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Optimizing V Antenna Arrays Using a Bayesian DOA Estimation Criterion

Houcem Gazzah
Dept. of Electrical and Computer Engineering
University of Sharjah
27272, Sharjah, UAE
Email: hgazzah@sharjah.ac.ae

Jean-Pierre Delmas
UMR CNRS 5157
Telecom SudParis
91011 Evry, France
Email: jean-pierre.delmas@it-sudparis.eu

***Abstract*—Experimentations have shown V-shaped uniform antenna arrays to be near-optimum for estimating the direction of arrival of a far-field source, whether the source position is fixed or random. We consider, as performance measure, the expected Cramer-Rao bound, normalized (for comparison purposes) to that of the commonly used uniform circular arrays. We study in details the behavior of V arrays in this context. For large-sized V arrays, the performance measure shows a simple expression, enabling analytical solution of the subsequent array-geometry optimization problem. We obtain closed-form expressions of the orientation, shape and performance of optimal V arrays and learn about their ability to benefit from the available prior about the source direction.**

I. Introduction

Direction finding is a major research field of statistical signal processing that has connections to a wide range of electronic systems and applications, civil and military. The direction of arrival (DOA) of an emitting source is estimated from the signals it induces at a set of sensors that form the antenna array. The most studied scenario is that of sensors displayed in a plane and a source anywhere in the array far-field.

The array geometry, i.e. sensors positions in the plane, has a notorious impact on DOA estimation, at least from two points-of-view. The first one is ambiguity, i.e. the ability to resolve sources with distinct DOAs. The second one is accuracy, i.e. the ability to distinguish two array output response vectors associated with different sources.

Array ambiguity is a difficult issue, already because there are many definitions of ambiguity: first-order vs higher order (two or more steering vectors are taken into consideration) and strict vs wide sense ambiguity (strict-singularity or near-singularity of the matrix of steering vectors). By ambiguity-free, it is meant that two steering vectors cannot be colinear if they correspond to two different DOAs. Still, they can be close to be colinear, a situation referred to as wide-sense ambiguity [1], [2]. There are no conditions to strictly avoid wide-sense ambiguity. However, many studies tend to confirm that uniform arrays suffer much less ambiguities [1], from this point-of-view. We adopt a conservative approach of array ambiguity where we focus on uniform antenna arrays, because "For most of the applications, uniform arrays are the first choice ..." [3, Sec. I]. For uniform arrays, adjacent sensors are separated by a constant (half-the-wavelength) spacing [3]. Most notably, they include the Uniform Linear Array used for sources coplanar with the array, and the Uniform Circular Array (UCA) used when sources may appear anywhere in the 3D space. Since we study the latter more general scenario, we will use the UCA as our reference antenna, to which we compare the proposed uniform V arrays.

The second aspect, however, has remained largely unquantified, resulting into a very few theoretical results [4], [5] and heuristic-only geometry optimization techniques [6], [7]. Only recently has a simplification of the Cramer-Rao Bound (CRB) been obtained for the single source case [8], telling a lot about the interaction between the array geometry and the estimation accuracy. It has been profitably used to determine the best antenna arrays, for instance, those that minimize the CRB [9], [10], [11].

The most noticeable outcome of previous optimizations of uniform antenna arrays is that optimum uniform arrays show to have a shape close to V. This fact is true in both deterministic and probabilistic scenarios, i.e. if the source DOA is fixed (and unknown) or random (with known distribution). The V array is also convenient in practice because it is easy to manoeuvre (for instance to adapt the V orientation and shape to the available source prior, as we explain later).

The objective of this paper is to study in details the performance of V arrays in a scenario where the source DOA is not deterministic, but randomly located, following a known probabilistic distribution. It is reasonable to expect the available prior to help improving DOA estimation accuracy. In cellular communication networks, for example, we collect DOA statistics of base station users. The information is used to improve the quality of service and/or offer location-based services [12]. A variety of modeling techniques [13], [14] are used for this purpose.

For such a scenario, the CRB is no more a valid performance criterion. W.r.t. the theory of Bayesian estimation, Expected CRB (ECRB) is more relevant, and opportunistically, inherits the simple structure of the CRB. Consequently, analysis conducted in [9], [10] for fixed DOA, can be extended leading to analytical results which allow rich interpretation and are of great applicability. The first contribution of this paper is to justify the ad-hoc ECRB criterion used in [6], [11]

in the Bayesian framework. The second one is to give a detailed proof of an important result (presented in [11] without justification) about the performance of V arrays. The analytical expressions are followed by a detailed discussion and original illustrations.

The proposed paper is only the second paper (after [6]) to address array geometry adaptation based on a known distribution of the random source DOA. In comparison to [6], this paper (i) does not ignore the array ambiguity issue; (ii) does not restrict the source to be in the array plane; (iii) does not restrict the sensors to be in some closed region (iv) includes many analytical results compared to [6] that contains only the result that, if sensors are restricted to be in a closed connected region, then the optimal sensors placement is along the boundary of the region, as one may expect; (v) shows, for instance, that the array performance depends on the characteristic function at -2, rather than on the complete distribution function; (vi) shows that a major claim [6, Beginning of Sec. V] (that isotropic arrays are optimal for uniformly distributed DOA) is misleading. This is true because the sensors are also required to be inside a disk. As we do not impose such a constraint, we are able to find antenna arrays with better performance.

