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QUASIHOMOGENEOUS ANALYTIC AFFINE CONNECTIONS ON SURFACES

SORIN DUMITRESCU AND ADOLFO GUILLOT

ABSTRACT. We classify torsion-free real-analytic affine connections on compact oriented real-analytic surfaces which are locally homogeneous on a nontrivial open set, without being locally homogeneous on all of the surface. In particular, we prove that such connections exist. This classification relies in a local result that classifies germs of torsion-free real-analytic affine connections on a neighborhood of the origin in the plane which are quasihomogeneous, in the sense that they are locally homogeneous on an open set containing the origin in its closure, but not locally homogeneous in the neighborhood of the origin.

1. INTRODUCTION

This article deals with the classification of torsion-free real-analytic affine connections on compact orientable surfaces which are *locally homogeneous* on a nontrivial open set, but not on all of the surface. They are here called *quasihomogeneous*. Our main results are stated in Theorem 1 (the local classification) and in Theorem 3 (the global classification). In particular, we show that such (strictly) quasihomogeneous connections exist.

The study of *locally homogeneous* geometric structures is a classical subject in differential geometry. It has its roots in the seminal work of Lie [Lie80] and took shape through Ehresmann's article [Ehr36]. In the Riemannian setting, these locally homogeneous spaces are the context of Thurston's 3-dimensional geometrization program [Thu97]. Locally homogeneous torsion-free connections on surfaces were studied by Opozda in [Opo04]. In [BMM08], answering a question of Lie, the authors classified Riemannian metrics on surfaces whose underlying Levi-Civita connections are projectively locally homogeneous. After the locally homogeneous geometric structures, the most symmetric geometric structures are those that are *quasihomogeneous*, this is, those that are locally homogeneous in an open and dense set. These quasihomogeneous connections are the object of our article.

Our results are also motivated by Gromov's *open-dense orbit theorem* [DG91, Gro88] (see also [Ben97, CQB03, Fer02]). Gromov's result asserts that, if the pseudogroup of local automorphisms of a *rigid geometric structure* (an analytic connection, for example) acts with a dense orbit, then this orbit is open. In this case, the geometric structure is locally homogeneous on a open dense set. Gromov's theorem says nothing about this maximal open and dense set of local homogeneity. In many interesting geometric situations, it may be all of the (connected) manifold. This was proved, for instance, for Anosov flows with differentiable stable and unstable foliations and transverse contact structure [BFL92] and for three-dimensional compact Lorentz manifolds admitting a nonproper one-parameter group of automorphisms [Zeg96].

In [BF05], the authors deal with this question and their results indicate ways in which some rigid geometric structures cannot degenerate off the open dense set. In [Fis11], Fisher conjectured that the maximal open

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set of local homogeneity is all of the manifold as soon as the following three conditions are fulfilled: the automorphism group of the manifold acts with a dense orbit, the geometric structure is a G -structure (meaning that it is locally homogeneous at the first order) and the manifold is compact.

Surprisingly, for some specific geometric structures, if the subset of local homogeneity is not empty, a maximal open set of local homogeneity is all of the (connected) manifold, *even if we drop the assumption that the automorphism group acts with a dense orbit*. This is known to be true in the Riemannian setting [PTV96], as a consequence of the fact that all scalar invariants are constant (see also section 3 in [Dum12]). This was also proved for three dimensional real-analytic Lorentz metrics [Dum08] and, in higher dimension, for complete real-analytic Lorentz metrics having semisimple Killing Lie algebra [Mel09]. In [Dum12], the first author proved that a real-analytic unimodular affine connection on a real-analytic compact surface which is locally homogeneous on a nontrivial open set is locally homogeneous on all of the surface, and asks about the extent to which the unimodularity hypothesis is necessary.

Motivated by these results, Theorem 3 constructs and characterizes torsion-free real-analytic affine connections on compact surfaces which are quasihomogeneous (but not homogeneous).

The main local ingredient of Theorem 3 is the following classification of germs of torsion-free real-analytic affine connections which are quasihomogeneous. It is, in some sense, the quasihomogeneous analogue to the local results in [KOV04].

Theorem 1. *Let ∇ be a torsion-free real-analytic affine connection in a neighborhood of the origin in \mathbf{R}^2 . Suppose that the maximal open set where $\mathfrak{K}(\nabla)$, the Lie algebra of Killing vector fields of ∇ , is transitive, contains the origin in its closure, but does not contain the origin. Then, up to an analytic change of coordinates, the germ of ∇ at the origin is one of the following:*

Type I(n), $n \in \frac{1}{2}\mathbf{Z}$, $n \geq \frac{1}{2}$: *The germ at $(0,0)$ of*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = -\gamma x^n \frac{\partial}{\partial x}, \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{1}{n} \varepsilon x^{2n+1} \frac{\partial}{\partial x} - \phi x^n \frac{\partial}{\partial y},$$

with $\phi = 0$ and $\gamma = 0$, if $n \notin \mathbf{Z}$ and $(n, \phi, \varepsilon) \neq (1, -\gamma, -\gamma^2)$. For these, $\mathfrak{K}(\nabla) = \langle x\partial/\partial x - ny\partial/\partial y, \partial/\partial y \rangle$.

Type II(n), $n \in \frac{1}{2}\mathbf{Z}$, $n \geq \frac{5}{2}$: *The germ at $(0,0)$ —Type II⁰(n)—or the germ at $(0,1)$ —Type II¹(n)—of*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \left(-\frac{1}{n} \varepsilon x^{2n-3} y^2 + 2\gamma x^{n-2} y\right) \frac{\partial}{\partial x} + \left(-\frac{1}{n} \varepsilon x^{2n-4} y^3 + [2\gamma - \phi] x^{n-3} y^2\right) \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \left(\frac{1}{n} \varepsilon x^{2n-2} y - \gamma x^{n-1}\right) \frac{\partial}{\partial x} + \left(\frac{1}{n} \varepsilon x^{2n-3} y^2 + [\phi - \gamma] x^{n-2} y\right) \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{1}{n} \varepsilon x^{2n-1} \frac{\partial}{\partial x} - \left(\frac{1}{n} \varepsilon x^{2n-2} y + \phi x^{n-1}\right) \frac{\partial}{\partial y},$$

with $\phi = 0$ and $\gamma = 0$, if $n \notin \mathbf{Z}$. For these, $\mathfrak{K}(\nabla) = \langle x\partial/\partial x + (1-n)y\partial/\partial y, x\partial/\partial y \rangle$.

Type III: *The germ at $(0,0)$ of*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \left(-\frac{1}{2} \varepsilon x y^2 + 2\gamma y\right) \frac{\partial}{\partial x} - \frac{1}{2} \varepsilon y^3 \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \left(\frac{1}{2} \varepsilon x^2 y - \gamma x\right) \frac{\partial}{\partial x} + \left(\frac{1}{2} \varepsilon x y^2 + \gamma y\right) \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{1}{2} \varepsilon x^3 \frac{\partial}{\partial x} - \left(\frac{1}{2} \varepsilon x^2 y + 2\gamma x\right) \frac{\partial}{\partial y},$$

for which $\mathfrak{K}(\nabla)$ is the Lie algebra of divergence-free linear vector fields in \mathbf{R}^2 .

In Types I and II, $(\gamma, \phi, \varepsilon) \in \mathbf{R}^3 \setminus \{(0, 0, 0)\}$ and the connection with parameters $(\gamma, \phi, \varepsilon)$ is equivalent to the one with parameters $(\mu\gamma, \mu\phi, \mu^2\varepsilon)$, for $\mu > 0$ in the case of Type II^1 and, for $\mu \in \mathbf{R}^*$, in the other cases. In Type III, $(\gamma, \varepsilon) \in \mathbf{R}^2 \setminus \{(0, 0)\}$ and the connection with parameters (γ, ε) is equivalent to the one with parameters $(\mu\gamma, \mu^2\varepsilon)$ for $\mu \in \mathbf{R}^*$. Apart from this, all the above connections are inequivalent.

Remark 2. For the connections of Type III, $\mathfrak{K}(\nabla) \approx \mathfrak{sl}(2, \mathbf{R})$ and there is one two-dimensional orbit of the Killing algebra (the complement of the origin). In the other cases, $\mathfrak{K}(\nabla) \approx \mathfrak{aff}(\mathbf{R})$, the Lie algebra of the affine group of the real line, and the components of the complement of the geodesic $\{x = 0\}$ in \mathbf{R}^2 are the two-dimensional orbits of the Killing algebra. In particular, all these germs admit nontrivial open sets on which ∇ is locally isomorphic to a translation invariant connexion on the connected component of the affine group of the real line. *The closed set where ∇ is not locally homogeneous is either a geodesic, or a point.* Moreover, in all cases, every vector field in the Killing algebra $\mathfrak{K}(\nabla)$ is an affine one and thus these Killing Lie algebras also preserve a flat torsion-free affine connection.

Our global result is the following one:

- Theorem 3.** (1) For integers n_1, n_2 , with $n_2 \geq n_1 \geq 2$, there exists a unique (up to automorphism) real-analytic torsion-free affine connection ∇_{n_1, n_2} on \mathbf{R}^2 such that
- (a) ∇_{n_1, n_2} is locally homogeneous on a nontrivial open set, but not on all of \mathbf{R}^2 . For $i = 1, 2$, there exists a point $p_i \in \mathbf{R}^2$ such that ∇_{n_1, n_2} is, in a neighborhood of p_i , given by a normal form of type $\text{II}^1(n_i)$, if n_i is odd and by a normal form of type $\text{I}(n_i)$, if n_i is even (in particular, the Killing Lie algebra of ∇_{n_1, n_2} is isomorphic to that of the affine group of the real line).
 - (b) There exist groups of automorphisms of ∇_{n_1, n_2} acting freely, properly discontinuously and co-compactly on \mathbf{R}^2 .
- (2) Let S be a compact orientable analytic surface endowed with a real-analytic torsion-free affine connection that is locally homogeneous on some nontrivial open set, but not on all of S . Then (S, ∇) is isomorphic to a quotient of $(\mathbf{R}^2, \nabla_{n_1, n_2})$.
- (3) The moduli space of compact quotients of $(\mathbf{R}^2, \nabla_{n_1, n_2})$ is $\Xi = \mathbf{N} \times \mathbf{R} \times \mathbf{R}/\mathbf{Z}$. Every compact quotient of $(\mathbf{R}^2, \nabla_{n_1, n_2})$ is a torus. For the torus T corresponding to $(k, \tau, \theta) \in \Xi$ we have:
- (a) The open set of local homogeneity is dense and is a union of $2k$ (if $n_1 \neq n_2$) or k (if $n_1 = n_2$) cylinders bounded by simple closed geodesics.
 - (b) There exists a globally defined Killing field A on T , unique up to multiplication by a constant. When normalized such that there exists a Killing vector field B defined in some open subset such that $[A, B] = B$, A is periodic with period τ .
 - (c) If γ_1 and γ_2 are generators of the fundamental group of T and γ_1 is homotopic to an orbit of A , then the analytic continuations of B along γ_1 and γ_2 are, respectively, $e^\tau B$ and $e^{(\theta+k)\tau} B$.

Example 4. Let us describe explicitly the quotients of $\nabla_{2,2}$ with parameters $(1, \tau, [\theta]) \in \Xi$. This will also give a self-contained proof of the fact that tori admit quasihomogeneous connections. Consider, in \mathbf{R}^2 , the torsion-free affine connection ∇ such that

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \frac{3}{4}x^2 \frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{3}{8}x^5 \frac{\partial}{\partial x} + \frac{3}{4}x^2 \frac{\partial}{\partial y}$$

(it is a connection of type I for $n = 2$ and $\varepsilon = \phi = \gamma = -\frac{3}{4}$). Remark that $B = \partial/\partial y$ is a Killing vector field for ∇ . Consider the commuting meromorphic vector fields

$$A = \frac{1}{2}x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad Z = \frac{1}{2}x \frac{\partial}{\partial x} + x^{-2} \frac{\partial}{\partial y}.$$

If we let $h(x, y) = x^2y$, we have

$$(1) \quad \nabla_A Z = \frac{3}{4}A - Z, \quad \nabla_Z Z = -\frac{1}{4}Z, \quad \nabla_A A = \frac{3}{4}(h^2 + 2h)A + \left(\frac{1}{4} - \frac{3}{8}[h^2 + 2h]\right)Z.$$

Notice that, since $[A, Z] = 0$ and h is a first integral of A , the vector field A is a Killing field of ∇ . The Lie algebra of Killing vector fields of ∇ contains the subalgebra generated by A and by B . The rank of this Lie algebra of vector fields is one in $\{x = 0\}$ and two in its complement. A direct computation shows that the curvature tensor of ∇ vanishes exactly on $\{x = 0\}$. The connection is thus locally homogeneous in the half planes $\{x < 0\}$ and $\{x > 0\}$, but not on all of \mathbf{R}^2 .

Consider the orientation-preserving birational involution $\sigma(x, y) = (-x, -y - 2x^{-2})$. It preserves the vector fields A and Z . Moreover, $h \circ \sigma = -2 - h$ and hence $(h^2 + 2h) \circ \sigma = h^2 + 2h$. This implies, by (1), that ∇ is preserved by σ . Let $\Omega = \{(x, y); y < 0, h > -2\}$, $U^+ = \Omega \cap \{x > 0\}$, $U^- = \Omega \cap \{x < 0\}$. Notice that $\sigma|_{U^+} : U^+ \rightarrow U^-$ is an analytic diffeomorphism. Let $\phi : \Omega \rightarrow \Omega$ be the diffeomorphism generated by the flow of A in time τ . The diffeomorphism ϕ preserves ∇ and commutes with σ . The quotient of Ω under the group generated by ϕ is a cylinder containing a simple closed curve coming from $\{x = 0\}$, whose complement is the union of the two cylinders $U^+/\langle\phi\rangle$ and $U^-/\langle\phi\rangle$. Let $K : U^+/\langle\phi\rangle \rightarrow U^-/\langle\phi\rangle$ be given by, first, the restriction to $U^+/\langle\phi\rangle$ of the flow of A in time $\theta\tau$ and then composing with σ (notice that adding an integer to θ yields the same result). By identifying $U^+/\langle\phi\rangle$ and $U^-/\langle\phi\rangle$ (as open subsets of $\Omega/\langle\phi\rangle$), via K , we obtain a torus S , naturally endowed with a connection ∇_s , coming from ∇ , a globally defined Killing vector field for ∇_s , induced by A , and a multivalued one, induced by B . There is one simple closed curve in S coming from $\{x = 0\}/\langle\phi\rangle$. The rank of the Killing algebra of vector fields of ∇_s is one along this curve and two in the complement: the connection is not locally homogeneous everywhere (since the curvature tensor vanishes exactly on $\{x = 0\}$), but is locally homogeneous in a dense open subset.

Note that in the previous example the connected component of the automorphism group of ∇_s is the flow generated by A , all of whose orbits are closed. The proof of Theorem 3 shows that, in general, the automorphism group of a quasihomogeneous connection is, up to a finite group, the flow of a Killing field, all of whose orbits are simple closed curves. Hence, the automorphism group doesn't admit dense orbits (our quasihomogeneous connections are not counter-examples to Fisher's conjecture).

Let us describe the structure of the article and indicate briefly our methods. In section 2 we recall the background and the basic definitions. In particular, we recall Nomizu's result about extension of local Killing fields in the real-analytic setting.

The idea for the proof of Theorem 1 is the following. Lemma 9 proves that for a germ of quasihomogeneous real-analytic affine connection, we can always find a two-dimensional subalgebra of the Killing algebra which is transitive on a nontrivial open set, but not at the origin (Lemma 9). Consequently, there exists a nontrivial open set where the connection is locally isomorphic either to a translation-invariant connection on \mathbf{R}^2 , or to a left-invariant connection on the affine group. The Abelian case will be dealt with in Section 3 (Prop. 14): we show that a quasihomogeneous connection cannot be locally isomorphic (on a nontrivial open set) to a translation invariant connection on \mathbf{R}^2 (without being locally homogeneous everywhere). In order to deal with the affine case, we will have to study the invariant affine connections and their Killing algebras in the affine group (Section 4) in order to finish, via the same method, the proof of Theorem 1 (Prop. 19, Section 5). The method consists in considering normal forms at the origin of left invariant vector fields X, Y on the affine group, with respect to which we compute Christoffel coefficients (in general X and Y are meromorphic at the origin).

