



HAL
open science

An optimal Poincaré-Wirtinger inequality in Gauss space

Barbara Brandolini, Francesco Chiacchio, Antoine Henrot, Cristina Trombetti

► **To cite this version:**

Barbara Brandolini, Francesco Chiacchio, Antoine Henrot, Cristina Trombetti. An optimal Poincaré-Wirtinger inequality in Gauss space. *Mathematical Research Letters*, 2013, 20 (3), pp.449-457. hal-00814754

HAL Id: hal-00814754

<https://hal.science/hal-00814754>

Submitted on 17 Apr 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AN OPTIMAL POINCARÉ-WIRTINGER INEQUALITY IN GAUSS SPACE

BARBARA BRANDOLINI¹, FRANCESCO CHIACCHIO¹, ANTOINE HENROT²,
AND CRISTINA TROMBETTI¹

ABSTRACT. Let Ω be a smooth, convex, unbounded domain of \mathbb{R}^N . Denote by $\mu_1(\Omega)$ the first nontrivial Neumann eigenvalue of the Hermite operator in Ω ; we prove that $\mu_1(\Omega) \geq 1$. The result is sharp since equality sign is achieved when Ω is a N -dimensional strip. Our estimate can be equivalently viewed as an optimal Poincaré-Wirtinger inequality for functions belonging to the weighted Sobolev space $H^1(\Omega, d\gamma_N)$, where γ_N is the N -dimensional Gaussian measure.

1. INTRODUCTION

Let Ω be a convex domain of \mathbb{R}^N ($N \geq 2$) and let us denote by $d\gamma_N$ the standard Gaussian measure in \mathbb{R}^N , that is

$$d\gamma_N = \frac{1}{(2\pi)^{N/2}} e^{-\frac{|x|^2}{2}} dx.$$

In [3] (see also [36] and [2]) the authors prove, among other things, that, if Ω is a convex, bounded domain, the first nontrivial eigenvalue $\mu_1(\Omega)$ of the following problem

$$(1.1) \quad \begin{cases} -\operatorname{div} \left(\exp \left(-\frac{|x|^2}{2} \right) Du \right) = \mu \exp \left(-\frac{|x|^2}{2} \right) u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$(1.2) \quad \mu_1(\Omega) \geq \max \left\{ 1, \frac{1}{2} + \frac{\pi^2}{\operatorname{diam}(\Omega)^2} \right\}.$$

Here ν stands for the outward normal to $\partial\Omega$ and $\operatorname{diam}(\Omega)$ for the diameter of Ω . It is well-known that $\mu_1(\Omega)$ can be characterized in the following variational way

$$\mu_1(\Omega) = \min \left\{ \frac{\int_{\Omega} |D\psi|^2 d\gamma_N}{\int_{\Omega} \psi^2 d\gamma_N} : \psi \in H^1(\Omega, d\gamma_N) \setminus \{0\}, \int_{\Omega} \psi d\gamma_N = 0 \right\},$$

1991 *Mathematics Subject Classification.* 35B45; 35P15; 35J70.

Key words and phrases. Neumann eigenvalue, Hermite operator, sharp bounds.

¹ Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli “Federico II”, Complesso Monte S. Angelo, via Cintia - 80126 Napoli, Italy; email: brandolini@unina.it; fchiacch@unina.it; cristina@unina.it.

² Institut Elie Cartan, UMR CNRS 7502, Université de Lorraine, F-54506 Vandoeuvre-les-Nancy Cedex, France; antoine.henrot@univ-lorraine.fr.

where $H^1(\Omega, d\gamma_N)$ is the weighted Sobolev space defined as follows

$$H^1(\Omega, d\gamma_N) \equiv \left\{ u \in W_{\text{loc}}^{1,1}(\Omega) \text{ such that } (u, |Du|) \in L^2(\Omega, d\gamma_N) \times L^2(\Omega, d\gamma_N) \right\},$$

endowed with the norm

$$\|u\|_{H^1(\Omega, d\gamma_N)} = \|u\|_{L^2(\Omega, d\gamma_N)} + \|Du\|_{L^2(\Omega, d\gamma_N)}.$$

Incidentally note that the space $H^1(\Omega, d\gamma_N) \equiv H^1(\Omega)$ whenever Ω is a bounded domain.