The paper is organized as follows: First, in Sec. II, we introduce the observation model and recall previous results. We introduce our Bayesian performance criterion in Sec. III, then, apply it to optimize V-structured antenna arrays in Sec. IV. Finally, a conclusion is given in Sec. V.

II. Data Model and Previous Results

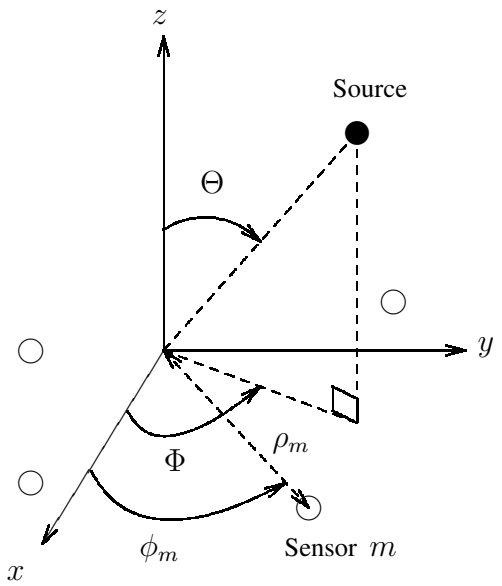

Fig. 1. Planar array and source DOAs.

As illustrated by Fig. 1, a planar antenna array is composed of M identical and omni-directional sensors disposed, each, in the (x, y) plane. The position of the m-th sensor is given by its cartesian coordinates x_m and y_m, or, equivalently, by the complex number $\gamma_m \hat{=} x_m + iy_m$. A source is radiating towards the antenna a narrow-band signal characterized by a wavelength λ. For a source located in the antenna far-field, its DOA is given by a set of two angular parameters: the azimuth angle Φ and the elevation angle Θ, as shown in Fig. 1. Snapshots outputted by the M sensors are collected and used to infer about the azimuth and elevation angles using a variety of algorithms. Some of them are capable of achieving the famous CRB.

Our analysis is conducted under the following assumptions: (i) circular, Gaussian, zero-mean and mutually independent source and noise signals with respective power σ_s^2 and σ_n^2; (ii) independent and identically distributed source snapshots and (iii) noise samples collected from different sensors and/or at different time indexes are mutually independent. Originally available in a very sophisticated expression [15], the CRB on (Φ, Θ) has been derived from the inverse of the Fisher Information Matrix (FIM) $\mathbf{J}_F$ associated with the unknown deterministic parameters $(\Phi, \Theta, \sigma_s^2, \sigma_n^2)$ in a compact expression that has been greatly simplified (under no additional assumptions) in [8] under the following convenient sinusoidal expression:

$$\mathbf{C}_{\Phi,\Theta} \hat{=} \begin{bmatrix} C_{\Phi\Phi} & C_{\Phi\Theta} \\ C_{\Theta\Phi} & C_{\Theta\Theta} \end{bmatrix},$$

with

$$\begin{aligned} C_{\Phi\Phi} &= \frac{A}{\sin^2(\Theta)} B(\Phi) \\ C_{\Theta\Theta} &= \frac{A}{\cos^2(\Theta)} B\left(\Phi + \frac{\pi}{2}\right) \\ C_{\Phi\Theta} &= \frac{A}{\sin(2\Theta)} \frac{\Im\left[S_1 \exp(-2j\Phi)\right]}{|S_1|^2 - S_0^2} \end{aligned}$$

where constant $A \hat{=} \frac{1}{4\pi^2} \frac{\sigma_n^2}{\sigma_s^2} \left(1 + \frac{\sigma_n^2}{M\sigma_s^2}\right)$ depends on the observation SNR, the antenna size M and the number of snapshots N; and

$$B(\Phi) \hat{=} \frac{S_0 + \Re\left[S_1 \exp(-2j\Phi)\right]}{S_0^2 - |S_1|^2}$$

is a function of the source azimuth angle and the array-geometry dependent constants

$$\begin{aligned} S_0 &\hat{=} \sum_{m=1}^{M} \left|\frac{\gamma_m}{\lambda}\right|^2 - \frac{1}{M} \left|\sum_{m=1}^{M} \frac{\gamma_m}{\lambda}\right|^2, \\ S_1 &\hat{=} \sum_{m=1}^{M} \left(\frac{\gamma_m}{\lambda}\right)^2 - \frac{1}{M} \left(\sum_{m=1}^{M} \frac{\gamma_m}{\lambda}\right)^2. \end{aligned}$$

III. Performance Criteria

In practice, there are situations where an *a priori* knowledge is available about the DOA parameters with (Φ, Θ) following some known joint distribution. The performance of any DOA estimator is measured by the associated MSE matrix [16, (18)

p. 4]

$$\mathbf{R}(\widehat{\Phi},\widehat{\Theta}) = \begin{bmatrix} \mathrm{E}(\widehat{\Phi}-\Phi)^2 & \mathrm{E}(\widehat{\Phi}-\Phi)(\widehat{\Theta}-\Theta) \\ \mathrm{E}(\widehat{\Theta}-\Theta)(\widehat{\Phi}-\Phi) & \mathrm{E}(\widehat{\Theta}-\Theta)^2 \end{bmatrix}. \tag{1}$$

