Locally optimal controllers and application to orbital transfer (long version)

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Abstract

In this paper we consider the problem of global asymptotic stabilization with prescribed local behavior. We show that this problem can formulated in terms of control Lyapunov functions. Moreover, we show that if the local control law has been synthesized employing a LQ approach, then the associated Lyapunov function can be seen has the value function of an optimal problem with some specific properties. We illustrate these result on two specific classes of systems: backstepping and feedforward systems. Finally, we show how this framework can be employed when considering an orbital transfer problem.

Key words: Lyapunov function, Nonlinear systems, optimal control, LQ

1 Introduction

The synthesis of a stabilizing control law for systems described by nonlinear differential equations has been the subject of great interest by the nonlinear control community during the last three decades. Depending on the structure of the model, some techniques are now available to synthesize control laws ensuring global and asymptotic stabilization of the equilibrium point.

For instance, we can refer to the popular backstepping approach (see [10,1] and the reference therein), or the forwarding approach (see [12,7,14]) and some others based on energy considerations or dissipativity properties (see [9] for a survey of the available approaches).

Although the global asymptotic stability of the steady point can be achieved in some specific cases, it remains difficult to address in the same control objective performance issues of a nonlinear system in a closed loop. However, when the first order approximation of the non-linear model is considered, some performance aspects can be addressed by using linear optimal control techniques (using LQ controller for instance).

Hence, it is interesting to raise the question of synthesizing a nonlinear control law which guarantees the global asymptotic stability of the origin while ensuring a prescribed local linear behavior.

In the present paper we consider this problem. In a first section we will motivate this control problem and we will consider a first strategy based on the design of a uniting control Lyapunov function. We will show that this is related to an equivalent problem which is the design of a control Lyapunov function with prescribed quadratic approximation around the origin. In a second part of this paper, we will consider the case in which the prescribed local behavior is an optimal LQ controller. In this framework, we investigate what type of performance is achieved by the control solution to the stabilization with prescribed local behavior. In a third part we consider two specific classes of systems and show how the control with prescribed local behavior can be solved. Finally in the third part of the paper, we consider a specific control problem which is the orbital transfer problem. Employing the Lyapunov approach of Kellet and Praly in [8] we will exhibit a class of costs for which the stabilization with local optimality can be achieved.

2 Stabilization with prescribed local behavior

To present the problem under consideration, we introduce a general controlled nonlinear system described by the following ordinary differential equation:

$$\dot{x} = \Phi(x, u) \ , \tag{1}$$

with the state x in $\mathbb{R}^n$ and $\Phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a C^1 function such that $\Phi(0,0) = 0$ and u is a scalar control input. For this system, we can introduce the two matrices describing its first order approximation which is assumed to be stabilizable:

$$\mathbb{A} := \frac{\partial \Phi}{\partial x}(0,0) \ , \ \mathbb{B} := \frac{\partial \Phi}{\partial u}(0,0) \ .$$

For system (1), the problem we intend to solve can be described as follows:

Global asymptotic stabilization with prescribed local behavior: Assume the linear state feedback law $u = K_o x$ stabilizes the first order approximation of system (1). We are looking for a stabilizing control law $u = \alpha_o(x)$, with $\alpha_o : \mathbb{R}^n \to \mathbb{R}^p$, differentiable at 0 such that:

(1) The origin of the closed-loop system $\dot{x} = \Phi(x, \alpha_o(x))$ is globally and asymptotically stable ;
(2) The first order approximation of the control law α_o satisfies the following equality.

$$\frac{\partial \alpha_o}{\partial x}(0) = K_o \ . \tag{2}$$

This problem has already been addressed in the literature. For instance, it is the topic of the papers [6,16,3]. Note moreover that this problem can be related to the problem of uniting a local and a global control laws as introduced in [19] (see also [15]).

In this paper, we restrict our attention to the particular case in which the system is input affine. More precisely we consider systems in the form

$$\dot{x} = a(x) + b(x)u \ , \tag{3}$$

with the two C^1 functions $a : \mathbb{R}^n \to \mathbb{R}^n$ and $b : \mathbb{R}^n \to \mathbb{R}^{n\times p}$. In this case we get $\mathbb{A} = \frac{\partial a}{\partial x}(0)$ and $\mathbb{B} = b(0)$.

