



HAL
open science

The real primes and the Riemann hypothesis

Jamel Ghannouchi

► **To cite this version:**

| Jamel Ghannouchi. The real primes and the Riemann hypothesis. 2013. hal-00815862v11

HAL Id: hal-00815862

<https://hal.science/hal-00815862v11>

Preprint submitted on 11 Oct 2013 (v11), last revised 14 Dec 2015 (v13)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The real primes and the Riemann hypothesis

Jamel Ghanouchi

Ecole Supérieure des Sciences et Techniques de Tunis

Jamel.ghanouchi@live.com

Abstract

In this paper, we present the Riemann problem and define the real primes. It allows to generalize the Riemann hypothesis to the reals. A calculus of integral solves the problem. We generalize the proof to the integers.

The Riemann hypothesis

The Riemann conjecture is a conjecture which has been formulated in 1859 by Bernard Riemann in the subject of the Riemann function zeta or ζ . It is called the zeta Riemann function.

This function is defined as follows

$$\zeta(s) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^s}\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

The first result is the divergence of the harmonic serie

$$\zeta(1) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

It has been proved in the middle age by Nicole Oresme.

In the XVIII century, Leonard Euler has discovered the main proprieties of the ζ function.

In the 1730's he conjectured after numerical calculus the following equality, which is often called the Basel problem.

$$\zeta(2) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^2}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Euler proved it in 1748 and introduced the ζ function. He calculated its value for the positive even numbers.

$$\zeta(2k) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^{2k}}\right) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \dots = \frac{|B_{2k}| (2\pi)^{2k}}{2(2k)!}$$

Where B_{2k} are the Bernoulli numbers.

Thereafter, he proved in 1744 the Euler identity where prime numbers are related to the ζ function.

$$\zeta(s) = \sum_{n=1}^{n=\infty} \left(\frac{1}{n^s}\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \prod_{primes} \frac{1}{1-p^{-s}}$$

Consequently he deduced the divergence of the serie of the inverse of primes.

With Bernard Riemann, s can be complex number. Riemann proved the following formula

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Where

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$$

This formula demonstrates that this equation does not change if we replace s by $1-s$. Thus it is symmetric | $s = \frac{1}{2}$

Riemann demonstrates that the only zeros in the $R(s) < 0$ are the trivial zeros negative even numbers and that there is no zero in the $R(s) > 1$.

The other zeros are the non trivial zeros. They are in the critical zone $0 \leq R(s) \leq 1$

. Riemann conjectured they are all in the critical line $R(s) = \frac{1}{2}$.

This conjecture is called the Riemann hypothesis.

They calculated numerically one billion zeros of the Riemann they are all located in the critical line.

Resolution of the Riemann hypothesis for the reals

Definition

A real number is compound if it can be written as $\prod_j p_j^{n_j}$ where p_j are primes and n_j are rationals. This decomposition in prime factors is unique. A prime real number or R-prime can be written only as $p = p \cdot 1$. Thus we define other real prime numbers like π , e , $\ln(2)$. Of course, it is a convention, because, we can consider π^2 as prime and π will be no more prime. It is equivalent in what will follow.

Thus $\sqrt[q]{p} = p^{\frac{1}{q}}$ is compound. Also $\sqrt[q]{p} + 1 = p^{\frac{1}{q}} + 1$ is prime when p is prime and we have

$\sqrt[2]{p} - 1 = (p-1)(\sqrt[2]{p} + 1)^{-1}(\sqrt[2^{i-1}]{p} + 1)^{-1} \dots (\sqrt{p} + 1)^{-1}$ compound for p prime, for example.

The approach of the Riemann hypothesis

The Riemann hypothesis states that the non trivial zeros of the Riemann zeta function

$$\zeta(z) = \sum_{t=1}^{\infty} \frac{1}{t^z} \text{ lie on the critical line } \frac{1}{2} + iy.$$

