



**HAL**  
open science

# Carleman Estimate and Inverse Source Problem for Biot's Equations Describing Wave Propagation in Porous Media

Mourad Bellassoued, Masahiro Yamamoto

► **To cite this version:**

Mourad Bellassoued, Masahiro Yamamoto. Carleman Estimate and Inverse Source Problem for Biot's Equations Describing Wave Propagation in Porous Media. 2013. hal-00816142

**HAL Id: hal-00816142**

**<https://hal.science/hal-00816142>**

Submitted on 20 Apr 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Carleman Estimate and Inverse Source Problem for Biot's Equations Describing Wave Propagation in Porous Media

Mourad Bellassoued<sup>1,2,3</sup> and Masahiro Yamamoto<sup>4</sup>

<sup>1</sup> University of Carthage, Faculty of Sciences of Bizerte,  
Department of Mathematics, 7021 Jarzouna Bizerte, Tunisia.

<sup>2</sup> Fédération Denis-Poisson,  
FR CNRS 2964, B.P. 6759, 45067 Orléans cedex 2, France.

<sup>3</sup> LE STUDIUM ®, Institute for Advanced Studies, Orléans, France  
mourad.bellassoued@fsb.rnu.tn

<sup>4</sup>Department of Mathematical Sciences,  
The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153, Japan.  
myama@ms.u-tokyo.ac.jp

## Abstract

According to Biot's paper in 1956, by using the Lagrangian equations in classical mechanics, we consider a problem of the filtration of a liquid in porous elastic-deformation media whose mechanical behavior is described by the Lamé system coupled with a hyperbolic equation. Assuming the null surface displacement on the whole boundary, we discuss an inverse source problem of determining a body force only by observation of surface traction on a suitable subdomain along a sufficiently large time interval. Our main result is a Hölder stability estimate for the inverse source problem, which is proved by a new Carleman estimate for Biot's system.

## 1 Introduction

In 1956, Biot [7] presented a three-dimensional theory for coupled frame-fluid wave propagation in fluid saturated porous media, treating the solid frame and the saturating fluid as two separate co-located coupled continua. Two second order coupled partial differential equations were derived from this theory. More precisely, let us consider an open and bounded domain  $\Omega$  of  $\mathbb{R}^3$  with  $C^\infty$  boundary  $\Gamma = \partial\Omega$ , and let  $\nu = \nu(x)$  be the unit outward normal vector to  $\partial\Omega$  at  $x$ . Given  $T > 0$ , Biot's equation is written as:

$$\begin{aligned} \varrho_{11}\partial_t^2\mathbf{u}^s + \varrho_{12}\partial_t^2\mathbf{u}^f - \Delta_{\mu,\lambda}\mathbf{u}^s(x,t) - \nabla(\mathbf{q}\operatorname{div}\mathbf{u}^f) &= F_1, \\ \varrho_{12}\partial_t^2\mathbf{u}^s + \varrho_{22}\partial_t^2\mathbf{u}^f - \nabla(\mathbf{q}\operatorname{div}\mathbf{u}^s) - \nabla(\mathbf{r}\operatorname{div}\mathbf{u}^f) &= F_2, \quad \text{in } Q := \Omega \times (-T, T) \end{aligned} \quad (1.1)$$

with the boundary condition

$$\mathbf{u}^s(x,t) = 0, \quad \mathbf{u}^f(x,t) \cdot \nu = 0, \quad (x,t) \in \Sigma := \Gamma \times (-T, T) \quad (1.2)$$

and the initial condition

$$(\mathbf{u}^s(x, 0), \partial_t \mathbf{u}^s(x, 0)) = (0, 0), \quad \left( \mathbf{u}^f(x, 0), \partial_t \mathbf{u}^f(x, 0) \right) = (0, 0), \quad x \in \Omega \quad (1.3)$$

where  $F = (F_1, F_2)^T$  is an external force with  $F_\ell = (F_\ell^1, F_\ell^2, F_\ell^3)^T$ ,  $\ell = 1, 2$ , and  $\Delta_{\mu, \lambda}$  is the elliptic second-order linear differential operator given by

$$\begin{aligned} \Delta_{\mu, \lambda} \mathbf{v}(x) &\equiv \mu \Delta \mathbf{v}(x) + (\mu + \lambda) (\nabla \operatorname{div} \mathbf{v}(x)) \\ &+ (\operatorname{div} \mathbf{v}(x)) \nabla \lambda(x) + (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \nabla \mu(x), \quad x \in \Omega. \end{aligned} \quad (1.4)$$

Throughout this paper,  $t$  and  $x = (x_1, x_2, x_3)$  denote the time variable and the spatial variable respectively, and  $\mathbf{u}^s = (u_1^s, u_2^s, u_3^s)^T$  and  $\mathbf{u}^f = (u_1^f, u_2^f, u_3^f)^T$  denote respectively the solid frame and fluid phase displacement vectors at the location  $x$  and the time  $t$ .

Here and henceforth  $\cdot^T$  denotes the transpose of matrices under consideration. We assume that the Lamé parameters  $\mu$  and  $\lambda$  satisfy

$$\mu(x) > 0, \quad \lambda(x) + \mu(x) > 0, \quad \forall x \in \overline{\Omega}.$$

The function  $q(x) > 0$ ,  $x \in \overline{\Omega}$ , is the dilatational coupling factor between the fluid phase and the solid frame. The coefficient  $r(x) > 0$ ,  $x \in \overline{\Omega}$  is the bulk modulus of the fluid phase and  $\varrho_{11}(x)$ ,  $\varrho_{22}(x) > 0$ ,  $x \in \overline{\Omega}$  are the corrected mass densities for the solid phase and the fluid phase porosity and  $\varrho_{12}(x)$  is the inertial coupling factor and see Hörlin and Peter [13].

We assume that the sources terms are given by

$$F_\ell(x, t) = p_\ell(x) R_\ell(x, t), \quad \ell = 1, 2, \quad (x, t) \in Q, \quad (1.5)$$

where  $p_\ell \in H^2(\Omega)$  is real-valued and  $R_\ell = (R_\ell^1, R_\ell^2, R_\ell^3)^T$  satisfy

$$\sum_{j=2}^3 \left( \|\partial_t^j R\|_{L^\infty(Q)}^2 + \|\partial_t^j \nabla R\|_{L^\infty(Q)}^2 \right) \leq C. \quad (1.6)$$

The main subject of this paper is the inverse problem of determining  $p = (p_1, p_2) \in (H^2(\Omega))^2$  uniquely from observed data of the displacement vector  $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$  in a subdomain  $\omega \subset \Omega$ . It is an important problem, for example, in mechanics to determine the source  $p$  inside a porous body from measurements of the slide frame and fluid phase displacements in  $\omega$ .

## 1.1 Inverse problem

Let  $\omega \subset \Omega$  be an arbitrarily given subdomain such that  $\partial\omega \supset \partial\Omega$ , i.e.,  $\omega = \Omega \cap \mathcal{V}$  where  $\mathcal{V}$  is a neighborhood of  $\Gamma$  in  $\mathbb{R}^3$  and let  $R(x, t) = (R_1(x, t), R_2(x, t))$  be appropriately given. Then we want to determine  $p(x) = (p_1(x), p_2(x))$ ,  $x \in \Omega$ , by measurements  $\mathbf{u}|_{\omega \times (-T, T)}$ .

Our formulation of the inverse problem requires only a finite number of observations. As for inverse problems for non-stationary Lamé system by infinitely many boundary observations (i.e., Dirichlet-to-Neumann map), we refer to Rachele [36], for example.

For the formulation with a finite number of observations, Bukhgeim and Klibanov [10] created a method based on a Carleman estimate and established the uniqueness for inverse problems of determining spatially varying coefficients for scalar partial differential equations. See also Bellassoued [1],

[2], Bellassoued and Yamamoto [3], [4], Benabdallah, Cristofol, Gaitan and Yamamoto [6], Bukhgeim [8], Bukhgeim, Cheng, Isakov and Yamamoto [9], Imanuvilov and Yamamoto [18] - [20], Isakov [24], [25], Khaïdarov [28], Klibanov [29], [30], Klibanov and Timonov [32], Klibanov and Yamamoto [33], Yamamoto [39]. In particular, as for inverse problems for the isotropic Lamé system, we can refer to Ikehata, Nakamura and Yamamoto [15], Imanuvilov, Isakov and Yamamoto [21], Imanuvilov and Yamamoto [22] - [23], Isakov [24], Isakov and Kim [26].

A Carleman estimate is an inequality for a solution to a partial differential equation with weighted  $L^2$ -norm and effectively yields the unique continuation for a partial differential equation with non-analytic coefficients. As a pioneering work concerning a Carleman estimate, we refer to Carleman's paper [11] where what is called a Carleman estimate was proved and applied it for proving the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since [11], the theory of Carleman estimates has been developed and we refer, for example, to Hörmander [14] and Isakov [25] for Carleman estimates for functions having compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). For Carleman estimates for functions without compact supports, we refer to Bellassoued and Yamamoto [5], Fursikov and Imanuvilov [12], [16], Lavrent'ev, Romanov and Shishat'skiĭ [34], Tataru [38]. Moreover Carleman estimates have been applied for estimating the energy and see e.g., Imanuvilov and Yamamoto [23], Kazemi and Klibanov [27], Klibanov and Malinsky [31], Klibanov and Timonov [32].

## 1.2 Notations and statement of main results

In order to formulate our results, we need to introduce some notations. For  $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ , we define the following set of the scalar coefficients

$$\mathcal{C}(m, \theta) = \left\{ c \in \mathcal{C}^2(\overline{\Omega}), c(x) > c^* > 0, x \in \overline{\Omega}, \|c\|_{\mathcal{C}^2(\overline{\Omega})} \leq m, \frac{\nabla c \cdot (x - x_0)}{2c} \leq 1 - \theta \right\}, \quad (1.7)$$

where the constants  $m > 0$  and  $\theta \in (0, 1)$  are given.

### Assumption A.1

Throughout this paper, we assume that the coefficients  $(\varrho_{ij})_{1 \leq i, j \leq 2}$ ,  $\mu$ ,  $\lambda$ ,  $q$ ,  $r \in \mathcal{C}^2(\overline{\Omega})$  satisfy the following conditions

$$\begin{aligned} \varrho(x) &= \varrho_{11}(x)\varrho_{22}(x) - \varrho_{12}^2(x) > 0, \quad \forall x \in \overline{\Omega}, \\ \lambda(x)r(x) - q^2(x) &> 0, \quad \forall x \in \overline{\Omega}. \end{aligned} \quad (1.8)$$

Let  $A(x) = (a_{ij}(x))_{1 \leq i, j \leq 2}$  be the  $2 \times 2$ -matrix given by

$$A(x) = \frac{1}{\varrho} \begin{pmatrix} \varrho_{22} & -\varrho_{12} \\ -\varrho_{12} & \varrho_{11} \end{pmatrix} \begin{pmatrix} 2\mu + \lambda & q \\ q & r \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (1.9)$$

By (1.8), we can prove that  $(a_{ij}(x))_{1 \leq i, j \leq 2}$  is a positive definite matrix on  $\overline{\Omega}$ .

