu}h + \frac{\tau}{\bar{\mu}}(x_2 + \varepsilon/2) \text{ in the glue} \\
 u_1 &= \frac{\tau}{\mu}(x_2 + h - \varepsilon/2) + \frac{\tau}{\bar{\mu}}\varepsilon \text{ in the upper adherent} \\
 u_2 &= 0 \text{ in the composite}
 \end{aligned} \tag{5}$$

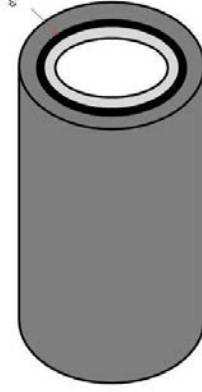


FIGURE 3. The case of a composite tube.

Based on the expansion presented in the previous section, we observe that

$$\begin{aligned}
 u_1^0 &= \frac{\tau}{\mu}(x_2 + h) \text{ in the lower adherent} \\
 u_1^0 &= \frac{\tau}{\mu}h + \frac{\tau}{\bar{\mu}}x_2 \text{ in the glue} \\
 u_1^0 &= \frac{\tau}{\mu}(x_2 + h) \text{ in the upper adherent}
 \end{aligned} \tag{6}$$

and

$$\begin{aligned}
 u_1^1 &= \frac{\tau}{2\mu} \text{ in the lower adherent} \\
 u_1^1 &= -\frac{\tau}{2\mu} + \frac{\tau}{\bar{\mu}} \text{ in the upper adherent}
 \end{aligned} \tag{7}$$

and hence,

$$\begin{aligned}
 [u^0] &= 0 \\
 [u_1^1] &= -\frac{\tau}{\mu} + \frac{\tau}{\bar{\mu}} \\
 [u_2^1] &= 0
 \end{aligned} \tag{8}$$

As in the previous section, the jump in the displacement is not equal to zero at the second order, and an imperfect interface law is therefore obtained.

2.3. An example in terms of curvilinear coordinates: A composite tube.

In this section, a two-dimensional case is analyzed and an analytical example is presented. We use the cylindrical coordinates (r, θ, z) . The composite consists of three tubes which are indexed by (1),(2), and (3), as shown in figure 3. The adhesion at the interfaces between the tubes is perfect. A pressure p_a is applied to

the internal surface $r = a$ and a pressure p_d is applied to the external surface $r = d$. We let $OA = a$, $OB = b$, $OC = c$ and $OD = d$. It is assumed that $c = b + \varepsilon$. The value of the parameter ε is assumed to be small (the internal tube is thin). The Lamé's coefficients are indexed depending on the number of tubes.

We take p_a , p_b , p_c and p_d to denote the pressures at points A , B , C and D respectively. The values of p_a and p_d are given, whereas the values of p_b and p_c are unknown.

The problem is assumed to be axially symmetric. The displacement fields in the radial direction are:

$$\begin{cases} u_1(r) = \frac{(p_a a^2 - p_b b^2)}{(b^2 - a^2)} \frac{r}{2(\lambda_1 + \mu_1)} + \frac{b^2 a^2 (p_a - p_b)}{2\mu_1 (b^2 - a^2) r} \\ u_2(r) = \frac{(p_b b^2 - p_c c^2)}{(c^2 - b^2)} \frac{r}{2(\lambda_2 + \mu_2)} + \frac{c^2 b^2 (p_b - p_c)}{2\mu_2 (c^2 - b^2) r} \\ u_3(r) = \frac{(p_c c^2 - p_d d^2)}{(d^2 - c^2)} \frac{r}{2(\lambda_3 + \mu_3)} + \frac{d^2 c^2 (p_c - p_d)}{2\mu_3 (d^2 - c^2) r} \end{cases} \quad (9)$$

To find the pressures p_b and p_c , we write the continuity of the displacement fields at points B and C [10]. A simple linear system is obtained:

$$\begin{cases} (M_a + M_{21})p_b - N_{21}p_c = K_a p_a \\ -N_{22}p_b + (M_d + M_{22})p_c = K_d p_d \end{cases} \quad (10)$$

where

$$\begin{cases} K_a = \frac{a^2 b}{b^2 - a^2} \frac{\lambda_1 + 2\mu_1}{\mu_1 (\lambda_1 + \mu_1)} \\ K_d = \frac{d^2 c}{d^2 - c^2} \frac{\lambda_3 + 2\mu_3}{\mu_3 (\lambda_3 + \mu_3)} \\ N_{21} = \frac{c^2 b}{c^2 - b^2} \frac{\mu_2 (\lambda_2 + \mu_2)}{\lambda_2 + 2\mu_2} \\ N_{22} = \frac{b^2 c}{b^2 - c^2} \frac{\mu_2 (\lambda_2 + \mu_2)}{\lambda_2 + 2\mu_2} \\ M_a = \frac{\mu_1 b^3 + (\lambda_1 + \mu_1) b a^2}{(b^2 - a^2) \mu_1 (\lambda_1 + \mu_1)} \\ M_d = \frac{\mu_3 c^3 + (\lambda_3 + \mu_3) c d^2}{(d^2 - c^2) \mu_3 (\lambda_3 + \mu_3)} \\ M_{21} = \frac{\mu_2 b^3 + (\lambda_2 + \mu_2) b c^2}{(c^2 - b^2) \mu_2 (\lambda_2 + \mu_2)} \\ M_{22} = \frac{\mu_2 c^3 + (\lambda_2 + \mu_2) c b^2}{(c^2 - b^2) \mu_2 (\lambda_2 + \mu_2)} \end{cases} \quad (11)$$

