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We study the fluctuations of the power variation of fractional Brownian motion in Brownian time.

Keywords: fractional Brownian motion; fractional Brownian motion in Brownian time; Hermite polynomials; iterated Brownian motion; limit theorem

1. Introduction

Studying the variations of a stochastic process is of fundamental importance in probability theory. In this paper, we are interested in the fractional Brownian motion in Brownian time, which is defined as follows. Consider a fractional Brownian motion X on $\mathbb{R}$ with Hurst parameter $H \in (0,1)$, as well as a standard Brownian motion Y on $\mathbb{R}_+$ independent from X. The process $Z = X \circ Y$ is the so-called *fractional Brownian motion in Brownian time* (F.B.M.B.T. in short). It is a self-similar process (of order $H/2$) with stationary increments, which is not Gaussian. When $H = 1/2$, one recovers the celebrated iterated Brownian motion.

In recent years, starting with the articles of Burdzy [3, 4], there has been an increased interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion, see, for example, [5, 8–11] to cite but a few. In the present paper, we are concerned with the study of the fluctuations of the pth variation of Z for any integer p, defined as

$$R_n^{(p)}(t) = \sum_{k=0}^{\lfloor 2^n t\rfloor -1} (Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p, \qquad n \in \mathbb{N}, t \geq 0.$$

At this stage, it is worthwhile noting that we are dealing with the pth variations of Z in the classical sense when p is even whereas, when p is odd, we are rather dealing with the *signed* pth variations of Z. The interested reader may read [1, 6] in order to find relevant information about power variations.

After proper normalization, we may expect the f.d.d. convergence to a non-degenerate limit (to be determined) of

$$S_n^{(p)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} ((Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p - E[(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p]), \qquad n \in \mathbb{N}, t \geq 0,$$

for some $\kappa > 0$ to be discovered. To reach this goal, a classical strategy consists in expanding the power function x^p in terms of Hermite polynomials. Doing so, our problem is reduced to the joint analysis of the following quantities:

$$U_n^{(r)}(t) = 2^{-n\widetilde{\kappa}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} H_r(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}), \qquad n \in \mathbb{N}, t \geq 0, r \in \mathbb{N}^*. \tag{1.1}$$

Here, $\widetilde{\kappa} > 0$ is some constant depending a priori on r, whereas H_r denotes the rth Hermite polynomial ($H_1(x) = x$, $H_2(x) = x^2 - 1$, etc.). Due to the fact that one cannot separate X from Y inside Z in the definition of $U_n^{(r)}$, working directly with (1.1) seems to be a difficult task (see also [10], Problem 5.1). This is why, following an idea introduced by Khosnevisan and Lewis [9] in the study of the case $H = 1/2$, we will rather analyze $U_n^{(r)}$ by means of certain stopping times for Y. The idea is quite simple: by stopping Y as it crosses certain levels, and by sampling Z at these times, one can effectively separate X from Y. To be more specific, let us introduce the following collection of stopping times (with respect to the natural filtration of Y), noted

$$\mathscr{T}_n = \{T_{k,n} \colon\ k \geq 0\}, \qquad n \geq 0, \tag{1.2}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} \colon\ j \in \mathbb{Z}\}$, $n \geq 0$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geq 0$, the stopping times $T_{k,n}$, appearing in (1.2), are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf\{s > T_{k-1,n} \colon\ Y(s) \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\}\}, \qquad k \geq 1.$$

Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y, and that, for every $n \geq 0$, the discrete stochastic process

$$\mathscr{Y}_n = \{Y(T_{k,n}) \colon\ k \geq 0\}$$

defines a simple random walk over $\mathscr{D}_n$. As shown in [9], as n tends to infinity the collection $\{T_{k,n} \colon\ 1 \leq k \leq 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} \colon\ 1 \leq k \leq 2^n t\}$ of order n of the time interval $[0, t]$ (see [9], Lemma 2.2, for a precise statement). Based on this fact, one can introduce the counterpart of (1.1) based on $\mathscr{T}_n$, namely,

$$V_n^{(r)}(t) = 2^{-n\widetilde{\kappa}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} H_r(Z_{T_{k+1,n}} - Z_{T_{k,n}}), \qquad n \in \mathbb{N}, t \geq 0, r \in \mathbb{N}^*. \tag{1.3}$$

We are now in a position to state the main result of the present paper.

Theorem 1.1. *The following two f.d.d. convergences in law take place as $n \to \infty$ for any integer $N \geq 1$.*

(1) *Assume that $H \leq \frac{1}{2}$. One has*

$$\begin{aligned} 2^{-n/4} &\left\{ \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{2r-1}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) \colon\ 1 \leq r \leq N \right\}_{t \geq 0} \\ &\xrightarrow{\text{f.d.d.}} \{\sigma_{2r-1} E_t^{(r)} \colon\ 1 \leq r \leq N\}_{t \geq 0}, \end{aligned} \tag{1.4}$$

where σ_{2r-1} is some (explicit) constant and $E = (B^{(1)} \circ Y, \dots, B^{(N)} \circ Y)$, with $B = (B^{(1)}, \dots, B^{(N)})$ a N-dimensional two-sided Brownian motion independent from Y.

(2) *Assume that $H < \frac{3}{4}$. One has*

$$\begin{aligned} 2^{-3n/4} &\left\{ \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{2r}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) \colon\ 1 \leq r \leq N \right\}_{t \geq 0} \\ &\xrightarrow{\text{f.d.d.}} \left\{ \sigma_{2r} \int_{-\infty}^{\infty} L_t^x(Y)\, \mathrm{d}B_x^{(r)} \colon\ 1 \leq r \leq N \right\}_{t \geq 0}, \end{aligned} \tag{1.5}$$

where σ_{2r} is some (explicit) constant, $B = (B^{(1)}, \dots, B^{(N)})$ is a N-dimensional two-sided Brownian motion independent from Y and $L_t^x(Y)$ stands for the local time of Y before time t at level x.

The process $\{\int_{\mathbb{R}} L_t^x(Y)\, \mathrm{d}B_x^{(r)}\}_{t \geq 0}$ appearing in (1.5) is nothing but the *Brownian motion in Random Scenery* introduced by Kesten and Spitzer (see [7]).

As a corollary of this theorem, we deduce the fluctuations of the power variation of Z.

Corollary 1.2. *The following two f.d.d. convergences in law take place as $n \to \infty$ for any integer $N \geq 1$.*

(1) *Assume that $H \leq \frac{1}{2}$. One has*

$$\begin{aligned} &\left\{ 2^{(-n/4)(1-(4r-2)H)} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r-1} \colon\ 1 \leq r \leq N \right\}_{t \geq 0} \\ &\xrightarrow{\text{f.d.d.}} \left\{ \sum_{k=1}^{r} a_{r,k} \sigma_{2k-1} E_t^{(k)} \colon\ 1 \leq r \leq N \right\}_{t \geq 0}, \end{aligned} \tag{1.6}$$

where $a_{r,k}$ is some constant given by: $a_{r,k} = \sum_{l=0}^{k-1} \frac{(-1)^l (2(r+k-l-1))!}{l!(2(k-l)-1)!(r+k-l-1)! 2^{r+k-1}}$.