### Assumption A.2:

Let  $A(x)$  have two distinct positive eigenvalues:  $\mu_2(x), \mu_3(x) > 0$ ,  $\mu_2(x) \neq \mu_3(x)$ . Moreover setting,  $\mu_1 := (\varrho^{-1}\varrho_{22})\mu$ , we assume

$$\mu_1, \mu_2, \mu_3 \in \mathcal{C}(m, \theta). \quad (1.10)$$

**Assumption A.3:**

We assume that the solution  $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$  satisfies the a priori boundedness and regularity:

$$\mathbf{u} \in H^5(Q), \quad \|\mathbf{u}\|_{H^5(Q)} \leq M_0, \quad (1.11)$$

for some positive constant  $M_0$ .

Before stating the main result on the stability for the inverse source problem, we present Theorem 1.1 on the unique existence of strong solution to (1.1)-(1.2) with initial condition:

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{and} \quad \partial_t \mathbf{u}(\cdot, 0) = \mathbf{u}_1.$$

Let  $V(\Omega) = (H^1(\Omega))^3 \times H(\text{div}, \Omega)$ , where

$$H(\text{div}, \Omega) = \{\mathbf{u} \in (L^2(\Omega))^3; \quad \text{div } \mathbf{u} \in L^2(\Omega)\}. \quad (1.12)$$

The norm in  $V(\Omega)$  is chosen as follows

$$\|(\mathbf{v}^1, \mathbf{v}^2)\|_{V(\Omega)}^2 = \|\mathbf{v}^1\|_{H^1(\Omega)}^2 + \|\mathbf{v}^2\|_{L^2(\Omega)}^2 + \|\text{div } \mathbf{v}^2\|_{L^2(\Omega)}^2, \quad \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V(\Omega).$$

**Theorem 1.1.** *Let  $F \in H^1(-T, T; L^2(\Omega))$ ,  $(\mathbf{u}_0, \mathbf{u}_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^6 \times (H^1(\Omega))^6$ . Then there exists a unique solution  $\mathbf{u}(x, t) = (\mathbf{u}^s(x, t), \mathbf{u}^f(x, t))$  of (1.1)-(1.2) with initial data  $(\mathbf{u}_0, \mathbf{u}_1)$  such that*

$$\begin{aligned} \mathbf{u}^s &\in \mathcal{C}([-T, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([-T, T]; H^1(\Omega)) \cap \mathcal{C}^2([-T, T]; L^2(\Omega)) \\ \mathbf{u}^f &\in \mathcal{C}^2([-T, T]; L^2(\Omega)), \quad \text{div } \mathbf{u}^f \in \mathcal{C}([-T, T]; H^1(\Omega)) \cap \mathcal{C}^1([-T, T]; L^2(\Omega)). \end{aligned} \quad (1.13)$$

*In particular there exists a constant  $C > 0$  such that*

$$\|\mathbf{u}\|_{\mathcal{C}^2([-T, T]; L^2(\Omega))} \leq C(\|F\|_{H^1(-T, T; L^2(\Omega))} + \|\mathbf{u}_0\|_{H^2(\Omega)} + \|\mathbf{u}_1\|_{H^1(\Omega)}).$$

*Moreover, if  $F = 0$ , then the energy of the solution  $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$  given by*

$$E(t) = \frac{1}{2} \int_{\Omega} \left( M(x) \partial_t \mathbf{u} \cdot \partial_t \mathbf{u} + \lambda |\text{div } \mathbf{u}^s|^2 + 2\mu |\varepsilon(\mathbf{u}^s)|^2 + r |\text{div } \mathbf{u}^f|^2 + 2q (\text{div } \mathbf{u}^f)(\text{div } \mathbf{u}^s) \right) dx$$

*is conserved, that is,*

$$E(t) = E(0), \quad \forall t \geq 0.$$

*Here  $M(x) = (\rho_{ij}(x)I_3)_{1 \leq i, j \leq 2}$  and  $\varepsilon(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$ .*

The proof is based on the Galerkin method and see Santos [37] for the case  $n = 2$ . For completeness we will give a proof for dimension 3 in Section 4.

In order to formulate our stability estimates for the inverse problem we introduce some notations. Let  $\vartheta : \overline{\Omega} \rightarrow \mathbb{R}$  be the strictly convex function given by

$$\vartheta(x) = |x - x_0|^2, \quad x \in \overline{\Omega}. \quad (1.14)$$

Set

$$D^2 = \max_{x \in \overline{\Omega}} \vartheta(x), \quad d^2 = \min_{x \in \overline{\Omega}} \vartheta(x), \quad D_0^2 = D^2 - d^2. \quad (1.15)$$

By Assumption A.2, there exist constants  $c_j^* > 0$ ,  $j = 1, 2, 3$  such that  $\mu_j(x) > c_j^* > 0$  for all  $x \in \overline{\Omega}$ ,  $j = 1, 2, 3$ . Let  $c_0^* = \min\{c_1^*, c_2^*, c_3^*\}$ . We choose  $\beta > 0$  such that

$$\beta + \frac{mD_0}{\sqrt{c_0^*}}\sqrt{\beta} < \theta c_0^*, \quad c_0^*d^2 - \beta D^2 > 0. \quad (1.16)$$

Here we note that since  $x_0 \notin \overline{\Omega}$ , such  $\beta > 0$  exists.

We set

$$T_0 = \frac{D_0}{\sqrt{\beta}}. \quad (1.17)$$

The main results of this paper can be stated as follows:

**Theorem 1.2.** (Stability) Assume (A.1), (A.2), and (A.3). Let  $T > T_0$  and  $\mathbf{u}$  be the solution of (1.1)-(1.2) and (1.3). Moreover let assume that  $\Phi_j(x) := R_j(x, 0)$  satisfy

$$\Phi_j(x) \cdot (x - x_0) \neq 0 \quad \text{for all } x \in \overline{\Omega}. \quad (1.18)$$

Let  $M > 0$ . Then there exist constants  $C > 0$  and  $\kappa \in (0, 1)$  such that the following estimate holds:

$$\|p_1\|_{H_0^1(\Omega)}^2 + \|p_2\|_{H_0^1(\Omega)}^2 \leq C\mathcal{E}_\omega(\mathbf{u})^\kappa \quad (1.19)$$

for any  $p_\ell \in H^2(\Omega)$ ,  $\ell = 1, 2$ , such that  $\|p_\ell\|_{H^2} \leq M$  and  $p_\ell = 0$ ,  $\nabla p_\ell = 0$  on  $\Gamma$ . Here

$$\mathcal{E}_\omega(\mathbf{u}) = \sum_{j=2}^3 \|\partial_t^j \mathbf{u}\|_{H^2(\omega \times (-T, T))}^2.$$

By Theorem 1.2, we can readily derive the uniqueness in the inverse problem:

**Corollary 1.1.** Under the assumptions in Theorem 1.2, we have the uniqueness:

Let  $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$  satisfy Biot's system (1.1)-(1.3) such that  $\mathbf{u}(x, t) = 0$ ,  $(x, t) \in \omega \times (-T, T)$ . Then  $p_1(x) = p_2(x) = 0$  for all  $x \in \Omega$  and  $\mathbf{u}(x, t) = 0$  in  $Q$ .

The remainder of the paper is organized as follows. In section 2, we give a Carleman estimate for the Biot's system. In section 3 we prove Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.1.

## 2 Carleman estimate for Biot's system

In this section we will prove a Carleman estimate for Biot's system, which is interesting of itself. In order to formulate our Carleman estimate, we introduce some notations. Let  $\vartheta : \overline{\Omega} \rightarrow \mathbb{R}$  be the strictly convex function given by (1.14), where  $x_0 \notin \overline{\Omega}$ .

We define two functions  $\psi, \varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^\infty$  by

$$\psi(x, t) = |x - x_0|^2 - \beta |t|^2 \quad \text{for all } x \in \Omega, \quad -T \leq t \leq T, \quad (2.1)$$

$$\varphi(x, t) = e^{\gamma\psi(x, t)}, \quad \gamma > 0,$$

where  $T > T_0$ . Therefore, by (1.17) and (1.15), we have

$$\varphi(x, 0) \geq d_0, \quad \varphi(x, \pm T) < d_0 \quad (2.2)$$

with  $d_0 = \exp(\gamma d^2)$ . Thus, for given  $\eta > 0$ , we can choose sufficiently small  $\varepsilon = \varepsilon(\eta)$  such that

$$\begin{aligned} \varphi(x, t) &\leq d_0 - \eta \equiv d_1 \quad \text{for all } (x, t) \in \{(x, t) \in Q; |t| > T - 2\varepsilon\}, \\ \varphi(x, t) &\geq d_0 - \frac{\eta}{2} \equiv d_2 \quad \text{for all } (x, t) \in \{(x, t) \in Q; |t| < \varepsilon\}. \end{aligned} \quad (2.3)$$

Let  $(\mathbf{u}^s, \mathbf{u}^f)$  satisfy Biot's system

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{u}^s(x, t) + \varrho_{12} \partial_t^2 \mathbf{u}^f - \Delta_{\mu, \lambda} \mathbf{u}^s(x, t) - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^f) &= F_1, \\ \varrho_{12} \partial_t^2 \mathbf{u}^s(x, t) + \varrho_{22} \partial_t^2 \mathbf{u}^f - \nabla (\mathbf{r} \operatorname{div} \mathbf{u}^s) - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^f) &= F_2, \quad \text{in } Q. \end{aligned} \quad (2.4)$$

The following theorem is a Carleman estimate for Biot's system (2.4).