The solution of this system is:

$$\begin{cases} p_b = \frac{1}{\Delta} (K_a p_a (M_d + M_{22}) + K_d p_d N_{21}) \\ p_c = \frac{1}{\Delta} (K_d p_d (M_a + M_{21}) + K_a p_a N_{22}) \end{cases} \quad (12)$$

where $\Delta = (M_a + M_{21})(M_d + M_{22}) - N_{21}N_{22}$.

We study p_b and p_c when ε tends to zero. We observe that p_b and p_c tend to the same value, and hence

$$\begin{aligned} p_c - p_b &\rightarrow [p^0] \\ &= 0 \end{aligned} \tag{13}$$

In the same way, we study $u_3(c) - u_1(b)$ when ε tends to zero.

This gives:

$$\begin{aligned} u_3(c) - u_1(b) &\rightarrow [u^0] \\ &= 0 \end{aligned} \tag{14}$$

We note that the interface law in the radial direction is a perfect interface law.

$$\begin{cases} [u^0] = 0 \\ [p^0] = 0 \end{cases} \tag{15}$$

It is observed that p_b/ε and p_c/ε do not tend to the same value, and we therefore study $(u_3(c) - u_1(b))/\varepsilon$ when ε goes to zero. In conclusion, at order one, an imperfect interface law is obtained. We have:

$$\begin{aligned} (p_c - p_b)/\varepsilon &\rightarrow [p^1] \\ &\neq 0 \end{aligned} \tag{16}$$

$$\begin{aligned} (u_3(c) - u_1(b)) &\rightarrow [u^1] \\ &\neq 0 \end{aligned} \tag{17}$$

3. Recalling mathematical results.

3.1. Recalling Γ -convergence. Let X be a topological space and let $F^\varepsilon : X \rightarrow [0, \infty]$ be a sequence of functionals on X . Then F^ε are said to Γ -converge to the Γ -limit $F^0 : X \rightarrow [0, \infty]$ if the following two conditions hold:

- For every sequence u^ε in X such that $u^\varepsilon \rightarrow u^0$ as $\varepsilon \rightarrow 0$, $F^0(u^0) \leq \liminf_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon)$ (lower bound inequality).
- For every $u^0 \in X$, there is a sequence u^ε converging to u^0 such that $F(u^0) \geq \limsup_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon)$ (upper bound inequality).

Note that the minimizers converge to the minimizers, i.e. if $F^\varepsilon \rightarrow F^0$, and u^ε is a minimizer for F^ε , then every cluster point in the sequence u^ε is a minimizer of F^0 . This theory will be applied in section 3.3.

3.2. The mechanical problem. Let us consider a body occupying an open bounded set Ω of R^3 , with a smooth boundary $\partial\Omega$, where the three dimensional space is referred to the orthonormal frame (O, e_1, e_2, e_3) . This set Ω is assumed to form a

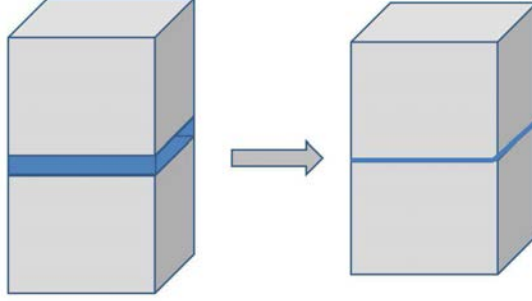


FIGURE 4. Initial configuration and geometrical limit of thin interfaces

non-empty intersection S with the plane $\{x_3 = 0\}$. We write $\hat{x} = (x_1, x_2)$. Let $\varepsilon > 0$ be a parameter tending to zero. We introduce the following domains:

$$\begin{aligned}
 B^\varepsilon &= \{(x_1, x_2, x_3) \in \Omega : |x_3| < \frac{\varepsilon}{2}\}, \\
 \Omega^\varepsilon &= \{(x_1, x_2, x_3) \in \Omega : |x_3| > \frac{\varepsilon}{2}\}, \\
 \Omega_\pm^\varepsilon &= \{(x_1, x_2, x_3) \in \Omega : \pm x_3 > \frac{\varepsilon}{2}\}, \\
 S_\pm^\varepsilon &= \{(x_1, x_2, x_3) \in \Omega : \pm x_3 = \frac{\varepsilon}{2}\}, \\
 \Omega_\pm &= \{(x_1, x_2, x_3) \in \Omega : \pm x_3 > 0\}, \\
 S &= \{(x_1, x_2, x_3) \in \Omega : x_3 = 0\}, \\
 \Omega_0 &= \Omega_+ \cup \Omega_-.
 \end{aligned} \tag{18}$$

The sets B^ε and Ω^ε are the domains occupied by the adhesive and the adherents, respectively (see fig. 4). The structure is subjected to a body force density φ and a surface force density g on part Γ_1 of the boundary, whereas it is clamped on the remaining part Γ_0 of the boundary. The two bodies and the joint are assumed to be linearly elastic. We take σ^ε and u^ε to denote the stress tensor and the displacement field, respectively, assuming the occurrence of small perturbations, and the strain tensor is written as follows:

$$e_{ij}(u^\varepsilon) = \frac{1}{2} \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right). \tag{19}$$

We take a_{ijkl}^\pm to denote the elasticity coefficients of the adherents, and a_{ijkl}^m to denote the elasticity coefficients of the glue.