(2) *Assume that* $H < \frac{3}{4}$. *One has*

$$\begin{aligned}&\left\{2^{(-3n/4)(1-4rH/3)}\sum_{k=0}^{\lfloor 2^n t\rfloor-1}((Z_{T_{k+1,n}}-Z_{T_{k,n}})^{2r}-2^{-nrH}b_{r,0})\colon\ 1\le r\le N\right\}_{t\ge 0}\\&\xrightarrow{\text{f.d.d.}}\left\{\sum_{k=1}^{r}b_{r,k}\sigma_{2k}\int_{-\infty}^{\infty}L_t^x(Y)\,\mathrm{d}B_x^{(k)}\colon\ 1\le r\le N\right\}_{t\ge 0},\end{aligned}\tag{1.7}$$

where $b_{r,k}$ *is some constant given by:* $b_{r,k}=\sum_{l=0}^{k}\frac{(-1)^l(2(r+k-l))!}{l!(2(k-l))!(r+k-l)!2^{r+k}}$.

Note that $b_{r,0}=E[(2^{nH/2}(Z_{T_{k+1,n}}-Z_{T_{k,n}}))^{2r}]=E[N^{2r}]$, with $N\sim\mathcal{N}(0,1)$.

In the particular case where $H=1/2$ (that is, when Z is an iterated Brownian motion) and $r=2,3,4$, we emphasize that Corollary 1.2 allows one to recovers Theorems 3.2, 4.4 and 4.5 from Khoshnevisan and Lewis [9].

Remark. To keep the length of this paper within bounds, I defer to future analysis the technical investigation of the tightness of the power variations of F.B.M.B.T given in the previous corollary.

The organisation of the paper is as follows. In Section 2, we provide some needed preliminaries. Theorem 1.1 and Corollary 1.2 are then shown in Section 3.

2. Preliminaries

In this section, we collect several results that are useful for the proof of Theorem 1.1.

2.1. An algebraic lemma and some local time estimates

For each integer $n\ge 0$, $k\in\mathbb{Z}$ and real number $t\ge 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2},(j+1)2^{-n/2}]$ within the first $\lfloor 2^n t\rfloor$ steps of the random walk $\{Y(T_{k,n})\}_{k\ge 1}$, that is,

$$\begin{aligned}U_{j,n}(t)=\sharp\{&k=0,\dots,\lfloor 2^n t\rfloor-1\colon\\&Y(T_{k,n})=j2^{-n/2}\text{ and }Y(T_{k+1,n})=(j+1)2^{-n/2}\};\end{aligned}\tag{2.1}$$

$$\begin{aligned}D_{j,n}(t)=\sharp\{&k=0,\dots,\lfloor 2^n t\rfloor-1\colon\\&Y(T_{k,n})=(j+1)2^{-n/2}\text{ and }Y(T_{k+1,n})=j2^{-n/2}\}.\end{aligned}\tag{2.2}$$

The following lemma will play a crucial role in our study of the asymptotic behavior of $V_n^{(r)}$. Its main feature is to separate X from Y, thus providing a representation of $V_n^{(r)}$ which is amenable to analysis.

Lemma 2.1 (See [9], Lemma 2.4). *Fix $t \geq 0$ and $r \in \mathbb{N}^*$. Then*

$$V_n^{(r)}(t) = 2^{-n\widetilde{\kappa}} \sum_{j\in\mathbb{Z}} H_r(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))(U_{j,n}(t) + (-1)^r D_{j,n}(t)). \tag{2.3}$$

Also, in order to prove the second point of Theorem 1.1 we will need estimates on the local time of Y taken from [9], that we collect in the following statement.

Proposition 2.2.

1. *For every $x \in \mathbb{R}$, $p \in \mathbb{N}^*$ and $t > 0$, we have*

$$E[(L_t^x(Y))^p] \leq 2E[(L_1^0(Y))^p] t^{p/2} \exp\left(-\frac{x^2}{2t}\right).$$

2. *There exists a positive constant μ such that, for every $a, b \in \mathbb{R}$ with $ab \geq 0$ and $t > 0$,*

$$E[|L_t^b(Y) - L_t^a(Y)|^2]^{1/2} \leq \mu\sqrt{|b-a|}\, t^{1/4} \exp\left(-\frac{a^2}{4t}\right).$$

3. *There exists a positive random variable $K \in L^8$ such that, for every $j \in \mathbb{Z}$, every $n \geq 0$ and every $t > 0$, one has that*

$$|\mathcal{L}_{j,n}(t) - L_t^{j2^{-n/2}}(Y)| \leq 2Kn2^{-n/4}\sqrt{L_t^{j2^{-n/2}}(Y)},$$

where $\mathcal{L}_{j,n}(t) = 2^{-n/2}(U_{j,n}(t) + D_{j,n}(t))$.

2.2. Breuer–Major Theorem

Let $\{G_k\}_{k\geq 1}$ be a centered stationary Gaussian sequence. In this Gaussian context, stationary just means that there exist $\rho : \mathbb{Z} \to \mathbb{R}$ such that $E[G_k G_l] = \rho(k-l)$, $k, l \geq 1$. Assume further that $\rho(0) = 1$, that is, each G_k is $\mathscr{N}(0,1)$ distributed. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$E[\varphi^2(G_1)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi^2(x) \mathrm{e}^{-x^2/2}\, \mathrm{d}x < +\infty. \tag{2.4}$$

The function φ may be expanded in $L^2(\mathbb{R}, \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\, \mathrm{d}x)$ (in a unique way) in terms of Hermite polynomials as follows:

$$\varphi(x) = \sum_{q=0}^{+\infty} a_q H_q(x). \tag{2.5}$$

Let $d \geq 0$ be the *Hermite rank* of φ, that is, the first integer $q \geq 0$ such that $a_q \neq 0$ in (2.5). We then have the celebrated Breuer–Major Theorem (see [2], see also [12] for a modern proof).

Theorem 2.3 (Breuer–Major). *Let* $\{G_k\}_{k\geq 1}$ *(with covariance* ρ*) and* $\varphi:\mathbb{R}\to\mathbb{R}$ *(with Hermite index* d*) be as above. Assume further that* $\sum_{k\in\mathbb{Z}}|\rho(k)|^d<+\infty$*. Then, as* $n\to+\infty$*,*

$$2^{-n/2}\left\{\sum_{k=1}^{\lfloor 2^n t\rfloor}(\varphi(G_k)-E[\varphi(G_k)])\right\}_{t\geq 0}\overset{\text{f.d.d.}}{\to}\{\sigma B_t\}_{t\geq 0}, \tag{2.6}$$

with B *a standard Brownian motion and* $\sigma>0$ *given by*

$$\sigma^2=\sum_{q=d}^{+\infty}q!a_q^2\sum_{k\in\mathbb{Z}}\rho(k)^q\in[0,+\infty[. \tag{2.7}$$

2.3. Peccati–Tudor Theorem

In a seminal paper of 2005, Nualart and Peccati [13] discovered a surprising central limit theorem (called the *Fourth Moment Theorem* nowadays) for sequences of multiple stochastic integrals of a fixed order: in this context, convergence in distribution to the standard normal law is actually equivalent to convergence of just the fourth moment. Shortly afterwards, Peccati and Tudor gave a multidimensional version of this characterization, making use of tools belonging to the Malliavin calculus. Since we will rely on this result in the present paper, let us give more details.