A necessary and sufficient condition to solve the global asymptotic stabilization with prescribed local behavior in terms of Lyapunov functions can be given as follows.
Theorem 1. *Given a linear state feedback law $u = K_o x$ which stabilizes the first order approximation of system (3). The following two statements are equivalent.*

(1) There exists a control law $u = \alpha_o(x)$ solution to the global asymptotic stabilization with prescribed local behavior problem.
(2) There exists a C^2 proper, positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that the following two properties are satisfied.
- *If we denote[1] $P := \frac{1}{2}H(V)(0)$, then P is a positive definite matrix. Moreover this inequality holds.*

$$(\mathbb{A} + \mathbb{B}K_o)'P + P(\mathbb{A} + \mathbb{B}K_o) < 0 \ ; \tag{4}$$

- *Artstein condition is satisfied. More precisely, this implication holds for all x in $\mathbb{R}^n \setminus \{0\}$.*

$$L_bV(x) = 0 \Rightarrow L_aV(x) < 0 \ . \tag{5}$$

[1] In the following, given a C^2 function $V : \mathbb{R}^n \to \mathbb{R}$, the notation $H(V)(x)$ is the Hessian matrix in $\mathbb{R}^{n\times n}$ evaluated at x of the function V. More precisely, it is the matrix

$$(H(V))_{i,j}(x) = \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \ .$$

Proof : 1 $\Rightarrow$ 2) The proof of this part of the theorem is based on recent results obtained in [2]. Indeed, the design of the function V is obtained from the uniting of a quadratic local control Lyapunov function (denoted V_0) and a global control Lyapunov function (denoted V_∞) obtained employing a converse Lyapunov theorem.

First of all, employing the converse Lyapunov theorem of Kurzweil in [11], there exists a C^∞ function $V_\infty : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\frac{\partial V_\infty}{\partial x}(x)[a(x) + b(x)\alpha_o(x)] < 0 \ , \ \forall \ x \neq 0 \ .$$

On the other hand, $\mathbb{A} + \mathbb{B}K_o$ being Hurwitz, there exists a matrix P such that the algebraic Lyapunov inequality (4) is satisfied. Let V_0 be the quadratic function $V_0(x) = x'Px$. Due to the fact that K_o satisfies equation (2) it yields that the matrix $\mathbb{A} + \mathbb{B}K_o$ is the first order approximation of the system (3) with the control law $u = \alpha_o(x)$. Consequently, it implies that there exists a positive real number ϵ_1 such that

$$\frac{\partial V_0}{\partial x}(x)[a(x) + b(x)\alpha_o(x)] < 0 \ , \ \forall \ |x| \leq \epsilon_1 \ .$$

This implies that the time derivative of the two control Lyapunov functions V_0 and V_∞ can be made negative definite with the same control law in a neighborhood of the origin. Employing [2, Theorem 2.1], it yields the existence of a C^2 at the origin function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a positive real number ϵ_2 such that the following two properties hold.

- For all x in $\mathbb{R}^n \setminus \{0\}$,

$$\frac{\partial V}{\partial x}(x)[a(x) + b(x)\alpha_o(x)] < 0 \ .$$

  Hence, Artstein condition (5) is satisfied ;
- For all x in $\mathbb{R}^n$ such that $|x| \leq \epsilon_2$, we have

$$V(x) = V_0(x) \ .$$

  Consequently $\mathcal{H}(V)(0) = 2P$.

2 $\Rightarrow$ 1) Let Q be the positive definite matrix defined as,

$$Q := -(\mathbb{A} + \mathbb{B}K_o)'P + P(\mathbb{A} + \mathbb{B}K_o) \ .$$

Employing the local approximation of the Lyapunov function V, it is possible to find r_0 such that

$$L_aV(x) + L_bV(x)K_o x < 0 \ , \ \forall x \in \{0 < V(x) \leq r_0\} \ . \tag{6}$$

This implies that the control Lyapunov function V satisfies the small control property (see [18]). Hence, we get the existence of a control law α_∞ (given by Sontag universal formulae introduced in [18]) such that this one satisfies for all $x \neq 0$

$$L_aV(x) + L_bV(x)\alpha_u(x) < 0 \ . \tag{7}$$

A solution to the stabilization with prescribed local problem can be given by the control law

$$\alpha_o(x) = \rho(V(x))\alpha_\infty(x) + (1 - \rho(V(x)))K_o x$$

where $\rho : \mathbb{R}_+ \to [0, 1]$ is any locally Lipschitz function such that

$$\rho(s) = \begin{cases} 0 \ , & s \leq \frac{r_0}{2} \ , \\ 1 \ , & s \geq r_0 \ . \end{cases}$$

Note that with this selection, it yields that equality (2) holds. Moreover, we have along the solution of the system (3)

$$\dot{V}(x)\Big|_{u=\alpha_o(x)} = \rho(V(x)) \ \dot{V}(x)\Big|_{u=\alpha_\infty} + (1 - \rho(V(x))) \ \dot{V}(x)\Big|_{u=K_o x} < 0$$

Hence, we get the result. □

From this theorem, we see that looking for a global control Lyapunov function locally assigned by the prescribed local behavior and looking for the controller itself are equivalent problems.

3 Locally optimal and globally inverse optimal control laws

If one wants to guarantee a specific behavior on the closed loop system, one might want to find a control law which minimizes a specific cost function. More precisely, we may look for a stabilizing control law which minimizes the criterium

$$J(x, u) = \int_0^{+\infty} q(X(x, t; u)) + u(x, t)' r(X(x, t; u)) u(x, t) dt \ , \tag{8}$$

where $X(x, t; u)$ is the solution of the system (3) initiated from x at $t = 0$ and employing the control $u(x, t)$, $q : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous function and r is a continuous function which values $r(x)$ are symmetric positive definite matrices.