Estimate (1.2) implies that, if Ω is a bounded, convex domain of \mathbb{R}^N , the following Poincaré-Wirtinger inequality holds

$$(1.3) \quad \int_{\Omega} \left(u - \int_{\Omega} u d\gamma_N \right)^2 d\gamma_N \leq \int_{\Omega} |Du|^2 d\gamma_N, \quad \forall u \in H^1(\Omega, d\gamma_N).$$

It is well-known (see for example [20]) that, when $\Omega = \mathbb{R}^N$, inequality (1.3) still holds true. The purpose of this paper is to fill the gap between convex, bounded sets and the whole \mathbb{R}^N by proving the following sharp lower bound.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a convex, C^2 domain whose boundary satisfies a uniform interior sphere condition (see (2.1) below). Then*

$$(1.4) \quad \mu_1(\Omega) \geq 1,$$

equality holding if Ω is any N -dimensional strip.

Remark 1.1. *We suspect that, up to a rotation, the strip is the unique minimizer of $\mu_1(\Omega)$, but it seems to be a challenging open problem to prove it.*

Our strategy consists into constructing a suitable sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ of bounded convex domains invading Ω and then passing to the limit in (1.2). To show that $\mu_1(\Omega_k)$ converge to $\mu_1(\Omega)$ one of the main ingredients is an extension theorem for functions belonging to $H^1(\Omega_k, d\gamma_N)$ with a constant independent of k (see Theorem 2.1, see also [19]).

The structure of the differential operator in (1.1) suggests the relevance of the case of unbounded sets since the density of Gaussian measure degenerates at infinity. Moreover, in such a case, the space $H^1(\Omega, d\gamma_N)$ does no longer coincide with $H^1(\Omega)$. Indeed, when $\Omega = \mathbb{R}^N$, the case mostly studied by physicists, the eigenvalues of problem (1.1) are the integers and the corresponding eigenfunctions are combinations of Hermite polynomials which clearly do not belong to $H^1(\mathbb{R}^N)$.

The Hermite operator appearing in problem (1.1) is widely studied in literature from many points of view. It is a classical subject in quantum mechanics (see for instance [20]) as well as in probability; indeed it is the generator of the Ornstein-Uhlenbeck semigroup (see for example [10]). Finally, problems of the kind (1.1) are related to some functional inequalities as the well-known Gross-Os Theorem on the Sobolev Logarithmic embedding (see e. g. [31, 27, 26, 38, 21, 11, 24]).

Note that the convexity assumption in Theorem 1.1 cannot be relaxed; it is enough to consider the classical example of a planar domain made by two equal squares connected by a thin corridor. Problems linking the geometry of a domain and the sequence of eigenvalues of a second order elliptic operator are classical since the estimates by Faber, Krahn or Pólya, Széögö concerning the first eigenvalue of the Laplacian with Dirichlet or Neumann boundary conditions respectively. Further developments of this topic can be found for instance in [4, 5, 1, 16, 22, 17, 15, 8], where estimates for Dirichlet eigenvalues and eigenfunctions of linear and nonlinear operator

are derived. Concerning Neumann boundary conditions we refer the reader to [5, 12, 14] and to [37, 6, 18, 28, 29] for lower bounds of Neumann eigenvalues in different contexts (Laplacian, p -Laplacian, manifolds of constant curvature). For results in Gauss space we mention, for instance, [9, 23, 13]. Clearly the above list of references is far from being exhaustive; more papers in this growing field of research are cited in [34, 35, 33, 32].