For t integer, Euler has proved that $\zeta(z) = \sum_{t=1}^{\infty} \frac{1}{t^z} = \prod_{primes} \frac{1}{1-p^{-z}} = \prod_{primes} (\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha z}})$, it is

the Euler identity. For t real, it is still true and it becomes

$$\prod_{R\text{-primes}} (\sum_{\alpha \in \mathbb{Q}} \frac{1}{p^{\alpha z}}) = \int_1^{\infty} \frac{dt}{t^z} = \frac{1}{1-z} [t^{1-z}]_1^{\infty} \text{ But But let } \zeta_1(z) \text{ the Riemann function for the}$$

reals and $\zeta(z)$ the Riemann function for the integers

$$\zeta_1(0) = B_0 \zeta(0) = \zeta(0) + A_0$$

$$\zeta_1(0) = \prod_{R\text{-primes}} (\sum_{\alpha \in \mathbb{Q}} \frac{1}{p^{\alpha \cdot 0}}) = \prod_{R\text{-primes}} (\sum_{\alpha \in \mathbb{Q}} 1) = \prod_{primes} (\sum_{\alpha \in \mathbb{N}} 1) = \prod_{primes} (\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha \cdot 0}}) = \zeta(0) = 0$$

$$\Rightarrow \zeta_1(0) = \left[\frac{t}{1} \right]_1^{\infty} = 0 = \lim_{t \rightarrow \infty} (t-1) \Rightarrow \lim_{t \rightarrow \infty} (t) = 1$$

$$\lim_{t \rightarrow \infty} (t^{\frac{1}{2}+iy} t^{\frac{1}{2}-iy}) = \lim_{t \rightarrow \infty} (t) = 1 \Rightarrow \lim_{t \rightarrow \infty} (t^{\frac{1}{2}+iy}) = \lim_{t \rightarrow \infty} \left(\frac{1}{t^{\frac{1}{2}-iy}} \right)$$

$$\Rightarrow \lim_{t \rightarrow \infty} \left((t^{\frac{1}{2}+iy})^{\frac{1}{2}-iy} \right) = \lim_{t \rightarrow \infty} \left(\frac{1}{(t^{\frac{1}{2}-iy})^{\frac{1}{2}-iy}} \right) = \lim_{t \rightarrow \infty} (t^{\frac{1}{4}+y^2}) = \lim_{t \rightarrow \infty} \left(\frac{1}{t^{\frac{1}{4}-y^2-iy}} \right)$$

$$\Rightarrow \lim_{t \rightarrow \infty} (t^{\frac{1}{4}+y^2} t^{\frac{1}{4}-y^2-iy}) = 1 = \lim_{t \rightarrow \infty} (t^{\frac{1}{2}-iy})$$

Anotherproof :

$$\lim_{i \rightarrow \infty} (t^{1-\frac{1}{2}-iy}) = \lim_{i \rightarrow \infty} (t^{\frac{1}{2}-iy}) = \lim_{i \rightarrow \infty} (t^{\frac{1}{2}+iy}) = \frac{1}{2} (\lim_{i \rightarrow \infty} (t^{\frac{1}{2}-iy}) + \lim_{i \rightarrow \infty} (t^{\frac{1}{2}+iy})) = a$$

$$\Rightarrow \lim_{i \rightarrow \infty} (t^{2(\frac{1}{2}+iy)} - 2at^{\frac{1}{2}+iy} + t) = \lim_{i \rightarrow \infty} (t^{2(\frac{1}{2}+iy)} - 2at^{\frac{1}{2}+iy} + 1) = 0 \Rightarrow \lim_{i \rightarrow \infty} (t^{\frac{1}{2}+iy}) = a = a + \sqrt{a^2 - 1} = 1$$

$$\text{it means that } \zeta\left(\frac{1}{2} + iy\right) = \left[\frac{1}{1 - \frac{1}{2} - iy} t^{1 - \frac{1}{2} - iy} \right]_1^{\infty} = \lim_{t \rightarrow \infty} \left(\frac{1}{1 - \frac{1}{2} - iy} (t^{1 - \frac{1}{2} - iy} - 1) \right) = 0$$

$$\text{Let now } \zeta(x+iy) = 0 = \left[\frac{t^{1-x-iy}}{1-x-iy} \right]_1^\infty = \lim_{t \rightarrow \infty} \left(\frac{t^{\frac{1-iy}{2} - \frac{1-x}{2}} - 1}{1-x-iy} \right) = \lim_{t \rightarrow \infty} \left(\frac{t^{\frac{1-x}{2}} - 1}{1-x-iy} \right) = 0 \Rightarrow x = \frac{1}{2}$$

We have proved that the non trivial zeros of the Riemann function for the reals lie in the critical line ! So the hypothesis is proved for the real numbers. The Riemann hypothesis is important because it gives information about the zeros of the Riemann function and the distribution of those zeros are related to real primes !