**Theorem 2.1.** *There exist  $\tau_* > 0$  and  $C > 0$  such that the following estimate holds:*

$$\begin{aligned} \int_Q \tau \left( |\nabla_{x,t} \mathbf{u}^s|^2 + |\nabla_{x,t} (\operatorname{div} \mathbf{u}^s)|^2 + |\nabla_{x,t} (\operatorname{div} \mathbf{u}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ + \int_Q \tau^3 \left( |\mathbf{u}^s|^2 + |\operatorname{div} \mathbf{u}^s|^2 + |\operatorname{div} \mathbf{u}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|F|^2 + |\nabla F|^2) e^{2\tau\varphi} dxdt \end{aligned} \quad (2.5)$$

for any  $\tau \geq \tau_*$  and any solution  $(\mathbf{u}^s, \mathbf{u}^f) \in (H^2(Q))^6$  to (2.4) which is supported in a fixed compact set  $K \subset \operatorname{int}(Q)$ .

In order to prove Theorem 2.1, we use a Carleman estimate for a coupling hyperbolic system, which we discuss in the next subsection.

## 2.1 Carleman estimate for a hyperbolic system

First we recall the following Carleman estimate for a scalar hyperbolic equation. As for the proof, we refer to Bellassoued and Yamamoto [5], and Imanuvilov and Yamamoto [20] for example.

**Lemma 2.1.** *Let  $c \in \mathcal{C}(m, \theta)$ . There exist constants  $C > 0$  and  $\tau_* > 0$  such that the following Carleman estimate holds:*

$$C \int_Q e^{2\tau\varphi} \left( \tau |\nabla_{x,t} y|^2 + \tau^3 |y|^2 \right) dxdt \leq \int_Q e^{2\tau\varphi} |(\partial_t^2 - c\Delta)y|^2 dxdt$$

whenever  $y \in H^2(Q)$  is supported in a fixed compact set  $K \subset \operatorname{int}(Q)$  and any  $\tau \geq \tau_*$ .

Let  $v = (v_1, v_2) \in (H^2(\Omega))^2$  satisfy the following hyperbolic system

$$\begin{cases} \partial_t^2 v_1 - b_{11}(x)\Delta v_1 - b_{12}(x)\Delta v_2 = g_1 & \text{in } Q \\ \partial_t^2 v_2 - b_{21}(x)\Delta v_1 - b_{22}(x)\Delta v_2 = g_2 & \text{in } Q, \end{cases} \quad (2.6)$$

for  $g = (g_1, g_2) \in (L^2(Q))^2$ . We assume that the matrix  $B(x) = (b_{ij}(x))_{1 \leq i, j \leq 2}$  has two distinct positive eigenvalues  $c_1, c_2 \in \mathcal{C}(m, \theta)$ . Then, by Lemma 2.1, we have the following Carleman estimate.

**Lemma 2.2.** *There exist constants  $C > 0$  and  $\tau_* > 0$  such that the following Carleman estimate holds*

$$C \int_Q e^{2\tau\varphi} \left( \tau |\nabla_{x,t} v|^2 + \tau^3 |v|^2 \right) dxdt \leq \int_Q e^{2\tau\varphi} |g|^2 dxdt$$

for any  $\tau \geq \tau_*$ , whenever  $v \in H^2(Q)$  is a solution of (2.6) and supported in a fixed compact set  $K \subset \text{int}(Q)$ .

**Proof.** The system (2.6) can be written in the equivalent form

$$\partial_t^2 v - B(x)\Delta v = g \quad \text{in } Q. \quad (2.7)$$

By the assumption on  $B(x)$ , there exists a matrix  $P(x)$  such that

$$(P^{-1}BP)(x) = \text{Diag}(c_1(x), c_2(x)) = \Lambda(x), \quad x \in \Omega.$$

Therefore system (2.7) can be written in an equivalent form:

$$\partial_t^2 \tilde{v} - \Lambda(x)\Delta \tilde{v} = \tilde{g} + \mathcal{B}_1(x, \partial)v,$$

where

$$\tilde{v}(x, t) = P^{-1}(x)v(x, t), \quad \tilde{g}(x, t) = P^{-1}(x)g(x, t), \quad (2.8)$$

and  $\mathcal{B}_1$  is a first-order differential operator.

Since  $c_j \in \mathcal{C}(m, \theta)$  for  $j = 1, 2$ , we can apply Lemma 2.1 for the two components of  $\tilde{v}$  and obtain

$$C \int_Q e^{2\tau\varphi} \left( \tau |\nabla_{x,t} \tilde{v}|^2 + \tau^3 |\tilde{v}|^2 \right) dxdt \leq \int_Q e^{2\tau\varphi} |\tilde{g}|^2 dxdt + \int_Q e^{2\tau\varphi} (|v|^2 + |\nabla v|^2) dxdt$$

and, by (2.8), we easily obtain

$$|v(x, t)| \leq C |\tilde{v}(x, t)|, \quad |\nabla v(x, t)| \leq C (|\nabla \tilde{v}(x, t)| + |\tilde{v}(x, t)|), \quad |\tilde{g}(x, t)| \leq C |g(x, t)|$$

for  $(x, t) \in Q$ . This completes the proof.  $\square$

## 2.2 Proof of the Carleman estimate for Biot's system

In this section, we derive a global Carleman estimate for a solutions of system (2.4). We consider the  $6 \times 6$ -matrix

$$M(x) = \begin{pmatrix} \varrho_{11}(x)I_3 & \varrho_{12}(x)I_3 \\ \varrho_{12}(x)I_3 & \varrho_{22}(x)I_3 \end{pmatrix}. \quad (2.9)$$

Here  $I_3$  is the  $3 \times 3$  identity matrix. Then by Assumption A.1, we have

$$M^{-1}(x) = \frac{1}{\varrho} \begin{pmatrix} \varrho_{22}(x)I_3 & -\varrho_{12}(x)I_3 \\ -\varrho_{12}(x)I_3 & \varrho_{11}(x)I_3 \end{pmatrix}.$$



Let  $v^s = \operatorname{div} \mathbf{u}^s$ ,  $v^f = \operatorname{div} \mathbf{u}^f$ ,  $v = (v^s, v^f)$  and  $\mathbf{w}^s = \operatorname{curl} \mathbf{u}^s$ . Put  $G = M^{-1}F$ ,  $F = (F_1, F_2)^T$  and apply  $M^{-1}$  to system (2.4), we obtain

$$\begin{aligned} \partial_t^2 \mathbf{u}^s - \mu_1 \Delta \mathbf{u}^s - (\mu_1 + \lambda_1) \nabla (\operatorname{div} \mathbf{u}^s) - q_1 \nabla (\operatorname{div} \mathbf{u}^f) &= G_1 + \mathcal{R}_1 \mathbf{u}^s + \mathcal{R}_0 \operatorname{div} \mathbf{u}^f, & \text{in } Q \\ \partial_t^2 \mathbf{u}^f + \mu_2 \Delta \mathbf{u}^s - r_2 \nabla (\operatorname{div} \mathbf{u}^f) - q_2 \nabla (\operatorname{div} \mathbf{u}^s) &= G_2 + \mathcal{R}'_1 \mathbf{u}^s + \mathcal{R}'_0 \operatorname{div} \mathbf{u}^f, & \text{in } Q. \end{aligned} \quad (2.10)$$

Here,  $\mathcal{R}_j, \mathcal{R}'_j, j = 0, 1$  are differential operators of order  $j$  with coefficients in  $L^\infty(Q)$ , and

$$\begin{aligned} \mu_1 &= \varrho^{-1} \mu \varrho_{22}, & \lambda_1 &= \varrho^{-1} (\lambda \varrho_{22} - q \varrho_{12}), & q_1 &= \varrho^{-1} (q \varrho_{22} - r \varrho_{12}) \\ \mu_2 &= \varrho^{-1} \mu \varrho_{12}, & q_2 &= \varrho^{-1} (q \varrho_{11} - (\mu + \lambda) \varrho_{12}), & r_2 &= \varrho^{-1} (r \varrho_{11} - q \varrho_{12}). \end{aligned} \quad (2.11)$$

Henceforth  $\mathcal{P}_j, j = 1, \dots, 4$  denote some first-order operators with  $L^\infty(Q)$ -coefficients.

We apply  $\operatorname{div}$  to the equations in (2.10), and can derive the following two equations:

$$\begin{aligned} \partial_t^2 v^s - a_{11} \Delta v^s - a_{12} \Delta v^f &= \operatorname{div} G_1 + \mathcal{P}_1(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s) \\ \partial_t^2 v^f - a_{21} \Delta v^s - a_{22} \Delta v^f &= \operatorname{div} G_2 + \mathcal{P}_2(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s), \end{aligned} \quad (2.12)$$

where  $(a_{ij})_{1 \leq i, j \leq 2}$  is given by (1.9). We apply the curl to the first equation (2.10) to obtain

$$\partial_t^2 \mathbf{w}^s - \mu_1 \Delta \mathbf{w}^s = \operatorname{curl} G_1 + \mathcal{P}_3(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s) \quad (2.13)$$

and

$$\partial_t^2 \mathbf{u}^s - \mu_1 \Delta \mathbf{u}^s = G_1 + \mathcal{P}_4(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s). \quad (2.14)$$

Applying Lemma 2.2 to system (2.12), we have for  $v = (v^s, v^f)$

$$\begin{aligned} C \int_Q e^{2\tau\varphi} \left( \tau |\nabla_{x,t} v|^2 + \tau^3 |v|^2 \right) dx dt &\leq \int_Q e^{2\tau\varphi} \left( |F|^2 + |\nabla F|^2 \right) dx dt \\ &\quad + \int_Q e^{2\tau\varphi} \left( |\mathbf{u}^s|^2 + |\mathbf{w}^s|^2 + |\nabla \mathbf{u}^s|^2 + |\nabla \mathbf{w}^s|^2 \right) dx dt. \end{aligned}$$

Applying Lemma 2.1 to (2.13) and (2.14), we obtain

$$\begin{aligned} C \int_Q e^{2\tau\varphi} \left( \tau |\nabla_{x,t} \mathbf{w}^s|^2 + \tau^3 |\mathbf{w}^s|^2 + \tau |\nabla_{x,t} \mathbf{u}^s|^2 + \tau^3 |\mathbf{u}^s|^2 \right) dx dt \\ \leq \int_Q e^{2\tau\varphi} \left( |F|^2 + |\nabla F|^2 \right) dx dt + \int_Q e^{2\tau\varphi} \left( |v|^2 + |\nabla v|^2 \right) dx dt. \end{aligned}$$

Therefore, for  $\tau$  sufficiently large, we obtain (2.5). This completes the proof of Theorem 2.1.

### 3 Proof of Theorem 1.2

In this section we prove the stability (Theorem 1.2) for the inverse source problem.

For the proof, we apply the method in Imanuvilov and Yamamoto [19] which modified the argument in [10] and proved the stability for an inverse coefficient problem for a hyperbolic equation. For it, the Carleman (Theorem 2.1) is a key.