The control law which solves this minimization problem (see [17]) is given as

$$u = -\frac{1}{2} r(x)^{-1} L_b V(x) \ , \tag{9}$$

where $V : \mathbb{R}^n \to \mathbb{R}_+$ is the solution with $V(0) = 0$ to the following Hamilton-Jacobi-Bellman equation for all x in $\mathbb{R}^n$

$$q(x) + L_a V(x) - \frac{1}{4} L_b V(x) r(x)^{-1} L_b V(x)' = 0 \ . \tag{10}$$

Given a function q and a function r, it is in general difficult or impossible to solve the so called HJB equation. However, for linear system, this might be solved easily. If we consider the first order approximation of the system (3), and given a positive definite matrix R and a positive semi definite matrix Q we can introduce the quadratic cost:

$$J(x, u) = \int_0^{+\infty} [X(x, t; u)' Q X(x, t; u) + u(x, t)' R u(x, t)] \, dt \ , \tag{11}$$

In this context, solving the HJB equation can be rephrased in solving the algebraic HJB equation given as

$$P\mathbb{A} + \mathbb{A}'P - P\mathbb{B}R^{-1}\mathbb{B}'P + Q = 0 \ . \tag{12}$$

It is well known that provided, the couple $(\mathbb{A}, \mathbb{B})$ is controllable, it is possible to find a solution to this equation. Hence, for the first order approximation, it is possible to solve the optimal control problem when considering a cost in the form of (11).

From this discussion, we see that an interesting control strategy is to solve the stabilization with prescribed local behavior with the local behavior obtained solving LQ control strategy. Note however that once we have solved this problem, one may wonder what type of performance has been achieved by this new control law. The following Theorem addresses this point and is inspired from [17] (see also [13]). Following Theorem 1, this one is given in terms of control Lyapunov functions.

Theorem 2 (Local optimality and global inverse optimality)**.** *Given two positive definite matrices R and Q. Assume there exists a C^2 proper positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that the following two properties hold.*

- *The matrix $P := H(V)(0)$ is positive definite matrix and satisfies the following equality.*

$$P\mathbb{A} + \mathbb{A}'P - P\mathbb{B}R^{-1}\mathbb{B}'P + Q = 0 \ ; \tag{13}$$

- *Artstein condition is satisfied (see (5)).*

Then there exist $q : \mathbb{R}^n \to \mathbb{R}_+$ *a continuous function,* C^2 *at zero and* r *a continuous function which values* $r(x)$ *are symmetric positive definite matrices such that the following properties are satisfied.*

- *The function* q *and* r *satisfy*

$$H(q)(0) = 2Q \ , \ r(0) = R \ ; \tag{14}$$

- *The function* V *is a value function associated to the cost (8). More precisely,* V *satisfies the HJB equation (10).*

Proof : This proof is inspired from some of the results of [13].

First of all, there exists a positive real number r_0 such that for all x such that $0 < V(x) \leq r_0$ we have

$$-L_f V(x) + \frac{1}{4} L_g V(x) R^{-1} L_g V(x)' > 0 \ .$$

Now, for all k in $\mathbb{N}$, we consider C_k the subset of $\mathbb{R}^n$ defined as

$$C_k = \{x, kr_0 \leq V(x) \leq (k+1)r_0\} \ .$$

Note that for all k the set C_k is a compact subset. Assume for the time being that for all k there exists ℓ_k in $\mathbb{R}_+$ such that :

$$L_a V(x) - \frac{\ell_k}{4} L_b V(x) R^{-1} L_g V(x)' < 0 \ , \ \forall x \in C_k \ . \tag{15}$$

Let μ be any continuous function such that,

$$\mu(s) \begin{cases} = 1 \ , & s \leq \frac{r_0}{2} \ , \\ \geq 1 \ , & \frac{r_0}{2} \leq s \leq r_0 \ , \\ \geq \ell_k \ , & kr_0 \leq s \leq (k+1)r_0 \ . \end{cases}$$

Moreover, let

$$r(x) := \frac{1}{\mu(V(x))} R \ ,$$

and

$$q(x) := -L_a V(x) + \frac{1}{4} L_b V(x) r(x)^{-1} L_b V(x)' \ .$$

With (15) and the definition of μ, it yields,

$$q(x) > 0 \ , \ \forall x \neq 0 \ .$$

Hence, V is solution to the associated HJB equation. Note moreover that we have $r(0) = R$ and

$$\frac{1}{2}H(q)(0) = \mathbb{A}'P + P\mathbb{A} - P\mathbb{B}R^{-1}\mathbb{B}'P = Q \ .$$

Hence, the result.