As the powers σ_s^2 and σ_n^2 remain in practice unknown deterministic parameters, the theoretical framework to derive lower bound on (1) is the *hybrid* CRB firstly introduced in [17], which states that the MSE matrix associated with $(\Phi,\Theta,\sigma_s^2,\sigma_n^2)$ is lower bounded by the so-called Bayesian CRB given by the inverse of the hybrid information matrix $\mathbf{J}_H = \mathbf{J}_D + \mathbf{J}_P$ [16, (74-75) p.13] with $\mathbf{J}_D = \mathrm{E}_{\Phi,\Theta}(\mathbf{J}_F)$ and $\mathbf{J}_P = \begin{bmatrix} \mathbf{J}_P(\Phi,\Theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\mathbf{J}_P(\Phi,\Theta)$ is the prior information matrix w.r.t. the DOA parameters. Consequently we have

$$\mathbf{R}(\widehat{\Phi},\widehat{\Theta}) \geq [(\mathbf{J}_D+\mathbf{J}_P)^{-1}]_{(1:2,1:2)}$$

with

$$[(\mathbf{J}_D+\mathbf{J}_P)^{-1}]_{(1:2,1:2)} \leq [\mathbf{J}_D^{-1}]_{(1:2,1:2)} \leq \mathrm{ECRB}, \tag{2}$$

where $\mathrm{ECRB} \hat{=} \mathrm{E}_{\Phi,\Theta}\left([\mathbf{J}_F^{-1}]_{(1:2,1:2)}\right) = \mathrm{E}_{\Phi,\Theta}(\mathbf{C}_{\Phi,\Theta})$ is the expectation of the conditional CRB (ECRB) [16, p. 6]. We note that the first inequality in (2) is tight for $N \gg 1$ if the prior source DOA are not too accurate because in our data model introduced in Sec. II, $\mathbf{J}_F$ and thus $\mathbf{J}_D$ is proportional to the number N of snapshots in contrast to $\mathbf{J}_P(\Phi,\Theta)$ which depends only on the prior distribution of (Φ,Θ). For example, for azimuth and elevation angles independently Gaussian distributed with prior variance σ_Φ^2 and σ_Θ^2, $\mathbf{J}_P(\Phi,\Theta) = \begin{bmatrix} \sigma_\Phi^{-2} & 0 \\ 0 & \sigma_\Theta^{-2} \end{bmatrix}$. Consequently in practice, the equivalence between $\mathbf{J}_H$ and $\mathbf{J}_D$ occurs each time the variance of the DOA Φ and Θ estimated by an efficient algorithm are much smaller than the respective prior variance σ_Φ^2 and σ_Θ^2. The second inequality in (2) is derived form the Jensen's inequality applied to the usual FIM $\mathbf{J}_F$, which is assumed definite positive, considered here as a random matrix

$$[\mathrm{E}_{\Phi,\Theta}(\mathbf{J}_F)]^{-1} \leq \mathrm{E}_{\Phi,\Theta}(\mathbf{J}_F^{-1}).$$

Noting that the MSE given by the maximum a posteriori estimator asymptotically (w.r.t. N) attains $\mathrm{E}_{\Phi,\Theta}(\mathbf{C}_{\Phi,\Theta})$, we propose to base the Bayesian optimization on this bound.

$$\mathrm{E}_{\Phi,\Theta}(\mathbf{C}_{\Phi,\Theta}) = \begin{bmatrix} \bar{C}_{\Phi\Phi} & \bar{C}_{\Phi\Theta} \\ \bar{C}_{\Theta\Phi} & \bar{C}_{\Theta\Theta} \end{bmatrix}.$$

Furthermore this bound inherits the convenient structure of the CRB displayed in Sec. II. in contrast to the 2×2 submatrix $[\mathrm{E}_{\Phi,\Theta}(\mathbf{J}_F]^{-1}_{(1:2,1:2)}$ which has not a simple compact form.

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Furthermore, this bound inherits the convenient structure of the CRB displayed in Sec. II, in contrast to matrix $[\mathrm{E}_{\Phi,\Theta}(\mathbf{J}_F)]^{-1}$ which does not have a simple compact form. Finally, we note that the ECRB was used in [6] as a cost function to be minimized, but only the azimuth DOA angle was considered.

We assume, as plausible in practice, that azimuth and elevation are independent. Then, we have

$$\begin{aligned} \bar{C}_{\Phi\Phi} &= A\,\mathrm{E}\left[\frac{1}{\sin^2(\Theta)}\right]\mathrm{E}\left[B\left(\Phi\right)\right] \\ \bar{C}_{\Theta,\Theta} &= A\,\mathrm{E}\left[\frac{1}{\cos^2(\Theta)}\right]\mathrm{E}\left[B\left(\Phi+\frac{\pi}{2}\right)\right], \end{aligned}$$

where Φ is taken in $[-\pi,\pi)$, in order to handle symmetrical distributions.

In the above expressions, terms that do not depend on the array geometry can be disposed of by normalizing to any reference antenna. For convenience, we choose the widely-used UCA, one with M sensors spaced by a distance d. It verifies $B_{\mathrm{UCA}} \hat{=} B\left(\Phi\right) = 4\lambda^2\sin^2(\pi/M)/(Md^2)$. If we denote $\varphi_\Phi(x) \hat{=} \mathrm{E}\left[\exp(j\Phi x)\right]$ as the characteristic function relative to the random parameter Φ, we straightforwardly obtain

$$\begin{aligned} \frac{\bar{C}_{\Phi\Phi}}{\bar{C}_{\Phi\Phi}|_{\mathrm{UCA}}} &= \frac{1}{B_{\mathrm{UCA}}}\frac{S_0+\Re\left[S_1\varphi_\Phi(-2)\right]}{S_0^2-|S_1|^2} \\ \frac{\bar{C}_{\Theta\Theta}}{\bar{C}_{\Theta\Theta}|_{\mathrm{UCA}}} &= \frac{1}{B_{\mathrm{UCA}}}\frac{S_0-\Re\left[S_1\varphi_\Phi(-2)\right]}{S_0^2-|S_1|^2}. \end{aligned}$$

Our criterion is the ECRB normalized to that of the UCA, as expressed above. The latter is a constant, so that minimizing the normalized ECRB is optimal in the general case. A planar array that minimizes this criterion will outperform any antenna array from the ECRB point-of-view.