Let $d\geq 2$ and $q_1,\ldots,q_d\geq 1$ be some fixed integers. Consider a sequence of random vectors $F_n=(F_{1,n},\ldots,F_{d,n})$ of the following form. Each $F_{i,n}$ can be written as

$$F_{i,n}=\sum_{j=0}^{N_n}a_{j,n}H_{q_i}(Y_{j,n}),$$

where N_n is an integer, $a_{j,n}$ are real numbers and $\{Y_{j,n}\}_{j\geq 0}$ is a centered stationary Gaussian family with unit variance. We then have the following result, shown in [14].

Theorem 2.4 (Peccati–Tudor). *Let* (F_n) *be a sequence as above. Let* $C\in\mathscr{M}_d(\mathbb{R})$ *be a symmetric and positive matrix, and let* N *be a centered Gaussian vector with covariance* C*. Assume that*

$$\lim_{n\to+\infty}E[F_{i,n}F_{j,n}]=C(i,j),\qquad 1\leq i,j\leq d. \tag{2.8}$$

Then, as $n\to+\infty$*, the following two conditions are equivalent:*

(a) F_n *converges in law to* N*;*
(b) *for every* $1\leq i\leq d$*,* $F_{i,n}$ *converges in law to* $\mathscr{N}(0,C(i,i))$*.*

3. Proof of Theorem 1.1

3.1. Proof of (1.4)

Recall the definition (1.3) of $V_n^{(r)}(t)$ and let us fix $\widetilde{\kappa} = 1/4$. First of all, let us apply Lemma 2.1. Because $2r-1$ is an odd number, we obtain that

$$V_n^{(2r-1)}(t) = 2^{-n/4} \sum_{j\in\mathbb{Z}} H_{2r-1}(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))(U_{j,n}(t) - D_{j,n}(t)). \quad (3.1)$$

Now, let us observe (see also [9], Lemma 2.5) that

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} \mathbf{1}(0 \le j < j^*(n,t)) & \text{if } j^*(n,t) > 0, \\ 0 & \text{if } j^* = 0, \\ -\mathbf{1}(j^*(n,t) \le j < 0) & \text{if } j^*(n,t) < 0, \end{cases}$$

where

$$j^*(n,t) = 2^{n/2} Y_{T_{\lfloor 2^n t\rfloor,n}}.$$

As a consequence,

$$V_n^{(2r-1)}(t) = \begin{cases} 2^{-n/4} \displaystyle\sum_{j=1}^{j^*(n,t)} H_{2r-1}(2^{nH/2}(X^+_{j2^{-n/2}} - X^+_{(j-1)2^{-n/2}})) & \text{if } j^*(n,t) > 0, \\ 0 & \text{if } j^* = 0, \\ 2^{-n/4} \displaystyle\sum_{j=1}^{|j^*(n,t)|} H_{2r-1}(2^{nH/2}(X^-_{j2^{-n/2}} - X^-_{(j-1)2^{-n/2}})) & \text{if } j^*(n,t) < 0, \end{cases}$$

where $X_t^+ = X_t$ for $t \ge 0$ and $X_{-t}^- = X_t$ for $t < 0$. Our analysis of $V_n^{(2r-1)}$ will become easier if one introduces the following sequence of processes $W_{\pm,n}^{(2r-1)}$, in which we have replaced $\sum_{j=1}^{\pm j^*(n,t)}$ by $\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}$, namely:

$$W_{+,n}^{(2r-1)}(t) = 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} H_{2r-1}(2^{nH/2}(X^+_{j2^{-n/2}} - X^+_{(j-1)2^{-n/2}})), \qquad t \ge 0,$$

$$W_{-,n}^{(2r-1)}(t) = 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} H_{2r-1}(2^{nH/2}(X^-_{j2^{-n/2}} - X^-_{(j-1)2^{-n/2}})), \qquad t \ge 0,$$

$$W_n^{(2r-1)}(t) = \begin{cases} W_{+,n}^{(2r-1)}(t) & \text{if } t \ge 0, \\ W_{-,n}^{(2r-1)}(-t) & \text{if } t < 0. \end{cases}$$

It is clear, using the self-similarity property of X, that the f.d.d. convergence in law of the vector $(W_{+,n}^{(2r-1)}, W_{-,n}^{(2r-1)}, 1 \le r \le N)$ is equivalent to the f.d.d. convergence in law of the vector $(\overline{W}_{+,n}^{(2r-1)}, \overline{W}_{-,n}^{(2r-1)}, 1 \le r \le N)$ defined as:

$$\overline{W}_{+,n}^{(2r-1)}(t) = 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} H_{2r-1}(X_j^+ - X_{j-1}^+), \qquad t \ge 0,$$

$$\overline{W}_{-,n}^{(2r-1)}(t) = 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} H_{2r-1}(X_j^- - X_{j-1}^-), \qquad t \ge 0.$$

Let $G_j = X_j^+ - X_{j-1}^+$. The family $\{G_j\}$ is Gaussian, stationary, centered, with variance 1; moreover its covariance ρ is given by

$$\rho(k) = E[G_j G_{j+k}] = \tfrac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}), \tag{3.2}$$

so that $\sum |\rho(k)| < \infty$ because $H \le \frac{1}{2}$. Hence, Breuer–Major Theorem 2.3 applies and yields that, as $n \to \infty$ and for any fixed r,

$$\{\overline{W}_{+,n}^{(2r-1)}(t)\colon\ t \ge 0\} \xrightarrow{\text{f.d.d.}} \sigma_{2r-1}\{B^{+,r}(t)\colon\ t \ge 0\},$$

with $B^{+,r}$ a standard Brownian motion and $\sigma_{2r-1} = \sqrt{(2r-1)! \sum_{a \in \mathbb{Z}} \rho(a)^{2r-1}}$. Note that $\sum_{a \in \mathbb{Z}} |\rho(a)|^{2r-1} < \infty$ if and only if $H < 1 - 1/(2(2r-1))$, which is satisfied for all $r \ge 1$ since we have supposed that $H \le 1/2$ (the case $H = 1/2$ may be treated separately). Similarly,

$$\{\overline{W}_{-,n}^{(2r-1)}(t)\colon\ t \ge 0\} \xrightarrow{\text{f.d.d.}} \sigma_{2r-1}\{B^{-,r}(t)\colon\ t \ge 0\},$$

with $B^{-,r}$ a standard Brownian motion and σ_{2r-1} as above. In order to deduce the joint convergence in law of $(\overline{W}_{+,n}^{(2r-1)}, \overline{W}_{-,n}^{(2r-1)}, 1 \le r \le N)$, from Peccati–Tudor Theorem 2.4 and taking into account that $E[\overline{W}_{\pm,n}^{(2r-1)}(t)\overline{W}_{\pm,n}^{(2l-1)}(t)] = 0$ for $l \ne r$ (since Hermite polynomials of different orders are orthogonal), it remains to check that, for any integer r and any real numbers $t, s \ge 0$,

$$\lim_{n \to \infty} E[\overline{W}_{+,n}^{(2r-1)}(t)\overline{W}_{-,n}^{(2r-1)}(s)] = 0. \tag{3.3}$$