### 3.1 Modified Carleman estimate for Biot's system

Let  $\omega_T = \omega \times (-T, T)$ . We modify Theorem 2.1 for functions which vanish at  $\pm T$  with first  $t$ -derivatives.

**Lemma 3.1.** *There exist positive constants  $\tau_*$ ,  $C > 0$  and  $C_0 > 0$  such that the following inequality holds:*

$$\begin{aligned} & \int_Q \tau \left( |\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ & + \int_Q \tau^3 \left( |\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau\varphi} dxdt \\ & + C e^{C_0\tau} \|\mathbf{v}\|_{H^2(\omega_T)}^2 \end{aligned} \quad (3.1)$$

for any  $\tau \geq \tau_*$  and any  $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in H^2(Q)$  satisfying, for  $G = (G_1, G_2)$

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{v}^s + \varrho_{12} \partial_t^2 \mathbf{v}^f - \Delta_{\mu,\lambda} \mathbf{v}^s - \nabla (q \operatorname{div} \mathbf{v}^f) &= G_1, \\ \varrho_{12} \partial_t^2 \mathbf{v}^s + \varrho_{22} \partial_t^2 \mathbf{v}^f - \nabla (q \operatorname{div} \mathbf{v}^s) - \nabla (r \operatorname{div} \mathbf{v}^f) &= G_2 \quad \text{in } Q \end{aligned} \quad (3.2)$$

such that

$$\partial_t^j \mathbf{v}(x, \pm T) = 0 \quad \text{for all } x \in \Omega, j = 0, 1. \quad (3.3)$$

**Proof.** Let  $\omega^0 \subset \omega$ . In order to apply Carleman estimate (2.5), we introduce a cut-off function  $\xi$  satisfying  $0 \leq \xi \leq 1$ ,  $\xi \in C^\infty(\mathbb{R}^3)$ ,  $\xi = 1$  in  $\overline{\Omega \setminus \omega^0}$  and  $\operatorname{Supp} \xi \subset \Omega$ . Let  $\mathbf{v} \in H^2(Q)$  satisfy (3.2) and (3.3). Put

$$\mathbf{w}(x, t) = \xi(x) \mathbf{v}(x, t), \quad (x, t) \in Q,$$

and let  $Q_0 = (\Omega \setminus \overline{\omega}) \times (-T, T)$ . Noting that  $\mathbf{w} \in H^2(Q)$  is compactly supported in  $Q$  and  $\mathbf{w} = \mathbf{v}$  in  $Q_0$  and applying Carleman estimate (2.5) to  $\mathbf{w}$ , we obtain

$$\begin{aligned} & \int_{Q_0} \tau \left( |\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ & + \int_Q \tau^3 \left( |\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau\varphi} dxdt \\ & + C \int_Q |\mathcal{Q}_2 \mathbf{v}|^2 e^{2\tau\varphi} dxdt \end{aligned}$$

for any  $\tau \geq \tau_*$ . Here  $\mathcal{Q}_2$  is a differential operator of order 2 whose coefficients are supported in  $\omega$ . Therefore

$$\begin{aligned} & \int_Q \tau \left( |\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ & + \int_Q \tau^3 \left( |\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau\varphi} dxdt \\ & + C e^{C_0\tau} \|\mathbf{v}\|_{H^2(\omega_T)}^2. \end{aligned}$$

This completes the proof of the lemma.  $\square$

By  $\mathcal{N}_{\tau,\varphi}(\mathbf{v})$  we denote

$$\begin{aligned} \mathcal{N}_{\tau,\varphi}(\mathbf{v}) &= \int_Q \tau \left( |\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ &\quad + \int_Q \tau^3 \left( |\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \end{aligned} \quad (3.4)$$

where  $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f)$ .

Now, we recall (2.2) and (2.3) for the definition of  $d_0$ ,  $\eta$  and  $\varepsilon$  and we introduce a cut-off function  $\zeta$  satisfying  $0 \leq \zeta \leq 1$ ,  $\zeta \in C^\infty(\mathbb{R})$  and

$$\zeta = 1 \quad \text{in } (-T + 2\varepsilon, T - 2\varepsilon), \quad \operatorname{Supp} \zeta \subset (-T + \varepsilon, T - \varepsilon). \quad (3.5)$$

Finally we denote by  $\tilde{\mathbf{v}}$  the function

$$\tilde{\mathbf{v}}(x, t) = \zeta(t)(\mathbf{v}^s, \mathbf{v}^f)(x, t), \quad (x, t) \in Q. \quad (3.6)$$

**Lemma 3.2.** *There exist positive constants  $\tau_*$ ,  $C$  and  $C_0$  such that the following inequality holds:*

$$C\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}) \leq \int_Q \left( |F|^2 + |\nabla F|^2 \right) e^{2\tau\varphi} dxdt + e^{C_0\tau} \|\mathbf{v}\|_{H^2(\omega_T)}^2 + e^{2d_1\tau} \|\mathbf{v}\|_{H^1(-T, T; H^1(\Omega))}^2$$

for any  $\tau \geq \tau_*$  and any  $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in (H^2(Q))^6$  satisfying

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{v}^s + \varrho_{12} \partial_t^2 \mathbf{v}^f - \Delta_{\mu,\lambda} \mathbf{v}^s - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}^f) &= F_1(x, t) \\ \varrho_{12} \partial_t^2 \mathbf{v}^s + \varrho_{22} \partial_t^2 \mathbf{v}^f - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}^s) - \nabla (\mathbf{r} \operatorname{div} \mathbf{v}^f) &= F_2(x, t), \quad (x, t) \in Q \end{aligned}$$

**Proof.** We note that  $\tilde{\mathbf{v}} \in (H^2(Q))^6$  and

$$\begin{aligned} \varrho_{11} \partial_t^2 \tilde{\mathbf{v}}^s + \varrho_{12} \partial_t^2 \tilde{\mathbf{v}}^f - \Delta_{\mu,\lambda} \tilde{\mathbf{v}}^s - \nabla (\mathbf{q} \operatorname{div} \tilde{\mathbf{v}}^f) &= \zeta(t) F_1(x, t) + P_1(\mathbf{v}, \partial_t \mathbf{v}), \\ \varrho_{12} \partial_t^2 \tilde{\mathbf{v}}^s + \varrho_{22} \partial_t^2 \tilde{\mathbf{v}}^f - \nabla (\mathbf{q} \operatorname{div} \tilde{\mathbf{v}}^s) - \nabla (\mathbf{r} \operatorname{div} \tilde{\mathbf{v}}^f) &= \zeta(t) F_2(x, t) + P_2(\mathbf{v}, \partial_t \mathbf{v}), \quad (x, t) \in Q, \end{aligned}$$

where  $P_1$  and  $P_2$  are zeroth-order operators and supported in  $|t| > T - 2\varepsilon$ . Therefore, applying Lemma 3.1 to  $\tilde{\mathbf{v}}$  and using (2.3), we complete the proof of the lemma.  $\square$

### 3.2 Preliminary estimates

Let  $\varphi(x, t)$  be the function defined by (2.1). Then

$$\varphi(x, t) = e^{\gamma\psi(x,t)} =: \rho(x)\alpha(t), \quad (3.7)$$

where  $\rho(x)$  and  $\alpha(t)$  are defined by

$$\rho(x) = e^{\gamma\vartheta(x)} \geq d_0, \quad \forall x \in \Omega \quad \text{and} \quad \alpha(t) = e^{-\beta\gamma t^2} \leq 1, \quad \forall t \in [-T, T]. \quad (3.8)$$

Next we present the following Carleman estimate of a first-order partial differential operator:

$$L(x, D)v = \sum_{i=1}^3 a_i(x) \partial_i v + a_0(x)v, \quad x \in \Omega$$

where

$$a_0 \in \mathcal{C}(\overline{\Omega}), \quad a = (a_1, a_2, a_3) \in [\mathcal{C}^1(\overline{\Omega})]^3 \quad (3.9)$$

and

$$|a(x) \cdot (x - x_0)| \geq c_0 > 0, \quad \text{on } \overline{\Omega} \quad (3.10)$$

with a constant  $c_0 > 0$ . Then

**Lemma 3.3.** *In addition to (3.9) and (3.10), we assume that  $\|a_0\|_{\mathcal{C}(\overline{\Omega})} \leq M$  and  $\|a_i\|_{\mathcal{C}^1(\overline{\Omega})} \leq M$ ,  $1 \leq i \leq 3$ . Then there exist constants  $\tau_* > 0$  and  $C > 0$  such that*

$$\tau \int_{\Omega} |v(x)|^2 e^{2\tau\rho(x)} dx \leq C \int_{\Omega} |L(x, D)v(x)|^2 e^{2\tau\rho(x)} dx$$

for all  $v \in H_0^1(\Omega)$  and all  $\tau > \tau_*$ .

The proof is direct by integration by parts and see e.g., [19].

Consider now the following system

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{u}^s + \varrho_{12} \partial_t^2 \mathbf{u}^f - \Delta_{\mu, \lambda} \mathbf{u}^s(x, t) - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^f) &= F_1(x, t), \\ \varrho_{12} \partial_t^2 \mathbf{u}^s + \varrho_{22} \partial_t^2 \mathbf{u}^f - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^s) - \nabla (\mathbf{r} \operatorname{div} \mathbf{u}^f) &= F_2(x, t), \quad (x, t) \in Q, \end{aligned} \quad (3.11)$$

with the boundary condition

$$\mathbf{u}^s(x, t) = 0, \quad \mathbf{u}^f(x, t) \cdot \nu = 0, \quad (x, t) \in \Sigma \quad (3.12)$$

and the initial condition

$$(\mathbf{u}^s(x, 0), \partial_t \mathbf{u}^s(x, 0)) = (0, 0), \quad (\mathbf{u}^f(x, 0), \partial_t \mathbf{u}^f(x, 0)) = (0, 0), \quad x \in \Omega, \quad (3.13)$$

where the functions  $F_1$  and  $F_2$  are given by

$$F_1(x, t) = p_1(x)R_1(x, t), \quad F_2(x, t) = p_2(x)R_2(x, t). \quad (3.14)$$

We introduce the following notations:

$$\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f), \quad \mathbf{v}_j(x, t) = \partial_t^j \mathbf{u}(x, t), \quad (x, t) \in Q, \quad j = 0, 1, 2, 3. \quad (3.15)$$

The functions  $\mathbf{v}_j$ ,  $j = 1, 2, 3$  solve the following system

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{v}_j^s + \varrho_{12} \partial_t^2 \mathbf{v}_j^f - \Delta_{\mu, \lambda} \mathbf{v}_j^s(x, t) - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}_j^f) &= \partial_t^j F_1(x, t), \\ \varrho_{12} \partial_t^2 \mathbf{v}_j^s + \varrho_{22} \partial_t^2 \mathbf{v}_j^f - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}_j^s) - \nabla (\mathbf{r} \operatorname{div} \mathbf{v}_j^f) &= \partial_t^j F_2(x, t), \quad (x, t) \in Q, \end{aligned} \quad (3.16)$$

with the boundary condition

$$\mathbf{v}_j^s(x, t) = 0, \quad \mathbf{v}_j^f(x, t) \cdot \nu = 0, \quad (x, t) \in \Sigma. \quad (3.17)$$

We set

$$\tilde{\mathbf{v}}_j = \zeta \mathbf{v}_j,$$

where  $\zeta(t)$  is given by (3.5). We apply Lemma 3.2 to obtain the following estimate:

$$C\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_j) \leq \int_Q \left( |\partial_t^j F|^2 + |\nabla \partial_t^j F|^2 \right) e^{2\tau\varphi} dxdt + e^{C_0\tau} \|\mathbf{v}_j\|_{H^2(\omega_T)}^2 + e^{2d_1\tau} \|\mathbf{v}_j\|_{H^1(-T,T;H^1(\Omega))}^2, \quad j = 0, 1, 2, 3, \quad (3.18)$$

provided that  $\tau > 0$  is large enough.

