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Carleman Estimate and Inverse Source Problem for Biot's Equations Describing Wave Propagation in Porous Media

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Abstract

According to Biot's paper in 1956, by using the Lagrangian equations in classical mechanics, we consider a problem of the filtration of a liquid in porous elastic-deformation media whose mechanical behavior is described by the Lamé system coupled with a hyperbolic equation. Assuming the null surface displacement on the whole boundary, we discuss an inverse source problem of determining a body force only by observation of surface traction on a suitable subdomain along a sufficiently large time interval. Our main result is a Hölder stability estimate for the inverse source problem, which is proved by a new Carleman estimate for Biot's system.

1 Introduction

In 1956, Biot [7] presented a three-dimensional theory for coupled frame-fluid wave propagation in fluid saturated porous media, treating the solid frame and the saturating fluid as two separate co-located coupled continua. Two second order coupled partial differential equations were derived from this theory. More precisely, let us consider an open and bounded domain Ω of \mathbb{R}^3 with C^∞ boundary $\Gamma = \partial\Omega$, and let $\nu = \nu(x)$ be the unit outward normal vector to $\partial\Omega$ at x . Given $T > 0$, Biot's equation is written as:

$$\begin{aligned} \varrho_{11}\partial_t^2\mathbf{u}^s + \varrho_{12}\partial_t^2\mathbf{u}^f - \Delta_{\mu,\lambda}\mathbf{u}^s(x,t) - \nabla(\mathfrak{q}\operatorname{div}\mathbf{u}^f) &= F_1, \\ \varrho_{12}\partial_t^2\mathbf{u}^s + \varrho_{22}\partial_t^2\mathbf{u}^f - \nabla(\mathfrak{q}\operatorname{div}\mathbf{u}^s) - \nabla(\mathfrak{r}\operatorname{div}\mathbf{u}^f) &= F_2, \quad \text{in } Q := \Omega \times (-T, T) \end{aligned} \quad (1.1)$$

with the boundary condition

$$\mathbf{u}^s(x,t) = 0, \quad \mathbf{u}^f(x,t) \cdot \nu = 0, \quad (x,t) \in \Sigma := \Gamma \times (-T, T) \quad (1.2)$$

and the initial condition

$$(\mathbf{u}^s(x, 0), \partial_t \mathbf{u}^s(x, 0)) = (0, 0), \quad \left(\mathbf{u}^f(x, 0), \partial_t \mathbf{u}^f(x, 0) \right) = (0, 0), \quad x \in \Omega \quad (1.3)$$

where $F = (F_1, F_2)^T$ is an external force with $F_\ell = (F_\ell^1, F_\ell^2, F_\ell^3)^T$, $\ell = 1, 2$, and $\Delta_{\mu, \lambda}$ is the elliptic second-order linear differential operator given by

$$\begin{aligned} \Delta_{\mu, \lambda} \mathbf{v}(x) &\equiv \mu \Delta \mathbf{v}(x) + (\mu + \lambda) (\nabla \operatorname{div} \mathbf{v}(x)) \\ &\quad + (\operatorname{div} \mathbf{v}(x)) \nabla \lambda(x) + (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \nabla \mu(x), \quad x \in \Omega. \end{aligned} \quad (1.4)$$

Throughout this paper, t and $x = (x_1, x_2, x_3)$ denote the time variable and the spatial variable respectively, and $\mathbf{u}^s = (u_1^s, u_2^s, u_3^s)^T$ and $\mathbf{u}^f = (u_1^f, u_2^f, u_3^f)^T$ denote respectively the solid frame and fluid phase displacement vectors at the location x and the time t .

Here and henceforth \cdot^T denotes the transpose of matrices under consideration. We assume that the Lamé parameters μ and λ satisfy

$$\mu(x) > 0, \quad \lambda(x) + \mu(x) > 0, \quad \forall x \in \overline{\Omega}.$$

The function $q(x) > 0$, $x \in \overline{\Omega}$, is the dilatational coupling factor between the fluid phase and the solid frame. The coefficient $r(x) > 0$, $x \in \overline{\Omega}$ is the bulk modulus of the fluid phase and $\varrho_{11}(x)$, $\varrho_{22}(x) > 0$, $x \in \overline{\Omega}$ are the corrected mass densities for the solid phase and the fluid phase porosity and $\varrho_{12}(x)$ is the inertial coupling factor and see Hörlin and Peter [13].

We assume that the sources terms are given by

$$F_\ell(x, t) = p_\ell(x) R_\ell(x, t), \quad \ell = 1, 2, \quad (x, t) \in Q, \quad (1.5)$$

where $p_\ell \in H^2(\Omega)$ is real-valued and $R_\ell = (R_\ell^1, R_\ell^2, R_\ell^3)^T$ satisfy

$$\sum_{j=2}^3 \left(\|\partial_t^j R\|_{L^\infty(Q)}^2 + \|\partial_t^j \nabla R\|_{L^\infty(Q)}^2 \right) \leq C. \quad (1.6)$$

The main subject of this paper is the inverse problem of determining $p = (p_1, p_2) \in (H^2(\Omega))^2$ uniquely from observed data of the displacement vector $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$ in a subdomain $\omega \subset \Omega$. It is an important problem, for example, in mechanics to determine the source p inside a porous body from measurements of the slide frame and fluid phase displacements in ω .

1.1 Inverse problem

Let $\omega \subset \Omega$ be an arbitrarily given subdomain such that $\partial\omega \supset \partial\Omega$, i.e., $\omega = \Omega \cap \mathcal{V}$ where \mathcal{V} is a neighborhood of Γ in \mathbb{R}^3 and let $R(x, t) = (R_1(x, t), R_2(x, t))$ be appropriately given. Then we want to determine $p(x) = (p_1(x), p_2(x))$, $x \in \Omega$, by measurements $\mathbf{u}|_{\omega \times (-T, T)}$.

Our formulation of the inverse problem requires only a finite number of observations. As for inverse problems for non-stationary Lamé system by infinitely many boundary observations (i.e., Dirichlet-to-Neumann map), we refer to Rachele [36], for example.

For the formulation with a finite number of observations, Bukhgeim and Klibanov [10] created a method based on a Carleman estimate and established the uniqueness for inverse problems of determining spatially varying coefficients for scalar partial differential equations. See also Bellassoued [1],

[2], Bellassoued and Yamamoto [3], [4], Benabdallah, Cristofol, Gaitan and Yamamoto [6], Bukhgeim [8], Bukhgeim, Cheng, Isakov and Yamamoto [9], Imanuvilov and Yamamoto [18] - [20], Isakov [24], [25], Khaïdarov [28], Klibanov [29], [30], Klibanov and Timonov [32], Klibanov and Yamamoto [33], Yamamoto [39]. In particular, as for inverse problems for the isotropic Lamé system, we can refer to Ikehata, Nakamura and Yamamoto [15], Imanuvilov, Isakov and Yamamoto [21], Imanuvilov and Yamamoto [22] - [23], Isakov [24], Isakov and Kim [26].

A Carleman estimate is an inequality for a solution to a partial differential equation with weighted L^2 -norm and effectively yields the unique continuation for a partial differential equation with non-analytic coefficients. As a pioneering work concerning a Carleman estimate, we refer to Carleman's paper [11] where what is called a Carleman estimate was proved and applied it for proving the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since [11], the theory of Carleman estimates has been developed and we refer, for example, to Hörmander [14] and Isakov [25] for Carleman estimates for functions having compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). For Carleman estimates for functions without compact supports, we refer to Bellassoued and Yamamoto [5], Fursikov and Imanuvilov [12], [16], Lavrent'ev, Romanov and Shishat'skiĭ [34], Tataru [38]. Moreover Carleman estimates have been applied for estimating the energy and see e.g., Imanuvilov and Yamamoto [23], Kazemi and Klibanov [27], Klibanov and Malinsky [31], Klibanov and Timonov [32].

1.2 Notations and statement of main results

In order to formulate our results, we need to introduce some notations. For $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$, we define the following set of the scalar coefficients

$$\mathcal{C}(m, \theta) = \left\{ c \in C^2(\overline{\Omega}), c(x) > c^* > 0, x \in \overline{\Omega}, \|c\|_{C^2(\overline{\Omega})} \leq m, \frac{\nabla c \cdot (x - x_0)}{2c} \leq 1 - \theta \right\}, \quad (1.7)$$

where the constants $m > 0$ and $\theta \in (0, 1)$ are given.

Assumption A.1

Throughout this paper, we assume that the coefficients $(\varrho_{ij})_{1 \leq i, j \leq 2}$, μ , λ , q , $r \in C^2(\overline{\Omega})$ satisfy the following conditions

$$\begin{aligned} \varrho(x) &= \varrho_{11}(x)\varrho_{22}(x) - \varrho_{12}^2(x) > 0, \quad \forall x \in \overline{\Omega}, \\ \lambda(x)r(x) - q^2(x) &> 0, \quad \forall x \in \overline{\Omega}. \end{aligned} \quad (1.8)$$

Let $A(x) = (a_{ij}(x))_{1 \leq i, j \leq 2}$ be the 2×2 -matrix given by

$$A(x) = \frac{1}{\varrho} \begin{pmatrix} \varrho_{22} & -\varrho_{12} \\ -\varrho_{12} & \varrho_{11} \end{pmatrix} \begin{pmatrix} 2\mu + \lambda & q \\ q & r \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (1.9)$$

By (1.8), we can prove that $(a_{ij}(x))_{1 \leq i, j \leq 2}$ is a positive definite matrix on $\overline{\Omega}$.