Let us do it. One can write,

$$\begin{aligned} &E[\overline{W}_{+,n}^{(2r-1)}(t)\overline{W}_{-,n}^{(2r-1)}(s)] \\ &\quad = 2^{-n/2} \sum_{k=1}^{\lfloor 2^{n/2} t \rfloor} \sum_{l=1}^{\lfloor 2^{n/2} s \rfloor} E[H_{2r-1}(X_k^+ - X_{k-1}^+) H_{2r-1}(X_l^- - X_{l-1}^-)] \end{aligned}$$

$$= (2r-1)!2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor}\left(E[(X_k^+ - X_{k-1}^+)(X_l^- - X_{l-1}^-)]\right)^{2r-1}$$

$$= (2r-1)!2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor}\left(E[(X_k - X_{k-1})(X_{-l} - X_{-l+1})]\right)^{2r-1}$$

$$= (2r-1)!2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor}\left(\frac{1}{2}[2|k+l-1|^{2H} - |k+l|^{2H} - |k+l-2|^{2H}]\right)^{2r-1}.$$

Setting $a = k+l$, we deduce that

$$\begin{aligned}
&E[\overline{W}_{+,n}^{(2r-1)}(t)\overline{W}_{-,n}^{(2r-1)}(s)] \\
&\quad = 2^{-(2r-1)}(2r-1)!2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{a=k+1}^{k+\lfloor 2^{n/2}s\rfloor}(2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H})^{2r-1} \\
&\quad = 2^{-(2r-1)}(2r-1)!2^{-n/2}\sum_{a=2}^{\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor}\ \sum_{k=(a-\lfloor 2^{n/2}s\rfloor)\vee 1}^{(a-1)\wedge\lfloor 2^{n/2}t\rfloor}(2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H})^{2r-1} \\
&\quad = 2^{-(2r-1)}(2r-1)!2^{-n/2}\sum_{a\in\mathbb{N}} f_n(a),
\end{aligned}$$

where

$$\begin{aligned}
f_n(a) &:= (2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H})^{2r-1}((a-1)\wedge\lfloor 2^{n/2}t\rfloor - (a-\lfloor 2^{n/2}s\rfloor)\vee 1 + 1) \\
&\qquad \times \mathbf{1}_{\{2\le a\le\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}.
\end{aligned}$$

For any $a\in\{2,\dots,\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}$, observe that $\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\ge 1$. Also, we have

$$2^{-n/2}|(a-1)\wedge\lfloor 2^{n/2}t\rfloor|\mathbf{1}_{\{2\le a\le\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\le 2^{-n/2}\lfloor 2^{n/2}t\rfloor\le t,$$

as well as

$$\begin{aligned}
&2^{-n/2}|(a-\lfloor 2^{n/2}s\rfloor)\vee 1|\mathbf{1}_{\{2\le a\le\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}} \\
&\quad\le 2^{-n/2}(|a|+\lfloor 2^{n/2}s\rfloor+1)\mathbf{1}_{\{2\le a\le\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}} \\
&\quad\le 2^{-n/2}(\lfloor 2^{n/2}t\rfloor+2\lfloor 2^{n/2}s\rfloor)+2^{-n/2}\mathbf{1}_{\{2\le a\le\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}} \\
&\quad\le 2^{-n/2}(\lfloor 2^{n/2}t\rfloor+2\lfloor 2^{n/2}s\rfloor)+2^{-n/2}(\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor)\le 2t+3s,
\end{aligned}$$

and

$$2^{-n/2}\mathbf{1}_{\{2\le a\le\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\le 2^{-n/2}(\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor)\le t+s.$$

Plugging all these inequalities together leads to

$$2^{-n/2}|f_n(a)| \leq (4t+4s)|2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}|^{2r-1}$$

for all n, with $\sum_{a\in\mathbb{N}} |2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}|^{2r-1} < \infty$ (recall that $H \leq 1/2$). Moreover, $2^{-n/2} f_n(a) \underset{n\to\infty}{\longrightarrow} 0$ for any fixed a because

$$\frac{(a-1)}{2^{n/2}} \wedge \frac{\lfloor 2^{n/2} t\rfloor}{2^{n/2}} - \frac{(a - \lfloor 2^{n/2} s\rfloor)}{2^{n/2}} \vee 2^{-n/2} + 2^{-n/2} \underset{n\to\infty}{\longrightarrow} 0 \wedge t - (-s) \vee 0 = 0$$

since $t, s \geq 0$. Hence, the dominated convergence theorem applies and yields

$$2^{-n/2} \sum_{a\in\mathbb{N}} f_n(a) \underset{n\to\infty}{\longrightarrow} 0,$$

that is, (3.3) holds true. As we said, using Peccati–Tudor Theorem 2.4 one thus obtains that

$$(W_{+,n}^{(2r-1)}, W_{-,n}^{(2r-1)}, 1 \leq r \leq N) \xrightarrow{\text{f.d.d.}} (\sigma_{2r-1} B^{+,r}, \sigma_{2r-1} B^{-,r}, 1 \leq r \leq N), \tag{3.4}$$

with $(B^{+,r}, B^{-,r}, 1 \leq r \leq N)$ a $2N$-dimensional standard Brownian motion. As a consequence, we have

$$(W_n^{(2r-1)}(t), 1 \leq r \leq N)_{t\in\mathbb{R}} \xrightarrow{\text{f.d.d.}} (\sigma_{2r-1} B^{(r)}(t), 1 \leq r \leq N)_{t\in\mathbb{R}}, \tag{3.5}$$

with $(B^{(r)}, 1 \leq r \leq N)$ a N-dimensional two-sided Brownian motion.

On the other hand, let us prove for any $r \in \mathbb{N}^*$ the existence of $C_r > 0$ such that, for any n and any $s, t \in \mathbb{R}$,

$$E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(s))^2] \leq 8C_r(2^{-n/2} + |t-s|). \tag{3.6}$$

To do so, we distinguish three cases, according to the sign of $s, t \in \mathbb{R}$ (and reducing the problem by symmetry):

(1) if $0 \leq s \leq t$:

$$\begin{aligned}
&E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(s))^2] \\
&\quad = E[(\overline{W}_{+,n}^{(2r-1)}(t) - \overline{W}_{+,n}^{(2r-1)}(s))^2] = \left| 2^{-n/2} E\left[\left(\sum_{j=\lfloor 2^{n/2} s\rfloor+1}^{\lfloor 2^{n/2} t\rfloor} H_{2r-1}(X_j^+ - X_{j-1}^+)\right)^2\right]\right| \\
&\quad = \left|(2r-1)! 2^{-n/2} \sum_{j,k=\lfloor 2^n s\rfloor+1}^{\lfloor 2^n t\rfloor} (\rho(j-k))^{2r-1}\right| \\
&\quad = \left|(2r-1)! 2^{-n/2} \sum_{j=\lfloor 2^{n/2} s\rfloor+1}^{\lfloor 2^{n/2} t\rfloor} \sum_{a=j-\lfloor 2^{n/2} t\rfloor}^{j-\lfloor 2^{n/2} s\rfloor-1} (\rho(a))^{2r-1}\right|
\end{aligned}$$

$$\leq (2r-1)!2^{-n/2}$$
$$\times \sum_{a=\lfloor 2^{n/2}s\rfloor-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor-1} |\rho(a)|^{2r-1}|(a+\lfloor 2^n t\rfloor)\wedge\lfloor 2^n t\rfloor-(a+\lfloor 2^n s\rfloor)\vee(\lfloor 2^n s\rfloor)|$$
$$\leq (2r-1)!2^{-n/2}\sum_{a\in\mathbb{Z}}|\rho(a)|^{2r-1}|\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor| = C_r 2^{-n/2}|\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor|$$
$$\leq C_r(|2^{-n/2}\lfloor 2^{n/2}t\rfloor-t|+|t-s|+|2^{-n/2}\lfloor 2^{n/2}s\rfloor-s|)\leq C_r(22^{-n/2}+|t-s|)$$
$$\leq 2C_r(2^{-n/2}+|t-s|),$$

with $C_r=(2r-1)!\sum_{a\in\mathbb{Z}}|\rho(a)|^{2r-1}<\infty$, hence (3.6) holds true.