If $f : \Omega \mapsto \mathbb{R}^3$ is a given function, we take f_ε^\pm to denote the restrictions of f to the adherents. We also take f_ε^m to denote the restriction applying to the glue. We also denote the jumps of f , as follows:

$$[f]_\varepsilon^+ := f_\varepsilon^+(x_1, x_2, (\frac{\varepsilon}{2})^+) - f_\varepsilon^m(x_1, x_2, (\frac{\varepsilon}{2})^-), \tag{20}$$

$$[f]_\varepsilon^- := f_\varepsilon^-(x_1, x_2, (-\frac{\varepsilon}{2})^-) - f_\varepsilon^m(x_1, x_2, (-\frac{\varepsilon}{2})^+), \tag{21}$$

$$[f]_\varepsilon := f_\varepsilon^m(x_1, x_2, (\frac{\varepsilon}{2})^-) - f_\varepsilon^m(x_1, x_2, (-\frac{\varepsilon}{2})^+). \tag{22}$$

If $f : \Omega_0 \mapsto R^3$ is a given function, we denote the restrictions of f to Ω_{\pm} by f^{\pm} and we also denote the following jump of f in S

$$[f] := f^+(x_1, x_2, 0^+) - f^-(x_1, x_2, 0^-). \quad (23)$$

We therefore have to solve the following problem:

$$(P_{\varepsilon}) \begin{cases} \text{Find } (u^{\varepsilon}, \sigma^{\varepsilon}) \text{ such that :} \\ \sigma_{ij,j}^{\varepsilon} = -\phi_i & \text{in } \Omega \\ \sigma_{ij}^{\varepsilon} = a_{ijkh}^{\pm} e_{kh}(u^{\varepsilon}) & \text{in } \Omega_{\pm}^{\varepsilon} \\ \sigma_{ij}^{\varepsilon} = a_{ijkh}^m e_{kh}(u^{\varepsilon}) & \text{in } B^{\varepsilon} \\ u^{\varepsilon} = 0 & \text{on } \Gamma_0 \\ \sigma_{\varepsilon} n = g & \text{on } \Gamma_1 \\ [u^{\varepsilon}]_{\varepsilon}^{\pm} = 0, [\sigma^{\varepsilon} e_3]_{\varepsilon}^{\pm} = 0 & \text{on } S_{\pm}^{\varepsilon} \end{cases}$$

We make the following assumptions

$$\begin{aligned} H1) \quad & \begin{cases} a_{ijkl}^{\pm, m} \in L^{\infty}(\Omega), \\ a_{ijkl}^{\pm, m} = a_{klij}^{\pm, m} = a_{jilk}^{\pm, m} \\ \exists \eta^{\pm, m} > 0 : a_{ijkl}^{\pm, m} e_{ij} e_{kl} \geq \eta^{\pm, m} e_{ij} e_{ij} \quad \forall e_{ij} = e_{ji}, \end{cases} \\ H2) \quad & \exists \varepsilon_0 : B_{\varepsilon} \cap (\Gamma_1 \cup \text{supp}(\phi)) = \emptyset, \quad \forall \varepsilon < \varepsilon_0. \\ H3) \quad & \phi \in (L^2(\Omega))^3, g \in (L^2(\Gamma_1))^3. \end{aligned}$$

and introduce the space of kinematically admissible displacements

$$V^{\varepsilon} = \{u \in (W^{1,2}(\Omega))^3 : u = 0 \text{ on } \Gamma_0\}. \quad (24)$$

Note that (P_{ε}) is equivalent to the minimization V^{ε} of the energy:

$$F^{\varepsilon}(v^{\varepsilon}) = \frac{1}{2} \int_{\Omega_{\varepsilon}} a^{\pm} e(v^{\varepsilon}) \cdot e(v^{\varepsilon}) dV + \frac{1}{2} \int_{B^{\varepsilon}} a^m e(v^{\varepsilon}) \cdot e(v^{\varepsilon}) dV - \int_{\Omega_{\varepsilon}} \phi \cdot v^{\varepsilon} dV - \int_{\Gamma_1} g \cdot v^{\varepsilon} dS \quad (25)$$

Using the Lax-Milgram lemma, it is clearly established that these two problems have a unique solution in V^{ε} .

3.3. Results at first order. In this section, it is assumed that the elastic coefficients of the glue a_{ijkl}^m do not depend on the thickness of the glue ε , i.e., that the adhesive and the adherents show a similar rigidity. Here we study the behavior of the solutions to the problem P_{ε} when the thickness ε tends to zero.