**Lemma 3.4.** *There exists a positive constant  $C > 0$  such that the following estimate*

$$\int_{\Omega} |z(x, 0)|^2 dx \leq C \int_Q (\tau |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2) dxdt$$

for any  $z \in L^2(Q)$  such that  $\partial_t z \in L^2(Q)$ .

**Proof.** Let  $\zeta$  be the cut-off function given by (3.5). By direct computations, we have

$$\begin{aligned} \int_{\Omega} \zeta^2(0) |z(x, 0)|^2 dx &= \int_{-T}^0 \frac{d}{dt} \left( \int_{\Omega} \zeta^2(t) |z(x, t)|^2 dx \right) dt \\ &= 2 \int_{-T}^0 \int_{\Omega} \zeta^2(t) z(x, t) \partial_t z(x, t) dxdt \\ &\quad + 2 \int_{-T}^0 \int_{\Omega} \zeta'(t) \zeta(t) |z(x, t)|^2 dxdt. \end{aligned}$$

Then we have

$$\int_{\Omega} |z(x, 0)|^2 dx \leq C \int_Q (\tau |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2) dxdt.$$

This completes the proof of the lemma.  $\square$

**Lemma 3.5.** *Let  $\phi_{\ell}(x) = \operatorname{div} (p_{\ell}(x) \Phi_{\ell}(x))$ . Then there exists a constant  $C > 0$  such that*

$$\begin{aligned} \sum_{\ell=1}^2 \int_{\Omega} e^{2\tau\rho} \left( |\phi_{\ell}(x)|^2 + |\nabla \phi_{\ell}(x)|^2 \right) dx \\ \leq C (\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_3)) + \sum_{\ell=1}^2 \int_{\Omega} \left( |p_{\ell}|^2 + |\nabla p_{\ell}|^2 \right) e^{2\tau\rho} dx, \end{aligned}$$

provided that  $\tau$  is large.

**Proof.** We set  $\mathbf{v}^{(1)} = \mathbf{v}_2^s$  and  $\mathbf{v}^{(2)} = \mathbf{v}_2^f$ . Applying Lemma 3.4 for  $z_j(x, t) = e^{\tau\varphi(x,t)} \operatorname{div} \tilde{\mathbf{v}}_2^{(j)}(x, t)$ ,  $j = 1, 2$ , we obtain the following inequality:

$$\begin{aligned} C\tau^2 \int_{\Omega} e^{2\tau\rho} \sum_{j=1}^2 \left| \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 dx \leq \tau^3 \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left| \operatorname{div} \tilde{\mathbf{v}}^{(j)}(x, t) \right|^2 dxdt \\ + \tau \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left| \partial_t \operatorname{div} \tilde{\mathbf{v}}^{(j)}(x, t) \right|^2 dxdt \leq \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_2). \quad (3.19) \end{aligned}$$

Applying Lemma 3.4 again with  $w_j(x, t) = e^{\tau\varphi(x, t)} \nabla \operatorname{div} (\tilde{\mathbf{v}}^{(j)}(x, t))$ , we obtain

$$\begin{aligned} C \int_{\Omega} e^{2\tau\rho} \sum_{j=1}^2 \left| \nabla \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 dx &\leq \tau \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left| \nabla \operatorname{div} \tilde{\mathbf{v}}^{(j)}(x, t) \right|^2 dx dt \\ &\quad + \tau^{-1} \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left( \left| \nabla \operatorname{div} \partial_t^3 \tilde{\mathbf{v}}^s(x, t) \right|^2 + \left| \nabla \operatorname{div} \partial_t^3 \tilde{\mathbf{v}}^f(x, t) \right|^2 \right) dx dt \\ &\leq \mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_3). \end{aligned} \quad (3.20)$$

Adding (3.19) and (3.20), we find

$$\int_{\Omega} e^{2\tau\rho} \sum_{j=1}^2 \left( \left| \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 + \left| \nabla \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 \right) dx \leq C (\mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_3)). \quad (3.21)$$

Since

$$M(x) \left( \mathbf{v}_2^s(x, 0), \mathbf{v}_2^f(x, 0) \right)^T = (p_1(x) \Phi_1(x), p_2(x) \Phi_2(x))^T, \quad x \in \bar{\Omega}$$

we have

$$\begin{aligned} |\phi_\ell(x)|^2 + |\nabla \phi_\ell(x)|^2 &\leq C \left( |\mathbf{v}_2^s(x, 0)|^2 + |\nabla \mathbf{v}_2^s(x, 0)|^2 + |\mathbf{v}_2^f(x, 0)|^2 + |\nabla \mathbf{v}_2^f(x, 0)|^2 \right. \\ &\quad \left. + |\operatorname{div} \mathbf{v}_2^s(x, 0)|^2 + |\nabla \operatorname{div} \mathbf{v}_2^s(x, 0)|^2 + |\operatorname{div} \mathbf{v}_2^f(x, 0)|^2 + |\nabla \operatorname{div} \mathbf{v}_2^f(x, 0)|^2 \right) \end{aligned} \quad (3.22)$$

for  $x \in \bar{\Omega}$ . On the other hand, using (3.11), we obtain

$$|\mathbf{v}_2^s(x, 0)|^2 + |\nabla \mathbf{v}_2^s(x, 0)|^2 + |\mathbf{v}_2^f(x, 0)|^2 + |\nabla \mathbf{v}_2^f(x, 0)|^2 \leq C \sum_{\ell=1}^2 \left( |p_\ell|^2 + |\nabla p_\ell|^2 \right), \quad x \in \bar{\Omega}. \quad (3.23)$$

Combining (3.23), (3.22) and (3.21), we complete the proof of the lemma.  $\square$

**Lemma 3.6.** *There exists a constant  $C > 0$  such that*

$$\tau \int_{\Omega} \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho} dx \leq C \int_{\Omega} \left( |\nabla \phi_\ell(x)|^2 + |\phi_\ell(x)|^2 \right) e^{2\tau\rho(x)} dx$$

for all large  $\tau > 0$ ,  $\ell = 1, 2$ .

**Proof.** We have

$$\operatorname{div} ((\partial_k p_\ell)(x) \Phi_\ell(x)) = \partial_k \phi_\ell(x) - \operatorname{div} (p_\ell \partial_k \Phi_\ell(x)) \quad \text{for all } k = 1, 2, 3.$$

Therefore

$$\begin{aligned} \int_{\Omega} \left( |\operatorname{div} ((\partial_k p_\ell) \Phi_\ell)|^2 + |\operatorname{div} (p_\ell \Phi_\ell)|^2 \right) e^{2\tau\rho} dx &\leq \int_{\Omega} \left( |\nabla \phi_\ell|^2 + |\phi_\ell|^2 \right) e^{2\tau\rho(x)} dx \\ &\quad + C \int_{\Omega} \left( |p_\ell|^2 + |\nabla p_\ell|^2 \right) e^{2\tau\rho} dx. \end{aligned} \quad (3.24)$$

Since  $p_\ell = 0$  and  $\nabla p_\ell = 0$  on the boundary  $\Gamma$  and  $\nabla \Phi_\ell \cdot (x - x_0) \neq 0$ , we can apply Lemma 3.3 respectively with the choice  $v = p_\ell$  and  $v = \partial_k p_\ell$  to obtain

$$\tau \int_{\Omega} \left( |\partial_k p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho} dx \leq C \int_{\Omega} \left( |\operatorname{div}((\partial_k p_\ell)\Phi_\ell)|^2 + |\operatorname{div}(p_\ell\Phi_\ell)|^2 \right) e^{2\tau\rho} dx \quad (3.25)$$

for  $\ell = 1, 2$  and  $k = 1, 2, 3$ . Inserting (3.24) into the left-hand side of (3.25) and choosing  $\tau > 0$  large, we obtain

$$\tau \int_{\Omega} \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho} dx \leq C \int_{\Omega} \left( |\nabla \phi_\ell(x)|^2 + |\phi_\ell(x)|^2 \right) e^{2\tau\rho(x)} dx.$$

The proof is completed.  $\square$

### 3.3 Completion of the proof of Theorem 1.2

By Lemmata 3.5 and 3.6, we obtain

$$\begin{aligned} \tau \sum_{\ell=1}^2 \int_{\Omega} e^{2\tau\rho(x)} \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) dx &\leq C \sum_{\ell=1}^2 \int_{\Omega} \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho(x)} dx \\ &\quad + C \left( \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_3) \right). \end{aligned}$$

Therefore, choosing  $\tau > 0$  large to absorb the first term on the right-hand side into the left-hand side and applying (3.19), we obtain

$$\begin{aligned} \tau \sum_{\ell=1}^2 \int_{\Omega} e^{2\tau\rho(x)} \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) dx &\leq C \sum_{j=2}^3 \left( \int_Q \left( |\partial_t^j F|^2 + |\partial_t^j \nabla F|^2 \right) e^{2\tau\varphi} dx dt \right. \\ &\quad \left. + C e^{C_0\tau} \|\mathbf{v}_j\|_{H^2(\omega_T)}^2 + C e^{2d_1\tau} \|\mathbf{v}_j\|_{H^1(-T,T;H^1(\Omega))}^2 \right) \\ &\leq C \sum_{\ell=1}^2 \int_Q \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\varphi} dx dt + C e^{C_0\tau} \mathcal{E}_\omega(\mathbf{u}) + C e^{2d_1\tau} M_0. \quad (3.26) \end{aligned}$$

Then the first term of the right-hand side of (3.26) can be absorbed into the left-hand side if we take large  $\tau > 0$ .