(2) If $s\leq t\leq 0$: by the same argument as above

$$\begin{aligned}
&E[(W_n^{(2r-1)}(t)-W_n^{(2r-1)}(s))^2]\\
&\quad= E[(\overline{W}_{-,n}^{(2r-1)}(-s)-\overline{W}_{-,n}^{(2r-1)}(-t))^2]\leq C_r(2^{-n/2}|\lfloor 2^{n/2}|t|\rfloor-\lfloor 2^{n/2}|s|\rfloor|)\\
&\quad\leq C_r(|2^{-n/2}\lfloor 2^{n/2}|t|\rfloor-|t||+||t|-|s||+|2^{-n/2}\lfloor 2^{n/2}|s|\rfloor-|s||)\\
&\quad\leq C_r(22^{-n/2}+||t|-|s||)\\
&\quad\leq 2C_r(2^{-n/2}+||t|-|s||)\leq 2C_r(2^{-n/2}+|t-s|),
\end{aligned}$$

so that (3.6) holds true as well.

(3) If $s<0<t$: using the two previous inequality (point (1) and point (2)) one has

$$\begin{aligned}
E[(W_n^{(2r-1)}(t)-W_n^{(2r-1)}(s))^2] &\leq 2E[(W_n^{(2r-1)}(t)-W_n^{(2r-1)}(0))^2]\\
&\quad+2E[(W_n^{(2r-1)}(s)-W_n^{(2r-1)}(0))^2]\\
&\leq 4C_r(2^{-n/2}+t)+4C_r(2^{-n/2}+|s|)\\
&=8C_r2^{-n/2}+4C_r(t+|s|)\\
&=8C_r2^{-n/2}+4C_r|t-s|\\
&\leq 8C_r(2^{-n/2}+|t-s|).
\end{aligned}$$

This proves (3.6).

Now, let us go back to $V_n^{(2r-1)}$. Observe that

$$V_n^{(2r-1)}(t)=W_n^{(2r-1)}(Y_{T_{\lfloor 2^n t\rfloor,n}})$$

and recall from [9], Lemma 2.3, that $E[|Y_{T_{\lfloor 2^n t\rfloor,n}}-Y(t)|]\to 0$ as $n\to\infty$ for any $t>0$. We deduce, combining these two latter facts with (3.6), that

$$V_n^{(2r-1)}(t)-W_n^{(2r-1)}(Y_t)\xrightarrow{L^2}0 \qquad \text{as } n\to\infty.$$

But Y is independent from $W_n^{(2r-1)}$ so, from (3.5), it comes that

$$(W_n^{(2r-1)} \circ Y, 1 \le r \le N) \xrightarrow{\text{f.d.d.}} (\sigma_{2r-1} B^{(r)} \circ Y, 1 \le r \le N), \tag{3.7}$$

from which the desired conclusion (1.4) follows.

3.2. Proof of (1.5)

Recall the definition (1.3) and let us fix $\widetilde{\kappa} = 3/4$. First of all, let us apply Lemma 2.1. Because $2r$ is an even number, we obtain that

$$V_n^{(2r)}(t) = 2^{-3n/4} \sum_{j \in \mathbb{Z}} H_{2r}(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))(U_{j,n}(t) + D_{j,n}(t)). \tag{3.8}$$

Set $\mathcal{L}_{j,n}(t) = 2^{-n/2}(U_{j,n}(t) + D_{j,n}(t))$, so that

$$V_n^{(2r)}(t) = 2^{-n/4} \sum_{j \in \mathbb{Z}} H_{2r}(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))\mathcal{L}_{j,n}(t).$$

At this stage, to simplify the exposition, let us introduce the short-hand notation

$$X_j^{(n)} = 2^{nH/2} X_{j2^{-n/2}}.$$

Fix $t \ge 0$. In order to study the convergence in law of $V_n^{(2r)}(t)$ as n tends to infinity, we shall consider (separately) the cases when n is even and when n is odd.

When n is even, for any even integers $n \ge m \ge 0$ and any integer $p \ge 0$, by following Nourdin and Peccati (see [11]) one can decompose $V_n^{(2r)}(t)$ as

$$V_n^{(2r)}(t) = A_{m,n,p}^{(2r)}(t) + B_{m,n,p}^{(2r)}(t) + C_{m,n,p}^{(2r)}(t) + D_{m,n,p}^{(2r)}(t),$$

where

$$A_{m,n,p}^{(2r)}(t) = 2^{-n/4} \sum_{-p2^{m/2}+1 \le j \le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}^{(n)} - X_i^{(n)})(\mathcal{L}_{i,n}(t) - L_t^{i2^{-n/2}}(Y)),$$

$$B_{m,n,p}^{(2r)}(t) = 2^{-n/4} \sum_{-p2^{m/2}+1 \le j \le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}^{(n)} - X_i^{(n)}) \times (L_t^{i2^{-n/2}}(Y) - L_t^{j2^{-m/2}}(Y)),$$

$$C_{m,n,p}^{(2r)}(t) = 2^{-n/4} \sum_{-p2^{m/2}+1 \le j \le p2^{m/2}} L_t^{j2^{-m/2}}(Y) \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}^{(n)} - X_i^{(n)}),$$

$$D_{m,n,p}^{(2r)}(t) = 2^{-n/4} \sum_{i \ge p2^{n/2}} H_{2r}(X_{i+1}^{(n)} - X_i^{(n)})\mathcal{L}_{i,n}(t) + \sum_{i < -p2^{n/2}} H_{2r}(X_{i+1}^{(n)} - X_i^{(n)})\mathcal{L}_{i,n}(t).$$

We can see that since we have taken even integers $n \geq m \geq 0$ then $2^{m/2}$, $2^{(n-m)/2}$ and $2^{n/2}$ are integers as well. This justifies the validity of the previous decomposition.

When n is odd, for any odd integers $n \geq m \geq 0$ we can work with the same decomposition for $V_n^{(2r)}(t)$. The only difference is that we have to replace the sum $\sum_{-p2^{m/2}+1\leq j\leq p2^{m/2}}$ in $A_{m,n,p}^{(2r)}(t)$, $B_{m,n,p}^{(2r)}(t)$ and $C_{m,n,p}^{(2r)}(t)$ by $\sum_{-p2^{(m+1)/2}+1\leq j\leq p2^{(m+1)/2}}$. And instead of $\sum_{i\geq p2^{n/2}}$ and $\sum_{i<-p2^{n/2}}$ in $D_{m,n,p}^{(2r)}(t)$, we must consider $\sum_{i\geq p2^{(n+1)/2}}$ and $\sum_{i<-p2^{(n+1)/2}}$ respectively. The analysis can then be done mutatis mutandis.