In order to prove Theorem 3, we begin, in Section 6, to construct the connections ∇_{n_1, n_2} on \mathbf{R}^2 that appear in item (1) of the statement of the Theorem. We then study the symmetries of these connections and prove item (3). Section 7 proves item (2) of Theorem 3. The proof depends on all the previous material and uses (G, X) -structures and their relation to locally homogeneous connections as discussed in Section 2.

Finally, in Section 8 we study geodesically complete affine connections on \mathbf{R}^n which are invariant under the special linear group.

2. CONNECTIONS AND THEIR SYMMETRIES

Recall that an *affine connection* ∇ is an operator defined on a manifold M which enables one to consider the derivatives of a smooth vector field Y with respect to a smooth vector field X in order to obtain a smooth vector field $\nabla_X Y$ such that, for a function f , $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X fY = (X \cdot f)Y + f\nabla_X Y$ (it is tensorial in X and satisfies Leibniz's rule in Y). The reader will find several equivalent definitions and many details on affine connections in [KN96].

In local coordinates (x_1, x_2, \dots, x_n) , we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^1 \frac{\partial}{\partial x_1} + \Gamma_{ij}^2 \frac{\partial}{\partial x_2} + \dots + \Gamma_{ij}^n \frac{\partial}{\partial x_n}$$

for some smooth functions Γ_{ij}^k . These functions, which determine ∇ locally, are called *Christoffel coefficients* or *Christoffel symbols* of the connection with respect to the coordinates (x_1, x_2, \dots, x_n) . In a real-analytic manifold, ∇ is called *real-analytic* if, for any real-analytic local coordinates (x_1, x_2, \dots, x_n) , the corresponding Christoffel coefficients Γ_{ij}^k are real-analytic functions.

There are two tensors attached to a connection. The *curvature tensor*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

and the *torsion tensor* $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. A connection ∇ is said to be *torsion-free* or *symmetric* if the torsion tensor vanishes identically. *All the connections considered in this article are symmetric.* A connection is said to be *flat* if the curvature tensor vanishes. By a classical result due to É. Cartan, if a symmetric connection is flat then it is locally isomorphic to the *standard* connection (the one with vanishing Christoffel symbols) on \mathbf{R}^n [Sha97, Wol67].

2.1. Killing fields. Recall the following classical definition of an infinitesimal symmetry of a connection.

Definition 5. A (local) Killing field of a connection ∇ is a (local) vector field whose (local) flow preserves ∇ .

The terminology comes, by extension, from the Riemannian setting, even if here there is no invariant metric for a Killing field. The Killing vector fields of a connection form naturally a Lie algebra under the Lie bracket. For a connection ∇ , we will denote by $\mathfrak{K}(\nabla)$ the Lie algebra of its Killing vector fields. For example, we have the following result, whose proof we omit.

Lemma 6. *Let ∇ be a connection in the manifold M and $X \in \mathfrak{K}(\nabla)$. If Z_1 and Z_2 commute with X , so does $\nabla_{Z_1} Z_2$.*

Following [KN96, Ch. VI, Prop. 2.2], a (local) vector field X is a Killing field of the connection ∇ if for every pair of (local) vectors fields Y and Z we have

$$(2) \quad [X, \nabla_Y Z] - \nabla_Y [X, Z] = \nabla_{[X, Y]} Z.$$

In local coordinates, if

$$(3) \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = C \frac{\partial}{\partial x} + D \frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = E \frac{\partial}{\partial x} + F \frac{\partial}{\partial y},$$

the vector field $X = a\partial/\partial x + b\partial/\partial y$ will be a Killing vector field of ∇ if and only if its coefficients satisfy the following system of linear partial differential equations:

$$(4) \quad 0 = a_{xx} + Aa_x - Ba_y + 2Cb_x + A_x a + A_y b,$$

$$(5) \quad 0 = b_{xx} + 2Ba_x + (2D - A)b_x - Bb_y + B_x a + B_y b,$$

$$(6) \quad 0 = a_{xy} + (A - D)a_y + Eb_x + Cb_y + C_x a + C_y b,$$

$$(7) \quad 0 = b_{xy} + Da_x + Ba_y + (F - C)b_x + D_x a + D_y b,$$

$$(8) \quad 0 = a_{yy} - Ea_x + (2C - F)a_y + 2Eb_y + E_x a + E_y b,$$

$$(9) \quad 0 = b_{yy} + 2Da_y - Eb_x + Fb_y + F_x a + F_y b.$$

These correspond, in formula (2), to the couples $(Y, Z) \in \{(\partial/\partial x, \partial/\partial x), (\partial/\partial x, \partial/\partial y), (\partial/\partial y, \partial/\partial y)\}$.

The following Lemma is classical in the field (see, for instance, [Wol67, Thm. 1.6.20] or [Sza98, §3.1]).

Lemma 7 (Linearization in exponential coordinates). *In exponential coordinates around a given point m in M , every Killing vector field of ∇ vanishing at m is linear.*

The main idea in the proof of this linearization result is that geodesics are integral curves of a vector field in TM with quadratic homogeneous vertical part. In exponential coordinates at m , geodesics passing through m locally rectify on lines with constant speed in $T_m M$ passing through 0 and any local isometry preserving 0 will be linear. In exponential coordinates the Christoffel symbols satisfy $\Gamma_{ij}^k(0) = 0$.

Lemma 7 implies that a local Killing field of a connection is completely determined by its 1-jet at a given point. Consequently, in the neighborhood of any point of the manifold, the Lie algebra of Killing fields of a connection is finite-dimensional. This implies that connections are (particular) examples of *rigid geometric structures* in Gromov's sense [Gro88, DG91].

In the analytic setting, we will make use of an extendability result for local Killing fields proved first by Nomizu in the Riemannian setting [Nom60] and generalized later for rigid geometric structures (in particular, for analytic connections) by Amores and Gromov [Amo79, Gro88, DG91]. This phenomenon states roughly that a local Killing field of a *rigid analytic* geometric structure can be extended along any curve in the underlying manifold M . We then get a multivalued Killing field defined on all of M or, equivalently, a global Killing field defined on the universal cover.

In particular, for analytic connections, the Killing algebra $\mathfrak{K}(\nabla)$, defined as *the Lie algebra of all local Killing fields in the neighborhood of a given point*, does not depend upon the point (as long as M is connected). We will then call, without ambiguity, $\mathfrak{K}(\nabla)$, the *Killing algebra of ∇* .

Definition 8. The connection ∇ is said to be *locally homogeneous* (and $\mathfrak{K}(\nabla)$ is said to be *transitive*) on the open subset $U \subset M$, if for any $u \in U$ and for any tangent vector $V \in T_u U$, there exists a local Killing field X of ∇ such that $X(u) = V$.

Nomizu's extension phenomenon doesn't imply that the extension of a family of pointwise linearly independent Killing fields stays linearly independent. The rank of the Killing Lie algebra is not constant. In this setting, we may say that Theorem 1 classifies, in the case of surfaces, all the ways in which an analytic symmetric affine connection may degenerate and cease to be locally homogeneous. The following Lemma reduces Theorem 1 to the study of the case where there is a Killing algebra of dimension two:

Lemma 9. *Let ∇ be a real analytic affine connection in a neighborhood of the origin in \mathbf{R}^2 . Suppose that the Killing algebra $\mathfrak{K}(\nabla)$ has rank two in an open subset that does not contain the origin, but that contains it in its closure. Then $\mathfrak{K}(\nabla)$ contains a two-dimensional subalgebra which has rank two in some open subset that does not contain the origin, but that contains it in its closure.*

Proof. Let $\mathfrak{K}_0(\nabla) \subset \mathfrak{K}(\nabla)$ be the subalgebra consisting of those vector fields that vanish at 0 (it is either the full algebra or has codimension 1). By Lemma 7, elements of $\mathfrak{K}_0(\nabla)$ are linear in exponential coordinates at 0. Consequently, $\mathfrak{K}_0(\nabla)$ is embedded in the linear algebra $\mathfrak{gl}(2, \mathbf{R})$. Let d denote the dimension of $\mathfrak{K}_0(\nabla)$.

If $d = 1$, $\mathfrak{K}(\nabla)$ is the sought algebra.

If $d = 4$, $\mathfrak{K}_0(\nabla) = \mathfrak{gl}(2, \mathbf{R})$ and it contains, for example, the two-dimensional diagonal subalgebra (which is of rank two on an open set containing the origin in its closure).

If $d = 3$, consider the restriction to $\mathfrak{K}_0(\nabla)$ of the trace map $T : \mathfrak{gl}(2, \mathbf{R}) \rightarrow \mathbf{R}$. If the trace map is trivial then $\mathfrak{K}_0(\nabla) = \mathfrak{sl}(2, \mathbf{R})$, and $\mathfrak{K}_0(\nabla)$ contains the two-dimensional parabolic subalgebra which is transitive on an open set accumulating the origin. If the trace map is not trivial, $\dim(\ker(T)) = 2$. Here $\ker(T)$ is a two dimensional subalgebra of $\mathfrak{sl}(2, \mathbf{R})$: the rank is two in an open set accumulating the origin.

Suppose that $d = 2$, but that $\mathfrak{K}_0(\nabla) \neq \mathfrak{K}(\nabla)$ and that $\mathfrak{K}_0(\nabla)$ is of rank one in the neighborhood of the origin, for otherwise there is nothing to prove. Up to a linear change of coordinates, $\mathfrak{K}_0(\nabla)$ is generated by $X = x\partial/\partial x$ and $Y = y\partial/\partial x$ (the evaluation of any Killing vector field at the origin is tangent to the foliation \mathcal{F} generated by dy). We have $[X, Y] = Y$. Let $Z \in \mathfrak{K}(\nabla) \setminus \mathfrak{K}_0(\nabla)$ be an element transverse to the foliation \mathcal{F} in an open set containing the origin in its closure. Suppose that $[Z, X] = aX + bY + cZ$ and that $[Z, Y] = dX + eY + fZ$, for some $a, b, c, d, e, f \in \mathbf{R}$. The Jacobi relation applied to X, Y and Z reads $(d + af - cd)X + (fb - ce - a)Y + fZ = 0$. We must conclude that $f = 0$ and that $a = -ce$. The equation reduces to $d(1 - c) = 0$. If $d = 0$, then $[Z, Y] = dY$ and the Lie algebra generated by Z and Y has the required properties. If $c = 1$, then $W = -2eX + bY + 2Z$ satisfies $[X, W] = -W$ and the Lie algebra generated by X and W has the required properties. \square

2.2. Locally homogeneous connections and (G, X) -structures. We finish this section by stating some general results, well known by specialists, showing, in particular, that a locally homogeneous affine connection on a manifold gives rise to a (G, X) -geometry in Thurston's sense [Thu97]. The results will be stated and proved for locally homogeneous affine connections but hold, more generally, for locally homogeneous rigid geometric structures in Gromov's sense [DG91, Gro88].

Let ∇ be an affine connection in a neighborhood of 0 in \mathbf{R}^n and suppose that $\mathfrak{K}(\nabla)$ is transitive. Let G be the simply connected Lie group with Lie algebra $\mathfrak{K}(\nabla)$ and $G_0 \subset G$ the subgroup corresponding to the subalgebra of the Killing vector fields vanishing at 0. If G_0 is closed, there is a natural identification of a neighborhood of G_0/G_0 in G/G_0 with a neighborhood of the origin in \mathbf{R}^n . When pulling back ∇ to G/G_0 , via this identification, we find a left invariant connection ∇_0 in G/G_0 which is locally isomorphic to ∇ .

In our setting, Lemma 9 implies that a connection satisfying the hypothesis of the Lemma is, on a non-trivial open set, locally isomorphic, in the previous sense, to a connection which is translation-invariant on a two-dimensional Lie group. Since a two-dimensional Lie algebra is either Abelian or isomorphic to the Lie algebra of the affine group of the real line, we will have to deal with translation-invariant connections either on \mathbf{R}^2 , or on the affine group. These will be studied, respectively, in sections 3 and 4.

The following Proposition gives a proof of Nomizu's extension of local automorphisms in the homogeneous context.

Proposition 10. *Let G be a connected and simply connected Lie group G acting transitively on a simply connected manifold M and preserving an affine connection ∇ . Suppose that, at each point $p \in M$, the Lie algebra of Killing vector fields of ∇ is the Lie algebra of fundamental vector fields of the G -action. Let $U \subset M$ be a connected open set and $f : U \rightarrow M$ a local automorphism of ∇ . There exists a global automorphism $\widehat{f} : M \rightarrow M$ such that $\widehat{f}|_U \equiv f$. Moreover, G has finite index in the group of global automorphisms of ∇ .*

Proof. Let $\Phi : G \times M \rightarrow M$ be the transitive action of G on M . Let \mathfrak{g} be the Lie algebra of right-invariant vector fields in G and let $\Phi_* : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ the induced Lie algebra morphism. Since f is a automorphism, it maps Killing vector fields of ∇ into Killing vector fields of ∇ and thus induces an automorphism $f_* : \mathfrak{g} \rightarrow \mathfrak{g}$. In its turn, this induces an automorphism $f_*^\sharp : G \rightarrow G$. Let

$$F(g, p) = \Phi(f_*^\sharp(g), f(\Phi(g^{-1}, p))),$$

wherever it is defined (in a subset of $G \times M$ and taking values in M). For g fixed, $F(g, p)$ is a automorphism. Notice that, for $p \in U$, $F(e, p) = f(p)$. We will show that $F(g, p)$ is in fact independent of g . Since for every $p \in M$ there exists $g \in G$ such that (g, p) is in the domain of definition of F , we may define $\widetilde{f}(p)$ as $F(g, p)$. For $g_0 \in G$ and $v \in T_{g_0}G$, identified to the element of \mathfrak{g} whose value at g_0 is v , we have

$$\begin{aligned} \left. \frac{\partial F}{\partial g} \right|_{g_0} (v) &= \Phi_*(f_*v) + \frac{\partial \Phi}{\partial p} \circ Df \circ \Phi_*(\text{Ad}_{g_0^{-1}}(-v)) \\ &= \Phi_*(f_*v) - \frac{\partial \Phi}{\partial p} \circ \Phi_*(f_*\text{Ad}_{g_0^{-1}}(v)) \\ &= \Phi_*(f_*v) - \Phi_*(\text{Ad}_{f_*^\sharp(g_0)}(f_*\text{Ad}_{g_0^{-1}}(v))) \\ &= \Phi_*(f_*v) - \Phi_*(f_*\text{Ad}_{g_0}f_*^{-1}(f_*\text{Ad}_{g_0^{-1}}(v))) = 0. \end{aligned}$$

and F is thus independent of g . Hence, a local automorphism is in fact the restriction of a global one. Hence, the automorphism pseudogroup of ∇ is the pseudogroup associated to a group action. If $f : M \rightarrow M$ is a global isometry, there is an element $g \in G$ such that $g \circ f$ fixes a point $p \in M$. But the group of germs of automorphisms fixing a point contains the group generated by the elements of the Killing algebra that vanish at the point with finite index (see [DG91] 3.5, or [Gro88] 3.4.A). This implies that $[\text{Isom}(\nabla) : G] < \infty$. \square

Finally, the following proposition establishes the link between (G, X) -structures and locally homogeneous connections.

Proposition 11. *Let M be a manifold endowed with a locally homogeneous affine connection ∇ . Let $p \in M$, let \mathfrak{g} be the Lie algebra of Killing vector fields of ∇ in a neighborhood of p and let $\mathfrak{g}_0 \subset \mathfrak{g}$ the subalgebra of those Killing vector fields vanishing at p . Let G be the connected and simply connected Lie group corresponding to \mathfrak{g} and $G_0 \subset G$ the subgroup corresponding to \mathfrak{g}_0 . If G_0 is closed, then there exists a finite covering $\overline{\pi} : \overline{M} \rightarrow M$, a G -invariant connection ∇_0 on G/G_0 and a $(G, G/G_0)$ -structure on \overline{M} , such that $\overline{\pi}^*\nabla$ is locally isomorphic to ∇_0 .*

Proof. Let $X = G/G_0$ and $x_0 = G_0/G_0$. There is a neighborhood $V \subset M$ and a diffeomorphism $\psi : (V, p) \rightarrow (X, x_0)$ mapping the Killing vector fields of ∇ to the vector fields induced by G on X . Let ∇_0^V be the connection in $\psi(V)$ given by $\psi(\nabla)$. By construction, the vector fields in X inducing the action of G are Killing vector fields of ∇_0^V . Propagate ∇_0^V to all of X via the action of G . This produces a globally-defined G -invariant affine connection ∇_0 in X , which is locally isomorphic to ∇ .