IV. Optimization of Large Sized V Arrays

Array optimization has been conducted in [9], [10], [11] for curve-like uniform arrays formed of a sequence of regularly spaced sensors. Thanks to an adequate parametrization of the sensors positions using angular (so bounded) parameters, array optimization can be achieved by exhaustive search. For a number of M sensors, the total number of search points is extremely huge. Likely enough, the results of the exhaustive search has shown optimal array to have a shape close to V. Conveniently enough, V array geometry is function of the angles characterizing the orientations of the two branches of the V shape and of the length of the branches, i.e. M. The normalized ECRBs will depend on these three parameters only and optimization will be much easier. For instance, it can be achieved analytically. Furthermore, such optimization will be found to be independent from M.

As shown in Fig. 2, the V array is made of an odd number M of sensors. Sensor 1 is placed at the origin while, for $k = 1,\cdots,(M-1)/2$, sensors $2k$ and $2k+1$ are positioned such that $\gamma_{2k} = kd\exp(j\Delta_1)$ and $\gamma_{2k+1} = kd\exp(j\Delta_2)$. When M tends to infinity, it has been shown that [11] the normalized ECRBs become independent from M. For instance, they are

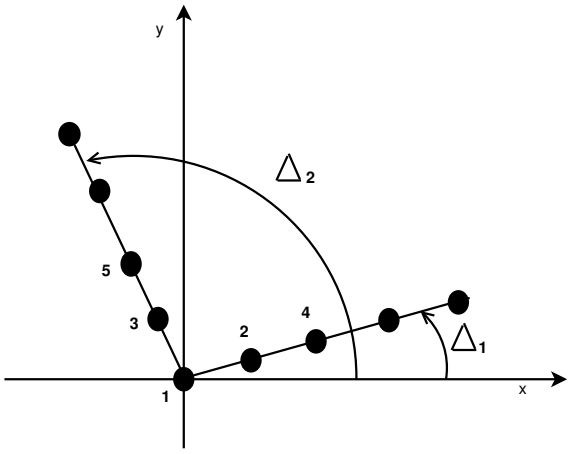

Fig. 2. The V-shaped antenna array.

equal to

$$r_\epsilon(\Delta_1,\Delta_2)\hat{=}\frac{3}{2\pi^2}\frac{5-3\cos(\Delta_1-\Delta_2)}{\sin^2(\Delta_2-\Delta_1)}\frac{+\epsilon\left[5\cos(\Delta_1-\Delta_2)-3\right]\Re\left[\exp[j(\Delta_1+\Delta_2)]\varphi_\Phi(-2)\right]}{}.$$

where $\epsilon=1$ (resp. -1) for the azimuth angle Φ (resp. elevation angle Θ). Our objective is to study in details functions $r_1(\Delta_1,\Delta_2)$ and $r_{-1}(\Delta_1,\Delta_2)$, and specifically, angles Δ_1 and Δ_2 for which the V array achieves a minimal $r_1(\Delta_1,\Delta_2)$ or $r_{-1}(\Delta_1,\Delta_2)$.

A. ECRB-Minimizing V arrays

We prove in Appendix that $\min_{\Delta_1,\Delta_2} r_\epsilon(\Delta_1,\Delta_2)$

$$= \frac{3}{4\pi^2}\frac{(3-5\alpha)^2}{5-3\alpha-4\sqrt{1-\alpha^2}}, \text{if } \alpha\neq\tfrac{3}{5} \quad (3)$$

$$= \frac{24}{5\pi^2}, \text{if } \alpha=\tfrac{3}{5}. \quad (4)$$

where the key-parameter $\alpha\hat{=}|\varphi_\Phi(2)|$ expresses the amount of prior available about the source azimuth angle. The proof of (3) is detailed in Appendix, while (4) is obtained by continuity of (3).

In Appendix, we also prove that $r_\epsilon(\Delta_1^\dagger,\Delta_2^\dagger)=\min_{\Delta_1,\Delta_2} r_\epsilon(\Delta_1,\Delta_2)$ iff

$$\cos(\Delta_1^\dagger-\Delta_2^\dagger) = \frac{4\sqrt{1-\alpha^2}+3\alpha-5}{5\alpha-3} \quad (5)$$

$$\cos(\Delta_1^\dagger+\Delta_2^\dagger) = \epsilon\frac{\mathrm{E}\left[\cos(2\Phi)\right]}{\alpha} \quad (6)$$

$$\sin(\Delta_1^\dagger+\Delta_2^\dagger) = \epsilon\frac{\mathrm{E}\left[\sin(2\Phi)\right]}{\alpha}. \quad (7)$$

W.r.t. the other cost function $r_{-\epsilon}(\Delta_1,\Delta_2)$, the V array characterized by the so-defined $\Delta_1^\dagger$ and $\Delta_2^\dagger$ achieves

$$r_{-\epsilon}(\Delta_1^\dagger,\Delta_2^\dagger)=\frac{3}{16\pi^2}\frac{5\alpha-3}{\sqrt{1-\alpha^2}}\times\left(3+5\alpha+\frac{(5+3\alpha)(3-5\alpha)}{3\alpha-5+4\sqrt{1-\alpha^2}}\right).$$

The proof of the above follows immediately.