Let us go back to our proof. First, we will prove that $A_{m,n,p}^{(2r)}(t)$, $B_{m,n,p}^{(2r)}(t)$ and $D_{m,n,p}^{(2r)}(t)$ converge to 0 in L^2 by letting n, then m, then p tend to infinity. Second, we will study the convergence in law (in the sense f.d.d.) of

$$(C_{m,n,p}^{(2r)}, 1 \leq r \leq N), \tag{3.9}$$

which will then be equivalent to the convergence in law (in the sense f.d.d.) of

$$(V_n^{(2r)}, 1 \leq r \leq N).$$

We will prove that $E[(A_{m,n,p}^{(2r)}(t))^2] \to 0$ as $n \to \infty$. We have, with ρ given by (3.2) (note that $\sum_{a\in\mathbb{Z}} |\rho(a)|^{2r} < \infty$ if and only if $H < 1 - 1/(4r)$, which is satisfied for any $r \geq 1$ because $H < 3/4$),

$$\begin{aligned}
&E[(A_{m,n,p}^{(2r)}(t))^2] \\
&\quad = \Bigg| 2^{-n/2} \\
&\qquad \times \sum_{-p2^{m/2}+1\leq j\leq p2^{m/2}} \sum_{-p2^{m/2}+1\leq j'\leq p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} E[H_{2r}(X_{i+1}^{(n)} - X_i^{(n)}) \\
&\qquad \times H_{2r}(X_{i'+1}^{(n)} - X_{i'}^{(n)})] E[(\mathcal{L}_{i,n}(t) - L_t^{i2^{-n/2}}(Y))(\mathcal{L}_{i',n}(t) - L_t^{i'2^{-n/2}}(Y))] \Bigg| \\
&\quad \leq (2r)! 2^{-n/2} \sum_{-p2^{m/2}+1\leq j\leq p2^{m/2}} \sum_{-p2^{m/2}+1\leq j'\leq p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r} \\
&\qquad \times \|\mathcal{L}_{i,n}(t) - L_t^{i2^{-n/2}}(Y)\|_2 \times \|\mathcal{L}_{i',n}(t) - L_t^{i'2^{-n/2}}(Y)\|_2,
\end{aligned}$$

where, in the first equality, we used the independence between X and Y. By the point 3 of Proposition 2.2, we have

$$\|\mathcal{L}_{i,n}(t) - L_t^{i2^{-n/2}}(Y)\|_2 \leq 2n2^{-n/4}\|K\|_4 \|L_t^{i2^{-n/2}}(Y)\|_2^{1/2}. \tag{3.10}$$

On the other hand

$$\|L_t^{i2^{-n/2}}(Y)\|_2 \le \|L_t^{i2^{-n/2}}(Y) - L_t^0(Y)\|_2 + \|L_t^0(Y)\|_2. \tag{3.11}$$

By the point 2 of Proposition 2.2, we have

$$\|L_t^{i2^{-n/2}}(Y) - L_t^0(Y)\|_2 \le \mu\sqrt{|i|2^{-n/2}}t^{1/4}. \tag{3.12}$$

By combining (3.11) and (3.12), we get that $\|L_t^{i2^{-n/2}}(Y)\|_2 \le \mu\sqrt{|i|}2^{-n/4}t^{1/4} + \|L_t^0(Y)\|_2$.
Since $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for all $a, b \ge 0$, we deduce that

$$\|L_t^{i2^{-n/2}}(Y)\|_2^{1/2} \le \sqrt{\mu}|i|^{1/4}2^{-n/8}t^{1/8} + \|L_t^0(Y)\|_2^{1/2}. \tag{3.13}$$

Finally, (3.13) together with (3.10) show that

$$\begin{aligned} \|\mathcal{L}_{(i,n)}(t) - L_t^{i2^{-n/2}}(Y)\|_2 \le{}& 2\sqrt{\mu}\|K\|_4 t^{1/8} n 2^{-n/4} 2^{-n/8}|i|^{1/4} \\ &+ 2\|K\|_4\|L_t^0(Y)\|_2^{1/2} n 2^{-n/4}. \end{aligned} \tag{3.14}$$

As a result,

$$\begin{aligned}
&E[(A_{m,n,p}^{(2r)}(t))^2] \\
&\quad\le 4(2r)!\mu t^{1/8}t^{1/8}\|K\|_4^2 2^{-n}2^{-n/4}n^2 \qquad (3.15)\\
&\qquad \times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \; \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \; \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \; \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}|ii'|^{1/4} \\
&\qquad + 4(2r)!\sqrt{\mu}t^{1/8}\|K\|_4^2\|L_t^0(Y)\|_2^{1/2}2^{-n}2^{-n/8}n^2 \qquad (3.16)\\
&\qquad \times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \; \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \; \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \; \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}|i|^{1/4} \\
&\qquad + 4(2r)!\sqrt{\mu}t^{1/8}\|K\|_4^2\|L_t^0(Y)\|_2^{1/2}2^{-n}2^{-n/8}n^2 \qquad (3.17)\\
&\qquad \times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \; \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \; \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \; \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}|i'|^{1/4} \\
&\qquad + 4(2r)!\|K\|_4^2\|L_t^0(Y)\|_2^{1/2}\|L_t^0(Y)\|_2^{1/2}2^{-n}n^2 \qquad (3.18)\\
&\qquad \times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \; \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \; \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \; \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r},
\end{aligned}$$

and we are thus left to prove the convergence to 0 of (3.15)–(3.18) as $n\to\infty$. Let us do it.

(a) We have

$$2^{-n}n^2 \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}$$

$$= 2^{-n}n^2 \sum_{i=-p2^{n/2}}^{p2^{n/2}-1} \sum_{i'=-p2^{n/2}}^{p2^{n/2}-1} \rho(i-i')^{2r}$$

$$\le 2^{-n}n^2 \sum_{i=-p2^{n/2}}^{p2^{n/2}-1} \sum_{i'\in\mathbb{Z}} \rho(i')^{2r} = \sum_{i'\in\mathbb{Z}} \rho(i')^{2r} n^2 2^{-n}(2p2^{n/2}).$$

Since it is clear that the last quantity converges to 0 as $n\to\infty$, one deduces that (3.18) tends to zero.

(b) Since $-p2^{m/2}+1\le j'\le p2^{m/2}$ and $(j'-1)2^{(n-m)/2}\le i'\le j'2^{(n-m)/2}-1$, we deduce that $-p2^{n/2}\le i'\le p2^{n/2}-1$. So, $|i'|\le p2^{n/2}$. Consequently we have that $|i'|^{1/4}\le p^{1/4}2^{n/8}$, which shows that

$$2^{-n}2^{-n/8}n^2$$

$$\times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}|i'|^{1/4}$$

$$\le p^{1/4}2^{-n}n^2$$

$$\times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}$$

and this last quantity converges to 0 by the same argument as above. This shows that (3.17) tends to zero.

(c) Following the same strategy as in point (b), one deduces that (3.16) tends to zero. Details are left to the reader.

(d) By the same arguments as above, one can see that $|ii'|^{1/4}\le p^{1/2}2^{n/4}$. It follows that

$$2^{-n}2^{-n/4}n^2$$

$$\times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r}|ii'|^{1/4}$$

$$\le p^{1/2}2^{-n}n^2$$

$$\times \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} \rho(i-i')^{2r},$$

which converges to 0 by the same arguments as above. Hence, (3.15) tends to zero. The proof of $E[(A^{(2r)}_{m,n,p}(t))^2]\to 0$ as $n\to\infty$ is complete.