Let $q \in M$. Because the Killing algebra of ∇ has full rank, by composing finitely many local automorphisms of those generated by integrating the Killing algebra, we may find a local automorphism $f : (U, q) \rightarrow$

(M, p) . Hence, for $h = \psi \circ f$, $h : U \rightarrow X$ maps ∇ to ∇_0 . We thus have an atlas $\{(U_i, h_i); i \in I\}$ of M taking values in X whose changes of coordinates lie within $\text{Isom}(\nabla_0)$. We will now proceed in a manner akin to the construction of the orientable double cover of a manifold. Let $\text{Isom}(\nabla_0)/G = \{[k_1], \dots, [k_n]\}$. For $j \in \{1, \dots, n\}$, let $V_i^j = U_i$. The manifold $W = \sqcup V_i^j$ is naturally endowed with a canonical projection $\pi : W \rightarrow M$ and charts $k_j \circ f_i : V_i^j \rightarrow X$. Within W , identify $x \in V_i^j$ with $x' \in V_{i'}^{j'}$, if $V_i^j \cap V_{i'}^{j'} \neq \emptyset$ and if $(k_j \circ h_i) \circ (k_{j'} \circ h_{i'})^{-1}$ is the restriction of an element of G . After the identification, we have a manifold \overline{M} and an n -fold covering $\overline{\pi} : \overline{M} \rightarrow M$ that is canonically endowed with an atlas taking values in X whose changes of coordinates are in G . (If \overline{M} is not connected we may replace it by one of its connected components). \square

Remark 12. In the past proof, if M is oriented and X is orientable, X may be oriented by ψ . In this case, by construction, the charts $h_i : U_i \rightarrow X$ preserve the orientation and we may replace, in the above construction, $\text{Isom}(\nabla_0)$, by its subgroup of orientation-preserving automorphisms $\text{Isom}^+(\nabla_0)$. In particular, if $G = \text{Isom}^+(\nabla_0)$, the finite covering of Proposition 11 may be chosen to be trivial.

Remark 13. The (G, X) -structure associated to M in Proposition 11 is not canonically associated to ∇ . The subgroup S_{G_0} of automorphisms of G that preserve G_0 acts naturally upon the G -invariant connections on G/G_0 and the connection ∇_0 of Proposition 11 may be replaced by any one along the same orbit of S_{G_0} , producing a different (G, X) -structure.

3. QUASIHOMOGENEOUS CONNECTIONS, COMMUTATIVE KILLING SUBALGEBRA

The aim of this section is to prove Theorem 1 in the case where the Killing algebra of dimension two constructed in Lemma 9 is commutative. We will prove:

Proposition 14. *Let ∇ be a real-analytic torsion-free affine connection defined in a neighborhood of the origin in \mathbf{R}^2 and suppose that ∇ admits Killing vector fields X and Y which commute and that are linearly independent in an open set having the origin in its closure. Then ∇ is locally homogeneous in the neighborhood of the origin.*

Let ∇ be a connection defined in a neighborhood of the origin of \mathbf{R}^2 with Christoffel symbols $\{\Gamma_{ij}^k\}$ and let X and Y be commuting Killing fields as in the statement of Proposition 14. By Lemma 6 and the fact that X and Y generate the Lie algebra of vector fields that commute with X and Y ,

$$(10) \quad \nabla_X X = \alpha X + \beta Y, \quad \nabla_X Y = \gamma X + \delta Y, \quad \nabla_Y Y = \varepsilon X + \phi Y.$$

for some constants $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$.

3.1. One-dimensional orbits. We will suppose, without loss of generality, that $X(0) = 0$ and that it is, following Lemma 7, linear. Since X and Y commute, X must be a constant multiple of Y along its common orbit and, since X vanishes at the origin, it must have one vanishing eigenvalue (and thus both of its eigenvalues are real). There are two cases to be considered, according to the diagonalizability of X .

Diagonalizable. We may assume that $X = x\partial/\partial x$. It vanishes along $x = 0$ and thus this curve is invariant by Y . If $Y = xf\partial/\partial x + g\partial/\partial y$ commutes with X , then $f_x = 0$ and $g_x = 0$. Hence, $f = f(y)$ and $g = g(y)$ with $g(0) \neq 0$. We may suppose, up to a change of variables in y , that $g \equiv 1$. In the coordinates $(e^{-\int_0^y h(s) ds} x, y)$ the vector fields are

$$X = x \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}.$$

On the other hand, from the expressions of X and Y in the coordinates (x, y) , we can compute $\nabla_X X, \nabla_X Y$ and $\nabla_Y Y$ in terms of the Christoffel symbols Γ_{ij}^k . Solving for the Christoffel symbols, we have

$$\Gamma_{11}^1 = \frac{\alpha - 1}{x}, \Gamma_{11}^2 = \frac{\beta}{x^2}, \Gamma_{12}^1 = \gamma, \Gamma_{12}^2 = \frac{\delta}{x}, \Gamma_{22}^1 = \varepsilon x, \Gamma_{22}^2 = \phi.$$

Since the connection is analytic at the origin, $\alpha = 1, \beta = 0, \delta = 0$ and the connection is

$$(11) \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \gamma \frac{\partial}{\partial x}, \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \varepsilon x \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y}.$$

We will prove that such a connection is locally homogeneous at the origin. A vector field $h(y)\partial/\partial x$ is a Killing vector field of (11) if and only if it satisfies the system of equations (4–9), which reduce to the single ordinary differential equation $h'' + (2\gamma - \phi)h' + \varepsilon h = 0$. If $h_0(y)$ is a solution to this equation that does not vanish at the origin, the Killing vector fields $\partial/\partial y$ and $h_0(y)\partial/\partial x$ are linearly independent at the origin.

Non-diagonalizable case. We have $X = x\partial/\partial y$. If $Y = f\partial/\partial x + g\partial/\partial y$, with f and g analytic functions, then, from $[X, Y] = 0, f_y = 0$ and $xg_y - f = 0$. Hence, there exist analytic functions $h(x), k(x)$ such that $f = xh, g = yh + k$ with $k(0) \neq 0$. This leads to the following expression:

$$X = x \frac{\partial}{\partial y}, Y = xh(x) \frac{\partial}{\partial x} + (yh(x) + k(x)) \frac{\partial}{\partial y}.$$

As for the equations defining the Christoffel symbols in the coordinates (x, y) , we have

$$\nabla_X X = x^2 \left(\Gamma_{22}^1 \frac{\partial}{\partial x} + \Gamma_{22}^2 \frac{\partial}{\partial y} \right) = \alpha x \frac{\partial}{\partial y} + \beta \left[xh \frac{\partial}{\partial x} + (yh + k) \frac{\partial}{\partial y} \right].$$

Evaluating at 0, the equation for $\partial/\partial y$ reads $\beta k(0) = 0$ and thus $\beta = 0$. This implies that $\alpha = 0, \Gamma_{22}^1 \equiv 0$ and $\Gamma_{22}^2 \equiv 0$. We also have

$$\nabla_X Y = x^2 h \left(\Gamma_{12}^1 \frac{\partial}{\partial x} + \Gamma_{12}^2 \frac{\partial}{\partial y} \right) + xh \frac{\partial}{\partial y} = \gamma x \frac{\partial}{\partial y} + \delta \left[xh \frac{\partial}{\partial x} + (yh + k) \frac{\partial}{\partial y} \right].$$

Again, evaluating at 0 in the equation for $\partial/\partial y$ gives $\delta = 0$ and thus $\Gamma_{12}^1 \equiv 0$. We obtain

$$(12) \quad \Gamma_{12}^2 = (\gamma h^{-1} - 1)x^{-1}.$$

Since Γ_{12}^2 must be analytic at the origin, $h(0) \neq 0$. Up to multiplying Y by a constant we will suppose that $h(0) = 1$.

We show now that we can find new local coordinates in the neighborhood of the origin, with respect to which X remains as before, but where we can suppose that $h \equiv 1$ in the expression for Y . For this, let ξ be an analytic function in x , with $\xi(0) \neq 0$ and such that, in the coordinate $\bar{x} = x\xi(x)$ the vector field $xh(x)\partial/\partial x$ reads $\bar{x}\partial/\partial\bar{x}$. Then, in the coordinates $(\bar{x}, \bar{y}) = (x\xi(x), y\xi(x))$, we have

$$X = \bar{x} \frac{\partial}{\partial \bar{y}}, Y = \bar{x} \frac{\partial}{\partial \bar{x}} + (\bar{y} + k(\bar{x})) \frac{\partial}{\partial \bar{y}}.$$

Let us keep the notation (x, y) for these new coordinates. We thus have, from (12), $\gamma = 1$ and $\Gamma_{12}^2 \equiv 0$. For the last coefficients of the connection we have

$$\nabla_Y Y - Y = x^2 \left(\Gamma_{11}^1 \frac{\partial}{\partial x} + \Gamma_{11}^2 \frac{\partial}{\partial y} \right) + xk' \frac{\partial}{\partial y} = \varepsilon x \frac{\partial}{\partial y} + (\phi - 1) \left[x \frac{\partial}{\partial x} + (y + k) \frac{\partial}{\partial y} \right].$$

Evaluating at the origin, we must have $\phi = 1$ and hence $\Gamma_{11}^1 \equiv 0$. All the Christoffel symbols vanish except for $\Gamma_{11}^2 = (\varepsilon - k')x^{-1}$. Since $\partial\Gamma_{11}^2/\partial y = 0$, the curvature of ∇ vanishes and ∇ is flat and, in particular, locally homogeneous at the origin.

3.2. Fixed points. We now assume that the Lie algebra generated by X and Y is of rank 0 at the origin. By Lemma 7, there are coordinates (x, y) where both vector fields are linear. If Z and W are linear vector fields,

$$\nabla_Z W = L + \sum Q_{ij}^k \Gamma_{ij}^k,$$

for some linear vector field L and some quadratic and homogeneous functions Q_{ij}^k . In this way, equations (10) become a system of the form

$$\sum_{i,j,k} Q_{ij}^k \Gamma_{ij}^k = L_l$$

for $l = 1, \dots, 3$, corresponding to the pairs $(Z, W) \in \{(X, Y), (X, X), (Y, Y)\}$. But since Γ_{ij}^k is analytic at the origin and the left-hand side of each of these equations has a trivial linear part at the origin, we must have $L_l = 0$. The system has a unique solution since the connection is determined by (10) and thus $\Gamma_{ij}^k \equiv 0$. The connection is flat and thus locally homogeneous. This finishes the proof of Proposition 14.

Remark that the above argument works also in higher dimensions. It shows that if ∇ is an analytic connection in a neighborhood of the origin of \mathbf{R}^n and X_1, \dots, X_n are commuting Killing vector fields of ∇ vanishing at the origin, but of rank n on some nontrivial open set, then ∇ is flat.

4. INVARIANT CONNECTIONS ON THE AFFINE GROUP

The group of affine transformations of the real line or, simply, the affine group is the group

$$\text{Aff}(\mathbf{R}) = \{x \mapsto ux + v, u, v \in \mathbf{R}, u \neq 0\},$$

with the product

$$(u_1, v_1) \cdot (u_2, v_2) = (u_1 u_2, u_1 v_2 + v_1)$$

and Lie algebra $\text{aff}(\mathbf{R})$. We will denote by $\text{Aff}_0(\mathbf{R})$ the connected component of the identity (the couples (u, v) with $u > 0$). In $\text{Aff}_0(\mathbf{R})$, the Lie algebra of *right-invariant* vector fields is generated by

$$(13) \quad A_0 = -\left(u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}\right), \quad B_0 = \frac{\partial}{\partial v},$$

which satisfy $[A_0, B_0] = B_0$. The Lie algebra of *left-invariant* vector fields is generated by

$$X_0 = u \frac{\partial}{\partial u}, \quad Y_0 = -u \frac{\partial}{\partial v},$$

which satisfy $[X_0, Y_0] = Y_0$. The Lie algebra of left-invariant vector fields may be locally characterized as that of the vector fields which simultaneously commute with A_0 and B_0 . A left-invariant vector field will be called *semisimple* if it is conjugate to X_0 and *unipotent* if it is conjugate to Y_0 . The group of automorphisms of $\text{Aff}_0(\mathbf{R})$ is the restriction to $\text{Aff}_0(\mathbf{R})$ of the group of inner automorphisms of $\text{Aff}(\mathbf{R})$. Via the action by inner automorphisms (adjoint action) we have

$$(14) \quad (u, v)_* \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} B_0 \\ A_0 \end{pmatrix}, \quad (u, v)_* \begin{pmatrix} Y_0 \\ X_0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} Y_0 \\ X_0 \end{pmatrix}.$$

In particular, the orientation-preserving automorphisms of $\text{Aff}_0(\mathbf{R})$ are inner ones.

If a connection ∇ in $\text{Aff}_0(\mathbf{R})$ is invariant by left-translations, its Killing algebra $\mathfrak{K}(\nabla)$ contains the Lie algebra generated by A_0 and B_0 . Since A_0 and B_0 commute with X_0 and Y_0 , ∇ must be constant with respect to these left-invariant vector fields in the sense that there must exist constants $\alpha, \beta, \gamma, \delta, \varepsilon$ and ϕ such that

$$(15) \quad \nabla_{X_0} X_0 = \alpha X_0 + \beta Y_0, \quad \nabla_{X_0} Y_0 = \gamma X_0 + \delta Y_0, \quad \nabla_{Y_0} Y_0 = \varepsilon X_0 + \phi Y_0.$$

The group of automorphisms of $\text{Aff}_0(\mathbf{R})$ acts upon the left-invariant connections (see formula (14) for the action on left-invariant vector fields): when replacing (X_0, Y_0) by $(X, Y) = (X_0 + \lambda Y_0, \mu Y_0)$,

$$(16) \quad \begin{aligned} \nabla_X X &= (\alpha + 2\lambda\gamma + \varepsilon\lambda^2)X + \mu^{-1}(\beta + [2\delta - \alpha - 1]\lambda + [\phi - 2\gamma]\lambda^2 - \lambda^3\varepsilon)Y, \\ \nabla_X Y &= \mu(\gamma + \lambda\varepsilon)X + (\delta + [\phi - \gamma]\lambda - \varepsilon\lambda^2)Y, \\ \nabla_Y Y &= \mu^2\varepsilon X + \mu(\phi - \lambda\varepsilon)Y. \end{aligned}$$

Definition 15. A *marked* connection on $\text{Aff}_0(\mathbf{R})$ is a couple (∇, Z) of a left-invariant connection ∇ and a left-invariant semisimple vector field Z such that $\nabla_Z Z \wedge Z = 0$. We say that Z is a *marking* of ∇ . In a marked connection (∇, Z) , $\alpha(\nabla, Z)$ is the unique real such that $\nabla_Z Z = \alpha(\nabla, Z)Z$ and $\delta(\nabla, Z)$ is the unique real such that $\nabla_Z Y_0 = \gamma Z + \delta(\nabla, Z)Y_0$. A marked connection (∇, Z) is said to be

- of Type $I_0(n)$, if $\alpha(\nabla, Z) = -1/n$ and $\delta(\nabla, Z) = 1$;
- of Type $II_0(n)$, if $\alpha(\nabla, Z) = -1/n$ and $\delta(\nabla, Z) = 1 + \alpha(\nabla, Z)$,

for some $n \in \mathbf{R}^*$.

A marked connection is said to be *special*, if it is either of Type $I_0(n)$, or of Type $II_0(n)$, with $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$.

The interest of this Definition comes from the fact that quasihomogeneous analytic connections on compact surfaces will be locally modeled, in the locus of local homogeneity, by special connections of Type $I_0(n)$, or of Type $II_0(n)$, with respect to some marking. Formula (16) guarantees that, in a marked connection (∇, Z) , the quantities $\alpha(\nabla, Z)$ and $\delta(\nabla, Z)$ are well-defined. From the same formula, it follows that a connection has, if finitely many, at most three markings.