2. PROOF OF THEOREM 1.1

We recall that, given a subset Ω of \mathbb{R}^N , $\partial\Omega$ satisfies a uniform interior sphere condition if

$$(2.1) \quad \exists \bar{r} > 0 : \quad \forall x \in \partial\Omega \quad \exists B_{\bar{r}} \subset \Omega \quad \text{such that} \quad \overline{B_{\bar{r}}} \cap \overline{\Omega} = \{x\},$$

where $B_{\bar{r}}$ denotes a ball with radius $\bar{r} > 0$.

The proof of our result is divided in two steps. The first one provides an extension theorem that may have an interest by its own. In the second one we consider a sequence of convex, bounded domains $\{\Omega_k\}$ invading Ω satisfying $\mu_1(\Omega_k) \geq 1$ and we show that $\lim_k \mu_1(\Omega_k) = \mu_1(\Omega)$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a convex, C^2 domain whose boundary satisfies (2.1) and let us denote $d_0 = \text{dist}(0, \partial\Omega)$. Let $u \in H^1(\Omega, d\gamma_N)$; there exist a function $\tilde{u} \in H^1(\mathbb{R}^N, d\gamma_N)$ extending u to the whole \mathbb{R}^N and a constant C such that*

$$(2.2) \quad \|\tilde{u}\|_{H^1(\mathbb{R}^N, d\gamma_N)} \leq C \|u\|_{H^1(\Omega, d\gamma_N)}.$$

In (2.2) $C = C(\bar{r}, N)$ if $0 \in \Omega$, while $C = C(\bar{r}, N, d_0)$ if $0 \notin \Omega$.

Proof of Theorem 2.1. We distinguish two cases: $0 \in \Omega$ and $0 \notin \Omega$ and we fix $\tilde{r} = \frac{\bar{r}}{2}$.

Suppose first that $0 \in \Omega$. Let us denote by $d(x) = \text{dist}(x, \partial\Omega)$ the distance of a point $x \in \mathbb{R}^N$ from $\partial\Omega$ and

$$\Omega^{\tilde{r}} = \{x \in \mathbb{R}^N \setminus \overline{\Omega} : d(x) < \tilde{r}\}, \quad \Omega_{\tilde{r}} = \{x \in \Omega : d(x) < \tilde{r}\}.$$

Let $u \in H^1(\Omega, d\gamma_N)$; we want to extend u to \mathbb{R}^N by reflection along the normal to $\partial\Omega$. Define

$$\Phi : x \in \Omega_{\tilde{r}} \longrightarrow \Phi(x) = x - 2d(x)Dd(x) \in \Omega^{\tilde{r}}.$$

By construction Φ is a C^1 one-to-one map; we claim that

$$(2.3) \quad 1 \leq |J_{\Phi}(x)| \leq 3^{N-1}, \quad \forall x \in \Omega_{\tilde{r}}.$$

A straightforward computation yields

$$\frac{\partial \Phi_i(x)}{\partial x_j} = \delta_{ij} - 2 \frac{\partial d(x)}{\partial x_j} \frac{\partial d(x)}{\partial x_i} - 2d(x) \frac{\partial^2 d(x)}{\partial x_i \partial x_j}.$$

By a rotation of coordinates we can assume that the x_N -axis lies in the direction $Dd(x)$. By a further rotation of the first $N-1$ coordinates we can also assume that the x_1, \dots, x_{N-1} axes lie along the principal directions corresponding to the principal curvatures $\kappa_1, \dots, \kappa_{N-1}$ of $\partial\Omega$ at $p(x) = \frac{x + \Phi(x)}{2}$. Clearly $p(x)$ is the projection of x on $\partial\Omega$. In this coordinate system, known as principal coordinate system at $p(x)$, it is immediate to verify that

$$|J_{\Phi}(x)| = \prod_{i=1}^{N-1} \left(1 + \frac{2d(x)\kappa_i}{1 - d(x)\kappa_i} \right).$$

Claim (2.3) follows recalling that $d(x) < \tilde{r} = \frac{\bar{r}}{2}$ and $\kappa_i \leq \frac{1}{\bar{r}}$.