Now, let us prove the convergence of $B^{(2r)}_{m,n,p}(t)$ to 0 in L^2 as $m\to\infty$, uniformly in n. We have

$$\begin{aligned}
&E[(B^{(2r)}_{m,n,p}(t))^2]\\
&\quad = 2^{-n/2} \sum_{-p2^{m/2}+1\le j\le p2^{m/2}} \sum_{-p2^{m/2}+1\le j'\le p2^{m/2}} \sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} \sum_{i'=(j'-1)2^{(n-m)/2}}^{j'2^{(n-m)/2}-1} (2r)!\rho(i-i')^{2r}\\
&\qquad \times E[(L_t^{i2^{-n/2}}(Y)-L_t^{j2^{-m/2}}(Y))(L_t^{i'2^{-n/2}}(Y)-L_t^{j'2^{-m/2}}(Y))].
\end{aligned}$$

By Proposition 2.2 (point 2) and Cauchy–Schwarz, there is a universal constant μ such that

$$\begin{aligned}
&|E[(L_t^{i2^{-n/2}}(Y)-L_t^{j2^{-m/2}}(Y))(L_t^{i'2^{-n/2}}(Y)-L_t^{j'2^{-m/2}}(Y))]|\\
&\quad \le \mu^2\sqrt{t}\sqrt{|i2^{-n/2}-j2^{-m/2}||i'2^{-n/2}-j'2^{-m/2}|}\le \mu^2\sqrt{t}2^{-m/2}.
\end{aligned}$$

This yields

$$\begin{aligned}
\sup_n E[(B^{(2r)}_{m,n,p}(t))^2] &\le \mu^2(2r)!2^{-m/2}\sqrt{t}\\
&\qquad \times \sup_n\left\{2^{-n/2}\sum_{i=-p2^{n/2}}^{p2^{n/2}-1}\sum_{i'=-p2^{n/2}}^{p2^{n/2}-1}\rho(i-i')^{2r}\right\}\\
&\le \mu^2(2r)!2^{-m/2}\sqrt{t}2p\sum_{i\in\mathbb{Z}}\rho(i)^{2r},
\end{aligned}$$

which converges to 0 as $m\to\infty$.

Finally, let us prove that $D^{(2r)}_{m,n,p}(t)$ converges to 0 in L^2 as $p\to\infty$, uniformly in m and n. We have

$$E[(D^{(2r)}_{m,n,p}(t))^2]$$

$$= 2^{-n/2}\sum_{i\ge p2^{n/2}}\sum_{j\ge p2^{n/2}} E[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j)\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)] \tag{3.19}$$

$$+ 22^{-n/2}\sum_{i\ge p2^{n/2}}\sum_{j< -p2^{n/2}} E[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j)\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)] \tag{3.20}$$

$$+\,2^{-n/2}\sum_{i<-p2^{n/2}}\sum_{j<-p2^{n/2}}E[H_{2r}(X_{i+1}^{(n)}-X_i^{(n)})H_{2r}(X_{j+1}^{(n)}-X_j^{(n)})\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)],\tag{3.21}$$

and we are thus left to prove the convergence to 0 of (3.19)–(3.21) as $p\to\infty$, uniformly in m and n. Let us do it.

(a) We have

$$\begin{aligned}&\left|2^{-n/2}\sum_{i\geq p2^{n/2}}\sum_{j\geq p2^{n/2}}E[H_{2r}(X_{i+1}^{(n)}-X_i^{(n)})H_{2r}(X_{j+1}^{(n)}-X_j^{(n)})\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)]\right|\\&=\left|2^{-n/2}\sum_{i\geq p2^{n/2}}\sum_{j\geq p2^{n/2}}E[H_{2r}(X_{i+1}^{(n)}-X_i^{(n)})H_{2r}(X_{j+1}^{(n)}-X_j^{(n)})]E[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)]\right|\\&=(2r)!2^{-n/2}\sum_{i\geq p2^{n/2}}\sum_{j\geq p2^{n/2}}\rho(i-j)^{2r}E[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)],\end{aligned}$$

where, in the second equality, we used the independence between X and Y. It is enough to prove that, uniformly in n and m, and as $p\to\infty$:

$$2^{-n/2}\sum_{i\geq p2^{n/2}}\sum_{j\geq p2^{n/2}}\rho(i-j)^{2r}E[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)]\to 0.\tag{3.22}$$

We can write

$$\begin{aligned}&2^{-n/2}\sum_{i\geq p2^{n/2}}\sum_{j\geq p2^{n/2}}\rho(i-j)^{2r}E[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)]\\&\leq 2^{-n/2}\sum_{i\geq p2^{n/2}}\sum_{j\geq p2^{n/2}}\rho(i-j)^{2r}E\left[\frac{1}{2}(\mathcal{L}_{i,n}(t)^2+\mathcal{L}_{j,n}(t)^2)\right]\\&=2^{-n/2}\sum_{i\geq p2^{n/2}}E[\mathcal{L}_{i,n}(t)^2]\sum_{j\geq p2^{n/2}}\rho(i-j)^{2r}\\&\leq 2^{-n/2}\sum_{i\geq p2^{n/2}}E[\mathcal{L}_{i,n}(t)^2]\sum_{j\in\mathbb{Z}}\rho(j)^{2r}=C_r2^{-n/2}\sum_{i\geq p2^{n/2}}E[\mathcal{L}_{i,n}(t)^2],\end{aligned}$$

where $C_r:=\sum_{j\in\mathbb{Z}}\rho(j)^{2r}<\infty$. By the third point of Proposition 2.2, we have

$$|\mathcal{L}_{i,n}(t)|\leq L_t^{i2^{-n/2}}(Y)+2Kn2^{-n/4}\sqrt{L_t^{i2^{-n/2}}(Y)}$$

so that

$$E[\mathcal{L}_{i,n}(t)^2]\leq 2E[L_t^{i2^{-n/2}}(Y)^2]+8n^22^{-n/2}\|K^2\|_2\|L_t^{i2^{-n/2}}(Y)\|_2.\tag{3.23}$$

On the other hand, thanks to the point 1 of Proposition 2.2, we have

$$E[L_t^{i2^{-n/2}}(Y)^2] \le Ct\exp\left(-\frac{(i2^{-n/2})^2}{2t}\right). \tag{3.24}$$

Consequently, we get

$$\|L_{t_{a_2}}^{i2^{-n/2}}(Y)\|_2 \le C^{1/2}t^{1/2}\exp\left(-\frac{(i2^{-n/2})^2}{4t}\right). \tag{3.25}$$

By combining (3.23) with (3.24) and (3.25), we deduce that

$$\begin{aligned} 2^{-n/2}\sum_{i\ge p2^{n/2}} E[\mathcal{L}_{i,n}(t)^2] \le 2Ct2^{-n/2} &\sum_{i\ge p2^{n/2}} \exp\left(-\frac{(i2^{-n/2})^2}{2t}\right) \\ &+ 8C^{1/2}t^{1/2}\|K^2\|_2 n^2 2^{-n/2} \\ &\times 2^{-n/2}\sum_{i\ge p2^{n/2}} \exp\left(-\frac{(i2^{-n/2})^2}{4t}\right). \end{aligned}$$

But, for $a\in\{2,4\}$,

$$2^{-n/2}\sum_{i\ge p2^{n/2}} \exp\left(-\frac{(i2^{-n/2})^2}{at}\right) \le \int_{p-1}^{\infty} \exp\left(\frac{-x^2}{at}\right)\mathrm{d}x \underset{p\to\infty}{\longrightarrow} 0.$$

This proves (3.22). Hence, we deduce that (3.19) converges to 0 as $p\to\infty$ uniformly in n and m.