Proof Using the same notations as Appendix, and bearing in mind that $\alpha_{\epsilon,u}=\alpha$, and following steps similar to those that led to (9), we can prove that

$$r_{-\epsilon}(\delta_3,\delta_4)=\frac{3}{2\pi^2}\frac{5+3\alpha-(5\alpha+3)\cos(\delta_3)}{1-\cos^2(\delta_3)}.$$

Implementing (13) and (14), where we denote $\beta\hat{=}\beta_{\epsilon,u}$, we get $\frac{2\pi^2}{3}r_{-\epsilon}(\delta_3,\delta_4)=\frac{5+3\alpha}{2\sqrt{\beta^2-1}\left(-\sqrt{\beta^2-1}+\mathrm{sign}(5\alpha-3)\beta\right)}$ $+\frac{3+5\alpha}{2\mathrm{sign}(5\alpha-3)\sqrt{\beta^2-1}}$. As a consequence,

$$\begin{aligned}
&\mathrm{sign}\,(5\alpha-3)\sqrt{\beta^2-1}\frac{4\pi^2}{3}r_{-\epsilon}(\delta_3,\delta_4)\\
=\;&\frac{5+3\alpha}{\beta-\mathrm{sign}\,(5\alpha-3)\sqrt{\beta^2-1}}+3+5\alpha, \text{i.e.}\\
&\mathrm{sign}\,(5\alpha-3)\frac{\sqrt{1-\alpha^2}}{|5\alpha-3|}\frac{16\pi^2}{3}r_{-\epsilon}(\delta_3,\delta_4)\\
=\;&\frac{5+3\alpha}{\beta+\mathrm{sign}\,(3-5\alpha)\,4\frac{\sqrt{1-\alpha^2}}{|3-5\alpha|}}+3+5\alpha, \text{and finally}\\
&\frac{\sqrt{1-\alpha^2}}{5\alpha-3}\frac{16\pi^2}{3}r_{-\epsilon}(\delta_3,\delta_4)\\
=\;&\frac{5+3\alpha}{\frac{3\alpha-5}{3-5\alpha}+4\frac{\sqrt{1-\alpha^2}}{3-5\alpha}}+3+5\alpha\\
=\;&\frac{(5+3\alpha)(3-5\alpha)}{3\alpha-5+4\sqrt{1-\alpha^2}}+3+5\alpha \quad \blacksquare
\end{aligned}$$

B. Interpretation

First, notice that the angle $\Delta_1^\dagger-\Delta_2^\dagger$ between the two branches of the optimal V array does not depend on the angle of interest (i.e. not on ϵ). It is shown in Fig. 3 as function of the available azimuth prior, as expressed by α. The angle (parameter) of interest (or also ϵ) affects the orientation of the (optimal) V array, which is not unique by the way. To identify V arrays that satisfy (5)-(7), we let $\Delta_3\hat{=}\arccos\left[(4\sqrt{1-\alpha^2}+3\alpha-5)/(5\alpha-3)\right]$ and Δ_4 be the angle in $[0,2\pi[$ whose cosine and sine are given by the right-hand sides in (6)-(7), respectively. Let's also define $\Delta_5\hat{=}(\Delta_4-\Delta_3)/2$ and $\Delta_6\hat{=}(\Delta_3+\Delta_4)/2$.

On one hand, Δ_4 is, by definition, in $[0,\pi)$. Given that $\Delta_1+\Delta_2$ is in $[0,4\pi)$, the system with (6) and (7) has two possible solutions for $\Delta_1+\Delta_2$. If $\epsilon\mathrm{E}\left[\sin(2\Phi)\right]$ is positive, then $\Delta_1+\Delta_2=\Delta_4$. Otherwise, $\Delta_1+\Delta_2=\Delta_4+2\pi$. On the other hand, Δ_3 is, by definition, in $[0,\pi)$. Given that $\Delta_1-\Delta_2$ is in $[-2\pi,2\pi)$, based on (6), there are 4 possible values for $\Delta_1-\Delta_2$. It is equal to either Δ_3, $-\Delta_3$, $\Delta_3-2\pi$ or $2\pi-\Delta_3$.

Hence, we have 8 possible cases for the tuple (Δ_1,Δ_2) which are: (Δ_6,Δ_5), (Δ_5,Δ_6), $(\Delta_6-\pi,\Delta_5+\pi)$, $(\Delta_5+\pi,\Delta_6-\pi)$, $(\Delta_6+\pi,\Delta_5+\pi)$, $(\Delta_5+\pi,\Delta_6+\pi)$, $(\Delta_6,\Delta_5+2\pi)$ and $(\Delta_5+2\pi,\Delta_6)$. By definition, Δ_5 and Δ_6 are in $[-\pi,\pi)$ and $[0,2\pi)$, respectively. Hence, some of these solutions correspond to identical V arrays. Actually, among these V arrays, only two are different. They are (Δ_5,Δ_6) and $(\Delta_5+\pi,\Delta_6+\pi)$. These two V arrays are obtained one from the other by symmetry w.r.t. to the origin.