We observe that in the simplest case $N = 2$, (2.3) has been proven, in a different and more direct way, in [13].

Now define

$$\bar{u}(x) = u(\Phi^{-1}(x)) \quad \forall x \in \Omega^{\tilde{r}}.$$

Let $\theta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \leq \theta \leq 1$ in \mathbb{R}^N , $\theta = 1$ on Ω , $\theta = 0$ in $\mathbb{R}^N \setminus (\Omega \cup \Omega^{\tilde{r}})$ and $|D\theta| \leq C = C(\tilde{r})$. Set

$$(2.4) \quad \tilde{u} = \begin{cases} u & \text{in } \Omega \\ \theta \bar{u} & \text{in } \Omega^{\tilde{r}} \\ 0 & \text{in } \mathbb{R}^N \setminus (\Omega \cup \Omega^{\tilde{r}}). \end{cases}$$

Since Ω contains the origin, it is easy to verify that

$$(2.5) \quad \exp\left(\frac{-|\Phi(x)|^2}{2} + \frac{|x|^2}{2}\right) \leq 1 \quad \forall x \in \Omega^{\tilde{r}}.$$

Thus, by (2.3), (2.4) and (2.5) we get

$$(2.6) \quad \begin{aligned} \int_{\mathbb{R}^N} \tilde{u}^2 d\gamma_N &= \int_{\Omega} u^2 d\gamma_N + \int_{\Omega^{\tilde{r}}} \tilde{u}^2 d\gamma_N \\ &\leq \int_{\Omega} u^2 d\gamma_N + \int_{\Omega^{\tilde{r}}} u^2(x) \exp\left(-\frac{|\Phi(x)|^2}{2} + \frac{|x|^2}{2}\right) |J_{\Phi}| d\gamma_N \\ &\leq C(N) \int_{\Omega} u^2 d\gamma_N. \end{aligned}$$

On the other hand, (2.3), (2.4), (2.5) and (2.6) imply

$$\begin{aligned} \int_{\mathbb{R}^N} |D\tilde{u}|^2 d\gamma_N &\leq C(N, \tilde{r}) \left[\int_{\Omega^{\tilde{r}}} \tilde{u}^2 d\gamma_N + \int_{\Omega \cup \Omega^{\tilde{r}}} |D\tilde{u}|^2 d\gamma_N \right] \\ &\leq C(N, \tilde{r}) \left[\int_{\Omega} u^2 d\gamma_N + \int_{\Omega} |Du|^2 d\gamma_N + \int_{\Omega^{\tilde{r}}} |Du|^2 \exp\left(-\frac{|\Phi(x)|^2}{2} + \frac{|x|^2}{2}\right) |J_{\Phi}| d\gamma_N \right] \\ &\leq C(N, \tilde{r}) \left[\int_{\Omega} u^2 d\gamma_N + \int_{\Omega} |Du|^2 d\gamma_N \right]. \end{aligned}$$

Hence if Ω contains the origin (2.2) holds true.

Suppose now that $0 \notin \Omega$. Up to a rotation about the origin, the translation $T : x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \rightarrow (x_1 - \delta, x_2, \dots, x_N) \in \mathbb{R}^N$, for a fixed $\delta > d_0$, maps Ω onto a set $T(\Omega)$ containing the origin. Define

$$v(x) = v(x_1, x_2, \dots, x_N) = u(x_1 + \delta, x_2, \dots, x_N) \exp\left(-\frac{x_1 \delta}{2} - \frac{\delta^2}{4}\right), \quad x \in T(\Omega);$$

then

$$\int_{\Omega} u^2 d\gamma_N = \int_{T(\Omega)} v^2 d\gamma_N.$$

Since by construction $T(\Omega)$ contains the origin, there exists a function $\tilde{v} \in H^1(\mathbb{R}^N, d\gamma_N)$ such that $\tilde{v}|_{T(\Omega)} = v$ and

$$\|\tilde{v}\|_{H^1(\mathbb{R}^N, d\gamma_N)} \leq C(\tilde{r}, N) \|v\|_{H^1(T(\Omega), d\gamma_N)}.$$