In conclusion, to get the result, we only need to show that for all k in $\mathbb{N}$, there exists ℓ_k such that (15) is satisfied. Assume this is not the case for a specific k in $\mathbb{N}$. This implies that for all j in $\mathbb{N}$ there exists x_j in C_k such that

$$L_aV(x_j) - \frac{j}{4}L_bV(x_j)R^{-1}L_bV(x_j)' \geq 0 \ .$$

The sequence x_j being in a compact set, we know there exists a converging subsequence denoted $(x_{j_\ell})_{\ell\in\mathbb{N}}$ which converges toward a cluster point denoted x^* in C_k. The previous inequality can be rewritten as:

$$\frac{L_aV(x_{j_\ell})}{j_\ell} \geq \frac{1}{4}L_bV(x_{j_\ell})R^{-1}L_bV(x_{j_\ell})' \geq 0 \ .$$

Letting j_ℓ goes to infinity yields the following.

$$L_aV(x^*) \geq 0 \ , \ L_bV(x^*) = 0 \ .$$

With Artstein condition, this implies that $L_aV(x^*) < 0$ hence a contradiction. This ends the proof. □

This Theorem establishes that if we solve the stabilization with a prescribed local behavior, we may design a control law $u = \alpha_o(x)$ such that this one is solution to an optimal control problem and such that the local approximation of the associated cost is exactly the one of the local system. This framework has already been studied in the literature in [6]. In this paper is addressed the design of a backstepping with a prescribed local optimal control law. In our context, we get a Lyapunov sufficient condition to design a globally and asymptotically stabilizing optimal control law with prescribed local cost function.

Corollary 1 (Locally optimal control design)**.** *Consider two positive definite matrices R and Q respectively in $\mathbb{R}^{m\times m}$ and $\mathbb{R}^{n\times n}$. Assume there exists a C^2 proper positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that the following properties hold.*

- *The matrix $P := \frac{1}{2}H(V)(0)$ is positive definite matrix and satisfies*

$$P\mathbb{A} + \mathbb{A}'P - P\mathbb{B}R^{-1}\mathbb{B}'P + Q = 0 \ ; \qquad (16)$$

- *Artstein condition is satisfied (see (5)).*

Then there exist q, r and α_o which is solution to an optimal control problem with cost $J(\chi, u)$ defined in (8), with q and r which satisfy (14).

4 Some particular classes of systems

In this Section we consider several classes of system and show what type of local optimal control problem can be solved.

4.1 Strict feedback form

Following the work of [6], consider the case in which system (3) with state $\chi = (y, x)$ can be written in the following form

$$\dot{y} = h(y, x) \ , \ \dot{x} = f(y, x) + g(y, x)u \ . \tag{17}$$

with y in $\mathbb{R}^{n_y}$, x in $\mathbb{R}$ and $g(y, x) \neq 0$ for all (y, x).

In this case, the first order approximation of the system is

$$\mathbb{A} = \begin{bmatrix} H_1 & H_2 \\ F_1 & F_2 \end{bmatrix} \ , \ \mathbb{B} = \begin{bmatrix} 0 \\ G \end{bmatrix} \ , \tag{18}$$

with $H_1 = \frac{\partial h}{\partial y}(0,0)$, $H_2 = \frac{\partial h}{\partial x}(0,0)$, $F_1 = \frac{\partial f}{\partial y}(0,0)$, $F_2 = \frac{\partial f}{\partial x}(0,0)$, $G = g(0,0)$.

For this class of system we make the following assumption.
Assumption 1. *For all K_y in $\mathbb{R}^{n_y}$ such that $H_1 + H_2 K_y$ is Hurwitz, there exists a smooth function $\alpha_y : \mathbb{R}^{n_y} \to \mathbb{R}$ such that the following holds.*

- *The origin is a globally asymptotically stable equilibrium for*

$$\dot{y} = h(y, \alpha_y(y)) \ ;$$

- *The function α_y satisfies $\frac{\partial \alpha_y}{\partial y}(0) = K_y$.*

This assumption establishes that the stabilization with prescribed local behavior is satisfied for the y subsystem seeing x as the control input.

For this class of system, we have the following theorem which can already be found in [6].

Theorem 3 (Backstepping Case). *Let K_o in $\mathbb{R}^{p\times n}$ be a matrix such that $\mathbb{A}+\mathbb{B}K_o$ is Hurwitz with $\mathbb{A}$ and $\mathbb{B}$ defined in (18). Then there exists a smooth function $\alpha_o : \mathbb{R}^n \to \mathbb{R}^p$ which solves the global asymptotic stabilization with prescribed local behavior.*

Proof : Let P be a positive definite matrix such that the algebraic Lyapunov inequality (4) is satisfied. This matrix can be rewritten $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}$ with P_{22}, P_{12}, P_{22} matrices respectively in $\mathbb{R}^{n_y\times n_y}, \mathbb{R}^{n_y\times n}, \mathbb{R}$. Consider the Lyapunov function $V_0(\mathcal{x}) = \mathcal{x}'P\mathcal{x}$. Let T be the matrix in $\mathbb{R}^{(n_y+1)\times n_y}$ defined as [2]

$$T = \begin{bmatrix} \mathrm{Id}_{n_y} \\ -\frac{P_{12}'}{P_{22}} \end{bmatrix} .$$

Note that this matrix satisfies

$$T'P = \begin{bmatrix} P_y & 0 \end{bmatrix} \ , \ T'P\mathbb{B} = 0 \ ,$$

where $P_y = P_{11} - P_{12}P_{22}^{-1}P_{12}'$.