Assumption A.2:

Let $A(x)$ have two distinct positive eigenvalues: $\mu_2(x), \mu_3(x) > 0$, $\mu_2(x) \neq \mu_3(x)$. Moreover setting, $\mu_1 := (\varrho^{-1}\varrho_{22})\mu$, we assume

$$\mu_1, \mu_2, \mu_3 \in \mathcal{C}(m, \theta). \quad (1.10)$$

Assumption A.3:

We assume that the solution $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$ satisfies the a priori boundedness and regularity:

$$\mathbf{u} \in H^5(Q), \quad \|\mathbf{u}\|_{H^5(Q)} \leq M_0, \quad (1.11)$$

for some positive constant M_0 .

Before stating the main result on the stability for the inverse source problem, we present Theorem 1.1 on the unique existence of strong solution to (1.1)-(1.2) with initial condition:

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{and} \quad \partial_t \mathbf{u}(\cdot, 0) = \mathbf{u}_1.$$

Let $V(\Omega) = (H^1(\Omega))^3 \times H(\text{div}, \Omega)$, where

$$H(\text{div}, \Omega) = \{\mathbf{u} \in (L^2(\Omega))^3; \quad \text{div } \mathbf{u} \in L^2(\Omega)\}. \quad (1.12)$$

The norm in $V(\Omega)$ is chosen as follows

$$\|(\mathbf{v}^1, \mathbf{v}^2)\|_{V(\Omega)}^2 = \|\mathbf{v}^1\|_{H^1(\Omega)}^2 + \|\mathbf{v}^2\|_{L^2(\Omega)}^2 + \|\text{div } \mathbf{v}^2\|_{L^2(\Omega)}^2, \quad \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V(\Omega).$$

Theorem 1.1. *Let $F \in H^1(-T, T; L^2(\Omega))$, $(\mathbf{u}_0, \mathbf{u}_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^6 \times (H^1(\Omega))^6$. Then there exists a unique solution $\mathbf{u}(x, t) = (\mathbf{u}^s(x, t), \mathbf{u}^f(x, t))$ of (1.1)-(1.2) with initial data $(\mathbf{u}_0, \mathbf{u}_1)$ such that*

$$\begin{aligned} \mathbf{u}^s &\in \mathcal{C}([-T, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1([-T, T]; H^1(\Omega)) \cap \mathcal{C}^2([-T, T]; L^2(\Omega)) \\ \mathbf{u}^f &\in \mathcal{C}^2([-T, T]; L^2(\Omega)), \quad \text{div } \mathbf{u}^f \in \mathcal{C}([-T, T]; H^1(\Omega)) \cap \mathcal{C}^1([-T, T]; L^2(\Omega)). \end{aligned} \quad (1.13)$$

In particular there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{C}^2([-T, T]; L^2(\Omega))} \leq C(\|F\|_{H^1(-T, T; L^2(\Omega))} + \|\mathbf{u}_0\|_{H^2(\Omega)} + \|\mathbf{u}_1\|_{H^1(\Omega)}).$$

Moreover, if $F = 0$, then the energy of the solution $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$ given by

$$E(t) = \frac{1}{2} \int_{\Omega} \left(M(x) \partial_t \mathbf{u} \cdot \partial_t \mathbf{u} + \lambda |\text{div } \mathbf{u}^s|^2 + 2\mu |\varepsilon(\mathbf{u}^s)|^2 + r |\text{div } \mathbf{u}^f|^2 + 2q (\text{div } \mathbf{u}^f)(\text{div } \mathbf{u}^s) \right) dx$$

is conserved, that is,

$$E(t) = E(0), \quad \forall t \geq 0.$$

Here $M(x) = (\rho_{ij}(x) I_3)_{1 \leq i, j \leq 2}$ and $\varepsilon(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$.

The proof is based on the Galerkin method and see Santos [37] for the case $n = 2$. For completeness we will give a proof for dimension 3 in Section 4.

In order to formulate our stability estimates for the inverse problem we introduce some notations. Let $\vartheta : \overline{\Omega} \rightarrow \mathbb{R}$ be the strictly convex function given by

$$\vartheta(x) = |x - x_0|^2, \quad x \in \overline{\Omega}. \quad (1.14)$$

Set

$$D^2 = \max_{x \in \overline{\Omega}} \vartheta(x), \quad d^2 = \min_{x \in \overline{\Omega}} \vartheta(x), \quad D_0^2 = D^2 - d^2. \quad (1.15)$$

By Assumption A.2, there exist constants $c_j^* > 0$, $j = 1, 2, 3$ such that $\mu_j(x) > c_j^* > 0$ for all $x \in \overline{\Omega}$, $j = 1, 2, 3$. Let $c_0^* = \min\{c_1^*, c_2^*, c_3^*\}$. We choose $\beta > 0$ such that

$$\beta + \frac{mD_0}{\sqrt{c_0^*}}\sqrt{\beta} < \theta c_0^*, \quad c_0^*d^2 - \beta D^2 > 0. \quad (1.16)$$

Here we note that since $x_0 \notin \overline{\Omega}$, such $\beta > 0$ exists.

We set

$$T_0 = \frac{D_0}{\sqrt{\beta}}. \quad (1.17)$$

The main results of this paper can be stated as follows:

Theorem 1.2. (Stability) Assume (A.1), (A.2), and (A.3). Let $T > T_0$ and \mathbf{u} be the solution of (1.1)-(1.2) and (1.3). Moreover let assume that $\Phi_j(x) := R_j(x, 0)$ satisfy

$$\Phi_j(x) \cdot (x - x_0) \neq 0 \quad \text{for all } x \in \overline{\Omega}. \quad (1.18)$$

Let $M > 0$. Then there exist constants $C > 0$ and $\kappa \in (0, 1)$ such that the following estimate holds:

$$\|p_1\|_{H_0^1(\Omega)}^2 + \|p_2\|_{H_0^1(\Omega)}^2 \leq C\mathcal{E}_\omega(\mathbf{u})^\kappa \quad (1.19)$$

for any $p_\ell \in H^2(\Omega)$, $\ell = 1, 2$, such that $\|p_\ell\|_{H^2} \leq M$ and $p_\ell = 0$, $\nabla p_\ell = 0$ on Γ . Here

$$\mathcal{E}_\omega(\mathbf{u}) = \sum_{j=2}^3 \|\partial_t^j \mathbf{u}\|_{H^2(\omega \times (-T, T))}^2.$$

By Theorem 1.2, we can readily derive the uniqueness in the inverse problem:

Corollary 1.1. Under the assumptions in Theorem 1.2, we have the uniqueness:

Let $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$ satisfy Biot's system (1.1)-(1.3) such that $\mathbf{u}(x, t) = 0$, $(x, t) \in \omega \times (-T, T)$. Then $p_1(x) = p_2(x) = 0$ for all $x \in \Omega$ and $\mathbf{u}(x, t) = 0$ in Q .

The remainder of the paper is organized as follows. In section 2, we give a Carleman estimate for the Biot's system. In section 3 we prove Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.1.

2 Carleman estimate for Biot's system

In this section we will prove a Carleman estimate for Biot's system, which is interesting of itself. In order to formulate our Carleman estimate, we introduce some notations. Let $\vartheta : \overline{\Omega} \rightarrow \mathbb{R}$ be the strictly convex function given by (1.14), where $x_0 \notin \overline{\Omega}$.

We define two functions $\psi, \varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ of class C^∞ by

$$\psi(x, t) = |x - x_0|^2 - \beta |t|^2 \quad \text{for all } x \in \Omega, \quad -T \leq t \leq T, \quad (2.1)$$

$$\varphi(x, t) = e^{\gamma\psi(x, t)}, \quad \gamma > 0,$$

where $T > T_0$. Therefore, by (1.17) and (1.15), we have

$$\varphi(x, 0) \geq d_0, \quad \varphi(x, \pm T) < d_0 \quad (2.2)$$

with $d_0 = \exp(\gamma d^2)$. Thus, for given $\eta > 0$, we can choose sufficiently small $\varepsilon = \varepsilon(\eta)$ such that

$$\begin{aligned} \varphi(x, t) &\leq d_0 - \eta \equiv d_1 \quad \text{for all } (x, t) \in \{(x, t) \in Q; |t| > T - 2\varepsilon\}, \\ \varphi(x, t) &\geq d_0 - \frac{\eta}{2} \equiv d_2 \quad \text{for all } (x, t) \in \{(x, t) \in Q; |t| < \varepsilon\}. \end{aligned} \quad (2.3)$$

Let $(\mathbf{u}^s, \mathbf{u}^f)$ satisfy Biot's system

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{u}^s(x, t) + \varrho_{12} \partial_t^2 \mathbf{u}^f - \Delta_{\mu, \lambda} \mathbf{u}^s(x, t) - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^f) &= F_1, \\ \varrho_{12} \partial_t^2 \mathbf{u}^s(x, t) + \varrho_{22} \partial_t^2 \mathbf{u}^f - \nabla (\mathbf{r} \operatorname{div} \mathbf{u}^s) - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^f) &= F_2, \quad \text{in } Q. \end{aligned} \quad (2.4)$$

The following theorem is a Carleman estimate for Biot's system (2.4).