Adopting the same hypotheses as above and using Γ -convergence theory, it can be proved [15] that the unique solution u^{ε} to problem P_{ε} tends in $L^2(\Omega_0)$ to u^0 , which is the minimum value of the energy defined in V^0

$$F^0(v) = \frac{1}{2} \int_{\Omega_0} a^{\pm} e(v) \cdot e(v) dV + \int_{\Omega_0} \phi \cdot v dV - \int_{\Gamma_1} g \cdot v dS \quad (26)$$

where

$$V^0 = \{u \in (W^{1,2}(\Omega_+))^3 \cup (W^{1,2}(\Omega_-))^3 : u = 0 \text{ on } \Gamma_0 \text{ } [u] = 0 \text{ on } S\}. \quad (27)$$

Note that the minimizer u^0 is the unique solution of problem P_0 where

$$(P_0) \left\{ \begin{array}{ll} \text{Find } (u^0, \sigma^0) \text{ such that :} & \\ \sigma_{ij,j}^0 = -\varphi_i & \text{in } \Omega_0 \\ \sigma_{ij}^0 = a_{ijkh}^\pm e_{kh}(u^0) & \text{in } \Omega^\pm \\ u^0 = 0 & \text{on } \Gamma_0 \\ \sigma^0 n = g & \text{on } \Gamma_1 \\ [u^0] = 0 & \text{on } S \\ [\sigma^0 n] = 0 & \text{on } S. \end{array} \right.$$

In particular, we observe that perfect adhesion is obtained at the interface between the two adhesives. This result is proved rigourously in [15]. Other possible methods have been presented in [16].

3.4. Results at the second order. In the previous section, we have recalled that

$$u^\varepsilon \rightarrow u^0 \text{ in } L^2(\Omega_0). \quad (28)$$

We can therefore extract a subsequence, which is not relabeled, such that

$$\frac{u^\varepsilon - u^0}{\varepsilon} \rightharpoonup u^1 \text{ in } L^2(\Omega_0). \quad (29)$$

In this section, we recall some properties of this weak limit u^1 .

Under the previous hypotheses and using some analytical arguments it is possible to show [15] that the weak limit u^1 is the solution (in the distributional sense) to problem P_1 , where

$$(P_1) \left\{ \begin{array}{ll} \text{Find } (u^1, \sigma^1) \text{ such that :} & \\ \sigma_{ij,j}^1 = 0 & \text{in } \Omega_0 \\ \sigma_{ij}^1 = a_{ijkh}^\pm e_{kh}(u^1) & \text{in } \Omega^\pm \\ u^1 = 0 & \text{on } \Gamma_0 \\ \sigma^1 n = 0 & \text{on } \Gamma_1 \\ [u^1] = A_{uu} D_\alpha u^0 + A_{u\sigma} \sigma^0 n & \text{on } S \\ [\sigma^1 n] = A_{\sigma u} D_\alpha^2 u^0 + A_{\sigma\sigma} D_\alpha \sigma^0 n & \text{on } S \end{array} \right.$$

where $A_{..}$ are four fourth order tensors and D_α are tangential derivatives in the plane of S . In particular, if the glue is isotropic and if we take λ and μ to denote

the Lamé's coefficients of the glue, the coefficients of these tensors are given by

$$\begin{aligned}
[u_\alpha^1] &= \frac{1}{\mu} \sigma_{\alpha 3}^0 - u_{3\alpha}^0 - \frac{1}{2}(u_{\alpha,3}^0(0^+) + u_{\alpha,3}^0(0^-)), \quad \alpha = 1, 2, \\
[u_3^1] &= \frac{1}{\lambda + 2\mu} \sigma_{33}^0 - \frac{\lambda}{\lambda + 2\mu} (u_{1,1}^0 + u_{2,2}^0) - \frac{1}{2}(u_{3,3}^0(0^+) + u_{3,3}^0(0^-)), \\
[\sigma_{13}^1] &= -\left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{1,11}^0 + \mu u_{1,22}^0 + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{2,21}^0 + \frac{\lambda}{\lambda + 2\mu} \sigma_{33,1}^0 \right) \\
&\quad - \frac{1}{2}(\sigma_{13,3}^0(0^+) + \sigma_{13,3}^0(0^-)) \\
[\sigma_{23}^1] &= -\left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{2,22}^0 + \mu u_{2,11}^0 + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{1,12}^0 + \frac{\lambda}{\lambda + 2\mu} \sigma_{33,2}^0 \right) \\
&\quad - \frac{1}{2}(\sigma_{23,3}^0(0^+) + \sigma_{23,3}^0(0^-)) \\
[\sigma_{33}^1] &= -\sigma_{13,1}^0 - \sigma_{23,2}^0 - \frac{1}{2}(\sigma_{33,3}^0(0^+) + \sigma_{33,3}^0(0^-)).
\end{aligned}$$

3.5. A comment about elasto-dynamics. In [20], under specific conditions as regards the volumic mass of the glue, similar results are obtained in elastodynamic terms, i.e., the last equation in the solution to problem \bar{P}_0 corresponds to a constitutive equation. In fact, it emerges that the elastodynamic problem involving a thin adhesive layer can be approximated, with a convergence result, by another expression in which the layer is changed into a mechanical constraint, which is exactly the same as that occurring in the equilibrium case.