Let

$$\tilde{u}(x) = \tilde{u}(x_1, x_2, \dots, x_N) = \tilde{v}(x_1 - \delta, x_2, \dots, x_N) \exp\left(\frac{x_1 \delta}{2} - \frac{\delta^2}{4}\right),$$

we finally get

$$\|\tilde{u}\|_{H^1(\mathbb{R}^N, d\gamma_N)} \leq C(\bar{r}, N, d_0) \|u\|_{H^1(\Omega, d\gamma_N)}.$$

□

Using the fact that $H^1(\mathbb{R}^N, d\gamma_N)$ is compactly embedded into $L^2(\mathbb{R}^N, d\gamma_N)$ (see for example [25]) and the above extension theorem we deduce the compact embedding of $H^1(\Omega, d\gamma_N)$ into $L^2(\Omega, d\gamma_N)$. By classical arguments (see for instance Theorem 5.4.3 in [7], see also Proposition 3.4 in [30]) we can deduce a Poincaré-Wirtinger inequality with a constant depending on Ω . Therefore, as we said in Section 1, by the classical spectral theory on compact self-adjoint operators, $\mu_1(\Omega)$ satisfies the following variational characterization

$$\mu_1(\Omega) = \min \left\{ \frac{\int_{\Omega} |D\psi|^2 d\gamma_N}{\int_{\Omega} |\psi|^2 d\gamma_N} : \psi \in H^1(\Omega, d\gamma_N) \setminus \{0\}, \int_{\Omega} \psi d\gamma_N = 0 \right\}.$$

When Ω is a convex, bounded domain, estimate (1.4) is contained in [3]. Therefore, from now on Ω will denote a convex, unbounded domain. Let Ω_k be a sequence of convex, bounded, C^2 domains whose boundaries satisfy (2.1) for every $k \in \mathbb{N}$, and invading Ω in the sense that

$$\Omega_k \subset \Omega_{k+1} \quad \forall k \in \mathbb{N} \quad \text{and} \quad \bigcup_{k \in \mathbb{N}} \Omega_k = \Omega.$$

For the explicit construction of a sequence of this kind see for instance [13].

As proven in [3] we have that

$$(2.7) \quad \mu_1(\Omega_k) \geq 1, \quad \forall k \in \mathbb{N},$$

that can be equivalently written as

$$(2.8) \quad \int_{\Omega_k} \psi^2 d\gamma_N \leq \int_{\Omega_k} |D\psi|^2 d\gamma_N, \quad \forall \psi \in H^1(\Omega_k, d\gamma_N) : \int_{\Omega_k} \psi d\gamma_N = 0.$$

Now we want to pass to the limit in (2.7). To this aim consider the operator

$$A_k : f \in L^2(\Omega, d\gamma_N) : \int_{\Omega} f d\gamma_N = 0 \longrightarrow \tilde{u}_k \in H^1(\Omega, d\gamma_N),$$

where \tilde{u}_k is the extension provided in Theorem 2.1 of the solution $u_k \in H^1(\Omega_k, d\gamma_N)$ to the following problem

$$(2.9) \quad \begin{cases} -\operatorname{div} \left(\exp \left(-\frac{|x|^2}{2} \right) Du_k \right) = (f - c_k) \exp \left(-\frac{|x|^2}{2} \right) & \text{in } \Omega_k \\ \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial\Omega_k \\ \int_{\Omega_k} u_k d\gamma_N = 0, \end{cases}$$