Remark 16. Let ∇ be a connection of the form (15) with $\beta = 0$, this is, (∇, X) is a marked connection. We have $\alpha(\nabla, X) = \alpha$ and $\delta(\nabla, X) = \delta$. For $\lambda \neq 0$, $(\nabla, X + \lambda Y)$ is a marked connection with $\alpha(\nabla, X + \lambda Y) = \alpha'$ and $\delta(\nabla, X + \lambda Y) = \delta'$ if and only if

$$(17) \quad (\lambda\gamma, \lambda^2\varepsilon, \lambda\phi) = (\delta + \delta' - 1 - \alpha, \alpha + \alpha' - 2\delta - 2\delta' + 2, 1 - 2\delta + \alpha').$$

In particular, the connection is determined by the values of α, α', δ and δ' up to the natural equivalence (16). For the third marking $X + \lambda''Y$ we have

$$(18) \quad \alpha'' = \frac{\alpha\alpha' - 1 - 4\delta\delta' + 2\delta + 2\delta'}{2 - 2\delta - 2\delta' + \alpha + \alpha'}, \quad \delta'' = \frac{1 - \delta\alpha' - \alpha\delta' + \alpha\alpha' + \alpha + \alpha' - \delta - \delta'}{2 - 2\delta - 2\delta' + \alpha + \alpha'}.$$

A left-invariant connection on the affine group might admit some extra symmetries: the full isometry group might be bigger than the one generated by Killing algebra. In the cases that will interest us, we have:

Proposition 17. *Let ∇_0 be a left-invariant connection in $\text{Aff}_0(\mathbf{R})$ such that $\mathfrak{K}(\nabla_0)$ is the Lie algebra of right-invariant vector fields.*

- (1) *If ∇'_0 is a left-invariant connection in $\text{Aff}_0(\mathbf{R})$, then any germ of isometry between ∇_0 and ∇'_0 , fixing the identity, is given by an inner automorphism of $\text{Aff}(\mathbf{R})$.*
- (2) *Any germ of orientation-preserving self-isometry of ∇_0 is the germ of a left translation (equivalently, a germ of orientation-preserving isometry of ∇_0 fixing a point is the identity).*

Proof. (1) Since the two connections are isometric, the dimension of the killing algebra is the same and thus $\mathfrak{K}(\nabla'_0)$ is the Lie algebra of right-invariant vector fields. The isometry must map $\mathfrak{K}(\nabla_0)$ into $\mathfrak{K}(\nabla'_0)$, this is, must preserve the Lie algebra of right-invariant vector fields at e . Every automorphism of $\text{aff}(\mathbf{R})$ comes from the adjoint action of $\text{Aff}(\mathbf{R})$. Hence the germ of ϕ at the identity is the germ of an inner automorphism of $\text{Aff}(\mathbf{R})$. (2) Consider now the case where $\nabla'_0 = \nabla_0$ and suppose that the inner automorphism is the one associated to $(X_0, Y_0) \mapsto (X_0 + \lambda Y_0, \mu Y_0)$. The associated constants are related according to (16): if a connection of the form (15) is preserved by such an automorphism then the connection (15) must equal the connection (16) when the frames (X_0, Y_0) and (X, Y) are identified. The corresponding equations are easy to solve for they are linear in $\alpha, \beta, \gamma, \delta, \varepsilon$ and ϕ .

If $\mu \neq -1$, a straightforward computation shows that we must have $(\gamma, \varepsilon, \phi) = (0, 0, 0)$. In this case, $v\partial/\partial v$ is a Killing field (which is not right-invariant). If $\mu = -1$, then any potential isometry has order two and is orientation-reversing. \square

4.1. Killing algebras of connections of Types I and II. The Killing Lie algebra of a left-invariant connection may be strictly bigger than the Lie algebra of right-invariant vector fields. We will now deal with this question and compute, for the connections of special type of Definition 15, the full Killing algebra.

In the coordinates $(x, y) = (\log(u), v)$ of $\text{Aff}_0(\mathbf{R})$, we have $X = \partial/\partial x, Y = -e^x\partial/\partial y$. The connection is thus of the form (3) for

$$(19) \quad A = \alpha, B = -\beta e^x, C = -\gamma e^{-x}, D = \delta - 1, E = \varepsilon e^{-2x}, F = -\phi e^{-x}.$$

Equation (2) implies that $a(x, y)\partial/\partial x + b(x, y)\partial/\partial y$ is a Killing vector field of the connection if its coefficients satisfy the system (4–9) with the above values of the corresponding functions.

A two-dimensional space of solutions of these equations is given by the vector fields coming from (13) since, by construction, our connections are left-invariant and hence right-invariant vector fields (generating left translations) are Killing fields. This space is generated by $(a, b) = (1, y)$ and $(a, b) = (0, 1)$. We will find all the solutions to these equations in two particular cases, related to special connections of Type I_0 and II_0 of Definition 15.

Proposition 18. *For $\alpha = -1/n, n \in \frac{1}{2}\mathbf{N}^*, \beta = 0$ and parameters ε, γ and ϕ that do not vanish simultaneously, for the system (4–9) with (19) we have:*

- (1) *If $\delta = 1$, the space of solutions has dimension 2 except in the case when $\alpha = -1, \varepsilon = -\gamma^2$ and $\phi = -\gamma$.*
- (2) *If $\delta = 1 + \alpha$ and $\alpha \neq -1$, the space of solutions has dimension 2 except in the case when $\alpha = -1/2$ and $\phi = 2\gamma$.*

In both exceptional cases the space of solutions has dimension 3 and is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$.

Proof. Let us begin by the first case, $\delta = 1$. Equation (5) yields $b = f_1(y) + f_2(y)e^{\alpha x}$ for some functions f_1 and f_2 . Equation (7) becomes $e^x f'_2 = (\phi - \gamma)f_2$. Since f_2 is a function of y , both sides of the equation must vanish. Now we consider two cases: $\phi \neq \gamma$ and $\phi = \gamma$.

- (1) if $\phi \neq \gamma$, then $f_2 \equiv 0$. From (4), $a = f_3(y) + f_4(y)e^{-\alpha x}$. Equation (6) minus equation (9) becomes

$$[\alpha f'_3 - f''_1]e^{x(1+\alpha)} + [(\gamma - \phi)(f_3 - f'_1)]e^{\alpha x} + [(\gamma - \phi)f_4] = 0.$$

The expressions in brackets are functions of y and thus, if there are no linear relations between 1, $e^{x(1+\alpha)}$ and $e^{\alpha x}$ (this is, if $\alpha \neq -1$), the expressions in brackets must vanish. In particular, we must have $f_3 = f'_1$.

If $\alpha \neq -1, f_4 \equiv 0$ and the equation becomes $f''_1 \equiv 0$. The space of solutions is two-dimensional.

If $\alpha = -1$, $f_4 = 2f_1''(\gamma - \phi)^{-1}$. Equation (9) is now $(\gamma + \phi)f_1'' = 0$. If $\phi + \gamma \neq 0$, then $f_1'' = 0$ and $f_4 = 0$. As before, the space of solutions is two-dimensional. Otherwise, $\phi = -\gamma$. In this last case we are left with equation (8):

$$e^{2x}f_1^{(iv)} - 2e^x\gamma f_1''' - 3(\varepsilon + \gamma^2)f_1'' = 0.$$

We must then have $\varepsilon = -\gamma^2$, $f_1''' = 0$. The space of solutions has dimension 3. An extra solution is given by $(a, b) = (y + \gamma^{-1}e^x, \frac{1}{2}y^2)$ and the corresponding Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$.

(2) if $\phi = \gamma$, then f_2 is constant. Either

(a) $\gamma = 0$. Equation (4) becomes $a_{xx} + \alpha a_x = 0$ and thus $a = f_3(y) + f_4(y)e^{-\alpha x}$. Equation (9) reads $f_1'' = \alpha\varepsilon e^{(\alpha-2)x}f_2$ and equation (6) is $f_3' = -\varepsilon e^{(\alpha-2)x}f_2$. Since $\varepsilon \neq 0$ and $\alpha(\alpha-2) \neq 0$, $f_2 \equiv 0$, $f_3' = 0$ and $f_1'' = 0$ and we are left with equation (8):

$$e^{2x}f_4'' + \varepsilon(\alpha-2)f_4 + 2\varepsilon e^{\alpha x}(f_1' - f_3) = 0;$$

we must have $f_4 \equiv 0$ and $f_3 = f_1'$. The space of solutions is two-dimensional.

(b) $\gamma \neq 0$. We can solve for $a(x, y)$ in equation (9) and obtain

$$a(x, y) = f_1' - \gamma^{-1}e^x f_1'' + \alpha\varepsilon\gamma^{-1}e^{(\alpha-1)x}f_2.$$

Equation (6) is

$$e^{3x}(\alpha+1)f_1''' - \gamma(\alpha-1)e^{2x}f_1'' - 2\gamma\varepsilon\alpha e^{\alpha x}f_2.$$

and thus $f_1'' = 0$. The equation becomes $\varepsilon f_2 = 0$. If $\varepsilon \neq 0$ then $f_2 = 0$ and the space of solutions is two-dimensional. If $\varepsilon = 0$ the first equation reduces also to $f_2 = 0$.

This proves the Proposition in the case $\delta = 1$. Let us assume, from now on, that $\delta = 1 + \alpha$. From equation (5), $b = f_1(y) + f_2(y)e^{-\alpha x}$. Solving for a in (7) we get

$$a(x, y) = \frac{\phi - \gamma}{(\alpha + 1)e^{(\alpha+1)x}}f_2 - \frac{1}{\alpha e^{\alpha x}}f_2' + f_3(y).$$

Equation (4) is now $(\phi + 2\gamma\alpha - \gamma)f_2 = 0$; equations (6), (8) and (9) are

$$f_2'' + [c_1 f_2']e^{-x} + \left[\frac{\gamma\phi - \varepsilon\alpha - \varepsilon\alpha^2 - \gamma^2}{\alpha + 1} f_2 \right] e^{-2x} + [\gamma(f_3 - f_1')] e^{(\alpha-1)x} = 0,$$

$$[c_2 f_2''']e^x + c_3 f_2'' + [c_4 f_2']e^{-x} + \left[\frac{\varepsilon(\alpha-1)(\phi-\gamma)}{1+\alpha} f_2 \right] e^{-2x} + f_3'' e^{(\alpha+1)x} + [c_5 f_3']e^{\alpha x} - [2\varepsilon(f_3 - f_1')]e^{(\alpha-1)x} = 0,$$

$$f_2'' + [c_6 f_2']e^{-x} - \left[\frac{\varepsilon\alpha - \phi\gamma + \varepsilon\alpha^2 + \phi^2}{\alpha + 1} f_2 \right] e^{-2x} - [\phi(f_3 - f_1')]e^{(\alpha-1)x} - [2\alpha f_3' + f_1'']e^{\alpha x} = 0,$$

for some constants c_i . In this equation, each summand is a function of y multiplying some power of e^x . The conditions imposed on α guarantee that, in the last three equations, the power of e^x that appears as coefficient for $f_3 - f_1'$ is different from the powers appearing in the other summands. Since ε , γ and ϕ do not vanish simultaneously, $f_3 = f_1'$. By the same argument, the coefficients of f_2 in these four equations do not all vanish and thus $f_2 \equiv 0$. The system reduces to the equations

$$e^x f_1''' = (2\gamma - \phi)f_1'', \quad (1 + 2\alpha)f_1'' = 0.$$

Either $f_1'' = 0$ and the space of solutions has dimension 2, or $\alpha = -1/2$, $\phi = 2\gamma$ and $f_1''' = 0$: the space of solutions has dimension three and an extra solution is given by $(a, b) = (2y, y^2)$: the Killing algebra is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. \square

5. QUASIHOMOGENEOUS CONNECTIONS, AFFINE KILLING SUBALGEBRA

The aim of this section is to prove Theorem 1 in the case where the Killing algebra of dimension two constructed in Lemma 9 is isomorphic to $\text{aff}(\mathbf{R})$. Following the terminology of Theorem 1, we will prove:

Proposition 19. *Let ∇ be a symmetric analytic connection in a neighborhood of the origin of \mathbf{R}^2 and let A and B be Killing fields of ∇ such that $[A, B] = B$ and such that A and B are linearly independent in an open set that accumulates to the origin, but that does not contain the origin.*

- *If $\mathfrak{K}(\nabla)$ is the one generated by A and B then:*
 - (1) *If A and B vanish at the origin, then ∇ is locally isomorphic to a connection of Type II^0 .*
 - (2) *If the Lie algebra generated by A and B is of rank 1 at the origin and the isotropy algebra at the origin is generated by a semisimple element, then ∇ is locally isomorphic to a connection of Type I.*
 - (3) *If the Lie algebra generated by A and B is of rank 1 at the origin and the isotropy algebra at the origin is generated by a unipotent element, then ∇ is locally isomorphic to a connection of Type II^1 .*
- *If $\mathfrak{K}(\nabla)$ contains properly the Lie algebra generated by A and B , then either ∇ is flat, or ∇ is locally isomorphic to a connection of Type III. In this last case, $\mathfrak{K}(\nabla) \approx \mathfrak{sl}(2, \mathbf{R})$.*

In subsection 5.1 we prove part (1) of Proposition 19. Parts (2) and (3) will be respectively proved in sections 5.2.1 and 5.2.2.

5.1. Fixed points. In this subsection A and B vanish simultaneously at 0 (which is then a fixed point for the local action of the Lie algebra generated by A and B). By Lemma 7, A and B are both linear in exponential coordinates. Since the Lie algebra generated by A and B is solvable, they are simultaneously upper-triangular. Since B is a nonzero commutator, we may suppose that $B = x\partial/\partial y$. Up to replacing A by $A + \lambda B$, we have, for some $n \in \mathbf{R}^*$,

$$A = \frac{1}{n}x \frac{\partial}{\partial x} + \left(\frac{1}{n} - 1\right)y \frac{\partial}{\partial y}, \quad B = x \frac{\partial}{\partial y}.$$

These fields are linearly independent in the complement of $\{x = 0\}$. The vector fields that, in the half plane $\{x > 0\}$, commute with these two are linear combinations of

$$X = -\frac{1}{n} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad Y = -x^{1-n} \frac{\partial}{\partial y}.$$

They satisfy the relation $[X, Y] = Y$. By Lemma 6, the connection is of the form (15).

Solving for the Christoffel symbols, we have

$$\begin{aligned} \Gamma_{11}^1 &= -\frac{1}{n}\varepsilon x^{2n-3}y^2 + 2\gamma x^{n-2}y - (n\alpha + 1)x^{-1}, \\ \Gamma_{11}^2 &= -\frac{1}{n}\varepsilon x^{2n-4}y^3 + (2\gamma - \phi)x^{n-3}y^2 + (1 - 2n - n\alpha + 2n\delta)x^{-2}y - \beta n^2 x^{-n-1}, \\ \Gamma_{12}^1 &= \frac{1}{n}\varepsilon x^{2n-2}y - \gamma x^{n-1}, \quad \Gamma_{12}^2 = \frac{1}{n}\varepsilon x^{2n-3}y^2 + (\phi - \gamma)x^{n-2}y + (n-1 - n\delta)x^{-1}, \\ \Gamma_{22}^1 &= -\frac{1}{n}\varepsilon x^{2n-1}, \quad \Gamma_{22}^2 = -\frac{1}{n}\varepsilon x^{2n-2}y - \phi x^{n-1}. \end{aligned}$$

Since ∇ is supposed to be analytic in the neighborhood of the origin, we must have $\alpha = -1/n$ and $\delta = 1 - 1/n$.

If one of the Christoffel symbols does not vanish (if ∇ has a chance of not being flat), we must have $n \in \frac{1}{2}\mathbf{Z}$. If $n < 2$, then all of the Christoffel symbols must vanish except, possibly, for Γ_{11}^2 , which will be a function of x . In this case the curvature of ∇ vanishes and ∇ is flat. We are left with the case $n \geq 2$ which implies,

in particular, that $\beta = 0$. If ε , ϕ and γ vanish simultaneously, all the Christoffel symbols vanish and the connection is flat. Leaving the flat case aside, ∇ is analytic at the origin in the following cases:

- $n = 2$, $\phi = 2\gamma$.
- $n \in \mathbf{Z}$, $n \geq 3$;
- $n \notin \mathbf{Z}$, $\gamma = 0$, $\phi = 0$ and $n \geq 5/2$;

In the first case, we have a connection of Type III. According to Proposition 18, $\mathfrak{K}(\nabla) \approx \mathfrak{sl}(2, \mathbf{R})$ and is, in the chosen coordinates, the Lie algebra of divergence-free linear vector fields.