4. Asymptotic analysis in terms of cylindrical coordinates.

4.1. Recalling the equations giving the problem. We take (e_r, e_θ, e_z) to denote the orthonormal cylindrical basis and (r, θ, z) to denote the three coordinates of a particle. Without any volume forces, the equilibrium equations for a deformable body can be written in terms of cylindrical coordinates as follows:

$$\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\sigma_{r\theta}}{\partial \theta} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\partial \sigma_{rz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r} \sigma_{r\theta} + \frac{\partial \sigma_{\theta z}}{\partial z} &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{1}{r} \sigma_{rz} + \frac{\partial \sigma_{zz}}{\partial z} &= 0
\end{aligned} \tag{30}$$

where $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}$ and $\sigma_{\theta z}$ are the components of the stress field. In the same way, the components of the strain tensor $e(u)$ are written:

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} \\
e_{\theta\theta} &= \frac{1}{r} \left(u_r + \frac{\partial u_\theta}{\partial \theta} \right) \\
e_{zz} &= \frac{\partial u_z}{\partial z} \\
e_{r\theta} &= \frac{1}{2r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) + \frac{\partial u_\theta}{\partial r} \\
e_{rz} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\
e_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right)
\end{aligned} \tag{31}$$

where u_r , u_θ and u_z are the three components of the displacement field expressed in terms of cylindrical coordinates.

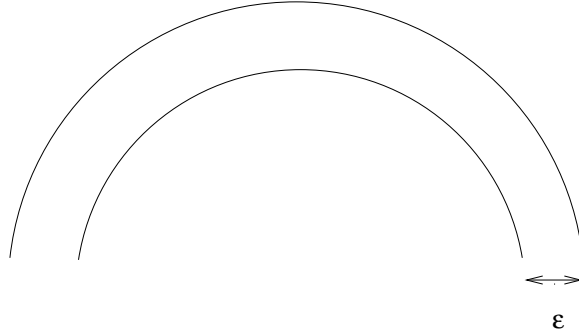


FIGURE 5. Gluing in normal direction.

4.2. A first analysis: Radial gluing. The gluing between the two adherents is assumed to be orthogonal to the radial direction. The thickness of the glue is also assumed to be constant and equal to ε . The glue lies in the interval $[r_0 - \varepsilon/2, r_0 + \varepsilon/2]$. A change of variable is introduced into the glue in the radial direction. We write $R = r_0 + \frac{r - r_0}{\varepsilon}$. Some expansions of the displacement u^ε and the stress σ^ε are performed to the power of ε :

$$\begin{aligned}
\sigma(r, \theta, z) &= \tau^0(R, \theta, z) + \varepsilon \tau^1(R, \theta, z) + \dots \\
u(r, \theta, z) &= v^0(R, \theta, z) + \varepsilon v^1(R, \theta, z) + \dots
\end{aligned} \tag{32}$$

It can be seen here that

$$\begin{aligned}
\partial R &= \frac{1}{\varepsilon} \partial r \\
\frac{1}{r} &= \frac{1}{r_0 + (R - r_0)\varepsilon} \\
\frac{1}{r} &\approx \frac{1}{r_0} \left(1 - \frac{R - r_0}{r_0} \varepsilon + \frac{(R - r_0)^2}{r_0^2} \varepsilon^2 + \dots \right)
\end{aligned} \tag{33}$$

The balance equation gives:

$$\begin{aligned}
\frac{1}{\varepsilon} \frac{\partial \tau_{rr}^0}{\partial R} + \frac{\partial \tau_{rr}^1}{\partial R} + \frac{1}{r_0} (\tau_{rr}^0 - \tau_{\theta\theta}^0) + \frac{\partial \tau_{rz}^0}{\partial z} + \frac{1}{r_0} \frac{\partial \tau_{r\theta}^0}{\partial \theta} \dots &= 0 \\
\frac{1}{\varepsilon} \frac{\partial \tau_{r\theta}^0}{\partial R} + \frac{\partial \tau_{r\theta}^1}{\partial R} + \frac{1}{r_0} \frac{\partial \tau_{\theta\theta}^0}{\partial \theta} + \frac{2}{r_0} \tau_{r\theta}^0 + \frac{\partial \tau_{\theta z}^0}{\partial z} + \dots &= 0 \\
\frac{1}{\varepsilon} \frac{\partial \tau_{rz}^0}{\partial R} + \frac{\partial \tau_{rz}^1}{\partial R} + \frac{1}{r_0} \frac{\partial \tau_{\theta z}^0}{\partial \theta} + \frac{1}{r_0} \tau_{rz}^0 + \frac{\partial \tau_{zz}^0}{\partial z} + \dots &= 0
\end{aligned} \tag{34}$$

We now focus on the term occurring in ε^{-1} in eq. (34). We obtain:

$$\begin{aligned}
\frac{\partial \tau_{rr}^0}{\partial R} &\approx 0 \\
\frac{\partial \tau_{r\theta}^0}{\partial R} &\approx 0 \\
\frac{\partial \tau_{rz}^0}{\partial R} &\approx 0
\end{aligned} \tag{35}$$

In conclusion, $\frac{\partial \tau_r^0}{\partial R} \approx 0$ i.e. $[[\tau_r^0]] = 0$, where $[[\cdot]]$ is the jump of a function between $\frac{1}{2}$ and $-\frac{1}{2}$.