By pre and post multiplying inequality (4) respectively by T' and T it yields the following inequality.

$$P_y\left(H_1 - H_2\frac{P_{12}'}{P_{22}}\right) + \left(H_1 - H_2\frac{P_{12}'}{P_{22}}\right)' P_y < 0 \ . \tag{19}$$

The matrix P being positive definite, its Schur complement P_y is also positive definite. Hence, inequality (19) can be seen as a Lyapunov inequality and $x = -\frac{P_{12}}{P_{22}}y$ as a stabilizing local controller for the y subsystem with P_y as associated Lyapunov matrix. With Assumption 1, and Theorem 1 we know there exist a smooth function $\alpha_y : \mathbb{R}^{n_y} \to \mathbb{R}$ and a smooth function $V_y : \mathbb{R}^{n_y} \to \mathbb{R}_+$ such that the following two properties hold.

- The origin of the system $\dot{y} = h(y, \alpha_y(y))$ is globally and asymptotically stable with associated Lyapunov function V_y. More precisely, we have

$$\frac{\partial V_y}{\partial y}(y)h(y,\alpha_y(y)) < 0 \ , \ \forall y \neq 0 \ ; \tag{20}$$

- We have the local properties

$$\frac{\partial \alpha_y}{\partial y}(0) = -\frac{P_{12}}{P_{22}} \ , \ H(V_y)(0) = 2P_y \ .$$

[2] Given a positive integer n, the notation Id_n is the identity matrix in $\mathbb{R}^{n\times n}$.

Consider now the function

$$V(x) = V_y(y) + P_{22}(x - \alpha_y(y))^2 \ . \tag{21}$$

Note that this function is proper and positive definite. Moreover, we have

$$L_b V(x) = 2P_{22}(x - \alpha_y(y))g(x, y) \ .$$

Since $g(x, y) \neq 0$ by assumption, this implies

$$L_b V(x) = 0, x \neq 0 \Rightarrow x = \alpha_y(y) \ .$$

Note that when $x = \alpha_y(y)$, with (20) we have for all $y \neq 0$

$$L_a V(x) = \frac{\partial V_y}{\partial y}(y)h(y, \alpha_y(y)) < 0 \ .$$

Hence, Artstein condition is satisfied. Finally, we have the following equality.

$$H(V)(0) = 2P \ .$$

Hence, with Theorem 1, we get the result. □

Note that with Corollary 1, this theorem establishes that given Q, a positive definite matrix in $\mathbb{R}^{n_y \times n_y}$, and R, a positive real number, then there exist q, r and α_o which is solution to an optimal control problem with cost $J(x, u)$ defined in (8), with q and r which satisfy (14). In other words we can design a globally and asymptotically stabilizing optimal control law with prescribed local cost function.

4.2 Feedforward form

Following our previous work in [3], consider the case in which the system with state $x = (y, x)$ can be written in the form

$$\dot{y} = h(x) \ , \ \dot{x} = f(x) + g(x)u \ , \tag{22}$$

with y in $\mathbb{R}$, x in $\mathbb{R}^{n_x}$. Note that to oppose to what has been done in the previous subsection, now the state component y is a scalar and x is a vector. Note moreover that the functions h, f and g do not depend of y. This restriction on h has been partially removed in our recent work in [4].

The first order approximation of the system is denoted by

$$\mathbb{A} = \begin{bmatrix} 0 & H \\ 0 & F \end{bmatrix} \ , \ \mathbb{B} = \begin{bmatrix} 0 \\ G \end{bmatrix} \ , \tag{23}$$

with $H = \frac{\partial h}{\partial x}(0)$, $F = \frac{\partial f}{\partial x}(0)$, $G = g(0)$.