In the other two, the connections are of Type $\Pi^0(n)$.

5.2. One-dimensional orbits. We will now analyze the case when the rank at the origin of the Killing algebra generated by A and B is one.

5.2.1. Semisimple stabiliser. We prove now part (2) of Proposition 19. We will now suppose that B does not vanish at the origin, but that A does. In exponential coordinates at the origin, the Killing vector field A is linear. Let $B = \sum_{i=0}^{\infty} B_i$ where B_i is a homogeneous polynomial vector field of degree i . By homogeneity, we have $[A, B_i] = B_i$ and, in particular, $[A, B_0] = B_0$. Thus, -1 is an eigenvalue of A and $B_0(0) = B(0)$ is the corresponding eigenvector. In particular, A has real eigenvalues. It may or may not be diagonalizable.

Diagonalizable case. If A is diagonalizable, we may suppose that it is of the form $A = \lambda x \partial / \partial x - y \partial / \partial y$, with $\lambda \in \mathbf{R}$ and that $B_0 = \partial / \partial y$. For the sake of computations, let $B = y^{-1} x f(x, y) \partial / \partial x + g(x, y) \partial / \partial y$ with g an analytic function such that $g(0) = 1$ and f a meromorphic function such that B is analytic (such that $x f$ is an analytic function which admits y as a factor). If

$$f = \sum_{i=-1, j=1} a_{ij} x^i y^j, \quad g = \sum_{i=0, j=0} b_{ij} x^i y^j$$

the equation $[A, B] = B$ reads, at the formal level, $(\lambda i - j) a_{ij} = 0$, $(\lambda i - j) b_{ij} = 0$ with i and j in the corresponding range. There are four possibilities

- $\lambda \notin \mathbf{Q}$. This implies that $f \equiv 0$, $g \equiv 1$ and thus

$$(20) \quad A = \lambda x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad B = \frac{\partial}{\partial y}.$$

- $\lambda \in \mathbf{Q}$, $\lambda < 0$. This implies that $g \equiv 1$, that $\lambda \in \mathbf{Z}$ and that $f = c x^{-1} y^{-\lambda}$ for some $c \in \mathbf{R}$ (we may suppose $c = 1$ up to rescaling x). In the coordinates $(\lambda x + y^{-\lambda}, y)$, we still have (20).
- $\lambda = 0$, $f \equiv 0$, $g = g(x)$, but in this case A and B are linearly independent nowhere.
- $\lambda = p/q$ for relatively prime and positive integers p and q . This implies that f and g are functions of $u = x^q y^p$, with $f(0) = 0$. If h and k are functions of u , with $k(0) = 1$, $h(0) = 1$, satisfying the differential equations

$$u h' = -\frac{f h}{q f + p g}, \quad u k' = \frac{1 - g k}{q f + p g},$$

then, in the coordinates (xh, yk) , we get the same normal form for A and B as in (20).

The existence of solutions to the last equations is guaranteed by the following Lemma, whose proof may be found in [Inc44, §12.6].

Lemma 20 (Briot-Bouquet). *For $i = 1, 2$, let $F_i(x_1, x_2, t)$ be analytic functions in the neighborhood of $p = (a_1, a_2, 0) \in \mathbf{R}^3$ such that $F_i(a_1, a_2, 0) = 0$. If neither of the eigenvalues of $\partial F_i / \partial x_j|_p$ is a strictly positive integer, then the system of ordinary differential equations $t x_i' = F_i(x_1, x_2, t)$ has an analytic solution $(x_1(t), x_2(t))$ with $x_i(0) = a_i$.*

We may thus suppose that the vector fields A and B are given by the expressions (20) with $\lambda \neq 0$. We introduce the notation $\lambda = 1/n$, $n \in \mathbf{R}^*$. The vector fields A and B are linearly independent in the half plane $\{x > 0\}$. The vector fields in this half-plane that simultaneously commute with them are linear combinations of

$$X = -\frac{1}{n}x \frac{\partial}{\partial x}, Y = -x^{-n} \frac{\partial}{\partial y}.$$

As in the previous case, ∇ admits the expression (15), with respect to the moving frame (X, Y) . Solving, as before, for the Christoffel symbols, we have

$$\begin{aligned} \Gamma_{11}^1 &= -(n\alpha + 1)x^{-1}, \Gamma_{11}^2 = -n^2\beta x^{-n-2}, \Gamma_{12}^1 = -\gamma x^n, \\ \Gamma_{12}^2 &= n(1 - \delta)x^{-1}, \Gamma_{22}^1 = -\frac{1}{n}\varepsilon x^{2n+1}, \Gamma_{22}^2 = -\phi x^n. \end{aligned}$$

Since Γ_{ij}^k are analytic functions at the origin, we conclude that $\delta = 1$ and $\alpha = -1/n$, which implies $\Gamma_{11}^1 = \Gamma_{12}^2 \equiv 0$. In fact we get more: either $\Gamma_{ij}^k \equiv 0$, for all $1 \leq i, j, k \leq 2$ or $n \in \frac{1}{2}\mathbf{Z}$. If $n < -1/2$, then all of the Christoffel symbols must vanish except, possibly, for Γ_{11}^2 , which will be a function of x . If $n = -1/2$, all of the Christoffel symbols must vanish except for Γ_{22}^1 , which will be constant. In both cases the curvature of ∇ vanishes. If $n = 0$ the connection cannot be analytic.

We thus conclude that $n > 0$ and, in consequence, that $\beta = 0$. If ε , γ and ϕ all vanish, the connection is flat, so we will suppose that one of these constants does not vanish. We have a non-flat analytic connection in the following situations:

- $n \in \mathbf{Z}$, $n \geq 1$,
- $n \in \frac{1}{2}\mathbf{Z}$, $n \geq \frac{1}{2}$, $\gamma = 0$, $\phi = 0$.

The connection induced in the right half-plane is an invariant connection of Type I, with respect to the marking X . By Proposition 18, the Killing algebra is two-dimensional (and hence the one generated by A and B) except in the case where $n = 1$ and $\phi = -\gamma$, $\varepsilon = -\gamma^2$, with $\gamma \neq 0$, where it has dimension three. In this last case we have the extra Killing field $2(xy + \gamma^{-1})\partial/\partial x - y^2\partial/\partial y$ and ∇ is thus locally homogeneous (we do not consider this last situation, since ∇ is supposed not to be locally homogeneous). We thus find the connections of Type I of Theorem 1.

Non-diagonalizable case. We may suppose that $A = (-x + y)\partial/\partial x - y\partial/\partial y$ and that $B_i = f_i\partial/\partial x + g_i\partial/\partial y$ with f_i and g_i homogeneous polynomials of degree i . Up to multiplying B by a constant, we assume $f_0 = 1$. The equation $[A, B_i] = B_i$ splits (via Euler's relation) into the equations $y\partial g_i/\partial x = ig_i$, $y\partial f_i/\partial x = if_i$. Hence, for every $k \in \mathbf{N}$, $y\partial^k g_i/\partial x^k = i^k g_i$. Since g_i and f_i are polynomials, $g_i \equiv 0$, for all i , and $f_i \equiv 0$, for all $i \neq 0$. Thus,

$$A = (y - x) \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, B = \frac{\partial}{\partial x}.$$

They are linearly independent in the complement of $\{y = 0\}$. The space of vector fields that commute with these two in $\{y > 0\}$ is generated by

$$X = -y \log y \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, Y = -y \frac{\partial}{\partial x}.$$

The connection ∇ admits the expression (15) with respect to (X, Y) . Solving for the Christoffel symbols,

$$\begin{aligned}\Gamma_{11}^1 &= -\frac{\phi}{y} - \frac{\varepsilon}{y} \log y, \Gamma_{11}^2 = \frac{\varepsilon}{y}, \Gamma_{12}^1 = \frac{\delta-1}{y} + \frac{\gamma-\phi}{y} \log y - \frac{\varepsilon}{y} \log^2 y, \\ \Gamma_{12}^2 &= -\frac{\gamma}{y} + \frac{\varepsilon}{y} \log y, \Gamma_{22}^1 = \frac{\beta-1}{y} + \frac{2\delta-1-\alpha}{y} \log y + \frac{2\gamma-\phi}{y} \log^2 y - \frac{\varepsilon}{y} \log^3 y, \\ \Gamma_{22}^2 &= \frac{\alpha-1}{y} - \frac{2\gamma}{y} \log y + \frac{\varepsilon}{y} \log^2 y.\end{aligned}$$

If the connection extends analytically to the origin, $\Gamma_{ij}^k \equiv 0$ and the connection is flat. This finishes the proof of item (2) in Proposition 19.

5.2.2. *Unipotent stabilizer.* We deal now with part (3) of Proposition 19. We suppose that B vanishes at 0, but that A does not. We may linearize B and, since it must vanish along the common one-dimensional orbit of A and B , it must have one vanishing eigenvalue. Supposing that B vanishes along $x = 0$, either $B = x\partial/\partial x$ or $B = x\partial/\partial y$. If $A = xf\partial/\partial x + h\partial/\partial y$, the bracket relations imply, in the first case, $xf_x = -1$, and thus $f(x) = -\log x + c$ and A cannot be analytic. In the second one, we have the relations $f_y = 0$ and $h_y = f - 1$ and thus $f = f(x)$ and $h = (f-1)y + g(x)$ with $f \neq 0$, $g(0) \neq 0$ (in particular, f is analytic at the origin). Hence,

$$(21) \quad A = xf(x) \frac{\partial}{\partial x} + (y[f(x) - 1] + g(x)) \frac{\partial}{\partial y}, \quad B = x \frac{\partial}{\partial y}.$$

They are linearly independent in the complement of $x = 0$. The vector fields that commute with these two in $\{x > 0\}$ are linear combinations of

$$X = -\left(xf(x) \frac{\partial}{\partial x} + [yf(x) + u(x)] \frac{\partial}{\partial y}\right), \quad Y = \frac{1}{v(x)} \frac{\partial}{\partial y},$$

where u and v are non-zero solutions to the differential equations

$$(22) \quad xu' = xg' - g + \left(\frac{f-1}{f}\right)u, \quad xv' = -\left(\frac{f-1}{f}\right)v,$$

defined in $\{x > 0\}$. By solving for the Christoffel symbols and using the above relations,

$$\begin{aligned}\Gamma_{11}^1 &= -\frac{\alpha}{xf} - \frac{1}{x} - \frac{f'}{f} - \frac{2\gamma v}{xf} (yf + u) - \frac{\varepsilon v^2}{xf} (yf + u)^2, \\ \Gamma_{11}^2 &= \frac{[2\delta - \alpha - 2 + f]}{x^2 f^2} (yf + u) + \frac{\beta}{x^2 f^2 v} + \frac{[\phi - 2\gamma]v}{x^2 f^2} (yf + u)^2 - \frac{\varepsilon v^2}{x^2 f^2} (yf + u)^3 - \frac{yf' + u'}{xf}, \\ \Gamma_{12}^1 &= \gamma v + \varepsilon v^2 (yf + u), \\ \Gamma_{12}^2 &= \frac{\varepsilon v^2}{xf} (yf + u)^2 + \frac{[\gamma - \phi]v}{xf} (yf + u) - \frac{\delta}{xf} + \frac{1}{xf} - \frac{1}{x}, \\ \Gamma_{22}^1 &= -\varepsilon x f v^2, \quad \Gamma_{22}^2 = \phi v - \varepsilon v^2 (yf + u).\end{aligned}$$

We will now prove that $f(0) \neq 0$. If $f(0) = 0$ then, from equation (22), v has an essential singularity at the origin, for its logarithmic derivative will have more than simple poles. Since Γ_{11}^1 is analytic at the origin and the coefficient of y^2 in its expression contains v^2 , this coefficient must vanish and thus $\varepsilon = 0$. Also the coefficient of y in Γ_{11}^1 must vanish, and thus $\gamma = 0$. We must have $\alpha = 0$ for otherwise Γ_{11}^1 would have pole of order greater than one at the origin. Finally, $\Gamma_{11}^1 = -1/x - f'/f$ which has a pole at the origin since f is analytic: a contradiction.

Hence, if the connection extends analytically to the origin, $f(0) \neq 0$. We will now find new coordinates where the expressions of X and Y simplify.

Choose a coordinate $z = x\xi(x)$ in which the vector field $xf(x)\partial/\partial x$ reads $n^{-1}z\partial/\partial z$ for some $n \in \mathbf{R}^*$. In the coordinates $(x\xi(x), y\xi(x))$, which preserve the form of X and Y , we have $f(x) \equiv n^{-1}$. Hence, from equation (22), $v(x) = -x^{n-1}$. We thus have

$$X = -\left(\frac{1}{n}x\frac{\partial}{\partial x} + \left[\frac{1}{n}y + u(x)\right]\frac{\partial}{\partial y}\right), Y = -x^{1-n}\frac{\partial}{\partial y}.$$

We will simplify even more the normal form of X and Y . For this, we will first prove that $n \in \frac{1}{2}\mathbf{Z}, n \geq 2$.

Assume, by contradiction, that v^2/x^2 is *not* analytic. From the coefficient of y^3 in Γ_{11}^2 , $\varepsilon = 0$; from the coefficient of y in Γ_{12}^2 and in Γ_{11}^1 , $\gamma = 0$ and $\phi = 0$; from the analyticity of Γ_{11}^1 and Γ_{12}^2 , $\alpha = -1/n$, $\delta = 1 - 1/n$. These conditions imply that all the Christoffel symbols vanish except (possibly) for Γ_{11}^2 which will be a function of x . Thus the connection ∇ is flat: a contradiction. We must conclude that v^2/x^2 is analytic and thus that $n \in \frac{1}{2}\mathbf{Z}, n \geq 2$.

We apply Lemma 20 and consider $\psi(x)$ a solution to the differential equation $x\psi' = [1 - n]\psi - n[g(x) - g(0)]$ that vanishes at the origin (the equation has a solution, for $n > 1$ implies $1 - n < 0$).

Now we replace y by $y + \psi(x)$. The vector field B in (21) is preserved; the vector field A changes to one where g is constant. Up to multiplying y by a constant, we will suppose that $g \equiv 1/n - 1$. The differential equation (22) for u becomes $xu' = (1 - 1/n)(1 - nu)$. The general solution of this equation is $u(x) = 1/n + cx^{1-n}$. Hence, up to the addition of a multiple of Y to X , we can suppose that $u \equiv 1/n$. We thus have

$$A = \frac{1}{n}\left(x\frac{\partial}{\partial x} - (n-1)(y+1)\frac{\partial}{\partial y}\right), B = x\frac{\partial}{\partial y}$$

$$X = -\frac{1}{n}\left(x\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}\right), Y = -x^{1-n}\frac{\partial}{\partial y}.$$

Once again, ∇ has the expression (15) with respect to (X, Y) and using the last expression of X and Y , we find the Christoffel coefficients:

$$\begin{aligned}\Gamma_{11}^1 &= -(n\alpha + 1)x^{-1} + 2\gamma x^{n-2}(y+1) - \frac{1}{n}\varepsilon x^{2n-3}(y+1)^2, \\ \Gamma_{11}^2 &= -n^2\beta x^{-n-1} - [2(n - n\delta - 1) + (n\alpha + 1)]x^{-2}(y+1) - [\phi - 2\gamma]x^{n-3}(y+1)^2 - \frac{1}{n}\varepsilon x^{2n-4}(y+1)^3, \\ \Gamma_{12}^1 &= -\gamma x^{n-1} + \frac{1}{n}\varepsilon x^{2n-2}(y+1), \\ \Gamma_{12}^2 &= (n - n\delta - 1)x^{-1} - [\gamma - \phi]x^{n-2}(y+1) + \frac{1}{n}\varepsilon x^{2n-3}(y+1)^2 \\ \Gamma_{22}^1 &= -\frac{1}{n}\varepsilon x^{2n-1}, \Gamma_{22}^2 = -\phi x^{n-1} - \frac{1}{n}\varepsilon x^{2n-2}(y+1).\end{aligned}$$

For Γ_{11}^2 and Γ_{11}^1 to be analytic at the origin, $\beta = 0$ and $\alpha = -1/n$. From the coefficient of x^{-2} in Γ_{11}^2 , $\delta = 1 - 1/n$. If ε, γ and ϕ vanish simultaneously, the connection is flat. Leaving this case aside, the connection extends analytically to the origin in the following cases (recall that we have already established $n \in \frac{1}{2}\mathbf{Z}, n \geq 2$)

- $n = 2, \phi = 2\gamma$.
- $n \in \mathbf{Z}, n \geq 3$
- $n = m/2, m \in \mathbf{Z}, m \geq 5$ odd, $\gamma = 0, \phi = 0$.