Theorem 2.1. *There exist $\tau_* > 0$ and $C > 0$ such that the following estimate holds:*

$$\begin{aligned} \int_Q \tau \left(|\nabla_{x,t} \mathbf{u}^s|^2 + |\nabla_{x,t} (\operatorname{div} \mathbf{u}^s)|^2 + |\nabla_{x,t} (\operatorname{div} \mathbf{u}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ + \int_Q \tau^3 \left(|\mathbf{u}^s|^2 + |\operatorname{div} \mathbf{u}^s|^2 + |\operatorname{div} \mathbf{u}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|F|^2 + |\nabla F|^2) e^{2\tau\varphi} dxdt \end{aligned} \quad (2.5)$$

for any $\tau \geq \tau_*$ and any solution $(\mathbf{u}^s, \mathbf{u}^f) \in (H^2(Q))^6$ to (2.4) which is supported in a fixed compact set $K \subset \operatorname{int}(Q)$.

In order to prove Theorem 2.1, we use a Carleman estimate for a coupling hyperbolic system, which we discuss in the next subsection.

2.1 Carleman estimate for a hyperbolic system

First we recall the following Carleman estimate for a scalar hyperbolic equation. As for the proof, we refer to Bellassoued and Yamamoto [5], and Imanuvilov and Yamamoto [20] for example.

Lemma 2.1. *Let $c \in \mathcal{C}(m, \theta)$. There exist constants $C > 0$ and $\tau_* > 0$ such that the following Carleman estimate holds:*

$$C \int_Q e^{2\tau\varphi} \left(\tau |\nabla_{x,t} y|^2 + \tau^3 |y|^2 \right) dxdt \leq \int_Q e^{2\tau\varphi} |(\partial_t^2 - c\Delta)y|^2 dxdt$$

whenever $y \in H^2(Q)$ is supported in a fixed compact set $K \subset \operatorname{int}(Q)$ and any $\tau \geq \tau_*$.

Let $v = (v_1, v_2) \in (H^2(\Omega))^2$ satisfy the following hyperbolic system

$$\begin{cases} \partial_t^2 v_1 - b_{11}(x)\Delta v_1 - b_{12}(x)\Delta v_2 = g_1 & \text{in } Q \\ \partial_t^2 v_2 - b_{21}(x)\Delta v_1 - b_{22}(x)\Delta v_2 = g_2 & \text{in } Q, \end{cases} \quad (2.6)$$

for $g = (g_1, g_2) \in (L^2(Q))^2$. We assume that the matrix $B(x) = (b_{ij}(x))_{1 \leq i, j \leq 2}$ has two distinct positive eigenvalues $c_1, c_2 \in \mathcal{C}(m, \theta)$. Then, by Lemma 2.1, we have the following Carleman estimate.

Lemma 2.2. *There exist constants $C > 0$ and $\tau_* > 0$ such that the following Carleman estimate holds*

$$C \int_Q e^{2\tau\varphi} \left(\tau |\nabla_{x,t} v|^2 + \tau^3 |v|^2 \right) dxdt \leq \int_Q e^{2\tau\varphi} |g|^2 dxdt$$

for any $\tau \geq \tau_*$, whenever $v \in H^2(Q)$ is a solution of (2.6) and supported in a fixed compact set $K \subset \text{int}(Q)$.

Proof. The system (2.6) can be written in the equivalent form

$$\partial_t^2 v - B(x)\Delta v = g \quad \text{in } Q. \quad (2.7)$$

By the assumption on $B(x)$, there exists a matrix $P(x)$ such that

$$(P^{-1}BP)(x) = \text{Diag}(c_1(x), c_2(x)) = \Lambda(x), \quad x \in \Omega.$$

Therefore system (2.7) can be written in an equivalent form:

$$\partial_t^2 \tilde{v} - \Lambda(x)\Delta \tilde{v} = \tilde{g} + \mathcal{B}_1(x, \partial)v,$$

where

$$\tilde{v}(x, t) = P^{-1}(x)v(x, t), \quad \tilde{g}(x, t) = P^{-1}(x)g(x, t), \quad (2.8)$$

and \mathcal{B}_1 is a first-order differential operator.

Since $c_j \in \mathcal{C}(m, \theta)$ for $j = 1, 2$, we can apply Lemma 2.1 for the two components of \tilde{v} and obtain

$$C \int_Q e^{2\tau\varphi} \left(\tau |\nabla_{x,t} \tilde{v}|^2 + \tau^3 |\tilde{v}|^2 \right) dxdt \leq \int_Q e^{2\tau\varphi} |\tilde{g}|^2 dxdt + \int_Q e^{2\tau\varphi} (|v|^2 + |\nabla v|^2) dxdt$$

and, by (2.8), we easily obtain

$$|v(x, t)| \leq C |\tilde{v}(x, t)|, \quad |\nabla v(x, t)| \leq C (|\nabla \tilde{v}(x, t)| + |\tilde{v}(x, t)|), \quad |\tilde{g}(x, t)| \leq C |g(x, t)|$$

for $(x, t) \in Q$. This completes the proof. \square

2.2 Proof of the Carleman estimate for Biot's system

In this section, we derive a global Carleman estimate for a solutions of system (2.4). We consider the 6×6 -matrix

$$M(x) = \begin{pmatrix} \varrho_{11}(x)I_3 & \varrho_{12}(x)I_3 \\ \varrho_{12}(x)I_3 & \varrho_{22}(x)I_3 \end{pmatrix}. \quad (2.9)$$

Here I_3 is the 3×3 identity matrix. Then by Assumption A.1, we have

$$M^{-1}(x) = \frac{1}{\varrho} \begin{pmatrix} \varrho_{22}(x)I_3 & -\varrho_{12}(x)I_3 \\ -\varrho_{12}(x)I_3 & \varrho_{11}(x)I_3 \end{pmatrix}.$$

Let $v^s = \operatorname{div} \mathbf{u}^s$, $v^f = \operatorname{div} \mathbf{u}^f$, $v = (v^s, v^f)$ and $\mathbf{w}^s = \operatorname{curl} \mathbf{u}^s$. Put $G = M^{-1}F$, $F = (F_1, F_2)^T$ and apply M^{-1} to system (2.4), we obtain

$$\begin{aligned} \partial_t^2 \mathbf{u}^s - \mu_1 \Delta \mathbf{u}^s - (\mu_1 + \lambda_1) \nabla (\operatorname{div} \mathbf{u}^s) - q_1 \nabla (\operatorname{div} \mathbf{u}^f) &= G_1 + \mathcal{R}_1 \mathbf{u}^s + \mathcal{R}_0 \operatorname{div} \mathbf{u}^f, & \text{in } Q \\ \partial_t^2 \mathbf{u}^f + \mu_2 \Delta \mathbf{u}^s - r_2 \nabla (\operatorname{div} \mathbf{u}^f) - q_2 \nabla (\operatorname{div} \mathbf{u}^s) &= G_2 + \mathcal{R}'_1 \mathbf{u}^s + \mathcal{R}'_0 \operatorname{div} \mathbf{u}^f, & \text{in } Q. \end{aligned} \quad (2.10)$$

Here, $\mathcal{R}_j, \mathcal{R}'_j, j = 0, 1$ are differential operators of order j with coefficients in $L^\infty(Q)$, and

$$\begin{aligned} \mu_1 &= \varrho^{-1} \mu \varrho_{22}, & \lambda_1 &= \varrho^{-1} (\lambda \varrho_{22} - q \varrho_{12}), & q_1 &= \varrho^{-1} (q \varrho_{22} - r \varrho_{12}) \\ \mu_2 &= \varrho^{-1} \mu \varrho_{12}, & q_2 &= \varrho^{-1} (q \varrho_{11} - (\mu + \lambda) \varrho_{12}), & r_2 &= \varrho^{-1} (r \varrho_{11} - q \varrho_{12}). \end{aligned} \quad (2.11)$$

Henceforth $\mathcal{P}_j, j = 1, \dots, 4$ denote some first-order operators with $L^\infty(Q)$ -coefficients.

We apply div to the equations in (2.10), and can derive the following two equations:

$$\begin{aligned} \partial_t^2 v^s - a_{11} \Delta v^s - a_{12} \Delta v^f &= \operatorname{div} G_1 + \mathcal{P}_1(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s) \\ \partial_t^2 v^f - a_{21} \Delta v^s - a_{22} \Delta v^f &= \operatorname{div} G_2 + \mathcal{P}_2(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s), \end{aligned} \quad (2.12)$$

where $(a_{ij})_{1 \leq i, j \leq 2}$ is given by (1.9). We apply the curl to the first equation (2.10) to obtain

$$\partial_t^2 \mathbf{w}^s - \mu_1 \Delta \mathbf{w}^s = \operatorname{curl} G_1 + \mathcal{P}_3(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s) \quad (2.13)$$

and

$$\partial_t^2 \mathbf{u}^s - \mu_1 \Delta \mathbf{u}^s = G_1 + \mathcal{P}_4(v^f, v^s, \mathbf{u}^s, \mathbf{w}^s). \quad (2.14)$$

Applying Lemma 2.2 to system (2.12), we have for $v = (v^s, v^f)$

$$\begin{aligned} C \int_Q e^{2\tau\varphi} \left(\tau |\nabla_{x,t} v|^2 + \tau^3 |v|^2 \right) dxdt &\leq \int_Q e^{2\tau\varphi} \left(|F|^2 + |\nabla F|^2 \right) dxdt \\ &\quad + \int_Q e^{2\tau\varphi} \left(|\mathbf{u}^s|^2 + |\mathbf{w}^s|^2 + |\nabla \mathbf{u}^s|^2 + |\nabla \mathbf{w}^s|^2 \right) dxdt. \end{aligned}$$

Applying Lemma 2.1 to (2.13) and (2.14), we obtain

$$\begin{aligned} C \int_Q e^{2\tau\varphi} \left(\tau |\nabla_{x,t} \mathbf{w}^s|^2 + \tau^3 |\mathbf{w}^s|^2 + \tau |\nabla_{x,t} \mathbf{u}^s|^2 + \tau^3 |\mathbf{u}^s|^2 \right) dxdt \\ \leq \int_Q e^{2\tau\varphi} \left(|F|^2 + |\nabla F|^2 \right) dxdt + \int_Q e^{2\tau\varphi} \left(|v|^2 + |\nabla v|^2 \right) dxdt. \end{aligned}$$

Therefore, for τ sufficiently large, we obtain (2.5). This completes the proof of Theorem 2.1.