The generalization to the integers

We have

$$\begin{aligned} \int_1^\infty \frac{dt}{t^z} &= \frac{1}{1-z} \left[t^{1-z} \right]_1^\infty = \sum_{t=1}^\infty \frac{1}{t^z} + \sum_{\alpha \in \mathbb{Q}^-} \frac{1}{t^{i\alpha z}} + \sum_{\alpha \in \mathbb{Q}^+ \setminus \mathbb{N}} \frac{1}{t^{i\alpha z}} \\ &= \sum_{t=1}^\infty \frac{1}{t^z} \cdot B = \prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha z}} \right) \cdot B = \sum_{t=1}^\infty \frac{1}{t^z} + A \end{aligned}$$

Let now

$$\prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha(x+iy)}} \right) = 0$$

We have : If B is finite, the zeta function for the reals will be equal to zero. Else

Let $\zeta_1(z)$ the Riemann function for the reals and $\zeta(z)$ the Riemann function for the integers, if $B = +\infty$ and if A is finite and

$$A > 1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^n \zeta(z) + A)^2 = A^2 = A^2 + (A^n \zeta(z))^2 + 2A^{n+1} \zeta(z) \Rightarrow A^{n+1} \zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square A^\alpha < A^n \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^n \zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow 1 \geq A$$

And if

$$1 \geq A > 0$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-n}\zeta(z) + A^{-1})^2 = A^{-2} = A^{-2} + (A^{-n}\zeta(z))^2 + 2A^{-n-1}\zeta(z) \Rightarrow A^{-n-1}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square A^{-\alpha} < A^{-n} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^{-n}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

Now if we suppose

$$0 \geq A \geq -1$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-2n-1}\zeta(z) + A^{-2})^2 = A^{-4} = A^{-4} + (A^{-2n-1}\zeta(z))^2 + 2A^{-2n-3}\zeta(z) \Rightarrow A^{-2n-3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square -A^{-\alpha} < -A^{-2n-1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < -A^{-2n-1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$A < -1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^{2n+1}\zeta(z) + A^2)^2 = A^4 = A^4 + (A^{2n+1}\zeta(z))^2 + 2A^{2n+3}\zeta(z) \Rightarrow A^{2n+3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square -A^{-\alpha} < -A^{2n+1} \Rightarrow 0 < \zeta_1(z) = B\zeta(z) < A^{2n+1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow A \geq -1$$

If now $B = -\infty$ and if A is finite and

$$A > 1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^n\zeta(z) + A)^2 = A^2 = A^2 + (A^n\zeta(z))^2 + 2A^{n+1}\zeta(z) \Rightarrow A^{n+1}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square -A^{-\alpha} > -A^n \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > -A^n\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow 1 \geq A$$

And if

$$1 \geq A \geq 0$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-n}\zeta(z) + A^{-1})^2 = A^{-2} = A^{-2} + (A^{-n}\zeta(z))^2 + 2A^{-n-1}\zeta(z) \Rightarrow A^{-n-1}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square -A^{-\alpha} > -A^{-n} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > -A^{-n}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$0 \geq A \geq -1$$

$$(A^{-1} + \zeta(z))^2 = A^{-2} = A^{-2} + \zeta(z)^2 + 2A^{-1}\zeta(z) = A^{-2} + 2A^{-1}\zeta(z) \Rightarrow A^{-1}\zeta(z) = 0$$

$$(A^{-2n-1}\zeta(z) + A^{-2})^2 = A^{-4} = A^{-4} + (A^{-2n-1}\zeta(z))^2 + 2A^{-2n-3}\zeta(z) \Rightarrow A^{-2n-3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square A^{-\alpha} > A^{-2n-1} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > A^{-2n-1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0$$

And if

$$A < -1$$

$$(A + \zeta(z))^2 = A^2 = A^2 + \zeta(z)^2 + 2A\zeta(z) = A^2 + 2A\zeta(z) \Rightarrow A\zeta(z) = 0$$

$$(A^{2n+1}\zeta(z) + A^2)^2 = A^4 = A^4 + (A^{2n+1}\zeta(z))^2 + 2A^{2n+3}\zeta(z) \Rightarrow A^{2n+3}\zeta(z) = 0, \forall n$$

$$\exists \alpha | B \square A^\alpha > A^{2n+1} \Rightarrow 0 > \zeta_1(z) = B\zeta(z) > A^{2n+1}\zeta(z) = 0 \Rightarrow \zeta_1(z) = 0 \Rightarrow A \geq -1$$