For this class of system we make the following assumption.
Assumption 2. *For all K_x in $\mathbb{R}^{n_x}$ such that $F + GK_x$ is Hurwitz, there exists a smooth function $\alpha_x : \mathbb{R}^{n_x} \to \mathbb{R}$ such that the following holds.*

- *The origin is a globally asymptotically stable equilibrium for*

$$\dot{x} = f(x) + g(x)\alpha_x(x) \ ;$$

- *The function α_x satisfies $\frac{\partial \alpha_x}{\partial x}(0) = K_x$.*

This assumption establishes that the stabilization with prescribed local behavior is satisfied for the x subsystem. With this Assumption we have the following theorem which proof can be found in [3].
Theorem 4 (Forwarding Case)**.** *Let K_o in $\mathbb{R}^{p \times n}$ be a vector such that with $\mathbb{A}$ and $\mathbb{B}$ defined in (23) the matrix $\mathbb{A} + \mathbb{B}K_o$ is Hurwitz. Then there exists a smooth function $\alpha_o : \mathbb{R}^n \to \mathbb{R}^p$ which solves the global asymptotic stabilization with prescribed local behavior.*

Similarly to the backstepping case, with Corollary 1), this theorem establishes that given Q, a positive definite matrix in $\mathbb{R}^{n \times n}$, and R, a positive real number, there exists q, r and α_o which is solution to an optimal control problem with cost $J(\chi, u)$ defined in (8), with q and r which satisfy (14). Consequently, similarly to the backstepping case, we can design a globally and asymptotically stabilizing optimal control law with prescribed local cost function.

5 Illustration on the orbital transfer problem

As an illustration of the results described in the previous sections, we consider the problem of designing a control law which ensures the orbital transfer of a satellite from one orbit to another. In this section we consider the approach developed in [8] where a bounded stabilizing control law was developed. More precisely, we study the class of optimal control law (in the LQ sense) that can be synthesized. This may be of interest since, as mentioned in [5], it is difficult to consider performance issues with this control law.

Following [8], let $(a, e, \omega, \Omega, i, f)$ be the orbital parameters of the space vehicle.

Consider the state variable

$$\begin{cases} p = a(1-e^2) \\ e_x = e\cos(\omega+\Omega) \\ e_y = e\sin(\omega+\Omega) \\ h_x = \tan(i/2)\cos(\Omega) \\ h_y = \tan(i/2)\sin(\Omega) \\ L = \omega+\Omega+f \end{cases}$$

Denoting u_r, u_θ and u_h the three components of the acceleration the propulsory of the spacecraft may provide, we get the following orbital transfer model described by the following sixth order system:

$$\begin{cases} \dot{p} = 2kpu_\theta \\ \dot{e}_x = k\left[Z\sin(L)u_r + Au_\theta - e_yYu_h\right] \\ \dot{e}_y = k\left[-Z\cos(L)u_r + Bu_\theta + e_xYu_h\right] \\ \dot{L} = \sqrt{\frac{\mu}{p^3}}Z^2 + kYu_h \\ \dot{h}_x = \frac{k}{2}X\cos(L)u_h \\ \dot{h}_y = \frac{k}{2}X\sin(L)u_h \end{cases} \tag{24}$$

where

$$\begin{aligned} k &= \sqrt{\tfrac{p}{\mu}}\tfrac{1}{Z}\ ,\ \ Z = 1+e\cos(f) \\ A &= e_x + (1+Z)\cos(L) \\ B &= e_y + (1+Z)\sin(L) \\ X &= 1+h_x^2+h_y^2 \\ Y &= h_x\sin(L) - h_y\cos(L) \end{aligned}$$

The control objective is to achieve asymptotic stabilization of the system to an equilibrium with parameter $p = p_0$, $e_x = e_y = h_x = h_y = 0$ and $L(t) = L_0(t)$ given by

$$L_0(t) = \sqrt{\frac{\mu}{p_0^3}}t\ (\mathtt{mod}2\pi)\ .$$

As mentioned in [8], this is a circular orbit in the equatorial plane. Contrary to what has been done in [8], in order to simplify the presentation, we do not consider input saturation constraint.

Consider the rotation matrix

$$R(L) = \begin{bmatrix} \cos(L) & \sin(L) \\ -\sin(L) & \cos(L) \end{bmatrix} ,$$

and the new coordinates

$$x_1 = L - L_0 \ , \ x_2 = \frac{p_0}{p}(1 + \bar{x}_2) - 1 \ , \ x_3 = -\sqrt{\frac{p_0}{p}}\bar{x}_3 \ ,$$

where

$$\begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = R(L) \begin{bmatrix} e_x \\ e_y \end{bmatrix} \ , \ x_4 = p \ , \ \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = R(L) \begin{bmatrix} h_x \\ h_y \end{bmatrix} \ .$$

With these new coordinates, the system can be rewritten as the following one.