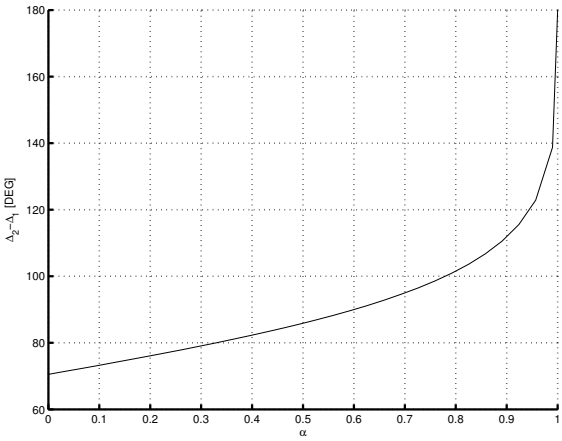

Fig. 3. The angle between the two branches of the optimal V array as function of parameter α.

C. Distributions of Φ such that $|\varphi_\Phi(2)| = 1$

The largest reduction of the ECRB is obtained for $|\varphi_\Phi(2)| = 1$. This is the case of a deterministic prior, but not only. More generally, we prove that α is equal to 1 iff the random azimuth has two possible values spaced by π, with arbitrary probabilities (hence, including the deterministic case). This is not surprising since the CRB is identical for two such directions. To establish the proof, let's, first, prove the following lemma.

Lemma Let I be a bounded interval and $f(x)$ be a summable complex-valued function over I. Then,
$\left|\int_I f(x)dx\right| = \int_I |f(x)|\,dx$ iff $\arg[f(x)]$ is constant over I almost everywhere.

Proof Consider the constant $z \hat{=} \int_I f(x)dx$. Then

$$\begin{aligned}\left|\int_I f(x)dx\right| &= e^{-j\arg(z)}z \\ &= \Re\left(e^{-j\arg(z)}z\right) \\ &= \int_I \Re\left(e^{-j\arg(z)}f(x)\right)dx\end{aligned}$$

that is meant, here, to be equal to $\int_I |f(x)|\,dx$. However, $\Re\left(e^{-j\arg(z)}f(x)\right)$ is lower than its magnitude, i.e., lower than $|f(x)|$. Both being real-valued functions, their respective integrals over I can be equal iff the two functions are identical almost everywhere over I. By writing that $\Re\left(e^{-j\arg(z)}f(x)\right) = |f(x)|\Re\left(e^{j\arg[f(x)]-j\arg(z)}\right)$ $= |f(x)|\cos\{\arg[f(x)] - \arg(z)\}$ must be equal to $|f(x)|$, we conclude that $\arg[f(x)] - \arg(z)$ must be a constant (0 modulo 2π), where $\arg(z)$ is itself a constant. This terminates the proof.

In order to apply the above lemma, we rewrite $|\varphi_\Phi(2)| = 1$ as $\left|\int_{-\pi}^{\pi} f(\Phi)\exp(2j\Phi)d\Phi\right| = \int_{-\pi}^{\pi}|f(\Phi)\exp(2j\Phi)|d\Phi = 1$, where $f(\Phi)$ is the PDF associated with the random variable Φ. Hence, the argument of $f(\Phi)\exp(2j\Phi)$, i.e., $2j\Phi$ is a constant. Let's denote it Φ_0, chosen in $[-\pi, \pi)$. Then, Φ is equal to any number of the form $\Phi_0/2 + k\pi$, where k is any integer. There are two such numbers in $[-\pi, \pi)$ ■

V. Conclusion

We study in details the performance of uniform V arrays w.r.t. the estimation of the azimuth and elevation angles of a randomly located source. The expected CRB, normalized to that of the UCA, is our performance measure. It depends on the direction of the source (in fact, of the only azimuth angle) and the orientation of the branches of the V shape. Analytical expressions are obtained and closed-form solution of the optimal V array: its shape, orientation and accuracy.

Appendix

We denote $\delta_3 \hat{=} \Delta_2 - \Delta_1$ and $\delta_4 \hat{=} \Delta_1 + \Delta_2$ and rewrite

$$r_\epsilon(\delta_3, \delta_4) \hat{=} \frac{3}{2\pi^2}\frac{5 - 3\cos(\delta_3) + \epsilon\left[5\cos(\delta_3) - 3\right]}{\times\left\{\cos(\delta_4)\,\mathrm{E}\left[\cos(2\Phi)\right] + \sin(\delta_4)\,\mathrm{E}\left[\sin(2\Phi)\right]\right\}}{\sin^2(\delta_3)}.$$

Minimization of r_ϵ w.r.t. δ_4 straightforwardly leads to

$$\tan(\delta_4) = \mathrm{E}\left[\sin(2\Phi)\right]/\mathrm{E}\left[\cos(2\Phi)\right]. \tag{8}$$