The invariant connection in $\text{Aff}_0(\mathbf{R})$ associated to the restriction of this connection in $\{x > 0\}$ is of Type II. From Proposition 18, $\mathfrak{K}(\nabla)$ has dimension two (and is generated by A and B) except in the first case, where it has dimension three (in this last case $(y+1)\partial/\partial x \in \mathfrak{K}(\nabla)$ and the connection is locally homogeneous). This

| Type | A | B | X | Y |
|---------|--|--------------------------------|--|---------------------------------------|
| $I(n)$ | $\frac{1}{n}x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ | $\frac{\partial}{\partial y}$ | $-\frac{1}{n}x\frac{\partial}{\partial x}$ | $-x^{-n}\frac{\partial}{\partial y}$ |
| $II(n)$ | $\frac{1}{n}\left(x\frac{\partial}{\partial x} + [1-n]y\frac{\partial}{\partial y}\right)$ | $x\frac{\partial}{\partial y}$ | $-\frac{1}{n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$ | $-x^{1-n}\frac{\partial}{\partial y}$ |

TABLE 1. Killing vector fields (A and B) and their centralizers (X and Y) in $\{x > 0\}$ for the connections of Theorem 1 whose Killing algebra is isomorphic to $\mathfrak{aff}(\mathbf{R})$ ($n \in \frac{1}{2}\mathbf{Z}$, $n \geq \frac{1}{2}$ for Type I and $n \geq \frac{5}{2}$ for Type II).

eliminates the case $n = 2$. These vector fields and connections are the germs at $(0, 1)$ of those we found in the proof of part (1) in Proposition 19 (section 5.1). The connections are thus of Type II^1 .

5.3. Local isometries and normal forms. So far we have proved that any germ of connection satisfying the hypothesis of Theorem 1, such that one of the two-dimensional algebras guaranteed by Proposition 9 is affine, is indeed the germ of a connection of Type I, II or III in the neighborhood of some point $O \in \mathbf{R}^2$ (the point $(0, 1)$ for the connections of Type II^1 , the origin for the other cases). In order to get normal forms we must investigate the parameters that yield isometric connections.

The isomorphism type of the Killing Lie algebra of a connection is an isometry invariant. Our discussion splits naturally in two cases:

5.3.1. *Types I, II^0 and II^1 .* By Proposition 19, the isometry class of a connection ∇ cannot belong simultaneously to two of the Types I, II^0 , II^1 . In all these cases the rank of $\mathfrak{K}(\nabla)$ is one along $\{x = 0\}$ and two on its complement. In restriction to each connected component of the complement, ∇ is locally modeled by a left-invariant connection in the affine group. For the convenience of the reader we reproduce, in Table 1, the vector fields associated to the normal forms of Theorem 1 (recall that for the connections of Type II^1 , one should consider germs at $(0, 1)$).

We will begin by proving that, within the connections of Type I, the value of n is an isometry invariant and that the same is true for the connections of Types II^0 and II^1 .

For Type $I(n)$, A generates the space of Killing vector field vanishing at 0. It is normalized by $[A, B] = B$. Its eigenvalue at the origin in the direction transverse to $\{x = 0\}$ is $1/n$.

For Type $II^0(n)$, the semisimple vector fields are those of the form $A + \lambda B$. All of them vanish at the origin and the eigenvalue in the direction transverse to $\{x = 0\}$ is $1/n - 1$.

For Type $II^1(n)$, the vector field X is the only analytic vector field in the centralizer of $\mathfrak{K}(\nabla)$ such that, on either side of $\{x = 0\}$, there exists a vector field Y on the centralizer of $\mathfrak{K}(\nabla)$ such that $[X, Y] = Y$. The most general semisimple vector field is $A + \lambda B$ and has, in the neighborhood of $(0, 1)$, the primitive first integral $h = x(y - \lambda x)^{1/(n-1)}$. We have $X \cdot h = (1 - n)^{-1}h$. The eigenvalue $(1 - n)^{-1}$ is intrinsically attached to ∇ .

If a diffeomorphism maps a connection ∇ of Type $I(n)$ to a connection ∇' of the same Type, it must map $\mathfrak{K}(\nabla)$ to $\mathfrak{K}(\nabla')$ and thus it must actually preserve $\mathfrak{K}(\nabla)$, which is independent of the parameters $(\gamma, \varepsilon, \delta)$. The same is true for connections of Types $II^0(n)$ and $II^1(n)$. We will study, in all cases, the group of germs of diffeomorphisms fixing O and preserving $\mathfrak{K}(\nabla)$.

Any germ of diffeomorphism $g : (\mathbf{R}^2, O) \rightarrow (\mathbf{R}^2, O)$ that preserves $\mathfrak{K}(\nabla)$ should preserve the line $\{x = 0\}$, where the rank of this algebra is one. It may do so either by preserving or by exchanging the connected

components of its complement. It may preserve or reverse the orientation. Hence, within the group of germs of automorphisms of $\mathfrak{R}(\nabla)$, the subgroup of those preserving the orientation and the half-plane $\{x = 0\}$ is a normal subgroup of index one, two or four.

If g preserves the half-plane $\{x > 0\}$, then it must preserve the centralizer of $\mathfrak{R}(\nabla)$ within the half-plane, the Lie algebra generated by X and Y . Since, within this algebra, X is the only element that extends analytically to the origin and is normalized by $[X, Y] = Y$, we must have $g_*X = X$. From (16), g has the effect of multiplying Y by some constant, positive, if g preserves orientation, negative, if it does not.

We claim that, in the case when g preserves orientation, the positive constant is arbitrary. For the connections of Types I and Π^0 , the flow of the vector field X fixes O , preserves $\mathfrak{R}(\nabla)$, preserves X and multiplies Y by an arbitrary positive constant. For the connections of Type Π^1 , the vector field $A + (1 - n)X$ vanishes at O , its flow preserves $\mathfrak{R}(\nabla)$ and X and multiplies Y by an arbitrary positive constant. Hence, the action upon the parameters of the connection of these orientation-preserving mappings is

$$(23) \quad (\gamma, \varepsilon, \phi) \mapsto (\mu\gamma, \mu^2\varepsilon, \mu\phi)$$

for $\mu > 0$.

Let us now consider the case where g preserves the half-plane $\{x > 0\}$, but not the orientation. We claim that this cannot happen for the family of connections of Type Π^1 and that the above negative constant is arbitrary for the connections of Types I and Π^0 . Let

$$(24) \quad \rho(x, y) = (x, -y).$$

For the connections of Types I and Π^0 , $\rho(O) = O$, ρ preserves the half-plane $\{x > 0\}$ and satisfies $\rho_*X = X$, $\rho_*Y = -Y$. In these cases, the action upon the parameters of the connection is (23) for $\mu = -1$. This, combined with the previous results, shows that the negative constant is arbitrary. If ∇ is a connection of Type Π^1 and if $g : (\mathbf{R}^2, O) \rightarrow (\mathbf{R}^2, O)$ preserves $\mathfrak{R}(\nabla)$ and the half-plane $\{x > 0\}$, but does not preserve the orientation, by the previous arguments, we may suppose that $g_*X = X$ and $g_*Y = -Y$. The most general mapping doing this is given by

$$(x, y) \mapsto (e^t x, e^{(1-n)t}(2x - y) + se^t x)$$

with $s, t \in \mathbf{R}$. But if such a map fixes $(0, 1)$ then $-e^{(1-n)t} = 1$, which is impossible.

Let us now consider the case where g does not preserve neither the half-plane $\{x > 0\}$, nor the orientation. Let

$$(25) \quad \sigma(x, y) = (-x, y).$$

The vector field X is well-defined in a neighborhood of O and we have $\sigma_*(X) = X$.

If $n \in \mathbf{Z}$, there is a natural extension of Y onto $\{x < 0\}$, that we will still call Y . We have $\sigma_*(X) = X$, $\sigma_*Y = (-1)^n Y$ (for Type I) and $\sigma_*Y = (-1)^{n+1} Y$ (for Types Π^0 and Π^1). In particular, σ is an isometry for the connections of Type I(n), when n is even, and for those of Type II(n), if n is odd. For the opposite parities, the action upon the parameters of the connection is (23) for $\mu = -1$.

If $n \notin \mathbf{Z}$, the vector field σ_*Y is well-defined in $\{x < 0\}$ and satisfies $[X, \sigma_*Y] = \sigma_*Y$. We have $\nabla_X X = \alpha X$, $\nabla_X(\sigma_*Y) = \delta(\sigma_*Y)$ and $\nabla_{\sigma_*Y}(\sigma_*Y) = -\varepsilon X$. The transformation has the effect of changing the sign of ε (in some sense, σ acts upon the coefficients as (23) for $\mu = \sqrt{-1}$).

| | preserves $\{x > 0\}$? | preserves orientation? | $I(n)$ | $\Pi^0(n)$ | $\Pi^1(n)$ |
|---------------------|----------------------------|---------------------------|--------------|--------------|--------------|
| 1 | yes | yes | 1 | 1 | 1 |
| σ | no | no | $(-1)^n$ | $(-1)^{n+1}$ | $(-1)^{n+1}$ |
| ρ | yes | no | -1 | -1 | \times |
| $\sigma \circ \rho$ | no | yes | $(-1)^{n+1}$ | $(-1)^n$ | \times |

TABLE 2. Action upon the family of connections of some transformations preserving $\mathfrak{R}(\nabla)$, in the case $n \in \mathbf{Z}$.

The last case, where g preserves the orientation, but not $\{x > 0\}$, is a composition of the two orientation-reversing cases (it takes place if and only if both of them take place).

The results for $n \in \mathbf{Z}$ are summarized in Table 2.

Remark 21. We proved that σ is an isometry for the connections of Type $I(n)$, if n is even, and for the connections of Type $II(n)$, if n is odd.

Remark 22. For all the connections of Theorem 1 there is a germ of isometry fixing O that does not preserve the half-plane $\{x > 0\}$.

Remark 23. If $n \notin \mathbf{Z}$, for the connections of Type I and II (seen as global connections in \mathbf{R}^2), the left-invariant connection in the affine group induced by the restriction of ∇ to $\{x > 0\}$ is *not* equivalent to the connection induced by the restriction to $\{x < 0\}$ (the sign of ε , which is an invariant, is not the same). These give examples of connections in \mathbf{R}^2 (with polynomial Cristoffel symbols) where the isometry pseudogroup acts with an open orbit but not with a dense one.

5.3.2. *Type III.* The local diffeomorphisms Φ preserving $\mathfrak{R}(\nabla)$ are necessarily linear, for they must preserve the space of divergence-free linear vector fields on $(\mathbf{R}^2, 0)$. They are generated by $\mathfrak{R}(\nabla)$, by the homotheties (the flow of X) and by σ . All these preserve X and the last two (which do not, a priori, preserve ∇) preserve Y . The effect of these diffeomorphisms is to multiply Y by an arbitrary constant and we thus obtain the normal form for the connections of Type III.

6. THE GLOBAL MODELS

We will begin by proving parts (1) and (3) of Theorem 3, by constructing the connections $\nabla_{n,m}$ in \mathbf{R}^2 announced in the Theorem and by studying their isometries.

6.1. **The connection.** The idea behind the construction of $\nabla_{n,m}$ may be sketched as follows:

- (1) to view \mathbf{R}^2 as the union of the sets $V_i = \{(x, y) | x \in (i, i + 1)\}$, $\ell_i = \{(x, y) | x = i\}$, $i \in \mathbf{Z}$.
- (2) to consider the action of $\tilde{\sigma}(x, y) = (-x, y)$ and $\tilde{\phi}(x, y) = (x + 2, y)$ on \mathbf{R}^2
- (3) to construct a connection ∇ in a neighborhood of the closure of V_0 whose germ at ℓ_0 is preserved by $\tilde{\sigma}$ and whose germ at ℓ_1 is preserved by $\tilde{\sigma} \circ \tilde{\phi}^{-1}$.
- (4) to propagate the connection to \mathbf{R}^2 by the group generated by $\tilde{\sigma}$ and $\tilde{\phi}$.

Within the left-invariant connections in $\text{Aff}_0(\mathbf{R})$, we will say that a marked connection is of Type $\Xi(n)$, if it is of Type $I_0(n)$ and n is even, or if it is of Type $II_0^1(n)$ and n is odd.

Let $(n, m) \in \mathbf{Z}^2$, $n, m \geq 2$. Let ∇_0 be a left-invariant connection in $\text{Aff}_0(\mathbf{R})$ such that X_0 and $X_0 - Y_0$ are markings of ∇ , such that $(\nabla_0, X_0 - Y_0)$ is of Type $\Xi(m)$ and that (∇_0, X_0) is of Type $\Xi(n)$. Such a connection exists and is unique by formula (17). Let $(\gamma, \phi, \varepsilon) = (\gamma_0, \phi_0, \varepsilon_0)$ be the corresponding parameters.

Lemma 24. *The parameters γ_0 and ϕ_0 do not vanish simultaneously.*

Proof. If γ_0 and ϕ_0 vanished simultaneously, we would have, from (16), $\delta + \delta' - 1 - \alpha = 0$ and $1 - 2\delta + \alpha' = 0$. If (∇, X_0) were of Type I, we would have $\delta = 1$ which would then imply $\alpha' = 1$, an impossibility. If (∇, X_0) were of Type II, we would have $\delta = 1 + \alpha$ and this would imply $\delta' = 0$, forcing $(\nabla, X_0 + \lambda Y_0)$ to be of type II, this is, $\delta' = 1 + \alpha'$ and thus $\alpha' = -1$. But this contradicts the fact that $m \geq 2$. \square

Let $V_0 = \{(u, v) \in \text{Aff}_0(\mathbf{R}), 0 < v/u < 1\}$. It is invariant by A_0 and is bounded by the common orbits of A_0 and the two special markings of ∇_0 just defined. Equivalently, V_0 is the set where

$$(26) \quad \frac{A_0 \wedge (X_0 - Y_0)}{A_0 \wedge X_0} = 1 - \frac{u}{v}$$

is negative. Let $\beta : \text{Aff}_0(\mathbf{R}) \rightarrow \text{Aff}_0(\mathbf{R})$ be the involutive mapping given by

$$(27) \quad \beta(u, v) = (u, u - v).$$

The image of ∇_0 under β is the unique left-invariant connection ∇'_0 in $\text{Aff}_0(\mathbf{R})$ such that X_0 and $X_0 - Y_0$ are markings of ∇'_0 , such that $(\nabla'_0, X_0 - Y_0)$ is of Type $\Xi(n)$ and such that (∇'_0, X_0) is of Type $\Xi(m)$. Moreover, β preserves V_0 and the vector field A_0 . In particular, if $n = m$, $\beta : V_0 \rightarrow V_0$ is an orientation-reversing isometry of ∇_0 . Within the affine group, β is the inner automorphism of $(-1, 0)$ followed by the right translation by $(1, 1)$.

We will add a boundary component to V_0 and extend ∇_0 analytically to it:

Case 1, n is odd: Let ∇ be a connection of type $\Pi^1(n)$ in \mathbf{R}^2 with the parameters $(\gamma, \phi, \varepsilon) = (\gamma_0, \phi_0, \varepsilon_0)$. Consider the embedding $\Psi : V_0 \rightarrow \mathbf{R}^2$ given by

$$\Psi(u, v) = \left(\frac{1}{\sqrt[n]{v/u}}, \frac{v}{\sqrt[n]{u}} \right) = (x, y).$$

It maps X_0 to X , Y_0 to Y and A_0 to A . In consequence, it maps ∇_0 to ∇ . The first integral v/u of A_0 is mapped to the first integral yx^{n-1} of A and thus $\Psi(V_0) = \{(x, y) | 0 < x, 0 < yx^{n-1} < 1\}$. Let $\ell = \{(x, y) \in \mathbf{R}^2, x = 0, y > 0\}$.