$$\begin{cases} \dot{x}_1 = \frac{\sqrt{\mu x_4}}{p_0^2}(1 + x_2)^2 - \sqrt{\frac{\mu}{p_0^3}} - \frac{x_4}{p_0}\sqrt{\frac{x_4}{\mu}}\frac{x_6}{1+x_2}u_h \\ \dot{x}_2 = -\sqrt{\frac{\mu}{p_0^3}}(1 + x_2)^2 x_3 \\ \dot{x}_3 = \sqrt{\frac{\mu}{p_0^3}}(1 + x_2)^2 \left[\frac{x_4}{p_0}(1 + x_2) - 1\right] + \sqrt{\frac{p_0}{\mu}}u_r \\ \dot{x}_4 = 2\frac{x_4}{p_0}\sqrt{\frac{x_4^3}{\mu}}\frac{1}{1+x_2}u_\theta \\ \dot{x}_5 = \frac{\sqrt{\mu x_4}}{p_0^2}(1 + x_2)^2 x_6 + \frac{p_0}{\sqrt{x_4 \mu}}\frac{1+x_5^2-x_6^2}{2+2x_2}u_h \\ \dot{x}_6 = -\frac{\sqrt{\mu x_4}}{p_0^2}(1 + x_2)^2 x_5 + \frac{p_0}{\sqrt{x_4 \mu}}\frac{x_5 x_6}{1+x_2}u_h \end{cases} \tag{25}$$

In compact form, the previous system is simply:

$$\dot{x} = a(x) + b_r(x)u_r + b_\theta(x)u_\theta + b_h(x)u_h \ .$$

The first order approximation of this system around the equilibrium is given as

$$\mathbb{A} = \sqrt{\frac{\mu}{p_0^3}} \begin{bmatrix} 0 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{p_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} ,$$

and

$$\mathbb{B} = \sqrt{\frac{p_0}{\mu}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2p_0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} .$$

Note that these matrices can be rewritten as

$$\mathbb{A} = \mathtt{diag}\{\tilde{\mathbb{A}}, A_1\} \ , \ \tilde{\mathbb{A}} = \begin{bmatrix} A_0 & A_2 \\ 0_{13} & 0 \end{bmatrix}$$

and

$$\mathbb{B} = \mathtt{diag}\{\tilde{\mathbb{B}}, B_2\} \ , \ \tilde{\mathbb{B}} = \begin{bmatrix} B_0 & 0_{31} \\ 0 & 2\sqrt{\frac{p_0^3}{\mu}} \end{bmatrix}$$

where

$$A_0 = \sqrt{\frac{\mu}{p_0^3}} \begin{bmatrix} -\frac{3}{2p_0} & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \ , \ A_1 = \begin{bmatrix} 0 & \sqrt{\frac{\mu}{p_0^3}} \\ -\sqrt{\frac{\mu}{p_0^3}} & 0 \end{bmatrix} \ ,$$

and,

$$A_2 = \sqrt{\frac{\mu}{p_0^3}} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{p_0} \end{bmatrix} \ , B_0 = \sqrt{\frac{p_0}{\mu}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ , \ B_2 = \sqrt{\frac{p_0}{\mu}} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} .$$

The control strategy developed in [8] was to successively apply backstepping, forwarding and dissipativity properties.

With the tools developed in the previous sections, we are able to solve the locally optimal control problem for a specific class of quadratic costs as described by the following theorem.

Theorem 5 (Locally optimal stabilizing control law)**.** *Given Q_0 a positive definite matrix in $\mathbb{R}^{3\times 3}$ and R_0 in $\mathbb{R}_+$. Let P_0 be the solution of the (partial) algebraic Riccati equation:*

$$A_0 P_0 + P_0 A_0 - P_0 B_0 R_0^{-1} B_0' P_0 = -Q_0 \ . \tag{26}$$

Then for all positive real numbers $R_0, R_1, R_2, \rho_1, \rho_2$ such that the matrix

$$Q = \boldsymbol{diag}\{\tilde{Q}, \rho_2^2 B_2 R_2^{-1} B_2'\} \ , \ \tilde{Q} = \begin{bmatrix} Q_0 & P_0 A_2 \\ A_2' P_0 & 4\frac{p_0^3}{\mu}\rho_1^2 R_1^{-1} \end{bmatrix}$$

is positive, there exists q and r and a globally asymptotically stabilizing control law $(u_r, u_\theta, u_h) = \alpha_o(x)$ which is solution to an optimal control problem with cost $J(x, u)$ defined in (8), with q and r which satisfy (14).

Proof : First of all, when $u_\theta = u_h = 0$ and when $x_4 = p_0$, then the dynamics of the (x_1, x_2, x_3) subsystem satisfies

$$\begin{cases} \dot{x}_1 = \sqrt{\frac{\mu}{p_0^3}}\,[(1+x_2)^2 - 1] \\ \dot{x}_2 = -\sqrt{\frac{\mu}{p_0^3}}(1+x_2)^2 x_3 \\ \dot{x}_3 = \sqrt{\frac{\mu}{p_0^3}}(1+x_2)^2 x_2 + \sqrt{\frac{p_0}{\mu}} u_r \end{cases} \tag{27}$$

It can be noticed setting $y := x_3$ and $x := x_2$ the (x_2, x_3) subsystem is in the strict feedback form (17). Note that employing Theorem 3, it yields that for this system all locally stabilizing linear behavior can be achieved.

Moreover, setting $y := x_1$ and $x := (x_2, x_3)$ the (x_1, x_2, x_3) subsystem is in the feedforward form (22). Note that employing Theorem 4, it yields that for this system all locally stabilizing linear behaviors can be achieved.