The coefficients of the strain tensor are now developed:

$$\begin{aligned}
e_{rr} &= \frac{1}{\varepsilon} \frac{\partial v_r^0}{\partial R} + \frac{\partial v_r^1}{\partial R} + \dots \\
e_{\theta\theta} &= \frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \frac{\varepsilon}{r_0} (-\frac{R-r_0}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + v_r^1 + \frac{\partial v_\theta^1}{\partial \theta}) + \dots \\
e_{zz} &= \frac{\partial v_z^0}{\partial z} + \varepsilon \frac{\partial v_z^1}{\partial z} + \dots \\
e_{r\theta} &= \frac{1}{2} (\frac{1}{\varepsilon r_0} \frac{\partial v_\theta^0}{\partial R} + \frac{1}{r_0} (\frac{\partial v_r^0}{\partial \theta} - v_\theta + \frac{\partial v_\theta^1}{\partial R})) + \dots \\
e_{rz} &= \frac{1}{2} (\frac{1}{\varepsilon} \frac{\partial v_z^0}{\partial R} + \frac{\partial v_r^0}{\partial z} + \frac{\partial v_z^1}{\partial R}) + \dots \\
e_{\theta z} &= \frac{1}{2} (\frac{\partial v_\theta^0}{\partial z} + \frac{\partial v_z^0}{\partial \theta}) + \frac{\varepsilon}{2} (\frac{\partial v_\theta^1}{\partial z} + \frac{\partial v_z^1}{\partial \theta}) + \dots
\end{aligned} \tag{36}$$

Since the glue is assumed to be elastic, the constitutive equation becomes:

$$\sigma = \lambda(e_{rr} + e_{\theta\theta} + e_{zz})Id + 2\mu e(u) \tag{37}$$

where λ and μ are the Lamé coefficients of the glue.

Equations (36) and (37) are now combined. The following term in ε^{-1} is obtained:

$$\begin{aligned}
\frac{\partial v_r^0}{\partial R} &= 0 \\
\frac{\partial v_\theta^0}{\partial R} &= 0 \\
\frac{\partial v_z^0}{\partial R} &= 0
\end{aligned} \tag{38}$$

In conclusion, $\frac{\partial v^0}{\partial R} \approx 0$ i.e. $[[v^0]] = 0$.

In the adherents, the material is assumed to be elastic and a change of variable is made in the radial direction. We write $R = r + \frac{1}{2} \pm \frac{\varepsilon}{2}$. Some expansions of the displacement u^ε and the stress σ^ε are now performed to the power of ε :

$$\begin{aligned}
\sigma(r, \theta, z) &= \sigma^0(R, \theta, z) + \varepsilon \sigma^1(R, \theta, z) + \dots \\
u(r, \theta, z) &= u^0(R, \theta, z) + \varepsilon u^1(R, \theta, z) + \dots
\end{aligned} \tag{39}$$

At level zero, the stress σ^0 satisfies the equilibrium conditions in the adherents and due to the continuity of the stress vector and the displacement along the interface glue/adherents, we have:

$$\begin{aligned}
\sigma^0(0^\pm, \theta, z) &\approx \tau^0(\pm \frac{1}{2}, \theta, z) \\
u^0(0^\pm, \theta, z) &\approx v^0(\pm \frac{1}{2}, \theta, z)
\end{aligned} \tag{40}$$

At level 0, we therefore obtain a perfect interface law:

$$\begin{aligned}
[\sigma^0] &= 0 \\
[u^0] &= 0
\end{aligned} \tag{41}$$

Now we deal with the term occurring in ε^0 in eqs. (36) and (37). This gives:

$$\begin{aligned}
\tau_{rr}^0 &= (\lambda + 2\mu) \frac{\partial v_r^1}{\partial R} + \lambda \left(\frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \frac{\partial v_z^0}{\partial z} \right) \\
\tau_{\theta\theta}^0 &= \frac{\lambda + 2\mu}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \lambda \left(\frac{\partial v_r^1}{\partial R} + \frac{\partial v_z^0}{\partial z} \right) \\
\tau_{zz}^0 &= (\lambda + 2\mu) \frac{\partial v_z^0}{\partial z} + \lambda \left(\frac{\partial v_r^1}{\partial R} + \frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) \right) \\
\tau_{r\theta}^0 &= \mu \left(\frac{1}{r_0} \left(\frac{\partial v_r^0}{\partial \theta} - v_\theta^0 \right) + \frac{\partial v_\theta^1}{\partial R} \right) \\
\tau_{rz}^0 &= \mu \left(\frac{\partial v_r^0}{\partial z} + \frac{\partial v_z^1}{\partial R} \right) \\
\tau_{\theta z}^0 &= \mu \left(\frac{\partial v_\theta^0}{\partial z} + \frac{1}{r_0} \frac{\partial v_z^0}{\partial \theta} \right)
\end{aligned} \tag{42}$$