3 Proof of Theorem 1.2

In this section we prove the stability (Theorem 1.2) for the inverse source problem.

For the proof, we apply the method in Imanuvilov and Yamamoto [19] which modified the argument in [10] and proved the stability for an inverse coefficient problem for a hyperbolic equation. For it, the Carleman (Theorem 2.1) is a key.

3.1 Modified Carleman estimate for Biot's system

Let $\omega_T = \omega \times (-T, T)$. We modify Theorem 2.1 for functions which vanish at $\pm T$ with first t -derivatives.

Lemma 3.1. *There exist positive constants τ_* , $C > 0$ and $C_0 > 0$ such that the following inequality holds:*

$$\begin{aligned} & \int_Q \tau \left(|\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ & + \int_Q \tau^3 \left(|\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau\varphi} dxdt \\ & + C e^{C_0\tau} \|\mathbf{v}\|_{H^2(\omega_T)}^2 \end{aligned} \quad (3.1)$$

for any $\tau \geq \tau_*$ and any $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in H^2(Q)$ satisfying, for $G = (G_1, G_2)$

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{v}^s + \varrho_{12} \partial_t^2 \mathbf{v}^f - \Delta_{\mu,\lambda} \mathbf{v}^s - \nabla (q \operatorname{div} \mathbf{v}^f) &= G_1, \\ \varrho_{12} \partial_t^2 \mathbf{v}^s + \varrho_{22} \partial_t^2 \mathbf{v}^f - \nabla (q \operatorname{div} \mathbf{v}^s) - \nabla (r \operatorname{div} \mathbf{v}^f) &= G_2 \quad \text{in } Q \end{aligned} \quad (3.2)$$

such that

$$\partial_t^j \mathbf{v}(x, \pm T) = 0 \quad \text{for all } x \in \Omega, j = 0, 1. \quad (3.3)$$

Proof. Let $\omega^0 \subset \omega$. In order to apply Carleman estimate (2.5), we introduce a cut-off function ξ satisfying $0 \leq \xi \leq 1$, $\xi \in C^\infty(\mathbb{R}^3)$, $\xi = 1$ in $\overline{\Omega \setminus \omega^0}$ and $\operatorname{Supp} \xi \subset \Omega$. Let $\mathbf{v} \in H^2(Q)$ satisfy (3.2) and (3.3). Put

$$\mathbf{w}(x, t) = \xi(x) \mathbf{v}(x, t), \quad (x, t) \in Q,$$

and let $Q_0 = (\Omega \setminus \overline{\omega}) \times (-T, T)$. Noting that $\mathbf{w} \in H^2(Q)$ is compactly supported in Q and $\mathbf{w} = \mathbf{v}$ in Q_0 and applying Carleman estimate (2.5) to \mathbf{w} , we obtain

$$\begin{aligned} & \int_{Q_0} \tau \left(|\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ & + \int_Q \tau^3 \left(|\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau\varphi} dxdt \\ & + C \int_Q |\mathcal{Q}_2 \mathbf{v}|^2 e^{2\tau\varphi} dxdt \end{aligned}$$

for any $\tau \geq \tau_*$. Here \mathcal{Q}_2 is a differential operator of order 2 whose coefficients are supported in ω . Therefore

$$\begin{aligned} & \int_Q \tau \left(|\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ & + \int_Q \tau^3 \left(|\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \leq C \int_Q (|G|^2 + |\nabla G|^2) e^{2\tau\varphi} dxdt \\ & + C e^{C_0\tau} \|\mathbf{v}\|_{H^2(\omega_T)}^2. \end{aligned}$$

This completes the proof of the lemma. \square

By $\mathcal{N}_{\tau,\varphi}(\mathbf{v})$ we denote

$$\begin{aligned} \mathcal{N}_{\tau,\varphi}(\mathbf{v}) &= \int_Q \tau \left(|\nabla_{x,t} \mathbf{v}^s|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^s)|^2 + |\nabla_{x,t}(\operatorname{div} \mathbf{v}^f)|^2 \right) e^{2\tau\varphi} dxdt \\ &\quad + \int_Q \tau^3 \left(|\mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^s|^2 + |\operatorname{div} \mathbf{v}^f|^2 \right) e^{2\tau\varphi} dxdt \end{aligned} \quad (3.4)$$

where $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f)$.

Now, we recall (2.2) and (2.3) for the definition of d_0 , η and ε and we introduce a cut-off function ζ satisfying $0 \leq \zeta \leq 1$, $\zeta \in C^\infty(\mathbb{R})$ and

$$\zeta = 1 \quad \text{in } (-T + 2\varepsilon, T - 2\varepsilon), \quad \operatorname{Supp} \zeta \subset (-T + \varepsilon, T - \varepsilon). \quad (3.5)$$

Finally we denote by $\tilde{\mathbf{v}}$ the function

$$\tilde{\mathbf{v}}(x, t) = \zeta(t)(\mathbf{v}^s, \mathbf{v}^f)(x, t), \quad (x, t) \in Q. \quad (3.6)$$

Lemma 3.2. *There exist positive constants τ_* , C and C_0 such that the following inequality holds:*

$$C\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}) \leq \int_Q \left(|F|^2 + |\nabla F|^2 \right) e^{2\tau\varphi} dxdt + e^{C_0\tau} \|\mathbf{v}\|_{H^2(\omega_T)}^2 + e^{2d_1\tau} \|\mathbf{v}\|_{H^1(-T, T; H^1(\Omega))}^2$$

for any $\tau \geq \tau_*$ and any $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in (H^2(Q))^6$ satisfying

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{v}^s + \varrho_{12} \partial_t^2 \mathbf{v}^f - \Delta_{\mu,\lambda} \mathbf{v}^s - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}^f) &= F_1(x, t) \\ \varrho_{12} \partial_t^2 \mathbf{v}^s + \varrho_{22} \partial_t^2 \mathbf{v}^f - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}^s) - \nabla (\mathbf{r} \operatorname{div} \mathbf{v}^f) &= F_2(x, t), \quad (x, t) \in Q \end{aligned}$$

Proof. We note that $\tilde{\mathbf{v}} \in (H^2(Q))^6$ and

$$\begin{aligned} \varrho_{11} \partial_t^2 \tilde{\mathbf{v}}^s + \varrho_{12} \partial_t^2 \tilde{\mathbf{v}}^f - \Delta_{\mu,\lambda} \tilde{\mathbf{v}}^s - \nabla (\mathbf{q} \operatorname{div} \tilde{\mathbf{v}}^f) &= \zeta(t) F_1(x, t) + P_1(\mathbf{v}, \partial_t \mathbf{v}), \\ \varrho_{12} \partial_t^2 \tilde{\mathbf{v}}^s + \varrho_{22} \partial_t^2 \tilde{\mathbf{v}}^f - \nabla (\mathbf{q} \operatorname{div} \tilde{\mathbf{v}}^s) - \nabla (\mathbf{r} \operatorname{div} \tilde{\mathbf{v}}^f) &= \zeta(t) F_2(x, t) + P_2(\mathbf{v}, \partial_t \mathbf{v}), \quad (x, t) \in Q, \end{aligned}$$

where P_1 and P_2 are zeroth-order operators and supported in $|t| > T - 2\varepsilon$. Therefore, applying Lemma 3.1 to $\tilde{\mathbf{v}}$ and using (2.3), we complete the proof of the lemma. \square

3.2 Preliminary estimates

Let $\varphi(x, t)$ be the function defined by (2.1). Then

$$\varphi(x, t) = e^{\gamma\psi(x,t)} =: \rho(x)\alpha(t), \quad (3.7)$$

where $\rho(x)$ and $\alpha(t)$ are defined by

$$\rho(x) = e^{\gamma\vartheta(x)} \geq d_0, \quad \forall x \in \Omega \quad \text{and} \quad \alpha(t) = e^{-\beta\gamma t^2} \leq 1, \quad \forall t \in [-T, T]. \quad (3.8)$$

Next we present the following Carleman estimate of a first-order partial differential operator:

$$L(x, D)v = \sum_{i=1}^3 a_i(x) \partial_i v + a_0(x)v, \quad x \in \Omega$$

where

$$a_0 \in \mathcal{C}(\overline{\Omega}), \quad a = (a_1, a_2, a_3) \in [\mathcal{C}^1(\overline{\Omega})]^3 \quad (3.9)$$

and

$$|a(x) \cdot (x - x_0)| \geq c_0 > 0, \quad \text{on } \overline{\Omega} \quad (3.10)$$

with a constant $c_0 > 0$. Then

Lemma 3.3. *In addition to (3.9) and (3.10), we assume that $\|a_0\|_{\mathcal{C}(\overline{\Omega})} \leq M$ and $\|a_i\|_{\mathcal{C}^1(\overline{\Omega})} \leq M$, $1 \leq i \leq 3$. Then there exist constants $\tau_* > 0$ and $C > 0$ such that*

$$\tau \int_{\Omega} |v(x)|^2 e^{2\tau\rho(x)} dx \leq C \int_{\Omega} |L(x, D)v(x)|^2 e^{2\tau\rho(x)} dx$$

for all $v \in H_0^1(\Omega)$ and all $\tau > \tau_*$.