where $c_k =: \int_{\Omega_k} f d\gamma_N$ and ν is the outward normal to $\partial\Omega_k$. Observe that Lax-Milgram theorem ensures the existence and uniqueness of u_k . Moreover $\gamma_N(\Omega \setminus \Omega_k) \rightarrow 0$ implies $c_k \rightarrow 0$. We also

introduce the operator

$$A : f \in L^2(\Omega, d\gamma_N) : \int_{\Omega} f d\gamma_N = 0 \longrightarrow u \in H^1(\Omega, d\gamma_N),$$

where u is the unique solution, whose existence is guaranteed by Lax-Milgram theorem, to the problem

$$(2.10) \quad \begin{cases} -\operatorname{div} \left(\exp \left(-\frac{|x|^2}{2} \right) Du \right) = f \exp \left(-\frac{|x|^2}{2} \right) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u d\gamma_N = 0. \end{cases}$$

Using u_k as test function in (2.9), from Schwarz inequality we deduce

$$\int_{\Omega_k} |Du_k|^2 d\gamma_N = \int_{\Omega_k} u_k (f - c_k) d\gamma_N \leq \left(\int_{\Omega_k} u_k^2 d\gamma_N \right)^{1/2} \left(\int_{\Omega_k} (f - c_k)^2 d\gamma_N \right)^{1/2}.$$

Using (2.8) and recalling that $c_k \rightarrow 0$, we get

$$\int_{\Omega_k} |Du_k|^2 d\gamma_N \leq C_1 \left(\int_{\Omega} f^2 d\gamma_N \right)^{1/2} + C_2,$$

where C_1, C_2 are positive constants whose values are independent of k . The above inequality together with (2.8) yield

$$\int_{\Omega_k} u_k^2 d\gamma_N + \int_{\Omega_k} |Du_k|^2 d\gamma_N \leq C,$$

where C is a positive constant whose value is independent of k . From (2.2) we deduce that the sequence $\{\tilde{u}_k\}_{k \in \mathbb{N}}$ is bounded in $H^1(\Omega, d\gamma_N)$. Since the embedding of $H^1(\Omega, d\gamma_N)$ into $L^2(\Omega, d\gamma_N)$ is compact, there exists a (not relabelled) subsequence \tilde{u}_k such that $\tilde{u}_k \rightharpoonup v$ in $H^1(\Omega, d\gamma_N)$, $\tilde{u}_k \rightarrow v$ in $L^2(\Omega, d\gamma_N)$ and a.e. in Ω . In fact v coincides with u since they both solve the same problem (2.10). Indeed, let $\phi \in C^\infty(\Omega)$. Recalling that $\gamma_N(\Omega \setminus \Omega_k) \rightarrow 0$ and $\Omega_k \subset \Omega$, we get

$$\begin{aligned} \int_{\Omega} Dv D\phi d\gamma_N &= \lim_k \int_{\Omega} D\tilde{u}_k D\phi d\gamma_N = \lim_k \left(\int_{\Omega_k} Du_k D\phi d\gamma_N + \int_{\Omega \setminus \Omega_k} D\tilde{u}_k D\phi d\gamma_N \right) \\ &= \lim_k \int_{\Omega_k} (f - c_k) \phi d\gamma_N = \int_{\Omega} f \phi d\gamma_N. \end{aligned}$$

Finally, as k goes to $+\infty$,

$$(2.11) \quad \|(A_k - A)f\|_{L^2(\Omega, d\gamma_N)} = \|\tilde{u}_k - u\|_{L^2(\Omega, d\gamma_N)} \rightarrow 0.$$

The compact embedding of $H^1(\Omega, d\gamma_N)$ into $L^2(\Omega, d\gamma_N)$ and (2.11) allow us to adapt Theorems 2.3.1 and 2.3.2 in [32] to conclude that the operators A_k strongly converge to A and hence

$$\mu_1(\Omega_k) \rightarrow \mu_1(\Omega).$$

Finally we prove the optimality of our estimate (1.4). Consider the N -dimensional strip

$$S_a = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : -a < x_1 < a, x_2, \dots, x_n \in \mathbb{R}\}, \quad a \in (0, +\infty).$$