Hence, with Theorem 1, it yields that given P_0 which by (26) is a CLF for the first order approximation of the system (27) there exists a smooth function $V_0 : \mathbb{R}^3 \to \mathbb{R}_+$ such that

- V_0 is a CLF for the (x_1, x_2, x_3) subsystem when considering the control u_r and when $x_4 = p_0$, i.e. for the system (27) ;
- V_0 is locally quadratic and satisfies $H(V_0)(0) = 2P_0$.

Let $\tilde{V} : \mathbb{R}^4 \to \mathbb{R}_+$ be the function defined by

$$\tilde{V}(x_1, x_2, x_3, x_4) = V_0(x_1, x_2, x_3) + V_1(x_4) \ ,$$

with $V_1(x_4) = \rho_1(p_0 - x_4)^2$. Note that this function is such that

$$H(\tilde{V})(0, 0, 0, p_0) = 2\tilde{P} \ , \ \tilde{P} = \mathtt{diag}\{P_0, \rho_1\} \ .$$

Employing (26), it can be checked that $\tilde{P}$ satisfies the (partial) algebraic HJB

$$\tilde{P}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}'\tilde{P} - \tilde{P}\tilde{\mathbb{B}}\tilde{R}^{-1}\tilde{\mathbb{B}}'\tilde{P} + \tilde{Q} = 0 \ ,$$

with $\tilde{R} = \mathtt{diag}\{R_1, R_2\}$. We will show that this function is also a control Lyapunov function when considering the (x_1, x_2, x_3, x_4) subsystem in (25) with the control inputs u_r and u_θ. Consider the set of point in $\mathbb{R}^4$ such that $L_{b_r}\tilde{V}(x) = L_{b_\theta}\tilde{V}(x) = 0$. Note that $L_{b_\theta}\tilde{V}(x) = 0$ implies that $x_4 = p_0$. With the CLF property for the system (27), it yields that in this set $L_a V_0(x) < 0$ for all $(x_1, x_2, x_3) \neq 0$. Consequently, $L_a(\tilde{V})(x) < 0$ for all $(x_1, x_2, x_3, x_4 - p_0) \neq 0$ such that $L_{b_r}\tilde{V}(x) = L_{b_\theta}\tilde{V}(x) = 0$. Hence with Theorem 2 we get the existence of $\tilde{q} : \mathbb{R}^4 \to \mathbb{R}_+$ a continuous function, C^2 at zero and $\tilde{r}$ a continuous function which values $r(x)$ are symmetric positive definite matrices such that:

- The function $\tilde{q}$ and $\tilde{r}$ satisfy the following property

$$H(\tilde{q})(0, 0, 0, p_0) = 2\tilde{Q} \ , \ r(0, 0, 0, p_0) = \tilde{R} \ . \tag{28}$$

- The function $\tilde{V}$ is a value function associated to the cost (8) with $\tilde{q}$ and $\tilde{r}$. More precisely, $\tilde{V}$ satisfies the HJB equation (10) when considering the (x_1, x_2, x_3, x_4) subsystem in (25).

Finally, let $V : \mathbb{R}^6 \to \mathbb{R}_+$ be defined by

$$V(x) = \tilde{V}(x_1, x_2, x_3, x_4) + V_2(x_5, x_6) \ ,$$

with $V_2(x_5, x_6) = \rho_2(x_5^2 + x_6^2)$. Moreover, consider q the positive semi definite function q defined as

$$q(x) = \tilde{q}(x_1, x_2, x_3, x_4) + \frac{1}{4}(L_{b_r}V(x))^2 R_2^{-1} \ ,$$

and r defined as

$$r(x) = \mathtt{diag}\{\tilde{r}(x), R_2\} \ .$$

Note that the following properties are satisfied.

- The function q and r satisfy

$$H(\tilde{q})(0) = 2Q \ , \ r(0) = \mathtt{diag}\{R_1, R_2, R_3\} \ ; \tag{29}$$

- The function V is a value function associated to the cost (8) with q and r.

Hence, the control law (9) makes the time derivative of the Lyapunov function V nondecreasing and is also optimal with respect to cost defined from q and r. Note however that we get a weak Lyapunov function. Nevertheless, following [8], it can be shown that employing this Lyapunov function in combination with LaSalle invariance principle, global asymptotic stabilization of the origin of the system (25) with the control law (9) is obtained. □

6 Conclusion

In this article we have developed a theory for constructing control laws having a predetermined local behavior. In a first step, we showed that this problem can be rewritten as an equivalent problem in terms of Lyapunov functions. In a second step we have demonstrated that when the local behavior comes from an (LQ) optimal approach, we can characterize a cost with specific local approximation that can be minimized. Finally, we have introduced two classes of system for which we know how to build these locally optimal control laws.

All this theory has been illustrated on the problem of orbital transfer.

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