The proof is direct by integration by parts and see e.g., [19].

Consider now the following system

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{u}^s + \varrho_{12} \partial_t^2 \mathbf{u}^f - \Delta_{\mu, \lambda} \mathbf{u}^s(x, t) - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^f) &= F_1(x, t), \\ \varrho_{12} \partial_t^2 \mathbf{u}^s + \varrho_{22} \partial_t^2 \mathbf{u}^f - \nabla (\mathbf{q} \operatorname{div} \mathbf{u}^s) - \nabla (\mathbf{r} \operatorname{div} \mathbf{u}^f) &= F_2(x, t), \quad (x, t) \in Q, \end{aligned} \quad (3.11)$$

with the boundary condition

$$\mathbf{u}^s(x, t) = 0, \quad \mathbf{u}^f(x, t) \cdot \nu = 0, \quad (x, t) \in \Sigma \quad (3.12)$$

and the initial condition

$$(\mathbf{u}^s(x, 0), \partial_t \mathbf{u}^s(x, 0)) = (0, 0), \quad (\mathbf{u}^f(x, 0), \partial_t \mathbf{u}^f(x, 0)) = (0, 0), \quad x \in \Omega, \quad (3.13)$$

where the functions F_1 and F_2 are given by

$$F_1(x, t) = p_1(x)R_1(x, t), \quad F_2(x, t) = p_2(x)R_2(x, t). \quad (3.14)$$

We introduce the following notations:

$$\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f), \quad \mathbf{v}_j(x, t) = \partial_t^j \mathbf{u}(x, t), \quad (x, t) \in Q, \quad j = 0, 1, 2, 3. \quad (3.15)$$

The functions \mathbf{v}_j , $j = 1, 2, 3$ solve the following system

$$\begin{aligned} \varrho_{11} \partial_t^2 \mathbf{v}_j^s + \varrho_{12} \partial_t^2 \mathbf{v}_j^f - \Delta_{\mu, \lambda} \mathbf{v}_j^s(x, t) - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}_j^f) &= \partial_t^j F_1(x, t), \\ \varrho_{12} \partial_t^2 \mathbf{v}_j^s + \varrho_{22} \partial_t^2 \mathbf{v}_j^f - \nabla (\mathbf{q} \operatorname{div} \mathbf{v}_j^s) - \nabla (\mathbf{r} \operatorname{div} \mathbf{v}_j^f) &= \partial_t^j F_2(x, t), \quad (x, t) \in Q, \end{aligned} \quad (3.16)$$

with the boundary condition

$$\mathbf{v}_j^s(x, t) = 0, \quad \mathbf{v}_j^f(x, t) \cdot \nu = 0, \quad (x, t) \in \Sigma. \quad (3.17)$$

We set

$$\tilde{\mathbf{v}}_j = \zeta \mathbf{v}_j,$$

where $\zeta(t)$ is given by (3.5). We apply Lemma 3.2 to obtain the following estimate:

$$C\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_j) \leq \int_Q \left(|\partial_t^j F|^2 + |\nabla \partial_t^j F|^2 \right) e^{2\tau\varphi} dxdt + e^{C_0\tau} \|\mathbf{v}_j\|_{H^2(\omega_T)}^2 + e^{2d_1\tau} \|\mathbf{v}_j\|_{H^1(-T,T;H^1(\Omega))}^2, \quad j = 0, 1, 2, 3, \quad (3.18)$$

provided that $\tau > 0$ is large enough.

Lemma 3.4. *There exists a positive constant $C > 0$ such that the following estimate*

$$\int_{\Omega} |z(x, 0)|^2 dx \leq C \int_Q (\tau |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2) dxdt$$

for any $z \in L^2(Q)$ such that $\partial_t z \in L^2(Q)$.

Proof. Let ζ be the cut-off function given by (3.5). By direct computations, we have

$$\begin{aligned} \int_{\Omega} \zeta^2(0) |z(x, 0)|^2 dx &= \int_{-T}^0 \frac{d}{dt} \left(\int_{\Omega} \zeta^2(t) |z(x, t)|^2 dx \right) dt \\ &= 2 \int_{-T}^0 \int_{\Omega} \zeta^2(t) z(x, t) \partial_t z(x, t) dxdt \\ &\quad + 2 \int_{-T}^0 \int_{\Omega} \zeta'(t) \zeta(t) |z(x, t)|^2 dxdt. \end{aligned}$$

Then we have

$$\int_{\Omega} |z(x, 0)|^2 dx \leq C \int_Q (\tau |z(x, t)|^2 + \tau^{-1} |\partial_t z(x, t)|^2) dxdt.$$

This completes the proof of the lemma. \square

Lemma 3.5. *Let $\phi_{\ell}(x) = \operatorname{div} (p_{\ell}(x) \Phi_{\ell}(x))$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} \sum_{\ell=1}^2 \int_{\Omega} e^{2\tau\rho} \left(|\phi_{\ell}(x)|^2 + |\nabla \phi_{\ell}(x)|^2 \right) dx \\ \leq C (\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_3)) + \sum_{\ell=1}^2 \int_{\Omega} \left(|p_{\ell}|^2 + |\nabla p_{\ell}|^2 \right) e^{2\tau\rho} dx, \end{aligned}$$

provided that τ is large.

Proof. We set $\mathbf{v}^{(1)} = \mathbf{v}_2^s$ and $\mathbf{v}^{(2)} = \mathbf{v}_2^f$. Applying Lemma 3.4 for $z_j(x, t) = e^{\tau\varphi(x,t)} \operatorname{div} \tilde{\mathbf{v}}_2^{(j)}(x, t)$, $j = 1, 2$, we obtain the following inequality:

$$\begin{aligned} C\tau^2 \int_{\Omega} e^{2\tau\rho} \sum_{j=1}^2 \left| \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 dx \leq \tau^3 \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left| \operatorname{div} \tilde{\mathbf{v}}^{(j)}(x, t) \right|^2 dxdt \\ + \tau \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left| \partial_t \operatorname{div} \tilde{\mathbf{v}}^{(j)}(x, t) \right|^2 dxdt \leq \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_2). \quad (3.19) \end{aligned}$$

Applying Lemma 3.4 again with $w_j(x, t) = e^{\tau\varphi(x, t)} \nabla \operatorname{div} (\tilde{\mathbf{v}}^{(j)}(x, t))$, we obtain

$$\begin{aligned} C \int_{\Omega} e^{2\tau\rho} \sum_{j=1}^2 \left| \nabla \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 dx &\leq \tau \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left| \nabla \operatorname{div} \tilde{\mathbf{v}}^{(j)}(x, t) \right|^2 dx dt \\ &\quad + \tau^{-1} \int_Q e^{2\tau\varphi} \sum_{j=1}^2 \left(\left| \nabla \operatorname{div} \partial_t^3 \tilde{\mathbf{v}}^s(x, t) \right|^2 + \left| \nabla \operatorname{div} \partial_t^3 \tilde{\mathbf{v}}^f(x, t) \right|^2 \right) dx dt \\ &\leq \mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_3). \end{aligned} \quad (3.20)$$

Adding (3.19) and (3.20), we find

$$\int_{\Omega} e^{2\tau\rho} \sum_{j=1}^2 \left(\left| \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 + \left| \nabla \operatorname{div} \mathbf{v}^{(j)}(x, 0) \right|^2 \right) dx \leq C (\mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau, \varphi}(\tilde{\mathbf{v}}_3)). \quad (3.21)$$

Since

$$M(x) \left(\mathbf{v}_2^s(x, 0), \mathbf{v}_2^f(x, 0) \right)^T = (p_1(x) \Phi_1(x), p_2(x) \Phi_2(x))^T, \quad x \in \bar{\Omega}$$

we have

$$\begin{aligned} |\phi_{\ell}(x)|^2 + |\nabla \phi_{\ell}(x)|^2 &\leq C \left(|\mathbf{v}_2^s(x, 0)|^2 + |\nabla \mathbf{v}_2^s(x, 0)|^2 + |\mathbf{v}_2^f(x, 0)|^2 + |\nabla \mathbf{v}_2^f(x, 0)|^2 \right. \\ &\quad \left. + |\operatorname{div} \mathbf{v}_2^s(x, 0)|^2 + |\nabla \operatorname{div} \mathbf{v}_2^s(x, 0)|^2 + |\operatorname{div} \mathbf{v}_2^f(x, 0)|^2 + |\nabla \operatorname{div} \mathbf{v}_2^f(x, 0)|^2 \right) \end{aligned} \quad (3.22)$$

for $x \in \bar{\Omega}$. On the other hand, using (3.11), we obtain

$$|\mathbf{v}_2^s(x, 0)|^2 + |\nabla \mathbf{v}_2^s(x, 0)|^2 + |\mathbf{v}_2^f(x, 0)|^2 + |\nabla \mathbf{v}_2^f(x, 0)|^2 \leq C \sum_{\ell=1}^2 \left(|p_{\ell}|^2 + |\nabla p_{\ell}|^2 \right), \quad x \in \bar{\Omega}. \quad (3.23)$$

Combining (3.23), (3.22) and (3.21), we complete the proof of the lemma. \square

Lemma 3.6. *There exists a constant $C > 0$ such that*

$$\tau \int_{\Omega} \left(|\nabla p_{\ell}(x)|^2 + |p_{\ell}(x)|^2 \right) e^{2\tau\rho} dx \leq C \int_{\Omega} \left(|\nabla \phi_{\ell}(x)|^2 + |\phi_{\ell}(x)|^2 \right) e^{2\tau\rho(x)} dx$$

for all large $\tau > 0$, $\ell = 1, 2$.