Since  $\rho(x) \geq d_0$ , we obtain

$$\sum_{\ell=1}^2 \int_{\Omega} \left( |\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) dx \leq C e^{2(d_1-d_0)\tau} + e^{C_0\tau} \mathcal{E}_\omega(\mathbf{u}) \leq C e^{-\epsilon\tau} + e^{C_0\tau} \mathcal{E}_\omega(\mathbf{u}). \quad (3.27)$$

At the last inequality, we used: By  $0 < d_1 < d_0$ , we can choose  $\epsilon > 0$  such that  $e^{2(d_1-d_0)\tau} \leq e^{-\epsilon\tau}$  for sufficiently large  $\tau > 0$ .

## 4 Well posedness of the direct problem

This section is devoted to the study the existence, uniqueness and regularity of solutions of the following system:

$$\begin{aligned} \varrho_{11}\partial_t^2\mathbf{u}^s + \varrho_{12}\partial_t^2\mathbf{u}^f - \Delta_{\mu,\lambda}\mathbf{u}^s(x,t) - \nabla(\mathbf{q}\operatorname{div}\mathbf{u}^f) &= F_1(x,t), \\ \varrho_{12}\partial_t^2\mathbf{u}^s + \varrho_{22}\partial_t^2\mathbf{u}^f - \nabla(\mathbf{q}\operatorname{div}\mathbf{u}^s) - \nabla(\mathbf{r}\operatorname{div}\mathbf{u}^f) &= F_2(x,t), \quad (x,t) \in Q \end{aligned} \quad (4.1)$$

with the boundary condition

$$\mathbf{u}^s(x,t) = 0, \quad \mathbf{u}^f(x,t) \cdot \nu = 0, \quad (x,t) \in \Sigma = \Gamma \times (-T,T) \quad (4.2)$$

and the initial condition

$$(\mathbf{u}^s(x,0), \mathbf{u}_t^s(x,0)) = (\mathbf{u}_0^s, \mathbf{u}_1^s), \quad (\mathbf{u}^f(x,0), \mathbf{u}_t^f(x,0)) = (\mathbf{u}_0^f, \mathbf{u}_1^f), \quad x \in \Omega. \quad (4.3)$$

### 4.1 Function spaces

We denote by  $\mathcal{D}(\Omega)$  the space of compactly supported, infinitely differentiable function in  $\Omega$  equipped with the inductive limit topology. We denote by  $\mathcal{D}'(\Omega)$  the space dual to  $\mathcal{D}(\Omega)$ . In general, we denote by  $X'$  the space dual to the function space  $X$ . We denote by  $(f, g)$  the inner product in  $L^2(\Omega)$  and by  $\langle f, g \rangle$  the value of  $f \in X'$  on  $g \in X$ . We use usual notations for Sobolev spaces. If  $X$  is a Banach space, then we denote by  $L^p(0, T; X)$  the space of functions  $f : (0, T) \rightarrow X$  which are measurable, take values in  $X$  and satisfy:

$$\left( \int_0^T \|f(t)\|_X^p dt \right)^{1/p} = \|f\|_{L^p(0,T;X)} < \infty$$

for  $1 \leq p < \infty$ , while

$$\|f\|_{L^\infty(0,T;X)} = \operatorname{esssup}_{t \in (0,T)} \|f(t)\|_X < \infty$$

for  $p = \infty$ . It is known that the space  $L^p(0, T; X)$  is complete.

We define the space

$$H(\operatorname{div}; \Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3; \operatorname{div} \mathbf{u} \in L^2(\Omega) \right\},$$

equipped with the norm

$$\|\mathbf{u}\|_{H(\operatorname{div}; \Omega)} = \left( \|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Let us consider the space

$$V(\Omega) = (H^1(\Omega)^3) \times H(\operatorname{div}; \Omega),$$

equipped with the norm

$$\|\mathbf{u}\|_{V(\Omega)} = \left( \|\mathbf{u}^2\|_{H^1(\Omega)}^2 + \|\mathbf{u}^2\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}^2\|_{L^2(\Omega)}^2 \right)^{1/2}.$$



## 4.2 Generalized solution

We introduce the bilinear form on  $V(\Omega)$  by

$$B(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_{\Omega} \left( \lambda \operatorname{div}(\mathbf{u}^s) \operatorname{div}(\mathbf{v}^s) + 2\mu (\varepsilon(\mathbf{u}^s) : \varepsilon(\mathbf{v}^s)) + r \operatorname{div}(\mathbf{u}^f) \operatorname{div}(\mathbf{v}^f) \right) dx \\ + \frac{1}{2} \int_{\Omega} q \left( \operatorname{div}(\mathbf{u}^f) \operatorname{div}(\mathbf{v}^s) + \operatorname{div}(\mathbf{v}^f) \operatorname{div}(\mathbf{u}^s) \right) dx \quad (4.4)$$

for any  $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f) \in V(\Omega)$ ,  $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in V(\Omega)$ . We recall that the matrix  $M$  is given by (2.9).

**Definition 4.1.** We say that  $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$  is a generalized solution of problem (4.1)-(4.2), if  $\mathbf{u} \in L^2(0, T; V(\Omega))$  satisfies the initial condition (4.3) and the following identity

$$(M \partial_t^2 \mathbf{u}(t), \mathbf{v}(t)) + B(\mathbf{u}(t), \mathbf{v}(t)) = (F(t), \mathbf{v}(t)), \quad \text{almost all } t \in (0, T) \quad (4.5)$$

for any  $\mathbf{v} \in L^2(0, T; V(\Omega))$ .

We note that in (4.5) the integration is only in  $x$ .

**Lemma 4.1.** For  $\eta > 0$ , we set

$$B_{\eta}(\mathbf{u}, \mathbf{v}) = B(\mathbf{u}, \mathbf{v}) + \eta (\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V(\Omega).$$

Then there exists sufficiently large constant  $\eta$  such that the symmetric bilinear form  $B_{\eta}$  satisfies

(i)  $|B_{\eta}(\mathbf{u}, \mathbf{v})| \leq C_1 \|\mathbf{u}\|_{V(\Omega)} \|\mathbf{v}\|_{V(\Omega)}$ , for any  $\mathbf{u}, \mathbf{v} \in V(\Omega)$ ,

(ii)  $B_{\eta}(\mathbf{u}, \mathbf{u}) \geq C_2 \|\mathbf{u}\|_{V(\Omega)}^2$ , for any  $\mathbf{u} \in V(\Omega)$ .

**Proof.** By (4.4) we obtain, for any  $\mathbf{u}, \mathbf{v} \in V(\Omega)$

$$|B(\mathbf{u}, \mathbf{v})| \leq \left( \|\mathbf{u}^s\|_{H^1(\Omega)} + \|\operatorname{div} \mathbf{u}^f\|_{L^2(\Omega)} \right) \left( \|\mathbf{v}^s\|_{H^1(\Omega)} + \|\operatorname{div} \mathbf{v}^f\|_{L^2(\Omega)} \right) \\ \leq C \|\mathbf{u}\|_V \|\mathbf{v}\|_V. \quad (4.6)$$

Then for any  $\eta$ , we can derive (i).

Now, we note that for a vector  $\mathbf{u}^s \in H_0^1(\Omega)$  we have the following Korn's inequality

$$C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 \leq \int_{\Omega} \varepsilon(\mathbf{u}^s) : \varepsilon(\mathbf{u}^s) dx.$$

Then, for  $W = (\operatorname{div} \mathbf{u}^s, \operatorname{div} \mathbf{u}^f)$ , we have

$$B(\mathbf{u}, \mathbf{u}) \geq \mu C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 + \frac{1}{2} \int_{\Omega} M_0 W \cdot W dx$$

where  $M_0$  is the symmetric  $2 \times 2$ -matrix given by

$$M_0(x) = \begin{pmatrix} \lambda & q \\ q & r \end{pmatrix} \geq \gamma_0 I_2.$$

Which implies

$$\begin{aligned}
B(\mathbf{u}, \mathbf{u}) &\geq \mu C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 + \frac{\gamma_0}{2} \left( \|\operatorname{div} \mathbf{u}^s\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}^f\|_{L^2(\Omega)}^2 \right) \\
&\geq \mu C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 + \frac{\gamma_0}{2} \|\operatorname{div} \mathbf{u}^f\|_{L^2(\Omega)}^2 - \frac{\gamma_0}{2} \|\mathbf{u}\|_{L^2(\Omega)}^2 \\
&\geq C_2 \|\mathbf{u}\|_V^2 - \eta \|\mathbf{u}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.7}$$

This completes the proof of the lemma.  $\square$

### 4.3 Construction of approximate solutions

Let  $\mathcal{A} : (L^2(\Omega))^6 \rightarrow (L^2(\Omega))^6$  be the self-adjoint operator defined by

$$\mathcal{A} \mathbf{u} = \begin{pmatrix} \Delta_{\mu, \lambda} \mathbf{u}^s + \nabla (q \operatorname{div} \mathbf{u}^f) \\ \nabla (q \operatorname{div} \mathbf{u}^s) + \nabla (r \operatorname{div} \mathbf{u}^f) \end{pmatrix}.$$

Then system (4.1) can be written as

$$M \partial_t^2 \mathbf{u} - \mathcal{A} \mathbf{u} = F, \quad (x, t) \in Q \tag{4.8}$$

with initial condition

$$\mathbf{u}(x, 0) = (\mathbf{u}_0^s(x), \mathbf{u}_0^f(x)), \quad \partial_t \mathbf{u}(x, 0) = (\mathbf{u}_1^s(x), \mathbf{u}_1^f(x)) \tag{4.9}$$

and the boundary condition

$$\mathbf{u}^s(x, t) = 0, \quad \mathbf{u}^f \cdot \nu = 0, \quad (x, t) \in \Sigma. \tag{4.10}$$

Let  $(\mathbf{w}_j)_{j \geq 1}$  be a sequence of solutions in  $(H^2(\Omega) \cap H_0^1(\Omega))^6$  such that for all  $m \in \mathbb{N}$ ,  $\mathbf{w}_1, \dots, \mathbf{w}_m$  are linearly independent and all the finite linear combinations of  $(\mathbf{w}_j)_{j \geq 1}$  are dense in  $(H^2(\Omega))^6$ .