The eigenfunctions are factorized and can be written as linear combinations of products of homogeneous Hermite polynomials $H_{n_1}(x_1), H_{n_2}(x_2), \dots, H_{n_N}(x_N)$. We recall that the Hermite polynomials in one variable are defined by

$$H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}, \quad n \in \mathbb{N} \cup \{0\},$$

and they constitute a complete set of eigenfunctions to problem (1.1) when $\Omega = \mathbb{R}$; more precisely

$$-\left(e^{-t^2/2} H'_n(t)\right)^{-t^2/2} H_n(t), \quad n \in \mathbb{N} \cup \{0\}.$$

Denote by $\lambda_1(-a, a)$ the first Dirichlet eigenvalue of the one-dimensional Hermite operator in the interval $(-a, a)$. One can easily verify that

$$\mu_1(-a, a) = \lambda_1(-a, a) + 1 > 1 = \mu_1(\mathbb{R}).$$

Therefore $\mu_1(S_a) = 1$ for every $a \in (0, +\infty)$ and a corresponding eigenfunction is, for instance, $H_1(x_2) = x_2$.

REFERENCES

- [1] A. Alvino - V. Ferone - G. Trombetti, *On the properties of some nonlinear eigenvalues*, SIAM J. Math. Anal. 29 (1998), no. 2, 437–451.
- [2] B. Andrews - J. Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. 24 (2011), no. 3, 899–916.
- [3] B. Andrews - L. Ni, *Eigenvalue comparison on Bakry-Emery manifolds*, Comm. Partial Differential Equations 37 (2012), 2081–2092.
- [4] M. S. Ashbaugh - R. D. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math. (2) 135 (1992), no. 3, 601–628.
- [5] M. S. Ashbaugh - R. D. Benguria, *Sharp upper bound to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature.*, J. London Math. Soc. (2) 52 (1995), no. 2, 402–416.
- [6] M. S. Ashbaugh - L. Hermi, *On extending the inequalities of Payne, Pólya, and Weinberger using spherical harmonics*, Rocky Mountain J. Math. 38 (2008), no. 4, 1037–1072.
- [7] H. Attouch - G. Buttazzo - G. Michaille, *Variational analysis in Sobolev and BV spaces, Applications to PDEs and optimization*. MPS/SIAM Series on Optimization, 6 SIAM, 2006.
- [8] R. D. Benguria - H. Linde - B. Loewe, *Isoperimetric inequalities for eigenvalues of the Laplacian and the Schrödinger operator*, Bulletin of Mathematical Sciences 2 (2012), 1–56.
- [9] M. F. Betta - F. Chiacchio - A. Ferone, *Isoperimetric estimates for the first eigenfunction of a class of linear elliptic problems*, Z. Angew. Math. Phys. 58 (2007), no. 1, 37–52.
- [10] V. I. Bogachev, *Gaussian Measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.
- [11] B. Brandolini - F. Chiacchio - C. Trombetti, *Hardy type inequalities and Gaussian measure*, Commun. Pure Appl. Anal. 6 (2007), no. 2, 411–428.
- [12] B. Brandolini - F. Chiacchio - C. Trombetti, *Sharp estimates for eigenfunctions of a Neumann problem*, Comm. Partial Differential Equations 34 (2009), no. 10-12, 1317–1337.
- [13] B. Brandolini - F. Chiacchio - C. Trombetti, *A sharp lower bound for some Neumann eigenvalues of the Hermite operator*, Differential and Integral Equations 26 (2013), no. 5-6, 639–654.
- [14] B. Brandolini - F. Chiacchio - C. Trombetti, *Optimal lower bounds for eigenvalues of linear and nonlinear Neumann problems*, arXiv:1302.1795.
- [15] B. Brandolini - P. Freitas - C. Nitsch - C. Trombetti, *Sharp estimates and saturation phenomena for a nonlocal eigenvalue problem*, Adv. Math. 228 (2011), no. 4, 2352–2365.
- [16] B. Brandolini - C. Nitsch - C. Trombetti, *New isoperimetric estimates for solutions to Monge-Ampère equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 4, 1265–1275.
- [17] B. Brandolini - C. Nitsch - C. Trombetti, *An upper bound for nonlinear eigenvalues on convex domains by means of the isoperimetric deficit*, Arch. Math. (Basel) 94 (2010), no. 4, 391–400.