Proof. We have

$$\operatorname{div} ((\partial_k p_{\ell})(x) \Phi_{\ell}(x)) = \partial_k \phi_{\ell}(x) - \operatorname{div} (p_{\ell} \partial_k \Phi_{\ell}(x)) \quad \text{for all } k = 1, 2, 3.$$

Therefore

$$\begin{aligned} \int_{\Omega} \left(|\operatorname{div} ((\partial_k p_{\ell}) \Phi_{\ell})|^2 + |\operatorname{div} (p_{\ell} \Phi_{\ell})|^2 \right) e^{2\tau\rho} dx &\leq \int_{\Omega} \left(|\nabla \phi_{\ell}|^2 + |\phi_{\ell}|^2 \right) e^{2\tau\rho(x)} dx \\ &\quad + C \int_{\Omega} \left(|p_{\ell}|^2 + |\nabla p_{\ell}|^2 \right) e^{2\tau\rho} dx. \end{aligned} \quad (3.24)$$

Since $p_\ell = 0$ and $\nabla p_\ell = 0$ on the boundary Γ and $\nabla \Phi_\ell \cdot (x - x_0) \neq 0$, we can apply Lemma 3.3 respectively with the choice $v = p_\ell$ and $v = \partial_k p_\ell$ to obtain

$$\tau \int_{\Omega} \left(|\partial_k p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho} dx \leq C \int_{\Omega} \left(|\operatorname{div}((\partial_k p_\ell)\Phi_\ell)|^2 + |\operatorname{div}(p_\ell\Phi_\ell)|^2 \right) e^{2\tau\rho} dx \quad (3.25)$$

for $\ell = 1, 2$ and $k = 1, 2, 3$. Inserting (3.24) into the left-hand side of (3.25) and choosing $\tau > 0$ large, we obtain

$$\tau \int_{\Omega} \left(|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho} dx \leq C \int_{\Omega} \left(|\nabla \phi_\ell(x)|^2 + |\phi_\ell(x)|^2 \right) e^{2\tau\rho(x)} dx.$$

The proof is completed. \square

3.3 Completion of the proof of Theorem 1.2

By Lemmata 3.5 and 3.6, we obtain

$$\begin{aligned} \tau \sum_{\ell=1}^2 \int_{\Omega} e^{2\tau\rho(x)} \left(|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) dx &\leq C \sum_{\ell=1}^2 \int_{\Omega} \left(|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\rho(x)} dx \\ &\quad + C \left(\mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_2) + \mathcal{N}_{\tau,\varphi}(\tilde{\mathbf{v}}_3) \right). \end{aligned}$$

Therefore, choosing $\tau > 0$ large to absorb the first term on the right-hand side into the left-hand side and applying (3.19), we obtain

$$\begin{aligned} \tau \sum_{\ell=1}^2 \int_{\Omega} e^{2\tau\rho(x)} \left(|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) dx &\leq C \sum_{j=2}^3 \left(\int_Q \left(|\partial_t^j F|^2 + |\partial_t^j \nabla F|^2 \right) e^{2\tau\varphi} dx dt \right. \\ &\quad \left. + C e^{C_0\tau} \|\mathbf{v}_j\|_{H^2(\omega_T)}^2 + C e^{2d_1\tau} \|\mathbf{v}_j\|_{H^1(-T,T;H^1(\Omega))}^2 \right) \\ &\leq C \sum_{\ell=1}^2 \int_Q \left(|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) e^{2\tau\varphi} dx dt + C e^{C_0\tau} \mathcal{E}_\omega(\mathbf{u}) + C e^{2d_1\tau} M_0. \quad (3.26) \end{aligned}$$

Then the first term of the right-hand side of (3.26) can be absorbed into the left-hand side if we take large $\tau > 0$.

Since $\rho(x) \geq d_0$, we obtain

$$\sum_{\ell=1}^2 \int_{\Omega} \left(|\nabla p_\ell(x)|^2 + |p_\ell(x)|^2 \right) dx \leq C e^{2(d_1-d_0)\tau} + e^{C_0\tau} \mathcal{E}_\omega(\mathbf{u}) \leq C e^{-\epsilon\tau} + e^{C_0\tau} \mathcal{E}_\omega(\mathbf{u}). \quad (3.27)$$

At the last inequality, we used: By $0 < d_1 < d_0$, we can choose $\epsilon > 0$ such that $e^{2(d_1-d_0)\tau} \leq e^{-\epsilon\tau}$ for sufficiently large $\tau > 0$.

4 Well posedness of the direct problem

This section is devoted to the study the existence, uniqueness and regularity of solutions of the following system:

$$\begin{aligned} \varrho_{11}\partial_t^2\mathbf{u}^s + \varrho_{12}\partial_t^2\mathbf{u}^f - \Delta_{\mu,\lambda}\mathbf{u}^s(x,t) - \nabla(\mathbf{q}\operatorname{div}\mathbf{u}^f) &= F_1(x,t), \\ \varrho_{12}\partial_t^2\mathbf{u}^s + \varrho_{22}\partial_t^2\mathbf{u}^f - \nabla(\mathbf{q}\operatorname{div}\mathbf{u}^s) - \nabla(\mathbf{r}\operatorname{div}\mathbf{u}^f) &= F_2(x,t), \quad (x,t) \in Q \end{aligned} \quad (4.1)$$

with the boundary condition

$$\mathbf{u}^s(x,t) = 0, \quad \mathbf{u}^f(x,t) \cdot \nu = 0, \quad (x,t) \in \Sigma = \Gamma \times (-T,T) \quad (4.2)$$

and the initial condition

$$(\mathbf{u}^s(x,0), \mathbf{u}_t^s(x,0)) = (\mathbf{u}_0^s, \mathbf{u}_1^s), \quad (\mathbf{u}^f(x,0), \mathbf{u}_t^f(x,0)) = (\mathbf{u}_0^f, \mathbf{u}_1^f), \quad x \in \Omega. \quad (4.3)$$

4.1 Function spaces

We denote by $\mathcal{D}(\Omega)$ the space of compactly supported, infinitely differentiable function in Ω equipped with the inductive limit topology. We denote by $\mathcal{D}'(\Omega)$ the space dual to $\mathcal{D}(\Omega)$. In general, we denote by X' the space dual to the function space X . We denote by (f, g) the inner product in $L^2(\Omega)$ and by $\langle f, g \rangle$ the value of $f \in X'$ on $g \in X$. We use usual notations for Sobolev spaces. If X is a Banach space, then we denote by $L^p(0, T; X)$ the space of functions $f : (0, T) \rightarrow X$ which are measurable, take values in X and satisfy:

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{1/p} = \|f\|_{L^p(0,T;X)} < \infty$$

for $1 \leq p < \infty$, while

$$\|f\|_{L^\infty(0,T;X)} = \operatorname{esssup}_{t \in (0,T)} \|f(t)\|_X < \infty$$

for $p = \infty$. It is known that the space $L^p(0, T; X)$ is complete.

We define the space

$$H(\operatorname{div}; \Omega) = \left\{ \mathbf{u} \in (L^2(\Omega))^3; \operatorname{div} \mathbf{u} \in L^2(\Omega) \right\},$$

equipped with the norm

$$\|\mathbf{u}\|_{H(\operatorname{div}; \Omega)} = \left(\|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Let us consider the space

$$V(\Omega) = (H^1(\Omega)^3) \times H(\operatorname{div}; \Omega),$$

equipped with the norm

$$\|\mathbf{u}\|_{V(\Omega)} = \left(\|\mathbf{u}^2\|_{H^1(\Omega)}^2 + \|\mathbf{u}^2\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}^2\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

4.2 Generalized solution

We introduce the bilinear form on $V(\Omega)$ by

$$B(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_{\Omega} \left(\lambda \operatorname{div}(\mathbf{u}^s) \operatorname{div}(\mathbf{v}^s) + 2\mu (\varepsilon(\mathbf{u}^s) : \varepsilon(\mathbf{v}^s)) + r \operatorname{div}(\mathbf{u}^f) \operatorname{div}(\mathbf{v}^f) \right) dx \\ + \frac{1}{2} \int_{\Omega} q \left(\operatorname{div}(\mathbf{u}^f) \operatorname{div}(\mathbf{v}^s) + \operatorname{div}(\mathbf{v}^f) \operatorname{div}(\mathbf{u}^s) \right) dx \quad (4.4)$$

for any $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f) \in V(\Omega)$, $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^f) \in V(\Omega)$. We recall that the matrix M is given by (2.9).

Definition 4.1. We say that $\mathbf{u} = (\mathbf{u}^s, \mathbf{u}^f)$ is a generalized solution of problem (4.1)-(4.2), if $\mathbf{u} \in L^2(0, T; V(\Omega))$ satisfies the initial condition (4.3) and the following identity

$$(M \partial_t^2 \mathbf{u}(t), \mathbf{v}(t)) + B(\mathbf{u}(t), \mathbf{v}(t)) = (F(t), \mathbf{v}(t)), \quad \text{almost all } t \in (0, T) \quad (4.5)$$

for any $\mathbf{v} \in L^2(0, T; V(\Omega))$.