We thus obtain:

$$\begin{aligned}
[[v_r^1]] &= \frac{\tau_{rr}^0}{\lambda + 2\mu} - \frac{\lambda}{\lambda + 2\mu} \left(\frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \frac{\partial v_z^0}{\partial z} \right) \\
[[v_\theta^1]] &= \frac{\tau_{r\theta}^0}{\mu} - \frac{1}{r_0} \left(\frac{\partial v_r^0}{\partial \theta} - v_\theta^0 \right) \\
[[v_z^1]] &= \frac{\tau_{rz}^0}{\mu} - \frac{\partial v_r^0}{\partial z}
\end{aligned} \tag{43}$$

Note that equation (43) introduces some tangential derivatives (i.e. in the z direction): the interface law will therefore be non local.

The equilibrium equation at order one gives:

$$\begin{aligned}
\frac{\partial \tau_{rr}^1}{\partial R} + \frac{1}{r_0} \left(\frac{\partial \tau_{r\theta}^0}{\partial \theta} + \tau_{rr}^0 - \tau_{\theta\theta}^0 \right) + \frac{\partial \tau_{rz}^0}{\partial z} &= 0 \\
\frac{\partial \tau_{r\theta}^1}{\partial R} + \frac{1}{r_0} \frac{\partial \tau_{\theta\theta}^0}{\partial \theta} + \frac{2}{r_0} \tau_{r\theta}^0 + \frac{\partial \tau_{\theta z}^0}{\partial z} &= 0 \\
\frac{\partial \tau_{rz}^1}{\partial R} + \frac{1}{r_0} \frac{\partial \tau_{\theta z}^0}{\partial \theta} + \frac{1}{r_0} \tau_{rz}^0 + \frac{\partial \tau_{zz}^0}{\partial z} &= 0
\end{aligned} \tag{44}$$

Based on the constitutive equation, we obtain:

$$\begin{aligned}
\tau_{\theta\theta}^0 &= \frac{\lambda + 2\mu}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \lambda \left(\frac{\tau_{rr}^0}{\lambda + 2\mu} - \frac{\lambda}{\lambda + 2\mu} \left(\frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \frac{\partial v_z^0}{\partial z} \right) + \frac{\partial v_z^0}{\partial z} \right) \\
\tau_{zz}^0 &= (\lambda + 2\mu) \frac{\partial v_z^0}{\partial z} \\
&\quad + \lambda \left(\frac{\tau_{rr}^0}{\lambda + 2\mu} - \frac{\lambda}{\lambda + 2\mu} \left(\frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) + \frac{\partial v_z^0}{\partial z} \right) + \frac{1}{r_0} (v_r^0 + \frac{\partial v_\theta^0}{\partial \theta}) \right) \\
\tau_{\theta z}^0 &= \mu \left(\frac{\partial v_\theta^0}{\partial z} + \frac{1}{r_0} \frac{\partial v_z^0}{\partial \theta} \right)
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
[[\tau_{rr}^1]] &= -\frac{1}{r_0} \left(\frac{\partial \tau_{r\theta}^0}{\partial \theta} - \frac{2\mu}{\lambda + 2\mu} \tau_{rr}^0 \right) - \frac{\partial \tau_{rz}^0}{\partial z} \\
&\quad + \frac{1}{r_0} \left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \left(\frac{v_r^0}{r_0} + \frac{1}{r_0} \frac{\partial v_\theta^0}{\partial \theta} \right) + \frac{2\lambda\mu}{\lambda + 2\mu} \frac{\partial v_z^0}{\partial z} \right) \\
[[\tau_{r\theta}^1]] &= -\frac{1}{r_0} \left(\frac{\lambda}{\lambda + 2\mu} \frac{\partial \tau_{rr}^0}{\partial \theta} + 2\tau_{r\theta}^0 \right) \\
&\quad - \frac{1}{r_0} \left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \left(\frac{1}{r_0} \frac{\partial v_r^0}{\partial \theta} + \frac{1}{r_0} \frac{\partial^2 v_\theta^0}{\partial \theta^2} \right) + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \frac{\partial^2 v_z^0}{\partial z \partial \theta} \right) - \mu \frac{\partial^2 v_\theta^0}{\partial z^2} \\
[[\tau_{rz}^1]] &= -\frac{\lambda}{\lambda + 2\mu} \frac{\partial \tau_{rr}^0}{\partial z} - \frac{1}{r_0} \tau_{rz}^0 \\
&\quad - \frac{1}{r_0} \left(\mu \left(\frac{\partial^2 v_\theta^0}{\partial z \partial \theta} + \frac{1}{r_0} \frac{\partial^2 v_z^0}{\partial \theta^2} \right) - \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial^2 v_z^0}{\partial z^2} - \frac{2\lambda\mu}{\lambda + 2\mu} \frac{1}{r_0} \frac{\partial v_r^0}{\partial z} \right)
\end{aligned} \tag{46}$$