Thus, $\cos(\delta_4) = u\mathrm{E}\left[\cos(2\Phi)\right]/\alpha$, where $u = \pm 1$ and

$$\alpha \hat{=} \sqrt{\left\{\mathrm{E}\left[\sin(2\Phi)\right]\right\}^2 + \left\{\mathrm{E}\left[\cos(2\Phi)\right]\right\}^2},$$

i.e., $|\varphi_\Phi(-2)| = |\varphi_\Phi(2)|$, which, by definition, is less than one. Replacing δ_4 by its expression, we rewrite r_ϵ as follows

$$r_\epsilon(\delta_3, \delta_4) = \frac{3}{2\pi^2}\frac{5 - 3\cos(\delta_3) + \alpha_{\epsilon,u}\left[5\cos(\delta_3) - 3\right]}{\sin^2(\delta_3)},$$

where $\alpha_{\epsilon,u} \hat{=} u\epsilon\alpha$. This is a function of $\cos(\delta_3)$,

$$r_\epsilon(\delta_3, \delta_4) = \frac{3}{2\pi^2}\frac{5 - 3\alpha_{\epsilon,u} + (5\alpha_{\epsilon,u} - 3)\cos(\delta_3)}{1 - \cos^2(\delta_3)}. \tag{9}$$

The minimization of r_ϵ w.r.t. $\cos(\delta_3)$ leads to the second order equation $\cos^2(\delta_3) + 2\beta_{\epsilon,u}\cos(\delta_3) + 1 = 0$, where

$$\beta_{\epsilon,u} \hat{=} (5 - 3\alpha_{\epsilon,u})/(5\alpha_{\epsilon,u} - 3). \tag{10}$$

Notice that

$$\beta_{\epsilon,u}^2 - 1 = 16(1 - \alpha^2)/((3 - 5\alpha_{\epsilon,u})^2) \tag{11}$$

is always positive because α is in $[0, 1]$, by definition. Hence, $\cos(\delta_3)$ is equal to the one among $-\beta_{\epsilon,u} + \sqrt{\beta_{\epsilon,u}^2 - 1}$ and $-\beta_{\epsilon,u} - \sqrt{\beta_{\epsilon,u}^2 - 1}$ that has an absolute value lower than 1. Notice that the product of these two values is 1, so one and only one is a valid cosine. Now, one can verify from (10) that $\beta_{\epsilon,u} - 1 = 8\left(1 - \alpha_{\epsilon,u}\right)/\left(5\alpha_{\epsilon,u} - 3\right)$ and $\beta_{\epsilon,u} + 1 = 8\left(1 + \alpha_{\epsilon,u}\right)/\left(5\alpha_{\epsilon,u} - 3\right)$. Both have the same sign, since their respective numerators are positive.

On one hand, if $\alpha_{\epsilon,u} \geq 3/5$, then $\beta_{\epsilon,u} - 1$ and $\beta_{\epsilon,u} + 1$ are both positive. Hence, we have $-\beta_{\epsilon,u} \leq -1 \leq \beta_{\epsilon,u}$, implying that $(\beta_{\epsilon,u} - 1)^2 \leq \beta_{\epsilon,u}^2 - 1 \leq (\beta_{\epsilon,u} + 1)^2$, or also that $\beta_{\epsilon,u} - 1 \leq \sqrt{\beta_{\epsilon,u}^2 - 1} \leq \beta_{\epsilon,u} + 1$. Hence, $\cos(\delta_3) = -\beta_{\epsilon,u} + \sqrt{\beta_{\epsilon,u}^2 - 1}$. On the other hand, if $\alpha_{\epsilon,u} \leq 3/5$, then $\beta_{\epsilon,u} - 1$ and $\beta_{\epsilon,u} + 1$ are both negative. Hence, we have $\beta_{\epsilon,u} \leq -1 \leq -\beta_{\epsilon,u}$, implying that $(\beta_{\epsilon,u} + 1)^2 \leq \beta_{\epsilon,u}^2 - 1 \leq (\beta_{\epsilon,u} - 1)^2$, or also

that $-\beta_{\epsilon,u}+1 \leq \sqrt{\beta_{\epsilon,u}^2-1} \leq -\beta_{\epsilon,u}-1$. Hence, $\cos(\delta_3) = -\beta_{\epsilon,u} - \sqrt{\beta_{\epsilon,u}^2-1}$. As a conclusion,

$$\cos(\delta_3) = \text{sign}\left(5\alpha_{\epsilon,u}-3\right)\sqrt{\beta_{\epsilon,u}^2-1} - \beta_{\epsilon,u}. \tag{12}$$

In order to update (9), we write that

$$\begin{aligned}
\cos^2(\delta_3) &= 2\beta_{\epsilon,u}^2 - 1 - 2\text{sign}\left(5\alpha_{\epsilon,u}-3\right)\beta_{\epsilon,u}\sqrt{\beta_{\epsilon,u}^2-1} \\
\frac{\sin^2(\delta_3)}{2} &= 1-\beta_{\epsilon,u}^2 \\
&\quad +\text{sign}\left(5\alpha_{\epsilon,u}-3\right)\beta_{\epsilon,u}\sqrt{\beta_{\epsilon,u}^2-1} \qquad (13)\\
&= \text{sign}\left(5\alpha_{\epsilon,u}-3\right)\sqrt{\beta_{\epsilon,u}^2-1} \\
&\quad \times\left[-\text{sign}\left(5\alpha_{\epsilon,u}-3\right)\sqrt{\beta_{\epsilon,u}^2-1}+\beta_{\epsilon,u}\right] \\
&= -\text{sign}\left(5\alpha_{\epsilon,u}-3\right)\sqrt{\beta_{\epsilon,u}^2-1}\cos(\delta_3), \qquad (14)
\end{aligned}$$

so that we rewrite (9) as $\frac{2\pi^2}{3}r_\epsilon(\delta_3,\delta_4) =$

$$\frac{5-3\alpha_{\epsilon,u}}{2\sqrt{\beta_{\epsilon,u}^2-1}\left(-\sqrt{\beta_{\epsilon,u}^2-1}+\text{sign}\left(5\alpha_{\epsilon,u}-3\right)\beta_{\epsilon,u}\right)} + \frac{3-5\alpha_{\epsilon,u}}{2\text{sign}\left(5\alpha_{\epsilon,u}-3\right)\sqrt{\beta_{\epsilon,u}^2-1}}.$$