We note that in (4.5) the integration is only in x .

Lemma 4.1. For $\eta > 0$, we set

$$B_{\eta}(\mathbf{u}, \mathbf{v}) = B(\mathbf{u}, \mathbf{v}) + \eta (\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V(\Omega).$$

Then there exists sufficiently large constant η such that the symmetric bilinear form B_{η} satisfies

(i) $|B_{\eta}(\mathbf{u}, \mathbf{v})| \leq C_1 \|\mathbf{u}\|_{V(\Omega)} \|\mathbf{v}\|_{V(\Omega)}$, for any $\mathbf{u}, \mathbf{v} \in V(\Omega)$,

(ii) $B_{\eta}(\mathbf{u}, \mathbf{u}) \geq C_2 \|\mathbf{u}\|_{V(\Omega)}^2$, for any $\mathbf{u} \in V(\Omega)$.

Proof. By (4.4) we obtain, for any $\mathbf{u}, \mathbf{v} \in V(\Omega)$

$$|B(\mathbf{u}, \mathbf{v})| \leq \left(\|\mathbf{u}^s\|_{H^1(\Omega)} + \|\operatorname{div} \mathbf{u}^f\|_{L^2(\Omega)} \right) \left(\|\mathbf{v}^s\|_{H^1(\Omega)} + \|\operatorname{div} \mathbf{v}^f\|_{L^2(\Omega)} \right) \\ \leq C \|\mathbf{u}\|_V \|\mathbf{v}\|_V. \quad (4.6)$$

Then for any η , we can derive (i).

Now, we note that for a vector $\mathbf{u}^s \in H_0^1(\Omega)$ we have the following Korn's inequality

$$C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 \leq \int_{\Omega} \varepsilon(\mathbf{u}^s) : \varepsilon(\mathbf{u}^s) dx.$$

Then, for $W = (\operatorname{div} \mathbf{u}^s, \operatorname{div} \mathbf{u}^f)$, we have

$$B(\mathbf{u}, \mathbf{u}) \geq \mu C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 + \frac{1}{2} \int_{\Omega} M_0 W \cdot W dx$$

where M_0 is the symmetric 2×2 -matrix given by

$$M_0(x) = \begin{pmatrix} \lambda & q \\ q & r \end{pmatrix} \geq \gamma_0 I_2.$$

Which implies

$$\begin{aligned}
B(\mathbf{u}, \mathbf{u}) &\geq \mu C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 + \frac{\gamma_0}{2} \left(\|\operatorname{div} \mathbf{u}^s\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{u}^f\|_{L^2(\Omega)}^2 \right) \\
&\geq \mu C_1 \|\mathbf{u}^s\|_{H^1(\Omega)}^2 + \frac{\gamma_0}{2} \|\operatorname{div} \mathbf{u}^f\|_{L^2(\Omega)}^2 - \frac{\gamma_0}{2} \|\mathbf{u}\|_{L^2(\Omega)}^2 \\
&\geq C_2 \|\mathbf{u}\|_V^2 - \eta \|\mathbf{u}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.7}$$

This completes the proof of the lemma. \square

4.3 Construction of approximate solutions

Let $\mathcal{A} : (L^2(\Omega))^6 \rightarrow (L^2(\Omega))^6$ be the self-adjoint operator defined by

$$\mathcal{A} \mathbf{u} = \begin{pmatrix} \Delta_{\mu, \lambda} \mathbf{u}^s + \nabla (q \operatorname{div} \mathbf{u}^f) \\ \nabla (q \operatorname{div} \mathbf{u}^s) + \nabla (r \operatorname{div} \mathbf{u}^f) \end{pmatrix}.$$

Then system (4.1) can be written as

$$M \partial_t^2 \mathbf{u} - \mathcal{A} \mathbf{u} = F, \quad (x, t) \in Q \tag{4.8}$$

with initial condition

$$\mathbf{u}(x, 0) = (\mathbf{u}_0^s(x), \mathbf{u}_0^f(x)), \quad \partial_t \mathbf{u}(x, 0) = (\mathbf{u}_1^s(x), \mathbf{u}_1^f(x)) \tag{4.9}$$

and the boundary condition

$$\mathbf{u}^s(x, t) = 0, \quad \mathbf{u}^f \cdot \nu = 0, \quad (x, t) \in \Sigma. \tag{4.10}$$

Let $(\mathbf{w}_j)_{j \geq 1}$ be a sequence of solutions in $(H^2(\Omega) \cap H_0^1(\Omega))^6$ such that for all $m \in \mathbb{N}$, $\mathbf{w}_1, \dots, \mathbf{w}_m$ are linearly independent and all the finite linear combinations of $(\mathbf{w}_j)_{j \geq 1}$ are dense in $(H^2(\Omega))^6$.

We seek approximate solutions of the problem in the form

$$\mathbf{u}_m(t) = \sum_{j=1}^m g_{jm}(t) \mathbf{w}_j. \tag{4.11}$$

The functions $g_{jm}(t)$ are defined by the solution of the system of ordinary differential equations

$$(M \partial_t^2 \mathbf{u}_m, \mathbf{w}_j) + B(\mathbf{u}_m, \mathbf{w}_j) = (F(t), \mathbf{w}_j), \quad 1 \leq j \leq m, \tag{4.12}$$

with the initial conditions

$$\begin{aligned}
\mathbf{u}_m(0) &= \mathbf{u}_{0m} \rightarrow \mathbf{u}_0 \quad \text{in} \quad (H^2(\Omega) \cap H_0^1(\Omega))^6, \\
\partial_t \mathbf{u}_m(0) &= \mathbf{u}_{1m} \rightarrow \mathbf{u}_1 \quad \text{in} \quad (H^1(\Omega))^6.
\end{aligned} \tag{4.13}$$

The system (4.12)-(4.13) depends on $g_{jm}(t)$ and therefore has a solution on some segment $[0, t_m]$; see [35]. From a priori estimates below and the theorem on continuation of a solution we deduce that it is possible to take $t_m = T$.

4.4 A priori estimates

Multiplying (4.8) by $g'_{jm}(t)$ and summing over j from 1 to m , we obtain

$$(M\partial_t^2 \mathbf{u}_m, \partial_t \mathbf{u}_m) + B(\mathbf{u}_m, \partial_t \mathbf{u}_m) = (F(t), \partial_t \mathbf{u}_m). \quad (4.14)$$

Hence

$$\frac{1}{2} \frac{d}{dt} \left[\|M^{1/2} \partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}_m(t), \mathbf{u}_m(t)) \right] = (F(t), \partial_t \mathbf{u}_m(t)) + \frac{\eta}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|_{L^2(\Omega)}^2. \quad (4.15)$$

Let

$$\Phi^2(t) = \|M^{1/2} \partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}_m(t), \mathbf{u}_m(t)).$$

From (4.15) we obtain

$$\frac{1}{2} \frac{d}{dt} \Phi^2(t) \leq C \left[\|F(t)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \right]. \quad (4.16)$$

Integrating with respect to τ from 0 to t , we obtain

$$\Phi^2(t) \leq C \left[\|F\|_{L^2(Q)}^2 + \Phi^2(0) + \int_0^t \left(\|\partial_t \mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 \right) \right]. \quad (4.17)$$

Since

$$\Phi^2(t) \geq C \left(\|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(t)\|_{V(\Omega)}^2 \right) \quad (4.18)$$

and

$$\Phi^2(0) \leq C + \|\mathbf{u}_0\|_{H^2(\Omega)}^2 + \|\mathbf{u}_1\|_{H^1(\Omega)}^2, \quad (4.19)$$

we have from (4.18)

$$\|\mathbf{u}_m(t)\|_{V(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \leq R_0 + \int_0^t \left(\|\partial_t \mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 \right), \quad (4.20)$$

where $R_0 = C + \|\mathbf{u}_0\|_{H^2(\Omega)}^2 + \|\mathbf{u}_1\|_{H^1(\Omega)}^2 + \|F\|_{L^2(Q)}^2$. By the Gronwall inequality, we conclude that

$$\|\mathbf{u}_m(t)\|_{V(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \leq R_0 \quad (4.21)$$

for all $t \in (0, T)$ and $m \geq 1$.