The stress σ^1 can be seen to meet the equilibrium conditions in the adherents at level one, and due to the continuity of the stress vector and the displacement along the glue/adherents interface:

$$\begin{aligned}
\sigma^1(0^\pm, \theta, z) + \frac{1}{2} \frac{\partial \sigma^0}{\partial r}(0^\pm, \theta, z) &\approx \tau^1(\pm \frac{1}{2}, \theta, z) \\
u^1(0^\pm, \theta, z) + \frac{1}{2} \frac{\partial u^0}{\partial r}(0^\pm, \theta, z) &\approx v^1(\pm \frac{1}{2}, \theta, z)
\end{aligned} \tag{47}$$

The interface law is therefore written:

$$\begin{aligned}
[\sigma_{rr}^1] &= -\frac{1}{r_0} \left(\frac{\partial \sigma_{r\theta}^0}{\partial \theta} - \frac{2\mu}{\lambda + 2\mu} \sigma_{rr}^0 \right) - \frac{\partial \sigma_{rz}^0}{\partial z} \\
&\quad + \frac{1}{r_0} \left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \left(\frac{u_r^0}{r_0} + \frac{1}{r_0} \frac{\partial u_\theta^0}{\partial \theta} \right) + \frac{2\lambda\mu}{\lambda + 2\mu} \frac{\partial u_z^0}{\partial z} \right) - S(\sigma_{rr}^1) \\
[\tau_{r\theta}^1] &= -\frac{1}{r_0} \left(\frac{\lambda}{\lambda + 2\mu} \frac{\partial \sigma_{rr}^0}{\partial \theta} + 2\sigma_{r\theta}^0 \right) \\
&\quad - \frac{1}{r_0} \left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \left(\frac{1}{r_0} \frac{\partial u_r^0}{\partial \theta} + \frac{1}{r_0} \frac{\partial^2 u_\theta^0}{\partial \theta^2} \right) + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \frac{\partial^2 u_z^0}{\partial z \partial \theta} \right) - \mu \frac{\partial^2 u_\theta^0}{\partial z^2} \\
&\quad - S(\sigma_{r\theta}^1) \\
[\sigma_{rz}^1] &= -\frac{\lambda}{\lambda + 2\mu} \frac{\partial \sigma_{rr}^0}{\partial z} - \frac{1}{r_0} \sigma_{rz}^0 \\
&\quad - \frac{1}{r_0} \left(\mu \left(\frac{\partial^2 u_\theta^0}{\partial z \partial \theta} + \frac{1}{r_0} \frac{\partial^2 u_z^0}{\partial \theta^2} \right) - \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \frac{\partial^2 u_z^0}{\partial z^2} - \frac{2\lambda\mu}{\lambda + 2\mu} \frac{1}{r_0} \frac{\partial u_r^0}{\partial z} \right) \\
&\quad - S(\sigma_{rz}^1)
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
[u_r^1] &= \frac{\sigma_{rr}^0}{\lambda + 2\mu} - \frac{\lambda}{\lambda + 2\mu} \left(\frac{1}{r_0} \left(u_r^0 + \frac{\partial u_\theta^0}{\partial \theta} \right) + \frac{\partial u_z^0}{\partial z} \right) - S(u_r^0) \\
[u_\theta^1] &= \frac{\sigma_{r\theta}^0}{\mu} - \frac{1}{r_0} \left(\frac{\partial u_r^0}{\partial \theta} - u_\theta^0 \right) - S(u_\theta^0) \\
[u_z^1] &= \frac{\sigma_{rz}^0}{\mu} - \frac{\partial u_r^0}{\partial z} - S(u_z^0)
\end{aligned} \tag{49}$$

where $S(f) = \frac{1}{2} \left(\frac{\partial f}{\partial r}(0^+, \theta, z) + \frac{\partial f}{\partial r}(0^-, \theta, z) \right)$.

The imperfect interface law is then obtained as follows:

- The jump in the displacement means that there is discontinuity between the two adherents. This jump is given by the solution at order zero. In particular, the law includes some tangential derivatives. The interfacial law is a non local law.
- The jump in stress vector means that a given load (depending on the zero order) is exerted along the interface.

Note that it is possible to solve the limit problem (involving equilibrium between the adherents) numerically at order zero with a perfect interface law. The displacement and the stress vector occurring at this order are the data inputs to the equilibrium equations for the adherents at order one in the framework of a non local interface law. The law obtained was found to depend on the curvature (which is given by the term r_0).

4.3. A comment about the dynamic processes. If we replace the equilibrium equations by elastodynamics equations, we have to consider various cases.

- If the density of the glue is low, ranging for example around ε , the results presented in the previous section will still hold true, i.e., there will be no inertial effects.
- If the density of the glue is not low enough, equation (44) will have to be changed, by adding inertial terms to the interfacial law at order one.

5. Conclusions. In this paper, an interface contact law was developed in terms of cylindrical coordinates. This non local interface law is of the imperfect interface kind. The non local law includes some terms accounting for the curvatures. It is now proposed to extend these findings to more general curvilinear interphases, accounting for roughness and cracks, and to implement these laws in a computational software program.

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