(b) Following the same strategy as in point (a), one deduces that (3.20) and (3.21) converge to 0 as $p\to\infty$ uniformly in n and m. Details are left to the reader.

This shows that $D_{m,n,p}^{(2r)}(t)$ converges to 0 in L^2 as $p\to\infty$, uniformly in m and n.

To finish our proof of (1.5), it remains to prove that, by letting n, then m, then p tend to infinity, we get

$$\{C_{m,n,p}^{(2r)}(t), 1\le r\le N\}_{t\ge 0} \overset{\text{f.d.d.}}{\longrightarrow} \left\{\sigma_{2r}\int_{-\infty}^{\infty} L_t^x(Y)\,\mathrm{d}B_x^{(r)}\colon\ 1\le r\le N\right\}_{t\ge 0}. \tag{3.26}$$

Since $H<3/4$, we claim that, as $n\to\infty$,

$$\begin{aligned} &\left(2^{-n/4}\sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}^{(n)}-X_i^{(n)}), 1\le r\le N\colon\ -p2^{m/2}+1\le j\le p2^{m/2}\right) \\ &\quad\overset{\text{law}}{\longrightarrow} (\sigma_{2r}(B_{(j+1)2^{-m/2}}^{(r)}-B_{j2^{-m/2}}^{(r)}), 1\le r\le N\colon\ -p2^{m/2}+1\le j\le p2^{m/2}), \end{aligned} \tag{3.27}$$

where $(B^{(1)},\dots,B^{(N)})$ is a N-dimensional two-sided Brownian motion.

Indeed, it is clear, using the self-similarity property of X, that the convergence in law of

$$\left(2^{-n/4}\sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}^{(n)}-X_i^{(n)}), 1\le r\le N\colon\ -p2^{m/2}+1\le j\le p2^{m/2}\right)$$

is equivalent to the convergence in law of

$$\left(2^{-n/4}\sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}-X_i), 1\le r\le N\colon\ -p2^{m/2}+1\le j\le p2^{m/2}\right).$$

Then, Breuer–Major Theorem 2.3 applies and yields that, as $n\to\infty$ and for any fixed $1\le r\le N$,

$$\left(2^{-n/4}\sum_{i=(j-1)2^{(n-m)/2}}^{j2^{(n-m)/2}-1} H_{2r}(X_{i+1}-X_i)\colon\ -p2^{m/2}+1\le j\le p2^{m/2}\right)$$
$$\xrightarrow{\text{law}} (\sigma_{2r}(B^{(r)}_{(j+1)2^{-m/2}}-B^{(r)}_{j2^{-m/2}})\colon\ -p2^{m/2}+1\le j\le p2^{m/2}).$$

In addition, from Peccati–Tudor Theorem 2.4 and taking into account the orthogonality of Hermite polynomial with different orders, we deduce (3.27). (The detailed proof of this result is similar to the proof of (3.5).)

As a consequence of (3.27), and thanks to the independence of X and Y, we have that as $n\to\infty$,

$$\{C^{(2r)}_{m,n,p}(t), 1\le r\le N\}_{t\ge 0}$$
$$\xrightarrow{\text{f.d.d.}} \left\{\sigma_{2r}\sum_{j=-p2^{m/2}+1}^{p2^{m/2}} L_t^{j2^{-m/2}}(Y)(B^{(r)}_{(j+1)2^{-m/2}}-B^{(r)}_{j2^{-m/2}}), 1\le r\le N\right\}_{t\ge 0}.$$

Since, for any fixed $t\ge 0$ and $1\le r\le N$ and as $m\to\infty$,

$$\sum_{j=-p2^{m/2}+1}^{p2^{m/2}} L_t^{j2^{-m/2}}(Y)(B^{(r)}_{(j+1)2^{-m/2}}-B^{(r)}_{j2^{-m/2}}) \xrightarrow{P} \int_{-p}^{p} L_t^x(Y)\,\mathrm{d}B_x^{(r)},$$

and since $\int_{-p}^{p} L_t^x(Y)\,\mathrm{d}B_x^{(r)} \xrightarrow{P} \int_{\mathbb{R}} L_t^x(Y)\,\mathrm{d}B_x^{(r)}$ as $p\to\infty$, we deduce finally that by letting m, then p tend to infinity, we get

$$\left\{\sigma_{2r}\sum_{j=-p2^{m/2}+1}^{p2^{m/2}} L_t^{j2^{-m/2}}(Y)(B^{(r)}_{(j+1)2^{-m/2}}-B^{(r)}_{j2^{-m/2}}), 1\le r\le N\right\}_{t\ge 0}$$

$$\xrightarrow{\text{f.d.d.}} \left\{ \sigma_{2r} \int_{-\infty}^{\infty} L_t^x(Y)\,\mathrm{d}B_x^{(r)}\colon\ 1 \le r \le N \right\}_{t\ge 0}.$$

This proves (3.26), and consequently (1.5).

3.3. Proof of Corollary 1.2

Let us decompose x^p in terms of Hermite polynomials. We have $x^p = \sum_{k=0}^{p} a_{p,k} H_k(x)$, where $a_{p,k}$ is some (explicit) integer. To calculate $a_{p,k}$, let N be a centred Gaussian variable with variance one. We have

$$N^p = \sum_{k=0}^{p} a_{p,k} H_k(N). \tag{3.28}$$

Thanks to the orthogonality property of Hermite polynomials with different orders and to the well known fact that $E[H_k(N)^2] = k!$, we get

$$a_{p,k} = \frac{1}{k!} E[N^p H_k(N)]. \tag{3.29}$$

On the other hand (see, e.g., [12], page 19) we have, for all $k \ge 1$,

$$H_k(x) = \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{k!(-1)^l}{l!(k-2l)!2^l} x^{k-2l}. \tag{3.30}$$

By combining (3.29) with (3.30), we deduce that

$$a_{p,k} = \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(-1)^l}{l!(k-2l)!2^l} E(N^{p+k-2l}). \tag{3.31}$$

Thus,

$$a_{p,k} = \begin{cases} \displaystyle\sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(-1)^l (p+k-2l)!}{l!(k-2l)!2^l 2^{(p+k-2l)/2}((p+k-2l)/2)!} & \text{if } p \text{ and } k \text{ are odd,} \\ \displaystyle\sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(-1)^l (p+k-2l)!}{l!(k-2l)!2^l 2^{(p+k-2l)/2}((p+k-2l)/2)!} & \text{if } p \text{ and } k \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we deduce that if p is odd, then

$$x^p = \sum_{k=1}^{\lfloor p/2 \rfloor + 1} a_{p,2k-1} H_{2k-1}(x), \tag{3.32}$$

whereas if p is even, then

$$x^p = \sum_{k=0}^{p/2} a_{p,2k} H_{2k}(x), \tag{3.33}$$

Finally, thanks to (3.32), (3.33), Theorem 1.1 and the Continuous Mapping theorem, we deduce the content of Corollary 1.2.

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