In order to obtain the second a priori estimate, we observe that

$$\|\partial_t^2 \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 \leq C \left[\|F(0)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_m(0)\|_{H^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 \right] \leq R_1. \quad (4.22)$$

Indeed, multiplying (4.8) by $g'_{jm}(0)$, summing over j and setting $t = 0$, we obtain

$$(M\partial_t^2 \mathbf{u}_m(0), \partial_t^2 \mathbf{u}_m(0)) + B(\mathbf{u}_m(0), \partial_t^2 \mathbf{u}_m(0)) = (F(0), \partial_t^2 \mathbf{u}_m(0)). \quad (4.23)$$

Consequently,

$$(M\partial_t^2 \mathbf{u}_m(0), \partial_t^2 \mathbf{u}_m(0)) = (F(0), \partial_t^2 \mathbf{u}_m(0)) + (\mathcal{A}\mathbf{u}_{0m}, \partial_t^2 \mathbf{u}_m(0)), \quad (4.24)$$

which implies

$$\|\partial_t^2 \mathbf{u}_m(0)\|^2 \leq C \left(\|F(0)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_{0m}\|_{H^2(\Omega)}^2 \right) \leq CR_2. \quad (4.25)$$

Differentiating (4.14) with respect to t , multiplying by g_{jm} and summing over j , we obtain the identity

$$\frac{1}{2} \frac{d}{dt} \left[\|M^{1/2} \partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}_m(t), \mathbf{u}_m(t)) \right] = (F(t), \partial_t^2 \mathbf{u}_m(t)) + \frac{\eta}{2} \frac{d}{dt} \|\partial_t \mathbf{u}_m(t)\|_{L^2(\Omega)}^2. \quad (4.26)$$

Then, we conclude that

$$\begin{aligned} \|\partial_t^2 \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_V^2 &\leq R_2 + \|\partial_t \mathbf{u}_m(0)\|_{L^2(\Omega)}^2 \\ &+ C \int_0^t \left(\|\partial_t^2 \mathbf{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(\tau)\|_V^2 \right) d\tau. \end{aligned} \quad (4.27)$$

By (4.27) and the Gronwall inequality, we obtain

$$\|\partial_t^2 \mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t \mathbf{u}_m(t)\|_V^2 \leq R_1. \quad (4.28)$$

Taking into consideration that $\mathbf{u}_m = 0$ in Σ , we see

$$\begin{aligned} \mathbf{u}_m &\in L^\infty(0, T; V(\Omega)), \quad \partial_t \mathbf{u}_m \in L^\infty(0, T; V(\Omega)), \\ \partial_t^2 \mathbf{u}_m &\in L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (4.29)$$

4.5 Passage to the limit

By (4.29), we can extract a sequence from $(\mathbf{u}_m)_{m \geq 0}$, which we denote again by $(\mathbf{u}_m)_m$, such that

$$\begin{aligned} \mathbf{u}_m &\rightarrow \mathbf{u} \quad \text{in the weak-star topology in } L^\infty(0, T; V(\Omega)) \\ \partial_t \mathbf{u}_m &\rightarrow \partial_t \mathbf{u} \quad \text{in the weak-star topology in } L^\infty(0, T; V(\Omega)) \\ \partial_t^2 \mathbf{u}_m &\rightarrow \partial_t^2 \mathbf{u} \quad \text{in the weak-star topology in } L^\infty(0, T; L^2(\Omega)) \end{aligned} \quad (4.30)$$

and

$$(\mathbf{u}_m, \partial_t \mathbf{u}_m) \rightarrow (\mathbf{u}, \partial_t \mathbf{u}) \quad \text{a.e., on } \Sigma.$$

Multiplying (4.8) by $\theta \in L^1(0, T)$ and integrating, we have

$$\int_0^T \left((M \partial_t^2 \mathbf{u}_m(t), \mathbf{w}_j) + B(\mathbf{u}_m, \mathbf{w}_j) \right) \theta(t) dt = \int_0^T (F(t), \mathbf{w}_j) \theta(t) dt. \quad (4.31)$$

On the other hand

$$\int_0^T B(\mathbf{u}_m, \mathbf{w}_j) \theta(t) dt = - \int_0^T (\mathbf{u}_m, \mathcal{A} \mathbf{w}_j) \theta(t) dt, \quad (4.32)$$

so that

$$\lim_{m \rightarrow \infty} \int_0^T B(\mathbf{u}_m, \mathbf{w}_j) \theta(t) dt = - \int_0^T (\mathbf{u}, \mathcal{A} \mathbf{w}_j) \theta(t) dt = \int_0^T B(\mathbf{u}, \mathbf{w}_j) \theta(t) dt. \quad (4.33)$$

Thus, we obtain

$$\int_0^T \left((M \partial_t^2 \mathbf{u}(t), \mathbf{w}_j) + B(\mathbf{u}, \mathbf{w}_j) \right) \theta(t) dt = \int_0^T (F(t), \mathbf{w}_j) \theta(t) dt. \quad (4.34)$$

Taking into account that \mathbf{w}_j are dense in $(H^2(\Omega) \cap H_0^1(\Omega))^6$ and therefore in V , we obtain

$$(M\partial_t^2 \mathbf{u}, \mathbf{v}) + B(\mathbf{u}(t), \mathbf{v}(t)) = (F(t), \mathbf{v}), \quad t \in (0, T) \quad (4.35)$$

for all $\mathbf{v} \in L^2(0, T; V(\Omega))$.

We have $B(\mathbf{u}(t), \mathbf{v}(t)) = -(\mathcal{A}\mathbf{u}(t), \mathbf{v}(t))$ for any $\mathbf{v} \in \mathcal{D}(\Omega)$, where the application of the differential operator A to \mathbf{u} is in the distributional sense in $\mathcal{D}'(\Omega)$. Hence we obtain

$$M\partial_t^2 \mathbf{u} - \mathcal{A}\mathbf{u} = F, \quad \text{in } \mathcal{D}'(\Omega), \text{ a.e. in } (0, T). \quad (4.36)$$

On the other hand, $\partial_t^2 \mathbf{u}, \partial_t \mathbf{u}, F \in L^\infty(0, T; L^2(\Omega))$. Hence (4.36) holds in $L^\infty(0, T; L^2(\Omega))$.

The boundary condition (4.2) is satisfied by the choice of the space $V(\Omega)$. We prove that the initial conditions are satisfied. Suppose $\theta \in C^1(0, T)$ and $\theta(T) = 0$. For any j we have

$$\int_0^T \left(\frac{\partial}{\partial t} (\mathbf{u}_m - \mathbf{u}), \mathbf{w}_j \right) \theta(t) dt = -(\mathbf{u}_m(0) - \mathbf{u}(0), \mathbf{w}_j) \theta(0) - \int_0^T (\mathbf{u}_m(t) - \mathbf{u}(t), \mathbf{w}_j) \theta'(t) dt. \quad (4.37)$$

Then, by (4.30), we have

$$\lim_{m \rightarrow \infty} |(\mathbf{u}_m(0) - \mathbf{u}(0), \mathbf{w}_j)| = 0.$$

Since $\mathbf{u}_{0m}(x) = \mathbf{u}_m(0, x)$ and $\mathbf{u}_{0m} \rightarrow \mathbf{u}_0$, we obtain $\mathbf{u}(0) = \mathbf{u}_0$, and can argue similarly for \mathbf{u}_1 .

Then, we conclude that, there exists a solution u of (4.1) such that

$$\partial_t \mathbf{u} \in L^\infty(0, T; V(\Omega)), \quad \text{and} \quad \partial_t^2 \mathbf{u} \in L^\infty(0, T; L^2(\Omega)), \quad (4.38)$$

which implies

$$\begin{aligned} \mathbf{u}^s &\in C^1(0, T; H_0^1(\Omega)), & \partial_t^2 \mathbf{u}^s &\in C(0, T; L^2(\Omega)) \\ \mathbf{u}^f &\in C^1(0, T; H(\text{div}, \Omega)), & \partial_t^2 \mathbf{u}^f &\in C(0, T; L^2(\Omega)). \end{aligned} \quad (4.39)$$

On the other hand

$$\begin{aligned} \nabla(\text{qdiv } \mathbf{u}^s) + \nabla(\text{rdiv } \mathbf{u}^f) &\in C(0, T; L^2(\Omega)), \\ \Delta_{\mu, \lambda} \mathbf{u}^s + \nabla(\text{qdiv } \mathbf{u}^f) &\in C(0, T; L^2(\Omega)). \end{aligned} \quad (4.40)$$

Consequently,

$$\Delta_{\mu, \tilde{\lambda}} \mathbf{u}^s \in C(0, T; L^2(\Omega)), \quad \tilde{\lambda} = \lambda - \frac{\mathbf{q}^2}{\mathbf{r}}.$$

Then by the elliptic regularity, $\mathbf{u}^s \in H_0^1(\Omega)$ yields

$$\mathbf{u}^s \in C(0, T; H^2(\Omega)).$$

By $\text{div } \mathbf{u}^f \in C(0, T; H^1(\Omega))$, we see

$$\begin{aligned} \mathbf{u}^s &\in C(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(0, T; H^1(\Omega)) \\ \mathbf{u}^f &\in C^2(0, T; L^2(\Omega)), \quad \text{div } \mathbf{u}^f \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)). \end{aligned} \quad (4.41)$$

4.6 Uniqueness

Let \mathbf{u}_1 and \mathbf{u}_2 be two solutions to (4.1)-(4.2) with the same initial data, and set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Then for every function $\mathbf{v} \in V(\Omega)$, we have

$$(M\partial_t^2 \mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{v}) = 0, \quad \forall t \in (0, T).$$

Since $\partial_t \mathbf{u} \in V(\Omega)$, we may take $\mathbf{v} = \partial_t \mathbf{u}$, and this equation can be reduced to equality

$$\frac{1}{2} \frac{d}{dt} \left[\|M^{1/2} \partial_t \mathbf{u}\|_{L^2(\Omega)}^2 + B_\eta(\mathbf{u}, \mathbf{u}) \right] = \frac{\eta}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(\Omega)}^2.$$

Then

$$\|\partial_t \mathbf{u}(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}(t)\|_{V(\Omega)}^2 \leq C \int_0^t \left(\|\partial_t \mathbf{u}(\tau)\|_{L^2(\Omega)}^2 + \|\mathbf{u}(\tau)\|_{L^2(\Omega)}^2 \right) d\tau.$$

This implies that $\|\mathbf{u}\|_{V(\Omega)} = 0 = \|\partial_t \mathbf{u}\|_{L^2(\Omega)}$ and $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in Q .

The proof of Theorem 1.1 is completed.

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