Case 2, n is even: Let ∇ be a connection of type $\text{I}(n)$ with parameters $(\gamma, \phi, \varepsilon) = (\gamma_0, \phi_0, \varepsilon_0)$. Consider the embedding $\Psi : V_0 \rightarrow \mathbf{R}^2$ given by

$$\Psi(u, v) = \left(\frac{1}{\sqrt[n]{u}}, v - 1 \right) = (x, y).$$

The vector fields X_0 and Y_0 are mapped, respectively, to the vector fields X and Y and thus, Ψ maps ∇_0 to ∇ . The vector field A_0 is mapped to $A - B$. The first integral v/u of A_0 is mapped to the first integral $(1 + y)x^n$ of $A - B$ and thus $\Psi(V_0) = \{(x, y) | x > 0, 1 > (1 + y)x^n > 0\}$. Let $\ell = \{(x, y) | x = 0, y > -1\}$.

The set $\Psi(V_0) \cup \ell$ is naturally a manifold-with-boundary and gives an embedding of V_0 into a manifold-with-boundary $V_0 \cup \partial^+ V_0$ to which ∇_0 extends. By exchanging the roles of n and m , via β , the above construction may be repeated in order to construct another boundary component $\partial^- V_0$ of V_0 and to extend analytically the connection to it.

We thus have a manifold-with-boundary \bar{V}_0 whose interior is V_0 and whose boundary components are $\partial^+ V_0$ and $\partial^- V_0$. There is an analytic connection ∇_0 on \bar{V}_0 .

For $j \in \mathbf{Z}$, let $\bar{V}_j = \bar{V}_0$ together with a tautological mapping $\pi_j : \bar{V}_0 \rightarrow \bar{V}_j$. Let ∇_j be the connection in \bar{V}_j such that π_j is an isometry. We will now glue \bar{V}_0 and \bar{V}_1 along ∂^+V_0 and ∂^+V_1 while gluing ∇_0 to ∇_1 . Consider the immersions $\Psi : (V_0 \cup \partial^+V_0) \rightarrow \mathbf{R}^2$ and $\sigma \circ \Psi \circ \pi_1^{-1} : (V_1 \cup \partial^+V_1) \rightarrow \mathbf{R}^2$, for the mapping σ defined in (25). By identifying the points that have the same image, we obtain an identification between ∂^+V_0 and ∂^+V_1 that produces a submanifold ℓ_1 of the gluing. For the connections of Type $\Xi(n)$, σ is an isometry of ∇ (Remark 21) and the connections ∇_0 and ∇_1 agree after the gluing. We can proceed in exactly the same way to identify ∂^+V_{2k} with ∂^+V_{2k+1} . By exchanging the roles of n and m , via β , we can now glue ∂^-V_{2k} with ∂^-V_{2k-1} .

In this way we obtain a manifold M , diffeomorphic to \mathbf{R}^2 , endowed with a connection $\nabla_{m,n}$. This is the connection of item (1) of Theorem 3.

6.2. The Isometries. We will now study the global isometries of the connection $\nabla_{m,n}$ just defined.

The vector field A is globally well-defined in M for $\sigma_*A = A$. It is complete, since it preserves V_i and ℓ_i , for every i , and is complete in restriction each of them. The flow of A gives thus a action of \mathbf{R} by isometries.

There is a unique orientation-preserving (tautological) isometry $\phi : M \rightarrow M$ such that $\phi(V_j) = V_{j+2}$ and such that $\phi \circ \pi_{-2} : V_0 \rightarrow V_0$ is the identity. This isometry gives an action of \mathbf{Z} on M by orientation-preserving isometries. The flow of A commutes with ϕ .

If $n = m$ there is, moreover, a unique orientation-preserving (tautological) isometry $\phi^{\frac{1}{2}} : M \rightarrow M$ such that $\phi^{\frac{1}{2}}(V_j) = V_{j+1}$, such that $\phi^{\frac{1}{2}}|_{V_0} \circ \pi_{-1}$ is given by the mapping β of formula (27). The flow of A commutes with $\phi^{\frac{1}{2}}$. We naturally have $\phi^{\frac{1}{2}} \circ \phi^{\frac{1}{2}} = \phi$.

Proposition 25. *The group of orientation-preserving isometries of $\nabla_{m,n}$ is isomorphic to $\mathbf{Z} \times \mathbf{R}$. If $m \neq n$, it is generated by ϕ and by the flow of A . If $m = n$, it is generated by $\phi^{\frac{1}{2}}$ and by the flow of A .*

Proof. Let us begin by studying the group of global isometries of ∇_0 in V_0 in order to prove that any orientation-preserving isometry of (V_0, ∇_0) embeds into the flow of A_0 and that V_0 admits an orientation-reversing isometry if and only if $n = m$.

Let $f : V_0 \rightarrow V_0$ be an isometry of ∇_0 . From Proposition 17 and the previously established fact that $(\gamma_0, \varepsilon_0, \phi_0) \neq (0, 0, 0)$, $\mathfrak{K}(\nabla_0)$ is the Lie algebra of right-invariant vector fields (this follows also from Proposition 18). In consequence, f should map the complete vector field A_0 to a complete vector field of the form $A_0 + \lambda B_0$. But such a vector field cannot preserve V_0 (be complete in restriction to V_0), unless $\lambda = 0$. We must conclude that $f_*A_0 = A_0$. Since B_0 generates the derived algebra of $\mathfrak{K}(\nabla_0)$, f must map B_0 to a multiple of it, positive if f preserves orientation, negative, if it does not.

If f preserves orientation, up to composing with the flow of A , we may suppose that $f_*B_0 = B_0$. By the orientation-preserving hypothesis and Proposition 17, f is a left translation. But the fact that it preserves the markings of ∇_0 implies that f must be the identity. This proves that the orientation-preserving isometries of (V_0, ∇_0) embed into the flow of A . If f does not preserve orientation, it must exchange the special markings and we have $n = m$. The mapping β of formula (27) is such an isometry and, after composing with it, we get an orientation-preserving isometry that embeds into the flow of A .

Let $f : M \rightarrow M$ be an orientation-preserving isometry. Since $\mathfrak{K}(\nabla)$ has rank two exactly in the union of the V_i , we should have $f(V_0) = V_k$ for some $k \in \mathbf{Z}$. The mapping $\pi_{-k} \circ f|_{V_0} : V_0 \rightarrow V_0$ is an isometry, orientation preserving if k is even, orientation-reversing if k is odd. If k is even (in particular, if $m \neq n$), up to composing f with a suitable time of the flow of A on M , we may suppose that $\pi_{-k} \circ f|_{V_0}$ is the identity: this implies that $f = \phi^k$. If k is odd, composition with $\phi^{\frac{1}{2}}$ reduces the claim to the previous case. \square

Let G be a discrete group acting by orientation-preserving isometries on M , such that the quotient is a compact surface S . Since every isometry preserves the nowhere vanishing vector field A , S admits a nowhere vanishing vector field and is thus a torus. Thus, in order to understand the compact quotients of M , we should understand the homomorphisms $\Phi : \mathbf{Z}^2 \rightarrow \text{Isom}^+(M, \nabla_{n,m})$. We already established an isomorphism between $\text{Isom}^+(M, \nabla_{n,m})$ and $\mathbf{Z} \times \mathbf{R}$.

We claim that we can choose an ordered set of generators (e_1, e_2) of \mathbf{Z}^2 such that $\Phi(e_1) = (0, \tau)$ and $\Phi(e_2) = (k, \theta)$, with $\tau > 0$ and $k > 0$. Let $\Phi(e_i) = (n_i, \tau_i) \in \mathbf{Z} \times \mathbf{R}$. We cannot have $n_1 = 0$ and $n_2 = 0$. Suppose first that $n_1 n_2 \neq 0$. Let $k > 0$ be the greatest common divisor of n_1 and n_2 , so that $n_i = k p_i$, with $p_1, p_2 \in \mathbf{Z}$. Let $q_1, q_2 \in \mathbf{Z}$, such that $p_1 q_1 + p_2 q_2 = 1$. In the basis $(p_2 e_1 - p_1 e_2, q_1 e_1 + q_2 e_2)$ of \mathbf{Z}^2 , the images of the generators read $(0, \tau)$ and (k, θ) , which reduces the case $n_1 n_2 \neq 0$ to the case $n_1 n_2 = 0$. Up to replacing the generators by their inverses, we have $\tau > 0$ and $k > 0$. In this setting, τ , k and the class of θ in $\mathbf{R}/\tau\mathbf{Z}$ are intrinsically defined and depend only upon the representation. This proves item (3) of Theorem 3.

7. QUASIHOMOGENEOUS CONNECTIONS ON COMPACT SURFACES

Let S be a compact oriented analytic surface endowed with a quasihomogeneous analytic connection ∇_s . Let $\Omega \subset S$ be the open set where the rank of the Killing algebra is at least two. Our main assumption is that Ω is neither empty, nor all of S . At the points in the boundary of Ω , ∇_s is locally isomorphic to one of the quasihomogeneous germs of connection classified in Theorem 1. Thus, the Killing algebra of ∇_s is isomorphic either to $\text{aff}(\mathbf{R})$, or to $\mathfrak{sl}(2, \mathbf{R})$. We shall analyze separately these two cases in order to completely classify the pairs (S, ∇_s) . In particular, we will prove, in Section 7.2, that the case $\mathfrak{sl}(2, \mathbf{R})$ does not occur. Section 7.1 achieves the proof of Theorem 3, in the case where the Killing algebra is isomorphic to $\text{aff}(\mathbf{R})$.

7.1. When the killing algebra is isomorphic to $\text{aff}(\mathbf{R})$. We will now prove part (2) of Theorem 3. We will prove that, under the hypothesis that the killing algebra is isomorphic to $\text{aff}(\mathbf{R})$, that (S, ∇_s) is isometric to a compact quotient of one of the affine manifolds $(M, \nabla_{n,m})$ defined in the previous section.

By Theorem 1, at the points in the boundary of Ω , ∇_s is of Type I, II^0 or II^1 .

Our first claim is that *S is a torus and that each connected component of $S \setminus \Omega$ is a cylinder*. We will begin by defining a non-singular foliation \mathcal{F} in S . In the neighborhood of every point we may find two Killing vector fields A and B of ∇_s such that $[A, B] = B$. The vector field B , the generator of the derived Lie algebra of $\mathfrak{K}(\nabla_s)$ is well-defined up to multiplication by a constant. At the points where B does not vanish, define \mathcal{F} as the foliation tangent to B . At the points where B vanishes, B is, in suitable coordinates, a nonzero multiple of $x\partial/\partial y$ (proof of Theorem 1; see Table 1). In this case, define \mathcal{F} locally as the kernel of dx . This gives a foliation without singularities in all of S , which is thus a torus.

Let p be a point in the boundary of Ω . If B does not vanish at p , then the orbit of B through p (a leaf of \mathcal{F}) gives, locally, the boundary of Ω . If B vanishes at p , in the coordinates of Theorem 1, Ω is the complement of $\{x = 0\}$ and \mathcal{F} is given by the kernel of dx . In all cases, the complement of Ω is a union of leaves of \mathcal{F} and is thus a finite union of circles. By the Poincaré-Hopf index Theorem, every connected component of Ω is a cylinder, as we claimed.

Thus, $S \setminus \Omega$ is a finite number of homologous simple closed curves. Let C be a connected component of Ω . We will denote by \overline{C} the manifold-with-boundary obtained by adding to each end of C the circle

in $S \setminus \Omega$ that compactifies it (this is, even if $S \setminus \Omega$ has only one connected component, \bar{C} will still be considered as a manifold with two boundary components). Let ∂C be a boundary component of \bar{C} . Let W be a tubular neighborhood of ∂C within S . Let $\Pi : \tilde{C} \rightarrow C$ and $P : \tilde{W} \rightarrow W$ be universal coverings of C and W . Since the inclusions of $C \cap W$ in C and in W are homotopy equivalences, there is a natural identification between $\Pi^{-1}(C \cap W)$ and $P^{-1}(C \cap W)$.

The restriction of ∇_s to C is locally homogeneous at every point. By Proposition 11, Remark 12 and item (2) of Proposition 17, there exists a left-invariant connection ∇_0 on $\text{Aff}_0(\mathbf{R})$, a *developing map* $\mathcal{D} : \tilde{C} \rightarrow \text{Aff}_0(\mathbf{R})$, which is a local isomorphism between $\Pi^{-1}(\nabla_s)$ and ∇_0 , and a *holonomy morphism* $\mu : \pi_1(C) \rightarrow \text{Aff}_0(\mathbf{R})$ such that for every $q \in \tilde{C}$ and $\gamma \in \pi_1(C)$,

$$(28) \quad \mathcal{D}(\gamma \cdot q) = \mu(\gamma)\mathcal{D}(q),$$

where $\gamma \cdot q$ is the image of q under the deck transformation associated to γ .

We now claim that *the image of the holonomy is non-trivial and belongs to a semisimple one-parameter group*. Let us prove this. If the holonomy is trivial, the developing map is, by formula (28), invariant by the action of $\pi_1(C)$ by deck transformations and, in consequence, induces a well-defined map $\mathcal{D}^b : C \rightarrow \text{Aff}_0(\mathbf{R})$. The Killing vector fields of ∇_s are then globally well-defined on C : they are pull-back of right-invariant vector fields on $\text{Aff}_0(\mathbf{R})$. The Killing vector fields of ∇_s in C are complete, for they preserve the boundary, and hence induce a (transitive) action of $\text{Aff}_0(\mathbf{R})$ on C . The map \mathcal{D}^b is equivariant with respect to this action of $\text{Aff}_0(\mathbf{R})$ on C and its action by left translations upon itself. But the first action has a non-trivial stabilizer and the second one is free. This contradiction shows that the holonomy cannot be trivial.

Assume, by contradiction, that the image of the holonomy is generated by an element of the form $(1, \tau)$ with $\tau \neq 0$. This is the flow of B_0 in time τ . In particular, B_0 is holonomy-invariant and induces a well-defined Killing vector field B_s in C , which preserves the boundary and is thus complete. Furthermore, its solutions are all periodic and have period τ (it is *isochronous*). The developing map \mathcal{D} maps $\Pi_*^{-1}(B_s)$ to a constant multiple of $B_0 = \partial/\partial v$. The completeness of B_s in C implies that the image of the developing map $\mathcal{D}(\tilde{C})$ is saturated by B_0 , and is thus of the form $V = \{u_0 < u < u_1\}$, with $0 \leq u_0 < u_1 \leq \infty$.

Let $p \in \partial C$. By Theorem 1, there exists a neighborhood U of p and a mapping $\Psi_0 : (U, p) \rightarrow (\mathbf{R}^2, O)$ that maps ∇_s to a connection ∇ of Type I or of Type II. The image of B_s under this mapping is a multiple of the vector field B , $\Psi_0(B_s) = \lambda B$ for some $\lambda \in \mathbf{R}$. (Notice that λB is a complete vector field in \mathbf{R}^2 .) By its isochronicity, B_s cannot vanish at p and thus B does not vanish at O : the connection ∇ cannot be of Type II and is necessarily of Type I (by Prop. 19).

Suppose, without loss of generality, that W (the tubular neighborhood of ∂C) is saturated by the flow of B_s . Let $T \subset \tilde{W}$ be a transversal intersecting once and only once each orbit of $P_*^{-1}B_s$ and such that $P(T) \subset U$. For each $q \in \tilde{W}$, let $t_q \in \mathbf{R}$ be such that $\Phi_{P_*^{-1}B_s}^{t_q}(q)$, the flow of $P_*^{-1}B_s$ in time t_q applied to q , belongs to T . Let $\Psi : \tilde{W} \rightarrow \mathbf{R}^2$ be the mapping defined by

$$(29) \quad \Psi(q) = \Phi_{\lambda B}^{-t_q} \circ \Psi_0 \circ P \circ \Phi_{P_*^{-1}B_s}^{t_q}(q).$$

It is a well-defined diffeomorphism onto its image which maps $P^{-1}(\nabla_s)$ to ∇ and $P_*^{-1}B_s$ to λB (it is, in some sense, a developing map).