- [18] F. Brock - F. Chiacchio - A. Mercaldo, *Weighted isoperimetric inequalities in cones and applications*, Non-linear Anal. 75 (2012), no. 15, 5737– 5755.
- [19] D. Chénais, *On the existence of a solution in a domain identification problem*, J. Math. Anal. Appl. 52 (1975), no. 2, 189–219.
- [20] R. Courant - D. Hilbert, *Methods of mathematical physics. Vol. I*. Interscience Publishers, Inc., New York, N.Y., 1953.
- [21] F. Chiacchio, *Comparison results for linear parabolic equations in unbounded domains via Gaussian symmetrization*, Differential and Integral Equations 17 (2004), 241–258.
- [22] F. Chiacchio, *Estimates for the first eigenfunction of linear eigenvalue problems via Steiner symmetrization*, Publ. Mat. 53 (2009), no. 1, 47–71.
- [23] F. Chiacchio - G. Di Blasio, *Isoperimetric inequalities for the first Neumann eigenvalue in Gauss space*, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 2, 199–216.
- [24] A. Cianchi - L. Pick, *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. 256 (2009), no. 11, 3588–3642.
- [25] G. Da Prato, *An introduction to infinite-dimensional analysis*. Revised and extended from the 2001 original by Da Prato. Universitext. Springer-Verlag, Berlin, 2006.
- [26] A. Ehrhard, *Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes*, Ann. Sci. Ecole Norm. Sup. 17 (1984), 317–332.
- [27] A. Ehrhard, *Éléments extrémaux pour les inégalités de Brunn-Minkowski gaussiennes*, Ann. Inst. H. Poincaré Probab. Statist. 22 (1986), no. 2, 149–168.
- [28] L. Esposito - C. Nitsch - C. Trombetti, *Best constants in Poincaré inequalities for convex domains*, J. Convex Anal. 20 (2013), no. 1, 253–264.
- [29] V. Ferone - C. Nitsch - C. Trombetti, *A remark on optimal weighted Poincaré inequalities for convex domains*, Rendiconti Lincei Matematica e Applicazioni 23 (2012), no. 4, 467–475.
- [30] F. Feo - M. R. Posteraro, *Logarithmic Sobolev trace inequalities*, The Asian Journal of Mathematics, to appear.
- [31] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97 (1975), no. 4, 1061–1083.
- [32] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
- [33] A. Henrot - M. Pierre, *Variation et optimisation de formes. Une analyse géométrique*, Mathematics & Applications, 48. Springer, Berlin, 2005.
- [34] B. Kawohl, *Rearrangements and convexity of level sets in PDE*. Lecture Notes in Mathematics, 1150. Springer-Verlag, Berlin, 1985.
- [35] S. Kesavan, *Symmetrization & applications*. Series in Analysis, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
- [36] Lei Ni, *Estimates on the modulus of expansion for vector fields solving nonlinear equations*, Journal de Mathématiques Pures et Appliquées 99, (2013), no. 1, 1–16.
- [37] L. E. Payne - H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. 5 (1960), 286–292.
- [38] E. Pelliccia - G. Talenti, *A proof of a logarithmic Sobolev inequality*, Calc. Var. Partial Differential Equations 1 (1993), no. 3, 237–242.