We seek approximate solutions of the problem in the form

$$\mathbf{u}_m(t) = \sum_{j=1}^m g_{jm}(t) \mathbf{w}_j. \tag{4.11}$$

The functions  $g_{jm}(t)$  are defined by the solution of the system of ordinary differential equations

$$(M \partial_t^2 \mathbf{u}_m, \mathbf{w}_j) + B(\mathbf{u}_m, \mathbf{w}_j) = (F(t), \mathbf{w}_j), \quad 1 \leq j \leq m, \tag{4.12}$$

with the initial conditions

$$\begin{aligned}
\mathbf{u}_m(0) &= \mathbf{u}_{0m} \rightarrow \mathbf{u}_0 \quad \text{in} \quad (H^2(\Omega) \cap H_0^1(\Omega))^6, \\
\partial_t \mathbf{u}_m(0) &= \mathbf{u}_{1m} \rightarrow \mathbf{u}_1 \quad \text{in} \quad (H^1(\Omega))^6.
\end{aligned} \tag{4.13}$$

The system (4.12)-(4.13) depends on  $g_{jm}(t)$  and therefore has a solution on some segment  $[0, t_m]$ ; see [35]. From a priori estimates below and the theorem on continuation of a solution we deduce that it is possible to take  $t_m = T$ .

#### 4.4 A priori estimates

Multiplying (4.8) by  $g'_{jm}(t)$  and summing over  $j$  from 1 to  $m$ , we obtain

$$(M\partial_t^2 \mathbf{u}_m, \partial_t \mathbf{u}_m) + B(\mathbf{u}_m, \partial_t \mathbf{u}_m) = (F(t), \partial_t \mathbf{u}_m). \quad (4.14)$$

Hence

$$\frac{1}{2} \frac{d}{dt} \left[ \|M^{1/2} \partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}_m(t), \mathbf{u}_m(t)) \right] = (F(t), \partial_t \mathbf{u}_m(t)) + \frac{\eta}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(\Omega)}^2. \quad (4.15)$$

Let

$$\Phi^2(t) = \|M^{1/2} \partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}_m(t), \mathbf{u}_m(t)).$$

From (4.15) we obtain

$$\frac{1}{2} \frac{d}{dt} \Phi^2(t) \leq C \left[ \|F(t)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \right]. \quad (4.16)$$

Integrating with respect to  $\tau$  from 0 to  $t$ , we obtain

$$\Phi^2(t) \leq C \left[ \|F\|_{L^2(Q)}^2 + \Phi^2(0) + \int_0^t \left( \|\partial_t \mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 \right) \right]. \quad (4.17)$$

Since

$$\Phi^2(t) \geq C \left( \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(t)\|_{V(\Omega)}^2 \right) \quad (4.18)$$

and

$$\Phi^2(0) \leq C + \|\mathbf{u}_0\|_{H^2(\Omega)}^2 + \|\mathbf{u}_1\|_{H^1(\Omega)}^2, \quad (4.19)$$

we have from (4.18)

$$\|\mathbf{u}_m(t)\|_{V(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \leq R_0 + \int_0^t \left( \|\partial_t \mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 \right), \quad (4.20)$$

where  $R_0 = C + \|\mathbf{u}_0\|_{H^2(\Omega)}^2 + \|\mathbf{u}_1\|_{H^1(\Omega)}^2 + \|F\|_{L^2(Q)}^2$ . By the Gronwall inequality, we conclude that

$$\|\mathbf{u}_m(t)\|_{V(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \leq R_0 \quad (4.21)$$

for all  $t \in (0, T)$  and  $m \geq 1$ .

In order to obtain the second a priori estimate, we observe that

$$\|\partial_t^2 \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 \leq C \left[ \|F(0)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(0)\|_{H^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 \right] \leq R_1. \quad (4.22)$$

Indeed, multiplying (4.8) by  $g'_{jm}(0)$ , summing over  $j$  and setting  $t = 0$ , we obtain

$$(M\partial_t^2 \mathbf{u}_m(0), \partial_t^2 \mathbf{u}_m(0)) + B(\mathbf{u}_m(0), \partial_t^2 \mathbf{u}_m(0)) = (F(0), \partial_t^2 \mathbf{u}_m(0)). \quad (4.23)$$

Consequently,

$$(M\partial_t^2 \mathbf{u}_m(0), \partial_t^2 \mathbf{u}_m(0)) = (F(0), \partial_t^2 \mathbf{u}_m(0)) + (\mathcal{A}\mathbf{u}_{0m}, \partial_t^2 \mathbf{u}_m(0)), \quad (4.24)$$

which implies

$$\|\partial_t^2 \mathbf{u}_m(0)\|^2 \leq C \left( \|F(0)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_{0m}\|_{H^2(\Omega)}^2 \right) \leq CR_2. \quad (4.25)$$

Differentiating (4.14) with respect to  $t$ , multiplying by  $g_{jm}$  and summing over  $j$ , we obtain the identity

$$\frac{1}{2} \frac{d}{dt} \left[ \|M^{1/2} \partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}_m(t), \mathbf{u}_m(t)) \right] = (F(t), \partial_t^2 \mathbf{u}_m(t)) + \frac{\eta}{2} \frac{d}{dt} \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2. \quad (4.26)$$

Then, we conclude that

$$\begin{aligned} \|\partial_t^2 \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_V^2 &\leq R_2 + \|\partial_t \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 \\ &+ C \int_0^t \left( \|\partial_t^2 \mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(\tau)\|_V^2 \right) d\tau. \end{aligned} \quad (4.27)$$

By (4.27) and the Gronwall inequality, we obtain

$$\|\partial_t^2 \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_V^2 \leq R_1. \quad (4.28)$$

Taking into consideration that  $\mathbf{u}_m = 0$  in  $\Sigma$ , we see

$$\begin{aligned} \mathbf{u}_m &\in L^\infty(0, T; V(\Omega)), \quad \partial_t \mathbf{u}_m \in L^\infty(0, T; V(\Omega)), \\ \partial_t^2 \mathbf{u}_m &\in L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (4.29)$$

#### 4.5 Passage to the limit

By (4.29), we can extract a sequence from  $(\mathbf{u}_m)_{m \geq 0}$ , which we denote again by  $(\mathbf{u}_m)_m$ , such that

$$\begin{aligned} \mathbf{u}_m &\rightarrow \mathbf{u} \quad \text{in the weak-star topology in } L^\infty(0, T; V(\Omega)) \\ \partial_t \mathbf{u}_m &\rightarrow \partial_t \mathbf{u} \quad \text{in the weak-star topology in } L^\infty(0, T; V(\Omega)) \\ \partial_t^2 \mathbf{u}_m &\rightarrow \partial_t^2 \mathbf{u} \quad \text{in the weak-star topology in } L^\infty(0, T; L^2(\Omega)) \end{aligned} \quad (4.30)$$

and

$$(\mathbf{u}_m, \partial_t \mathbf{u}_m) \rightarrow (\mathbf{u}, \partial_t \mathbf{u}) \quad \text{a.e., on } \Sigma.$$

Multiplying (4.8) by  $\theta \in L^1(0, T)$  and integrating, we have

$$\int_0^T \left( (M \partial_t^2 \mathbf{u}_m(t), \mathbf{w}_j) + B(\mathbf{u}_m, \mathbf{w}_j) \right) \theta(t) dt = \int_0^T (F(t), \mathbf{w}_j) \theta(t) dt. \quad (4.31)$$

On the other hand

$$\int_0^T B(\mathbf{u}_m, \mathbf{w}_j) \theta(t) dt = - \int_0^T (\mathbf{u}_m, \mathcal{A} \mathbf{w}_j) \theta(t) dt, \quad (4.32)$$

so that

$$\lim_{m \rightarrow \infty} \int_0^T B(\mathbf{u}_m, \mathbf{w}_j) \theta(t) dt = - \int_0^T (\mathbf{u}, \mathcal{A} \mathbf{w}_j) \theta(t) dt = \int_0^T B(\mathbf{u}, \mathbf{w}_j) \theta(t) dt. \quad (4.33)$$

Thus, we obtain

$$\int_0^T \left( (M \partial_t^2 \mathbf{u}(t), \mathbf{w}_j) + B(\mathbf{u}, \mathbf{w}_j) \right) \theta(t) dt = \int_0^T (F(t), \mathbf{w}_j) \theta(t) dt. \quad (4.34)$$

Taking into account that  $\mathbf{w}_j$  are dense in  $(H^2(\Omega) \cap H_0^1(\Omega))^6$  and therefore in  $V$ , we obtain

$$(M\partial_t^2 \mathbf{u}, \mathbf{v}) + B(\mathbf{u}(t), \mathbf{v}(t)) = (F(t), \mathbf{v}), \quad t \in (0, T) \quad (4.35)$$

for all  $\mathbf{v} \in L^2(0, T; V(\Omega))$ .

We have  $B(\mathbf{u}(t), \mathbf{v}(t)) = -(\mathcal{A}\mathbf{u}(t), \mathbf{v}(t))$  for any  $\mathbf{v} \in \mathcal{D}(\Omega)$ , where the application of the differential operator  $A$  to  $\mathbf{u}$  is in the distributional sense in  $\mathcal{D}'(\Omega)$ . Hence we obtain

$$M\partial_t^2 \mathbf{u} - \mathcal{A}\mathbf{u} = F, \quad \text{in } \mathcal{D}'(\Omega), \text{ a.e. in } (0, T). \quad (4.36)$$

On the other hand,  $\partial_t^2 \mathbf{u}, \partial_t \mathbf{u}, F \in L^\infty(0, T; L^2(\Omega))$ . Hence (4.36) holds in  $L^\infty(0, T; L^2(\Omega))$ .

The boundary condition (4.2) is satisfied by the choice of the space  $V(\Omega)$ . We prove that the initial conditions are satisfied. Suppose  $\theta \in C^1(0, T)$  and  $\theta(T) = 0$ . For any  $j$  we have

$$\int_0^T \left( \frac{\partial}{\partial t} (\mathbf{u}_m - \mathbf{u}), \mathbf{w}_j \right) \theta(t) dt = -(\mathbf{u}_m(0) - \mathbf{u}(0), \mathbf{w}_j) \theta(0) - \int_0^T (\mathbf{u}_m(t) - \mathbf{u}(t), \mathbf{w}_j) \theta'(t) dt. \quad (4.37)$$

Then, by (4.30), we have

$$\lim_{m \rightarrow \infty} |(\mathbf{u}_m(0) - \mathbf{u}(0), \mathbf{w}_j)| = 0.$$

Since  $\mathbf{u}_{0m}(x) = \mathbf{u}_m(0, x)$  and  $\mathbf{u}_{0m} \rightarrow \mathbf{u}_0$ , we obtain  $\mathbf{u}(0) = \mathbf{u}_0$ , and can argue similarly for  $\mathbf{u}_1$ .