And if $A = \pm\infty$: we have $B \square A^{\alpha+1}$ hence

$$A^{\alpha m} \zeta(z)^m = 1$$

$$A^{n\alpha} \zeta(z)^{n+1} = 0$$

$$B = A^{\alpha+1} = A^{(m+n)\alpha} \Rightarrow A^{(m+n)\alpha} \zeta(z)^{m+n+1} = B\zeta(z)^{m+n+1} = A\zeta(z)^{m+n} \square \frac{A^{m+n+1}}{A^{(\alpha+1)(m+n)}} = A^{1-(m+n)\alpha} = \frac{A}{B}$$

$$(m+n)\alpha = \alpha + 1$$

$$m+n-2 = \frac{1}{\alpha} - 1$$

$$B^2 \zeta(z)^{m+n+1} A^{-1} = 1$$

$$\square \frac{B^2 A^{m+n}}{B^{m+n}} \Rightarrow \frac{BA}{B} = A = B^{-2+m+n} A^{1-m-n} = B^{\frac{1}{\alpha}-1} A^{-\frac{1}{\alpha}}$$

$$\Rightarrow (A^{-1}B)^{\frac{1}{\alpha}-1} = A^{1-\alpha} = A^2 \Rightarrow B = A^{\alpha+1} = 1 < \infty$$

It is impossible !

$$\Rightarrow A \neq \infty, A \neq -\infty$$

Thus $A=0$ And

$$\int_1^\infty \frac{dt}{t^{x+iy}} = \frac{1}{1-x-iy} \left[t^{1-x-iy} \right]_1^\infty = \prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha(x+iy)}} \right) \cdot B = \prod_{primes} \left(\sum_{\alpha \in \mathbb{N}} \frac{1}{p^{\alpha(x+iy)}} \right) + A = 0 \Rightarrow x = \frac{1}{2}$$

Thus the non trivial zeros of the Riemann function zeta lie in the critical line like for the reals ! It is the proof of the Riemann hypothesis !

Conclusion

We have generalized the concept of prime to the reals. It allowed to prove the conjecture for the reals. Then, we have proved the Riemann hypothesis.

The Bibliography

[1]R. J. Backlund, « Sur les zéros de la fonction $\zeta(s)$ de Riemann », CRAS, vol. 158, 1914, p. 1979–1981.

[2]X. Gourdon, « The 10^{13} first zeros of the Riemann zeta function, and zeros computation at very large height »

[3]J.P.Gram, « Note sur les zéros de la fonction $\zeta(s)$ de Riemann », Acta Mathematica, vol. 27, 1903, p. 289–304.

[4]J.I.hutchinson « On the Roots of the Riemann Zeta-Function », Trans. AMS, vol. 27, n° 1, 1925, p. 49–60.

[5]A. M. Odlyzko, The 10^{20} -th zero of the Riemann zeta function and 175 million of its neighbors, 1992.

[6]J.Barkley Rosser, J. M. Yohe et Lowell Schoenfeld, « Rigorous computation and the zeros of the Riemann zeta-function.», Information Processing 68 (Proc. IFIP Congress, Edinburgh, 1968), Vol. 1: Mathematics, Software, Amsterdam, North-Holland, 1969, p. 70–76.

[7]E.C.Titchmarsh, « The Zeros of the Riemann Zeta-Function », Proceedings of the Royal Society, Series A, Mathematical and Physical Sciences, vol. 151, n° 873, 1935, p. 234–255.

[8]E. C. Titchmarsh, « The Zeros of the Riemann Zeta-Function », Proceedings of the Royal Society, Series A, Mathematical and Physical Sciences, The Royal Society, vol. 157, n° 891, 1936, p. 261–263.

[9]A.M.Turing, « Some calculations of the Riemann zeta-function », Proceedings of the LMS, Third Series, vol. 3, 1953, p. 99–117.

[10]J. van de Lune, H.te Riele et D. T. Winter, « On the zeros of the Riemann zeta function in the critical strip. IV », *Mathematics of Computation*, vol. 46, n° 174, 1986, p. 667–681.