The vector field λB has the first integral $h(x, y) = x$ and $(X \cdot h)/h = -1/n < 0$ (see Table 1). The function Ψ^*h and the vector field $\Psi_*^{-1}X$ are defined in \tilde{W} . They are invariant under the flow of $P^{-1}(B_s)$. In consequence, their images $h_s = (\Psi \circ P^{-1})^*h$ and $X_s = P_*\Psi_*^{-1}X$ are well-defined in W and satisfy the same relation.

Hence, for each point q in $C \cap W$, the orbit of X_s starting at q is defined for all positive time. Moreover, this orbit is properly embedded in C and, as time goes to infinity, accumulates to ∂C , the zero set of h_s .

The image of $\Pi_*^{-1}X_s$ under \mathcal{D} is a vector field of the form $X_0 + \lambda Y_0 = u\partial/\partial u + [\dots]\partial/\partial v$. By our previous arguments, for the initial conditions coming from $C \cap W$, the forward orbits of this vector field are complete in V . We conclude that the constant u_1 defining V is infinite and that p belongs to the end of C corresponding to u_1 . The normal forms do not allow us to describe the end of C corresponding to u_0 , this is, such an end is impossible to compactify. This shows, by contradiction, that the holonomy is not of the form $(1, \tau)$.

We conclude that the holonomy is generated by an element of the form (e^τ, v) with $\tau \neq 0$. It is conjugated to $(e^\tau, 0)$, the flow of A_0 in time τ . This proves our claim.

We will henceforth suppose that the holonomy is generated by the flow of A_0 in time τ . The holonomy preserves A_0 and induces, via \mathcal{D} , a well-defined Killing vector field A_s in C . As in the previous case, A_s will be complete and isochronous with period τ . The image of the developing map $\mathcal{D}(\tilde{C})$, being saturated by A_0 , is a cone of the form $V = \{c_0 < v/u < c_1\}$, with $-\infty \leq c_0 < c_1 \leq \infty$. There is a one-to-one mapping $\mathcal{D}^b : C \rightarrow \langle \Phi_{A_0}^\tau \rangle \setminus V$.

Let ∂C be a boundary component of C and let $p \in \partial C$. The vector field A_s does not vanish at p . Let $B_s \in \mathfrak{K}(\nabla_s)$ be a Killing vector field in a neighborhood of p satisfying $[A_s, B_s] = B_s$ (it is well-defined up to multiplication by a constant). Either B_s vanishes identically along ∂C or it is everywhere non-zero (by Prop. 19, the two cases lead to different normal forms for ∇_s at p).

- If B_s vanishes identically along ∂C , then there exists $n \in \frac{1}{2}\mathbf{Z}$, parameters $(\gamma, \varepsilon, \phi)$ and a map $\Psi_0 : (U, p) \rightarrow (\mathbf{R}^2, (0, 1))$ mapping ∇_s to the connection ∇ of Type $\text{II}^1(n)$, with parameters $(\gamma, \varepsilon, \phi)$, and mapping $C \cap U$ to $\{x > 0\}$ (Theorem 1 and Remark 22). Let E be the vector field in \mathbf{R}^2 that is mapped by $\mathcal{D} \circ \Psi_0^{-1}$ to A_0 . It is of the form $A + \lambda B$, for some $\lambda \in \mathbf{R}$. By post-composition of Ψ_0 with a suitable time of the flow of B , we will suppose that $E = A$. The vector field E has the primitive first integral $h(x, y) = xy^{1/(n-1)}$ and $X \cdot \log(h) = 1/(1-n) < 0$.
- If B_s does not vanish identically along ∂C , then there exists $n \in \frac{1}{2}\mathbf{Z}$, parameters $(\gamma, \varepsilon, \phi)$, and a map $\Psi_0 : (U, p) \rightarrow (\mathbf{R}^2, 0)$ mapping ∇_s to the connection ∇ of Type $\text{I}(n)$, with parameters $(\gamma, \varepsilon, \phi)$ and mapping $C \cap U$ to $\{x > 0\}$. Let $E = A + \lambda B$ ($\lambda \neq 0$) be the vector field that maps into A_0 via $\mathcal{D} \circ \Psi_0^{-1}$. Up to a change of coordinates given by the flow of A and since $\rho_* B = -B$ for (24), we may suppose that $\lambda = -1$, this is, $E = A - B$. It has the primitive first integral $h(x, y) = x(1-y)^{1/n}$ that satisfies $X \cdot \log(h) = -1/n < 0$.

Define $\Psi : \tilde{W} \rightarrow \mathbf{R}^2$ in a way analogue to (29). It is a diffeomorphism onto its image, maps $P^{-1}(\nabla_s)$ to ∇ and $P_*^{-1}A_s$ to E . Moreover, it is equivariant with respect to the flow of $P_*^{-1}A_s$ in \tilde{W} and the flow of E in $\Psi(\tilde{W}) \subset \mathbf{R}^2$. Hence, if $\Phi_E^\tau : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ denotes the flow of E in time τ , there is a one-to-one mapping $\Psi^b : W \rightarrow \Psi(\tilde{W})/\langle \Phi_E^\tau \rangle$.

As before, the function $h_s = (\Psi \circ P^{-1})^* h$ and the vector field $X_s = P_* \Psi_*^{-1} X$ are well-defined in W and $X_s \cdot \log(h_s)$ is a strictly negative real. The level curves of h_s are circles. Those in C are orbits of A_s and $\partial C = h_s^{-1}(0)$. For each point q in $C \cap W$, the orbit of X_s is, in positive time, *complete and properly embedded* in C .

Suppose that the end of C associated to ∂C corresponds to the end of V associated to c_1 . The image of $\Pi_*^{-1}X_s$ under \mathcal{D} is a vector field of the form $Z = X_0 + \lambda Y_0 = u(\partial/\partial u - \lambda\partial/\partial v)$. For the forward orbits of Z to be complete and properly embedded in $\langle \Phi_{A_0}^\tau \rangle \setminus V$, we must have that V is, on the corresponding side, bounded by $v/u = -\lambda$, the common orbit of Z and A_0 (otherwise, Z will not be complete in $\langle \Phi_{A_0}^\tau \rangle \setminus V$ or its orbits will not be properly embedded: see Figure 1). In particular, $c_1 < \infty$.

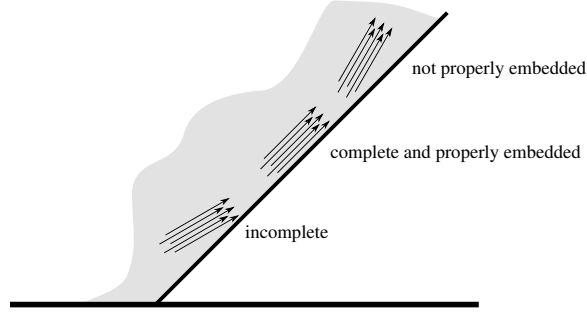


FIGURE 1. In the quotient of the cone V , the forward orbits of a semisimple left-invariant field may be incomplete, may not be properly embedded or, in only one case, may be complete and properly embedded.

Thus, the connection ∇_0 has a marking Z and V is bounded by the unique common orbit of Z and A_0 . Since Z is defined in all of V , the vector field X_s may be extended to all of C . Every orbit of X_s in C accumulates to ∂C as time goes to infinity.

For the other boundary component $\partial' C$ of \bar{C} we define analogue objects $X'_s, h'_s, Z', n', \nabla'_0, (\gamma', \varepsilon', \phi')$. Notice that X_s and X'_s are different, since their orbits in positive time converge to different boundary components of \bar{C} .

In the neighborhood of a point p in ∂C , $\Psi_0(X_s) = X$ and $\Psi_0(X'_s) = X + \lambda Y$ for some $\lambda \in \mathbf{R}$. By Remark 16, the nature of ∇_s at $\partial' C$ (if it is either of Type I or of Type II) and the value of n' determine the value of $(\gamma, \varepsilon, \phi)$ in ∇ , up to the natural equivalence. In particular, by (the proof of) Lemma 24, γ and ϕ cannot vanish simultaneously. We must conclude that $n \in \mathbf{Z}$. In an analogue way, $n' \in \mathbf{Z}$.

We now have that \bar{C} is very similar to the quotient of V_0 (as defined in the previous section) under the flow of A in time τ , except for the assumption upon the parity of n and n' . We will now prove that *if n is even (resp. if n is odd), ∇_0 is of Type I (resp. of Type II)*.

Let us consider the cylinder C' neighboring C across ∂C . It has the boundary components ∂C and $\partial'' C$. Consider the coordinate Ψ_0 around the point $p \in \partial C$ and suppose (Remark 22) that this coordinates map $C \cap U$ to $\{x > 0\}$. In $\{x > 0\}$ we have two vector fields X and $X + \lambda Y$ that give two special (Definition 15) markings for the corresponding connection in the affine plane. These two vector fields induce also two special markings for the connection in the affine plane induced by the restriction of ∇_s to C' . However, an invariant connection on the affine group does not have three special markings: indeed, by contradiction, from formula (18),

- if there are two markings of Type I, $\delta = 1$ and $\delta' = 1$, and thus $\alpha'' = \delta''$: the third special marking must be of Type II (for $\delta'' \neq \alpha'' + 1$), and thus $n = -1$. This is impossible: we conclude that the connection does not have three special markings;
- if there are two markings of Type II, $\delta = 1 + \alpha$ and $\delta' = 1 + \alpha$ and thus $3\delta'' = \alpha'' + 1$. If the third marking is of Type II, then $\delta'' = 0$ and $\alpha'' = -1$; if it is of Type I, then $\delta'' = 1$ and $\alpha'' = 2$. In both cases this is in contradiction with the possible values of n . We must conclude that the connection does not have three special markings.

We conclude that the vector field $X + \lambda Y$ induces both the special marking corresponding to $\partial' C$ and the one corresponding to $\partial'' C$. Hence, in C and in C' , the special markings are given by X and $X + \lambda Y$. For the connections of Type II¹, we have

$$\frac{E \wedge (X + \lambda Y)}{E \wedge X} = 1 + \lambda \frac{E \wedge Y}{E \wedge X} = 1 + \frac{\lambda}{x^{n-1}y},$$

whose sign, close to $(0, 1)$, is determined by the sign of λ/x^{n-1} . For connections of Type I, we have

$$\frac{E \wedge (X + \lambda Y)}{E \wedge X} = 1 + \lambda \frac{E \wedge Y}{E \wedge X} = 1 + \frac{\lambda}{x^n(y+1)}.$$

Its sign is controlled by λ/x^n . But we previously established in (26) that these expressions must be negative. Hence, n must be an even integer for Type I and an odd one for Type II¹, as claimed.

This implies that \bar{C} is covered by \bar{V}_0 for the connection $\nabla_{n,n'}$ of the previous section and that the same is true of \bar{C}' , since the connection in the universal covering is determined by the numbers attached to the special markings and since the period of A_s is the same in both C and C' . This implies that (S, ∇_s) is a global quotient of $(M, \nabla_{n,n'})$. This finishes the proof of Theorem 3 in the case where $\mathfrak{K}(\nabla) \approx \mathfrak{aff}(\mathbf{R})$.

7.2. When the killing algebra is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. We prove here:

Proposition 26. *Let S be a compact real analytic surface endowed with a real-analytic torsion-free affine connection ∇_s . If the Killing algebra of ∇_s is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ and ∇_s is locally homogeneous on a nontrivial open set in S , then ∇_s is locally homogeneous on all of S .*

Let $\Omega \subset S$ be the maximal open set where the Killing algebra is transitive. The normal form of ∇_s at points of the boundary of Ω is that of Type III. In particular, $\mathfrak{K}(\nabla_s) \approx \mathfrak{sl}(2, \mathbf{R})$ and the points of $S \setminus \Omega$ are isolated (Ω is connected). The restriction of ∇_s to Ω is locally homogeneous everywhere and is locally modeled in the homogeneous space of $\widetilde{\mathrm{SL}(2, \mathbf{R})}$ (the universal covering of $\mathrm{SL}(2, \mathbf{R})$) associated to the subgroup generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbf{R})$. This homogeneous space is, naturally, the universal covering of $\mathbf{R}^2 \setminus \{0\}$. Notice that it is also acted on transitively by $\widetilde{\mathrm{GL}(2, \mathbf{R})}$.

Let us define a foliation with singularities \mathcal{F} on S . For each point of Ω , define \mathcal{F} as the foliation tangent to the centralizer of $\mathfrak{K}(\nabla_s)$ in the Lie algebra generated by the $\widetilde{\mathrm{GL}(2, \mathbf{R})}$ -action. In $\mathbf{R}^2 \setminus \{0\}$, this centralizer is generated by $x\partial/\partial x + y\partial/\partial y$. For the points $p \notin \Omega$, in the coordinates of Theorem 1, the foliation extends as the foliation with singularities induced by the previous vector field at 0. These are the only singularities of \mathcal{F} . They have Poincaré-Hopf index 1. Hence, S is a sphere, the complement of Ω consists of two points and Ω is a cylinder.

Since S is simply connected, all the killing vector fields are globally well-defined in S and induce an action of $\widetilde{\mathrm{SL}(2, \mathbf{R})}$ that, in fact (from the normal forms in the neighborhood of the points that do not belong to Ω), factors through a faithful action of $\mathrm{SL}(2, \mathbf{R})$. By the normal forms, the restriction of this last action to Ω is transitive. Thus Ω identifies with $\mathbf{R}^2 \setminus \{0\}$, seen as a homogeneous space of $\mathrm{SL}(2, \mathbf{R})$.

We can see S as the identification of two copies of \mathbf{R}^2 glued along $\mathbf{R}^2 \setminus \{0\}$ in a $\mathrm{SL}(2, \mathbf{R})$ -equivariant way. In particular, there is an invariant area form on S (which is a constant multiple of the standard area form in restriction to each copy of \mathbf{R}^2). Since S is compact, the integral of the area form is finite. In the copies of \mathbf{R}^2 , the integral of the area form is infinite. This contradiction finishes the proof of the Proposition and of Theorem 3.

8. CONNECTIONS IN \mathbf{R}^2 THAT ARE INVARIANT UNDER THE SPECIAL LINEAR GROUP

We will now study the geodesics of the connection of Type III in \mathbf{R}^2 . We will prove that such a connection is geodesically complete if and only if $\gamma = 0$. We may describe the connections and their geodesics by making use of the *special affine curvature* of curves in the plane (see, for example, [Spi70, Chapter I]):

Proposition 27. *A non-flat torsion-free real-analytic geodesically complete connection ∇ in \mathbf{R}^2 , invariant under the action of $\mathrm{SL}(2, \mathbf{R})$, is of the form*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = ky^2E, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = -kxyE, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = kx^2E,$$

for $E = x\partial/\partial x + y\partial/\partial y$ and some $k \in \mathbf{R}^*$. The parametrized geodesics are either:

- lines through the origin at constant speed or
- conics of special affine curvature $\sqrt[3]{k}$, centered at the origin and parametrized at constant speed with respect to special affine arc length.

A straightforward calculation shows that the curve $s \mapsto (s, 0)$ is a geodesic of every connection of Type III and thus every line through the origin parametrized at constant speed is a geodesic. Let $v(s) = (x(s), y(s))$ be a non-constant geodesic of ∇ that is not a line through the origin, this is, such that $v(s)$ and $v'(s)$ are linearly independent. For each s , let $A(s) \in \mathrm{SL}(2, \mathbf{R})$ such that $A(s)v(s) = (1, 0)$ and such that $A(s)v'(s) = (0, u(s))$. We have $u = xy' - yx'$. The equation of the geodesics becomes the first-order non-linear equation $u' = 2\gamma\tau^2$. The solution to this equation with initial condition τ_0 is

$$u(s) = \frac{u_0}{1 - 2\gamma u_0 s}.$$

We must conclude that, if the connection is geodesically complete, $\gamma = 0$. In this case, if $\varepsilon = -2$, then a geodesic is given by $v(s) = (\cos(s), \sin(s))$. By invariance, the other geodesics are ellipses of area 1 centered at the origin (equivalently, ellipses centered at the origin whose special affine curvature is 1). If $\varepsilon = 2$, then a geodesic is given by $v(s) = (\cosh(s), \sinh(s))$ (a hyperbola centered at the origin with special affine curvature equal to 1). By the natural rescalings induced by homotheties, we obtain the proof of the Proposition.

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