Then, we conclude that, there exists a solution  $u$  of (4.1) such that

$$\partial_t \mathbf{u} \in L^\infty(0, T; V(\Omega)), \quad \text{and} \quad \partial_t^2 \mathbf{u} \in L^\infty(0, T; L^2(\Omega)), \quad (4.38)$$

which implies

$$\begin{aligned} \mathbf{u}^s &\in C^1(0, T; H_0^1(\Omega)), & \partial_t^2 \mathbf{u}^s &\in C(0, T; L^2(\Omega)) \\ \mathbf{u}^f &\in C^1(0, T; H(\text{div}, \Omega)), & \partial_t^2 \mathbf{u}^f &\in C(0, T; L^2(\Omega)). \end{aligned} \quad (4.39)$$

On the other hand

$$\begin{aligned} \nabla(\text{qdiv } \mathbf{u}^s) + \nabla(\text{rdiv } \mathbf{u}^f) &\in C(0, T; L^2(\Omega)), \\ \Delta_{\mu, \lambda} \mathbf{u}^s + \nabla(\text{qdiv } \mathbf{u}^f) &\in C(0, T; L^2(\Omega)). \end{aligned} \quad (4.40)$$

Consequently,

$$\Delta_{\mu, \tilde{\lambda}} \mathbf{u}^s \in C(0, T; L^2(\Omega)), \quad \tilde{\lambda} = \lambda - \frac{\mathbf{q}^2}{\mathbf{r}}.$$

Then by the elliptic regularity,  $\mathbf{u}^s \in H_0^1(\Omega)$  yields

$$\mathbf{u}^s \in C(0, T; H^2(\Omega)).$$

By  $\text{div } \mathbf{u}^f \in C(0, T; H^1(\Omega))$ , we see

$$\begin{aligned} \mathbf{u}^s &\in C(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(0, T; H^1(\Omega)) \\ \mathbf{u}^f &\in C^2(0, T; L^2(\Omega)), \quad \text{div } \mathbf{u}^f \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)). \end{aligned} \quad (4.41)$$

## 4.6 Uniqueness

Let  $\mathbf{u}_1$  and  $\mathbf{u}_2$  be two solutions to (4.1)-(4.2) with the same initial data, and set  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ . Then for every function  $\mathbf{v} \in V(\Omega)$ , we have

$$(M\partial_t^2 \mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{v}) = 0, \quad \forall t \in (0, T).$$

Since  $\partial_t \mathbf{u} \in V(\Omega)$ , we may take  $\mathbf{v} = \partial_t \mathbf{u}$ , and this equation can be reduced to equality

$$\frac{1}{2} \frac{d}{dt} \left[ \|M^{1/2} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}, \mathbf{u}) \right] = \frac{\eta}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(\Omega)}^2.$$

Then

$$\|\partial_t \mathbf{u}(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}(t)\|_{V(\Omega)}^2 \leq C \int_0^t \left( \|\partial_t \mathbf{u}(\tau)\|_{L^2(\Omega)}^2 + \|\mathbf{u}(\tau)\|_{L^2(\Omega)}^2 \right) d\tau.$$

This implies that  $\|\mathbf{u}\|_{V(\Omega)} = 0 = \|\partial_t \mathbf{u}\|_{L^2(\Omega)}$  and  $\mathbf{u}_1 = \mathbf{u}_2$  a.e. in  $Q$ .

The proof of Theorem 1.1 is completed.

**Acknowledgements:** Most of this article has been written during the stays of the first author at Graduate School of Mathematical Sciences of the University of Tokyo in 2010 and 2011. The author thanks the school for the hospitality.

## References

- [1] M.Bellassoued: *Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation*, Inverse Problems 20 (2004), 1033-1052.
- [2] M. Bellassoued: *Uniqueness and stability in determining the speed of propagation of second-order hyperbolic equation with variable coefficients*, Applicable Analysis 83 (2004), 983-1014.
- [3] M. Bellassoued and M.Yamamoto: *Logarithmic stability in determination of a coefficient in an acoustic equation by arbitrary boundary observation*, J. Math. Pures Appl. 85(2006), 193-224.
- [4] M. Bellassoued and M.Yamamoto: *Determination of a coefficient in the wave equation with a single measurment*, Applicable Analysis 87 (2008), 901-920.
- [5] M. Bellassoued and M.Yamamoto: *Carleman estimate with second large parameter for second order hyperbolic operators in a Riemannian manifold and applications in thermoelasticity cases*, Applicable Analysis 91 (2012), 35-67.
- [6] A. Benabdallah, M. Cristofol, P. Gaitan and M. Yamamoto: *Inverse problem for a parabolic system with two components by measurements of one component*. Applicable Analysis 88 (2009), 683 – 709.
- [7] M.A. Biot: *Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range*. J. Acoust. Soc. Amer. 28 (1956) 168-178, 73.99.
- [8] A.L.Bukhgeim: *Introduction to the Theory of Inverse Problems*, VSP, Utrecht (2000).
- [9] A.L.Bukhgeim, J.Cheng, V. Isakov and M.Yamamoto: *Uniqueness in determining damping coefficients in hyperbolic equations*, S.Saitoh et al. (eds), Analytic Extension Formulas and their Applications, 27-46 (2001).

- [10] A.L. Bukhgeim and M.V. Klibanov: *Global uniqueness of class of multidimensional inverse problems*, Soviet Math. Dokl. 24 (1981), 244-247.
- [11] T.Carleman: *Sur un problème d'unicité pour les système d'équations aux dérivées partielles à deux variables indépendents*, Ark. Mat. Astr. Fys. 2B (1939), 1-9.
- [12] A.V. Fursikov and O.Yu. Imanuvilov: *Controllability of Evolution Equations*, Seoul National University, Seoul (1996).
- [13] N.-E. Hörlin and G.Peter: *Weak, anisotropic symmetric formulations of Biot's equations for vibro-acoustic modelling of porous elastic materials*. Internat. J. Numer. Methods Engrg. 84 (2010), no. 12, 1519-1540.
- [14] L.Hörmander: *Linear Partial Differential Operators*, Springer-Verlag, Berlin (1963).
- [15] M. Ikehata, G.Nakamura and M.Yamamoto: *Uniqueness in inverse problems for the isotropic Lamé system*, J. Math. Sci. Univ. Tokyo 5 (1998), 627-692.
- [16] O.Yu. Imanuvilov: *On Carleman estimates for hyperbolic equations*, Asymptotic Analysis 32 (2002), 185–220.
- [17] O.Yu.Imanuvilov, V.Isakov and M.Yamamoto: *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math. 56 (2003), 1366-1382.
- [18] O.Yu.Imanuvilov and M.Yamamoto: *Lipshitz stability in inverse parabolic problems by Carleman estimate*, Inverse Problems 14 (1998), 1229-1249.
- [19] O.Yu.Imanuvilov and M.Yamamoto: *Global Lipschitz stability in an inverse hyperbolic problem by interior observations*, Inverse Problems 17 (2001), 717-728.
- [20] O.Yu.Imanuvilov and M.Yamamoto: *Determination of a coefficient in an acoustic equation with single measurement*, Inverse Problems 19 (2003), 157-171.
- [21] O.Yu.Imanuvilov and M.Yamamoto: *Carleman estimates for the non-stationary Lamé system and the application to an inverse problem*, ESIAM:COCV 11 (2005), 1-56.
- [22] O.Yu.Imanuvilov and M.Yamamoto: *Carleman estimates for the Lamé system with stress boundary condition*, Publ. Research Institute for Mathematical Sciences 43 (2007), 1023–1093.
- [23] O.Yu.Imanuvilov and M.Yamamoto: *An inverse problem and an observability inequality for the Lamé system with stress boundary condition*, Applicable Analysis 88 (2009), 711–733.
- [24] V.Isakov: *A nonhyperbolic Cauchy problem for  $\square_b \square_c$  and its applications to elasticity theory*, Comm. Pure and Appl. Math. 39 (1986), 747-767.
- [25] V.Isakov: *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin (1998, 2005).
- [26] V. Isakov and N. Kim: *Carleman estimates with second large parameter and applications to elasticity with residual stress*, Applicationes Mathematicae 35 (2008), 447-465.

- [27] M.A.Kazemi and M.V.Klibanov: *Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequality*, *Applicable Analysis* 50 (1993), 93-102.
- [28] A.Khaïdarov: *On stability estimates in multidimensional inverse problems for differential equation*, *Soviet Math. Dokl.* 38 (1989), 614-617.
- [29] M.V. Klibanov: *Inverse problems in the "large" and Carleman bounds*, *Differential Equations* 20 (1984), 755-760.
- [30] M.V.Klibanov: *Inverse problems and Carleman estimates*, *Inverse Problems* 8 (1992), 575-596.
- [31] M.V. Klibanov and J. Malinsky: *Newton-Kantorovich method for 3-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with time dependent data*, *Inverse Problems* 7 (1991), 577-595.
- [32] M.V. Klibanov and A. Timonov: *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht (2004).
- [33] M.V. Klibanov and M. Yamamoto: *Lipschitz stability of an inverse problem for an acoustic equation*, *Applicable Analysis* 85 (2006), 515-538.
- [34] M.M.Lavrent'ev, V.G. Romanov and S.P. Shishat'skiĭ: *Ill-posed Problems of Mathematics Physics and Analysis*, American Math. Soc., Providence (1986).
- [35] J.-L. Lions and E. Magenes: *Non-homogeneous Boundary Value Problems and Applications*, vol. I and II, Springer-Verlag, Berlin, 1972.
- [36] L.Rachele: *An inverse problem in elastodynamics: uniqueness of the wave speeds in the interior*, *J.Differential Equations* 162 (2000), 300-325.
- [37] J.-E.Santos: *Elastic wave propagation in fluid-saturated porous media. I. The existence and uniqueness theorems*. *RAIRO Modél. Math. Anal. Numér.* 20 (1986), no. 1, 113-128.
- [38] D.Tataru: *Carleman estimates and unique continuation for solutions to boundary value problems*, *J.Math.Pures Appl.* 75 (1996), 367-408.
- [39] M.Yamamoto: *Carleman estimates for parabolic equations and applications*, *Inverse Problems* 25 (2009) 123013 (75pp).