Using (10) and (11), we simplify the above as follows

$$\begin{aligned}
&2\sqrt{\beta_{\epsilon,u}^2-1}\frac{2\pi^2}{3}r_\epsilon(\delta_3,\delta_4) \\
&= \frac{5-3\alpha_{\epsilon,u}}{-4\frac{\sqrt{1-\alpha^2}}{|3-5\alpha_{\epsilon,u}|}+\frac{5-3\alpha_{\epsilon,u}}{|5\alpha_{\epsilon,u}-3|}} - |5\alpha_{\epsilon,u}-3| \\
&= \frac{\left(5-3\alpha_{\epsilon,u}\right)|5\alpha_{\epsilon,u}-3|}{-4\sqrt{1-\alpha^2}+5-3\alpha_{\epsilon,u}} - |5\alpha_{\epsilon,u}-3|,
\end{aligned}$$

so that $\frac{\sqrt{\beta_{\epsilon,u}^2-1}}{|5\alpha_{\epsilon,u}-3|}\frac{4\pi^2}{3}r_\epsilon(\delta_3,\delta_4) = \frac{5-3\alpha_{\epsilon,u}}{-4\sqrt{1-\alpha^2}+5-3\alpha_{\epsilon,u}} - 1$ or also $\frac{\sqrt{\beta_{\epsilon,u}^2-1}}{|5\alpha_{\epsilon,u}-3|}\frac{\pi^2}{3}r_\epsilon(\delta_3,\delta_4) = \frac{\sqrt{1-\alpha^2}}{-4\sqrt{1-\alpha^2}+5-3\alpha_{\epsilon,u}}$.
Using (11), $4\frac{\sqrt{1-\alpha^2}}{(5\alpha_{\epsilon,u}-3)^2}\frac{\pi^2}{3}r_\epsilon(\delta_3,\delta_4) = \frac{1}{\frac{5-3\alpha_{\epsilon,u}}{\sqrt{1-\alpha^2}}-4}$ and finally $16\pi^2\frac{\sqrt{1-\alpha^2}}{3}r_\epsilon(\delta_3,\delta_4) = \frac{(3-5\alpha_{\epsilon,u})^2}{\frac{5-3\alpha_{\epsilon,u}}{4\sqrt{1-\alpha^2}}-1}$.

To determine whether the minimum of r_ϵ is met with u equal to 1 or -1, we study the sign of $r_\epsilon(\delta_3,\delta_4)|_{u=1} - r_\epsilon(\delta_3,\delta_4)|_{u=-1}$, which, w.r.t. the above, is also that of

$$\frac{\left(3-5\alpha_{\epsilon,1}\right)^2}{\frac{5-3\alpha_{\epsilon,1}}{4\sqrt{1-\alpha^2}}-1} - \frac{\left(3-5\alpha_{\epsilon,-1}\right)^2}{\frac{5-3\alpha_{\epsilon,-1}}{4\sqrt{1-\alpha^2}}-1},$$

i.e., the same as the sign of $\frac{(3-5\epsilon\alpha)^2}{5-3\epsilon\alpha-4\sqrt{1-\alpha^2}} - \frac{(3+5\epsilon\alpha)^2}{5+3\epsilon\alpha-4\sqrt{1-\alpha^2}}$ which can be found to be equal to

$$\epsilon\alpha\frac{150\alpha^2+240\sqrt{1-\alpha^2}-246}{25-9\alpha^2-40\sqrt{1-\alpha^2}+16(1-\alpha^2)} = -6\epsilon\alpha.$$

As a conclusion, r_1 (resp. r_{-1}) is minimized by δ_3 and δ_4 for which $u=1$ (resp. $u=-1$). Hence $u=\epsilon$ and $\alpha_{\epsilon,u}=\alpha$. For such δ_3 and δ_4 minimizing r_ϵ we have, on one hand, by application of (12),
$\cos(\delta_3) = \text{sign}\left(5\alpha-3\right)\sqrt{\beta_{\epsilon,\epsilon}^2-1} - \beta_{\epsilon,\epsilon}$. By means of (10) and (11), this can be proved to be equal to (5). On the other hand, $\cos(\delta_4) = \epsilon\mathrm{E}\left[\cos\left(2\Phi\right)\right]/\alpha$, in addition to (8) lead to (6) and (7).

The minimum achieved by r_ϵ is, independently from ϵ,

$$16\pi^2\frac{\sqrt{1-\alpha^2}}{3}r_\epsilon(\delta_3,\delta_4) = \frac{\left(3-5\alpha_{\epsilon,\epsilon}\right)^2}{\frac{5-3\alpha_{\epsilon,\epsilon}}{4\sqrt{1-\alpha^2}}-1} = \frac{\left(3-5\alpha\right)^2}{\frac{5-3\alpha}{4\sqrt{1-\alpha^2}}-1},$$

which can be verified to be equal to (3).

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