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THE RING OF POLYNOMIAL FUNCTORS OF PRIME DEGREE

ALEXANDER ZIMMERMANN

ABSTRACT. Let $\hat{\mathbb{Z}}_p$ be the ring of p -adic integers. We prove in the present paper that the category of polynomial functors from finitely generated free abelian groups to $\hat{\mathbb{Z}}_p$ -modules of degree at most p is equivalent to the category of finitely generated modules over a particularly well understood ring, called Green order. That this is the case was conjectured by Yuri Drozd.

INTRODUCTION

Polynomial functors attained a lot of interest in recent years by at least two major discoveries. First, in [14] Henn, Lannes and Schwartz showed that the category of analytic functors from the category of finite dimensional vector spaces to the category of vector spaces over the same field of characteristic p is equivalent to the category of unstable modules over the mod p Steenrod algebra modulo nilpotent objects. Second, Franjou, Friedlander, Scorichenko and Suslin in [11] and Friedlander and Suslin in [12] use strict polynomial functor to prove the finite generation of cohomology of group schemes and to compute *Ext*-groups of modules over general linear groups. More recently Djament and Vespa studied stable homology of orthogonal, symplectic and unitary groups using some category of polynomial functors [28, 29, 30, 3, 4]. For definitions and more ample remarks of these concepts we refer to section 1.

The category \mathcal{A}_R of polynomial functors $\mathbb{Z} - free \rightarrow R - mod$ for a commutative ring R is a classical object in algebraic topology (cf Eilenberg, MacLane [7]). Let \mathcal{A}_R^n be the full subcategory of at most degree n polynomial functors in \mathcal{A}_R . Quadratic functors were characterized by Baues [1] as modules over a particular algebra. Baues, Dreckmann, Franjou and Pirashvili show in [2] that \mathcal{A}_R^n is a module category of finitely generated R -algebra Γ_R^n as well. This description was used by Drozd to show in [5] that $\mathcal{A}_{\hat{\mathbb{Z}}_2}^2$ and in [6] that $\mathcal{A}_{\hat{\mathbb{Z}}_3}^3$ are two very explicitly given classical orders over $\hat{\mathbb{Z}}_2$ and $\hat{\mathbb{Z}}_3$ respectively, whose representation theory is completely understood. In particular each of them admit only a finite number of indecomposable lattices. Here, and in the sequel, we denote by $\hat{\mathbb{Z}}_p$ the ring of p -adic integers, and by \mathbb{F}_q the field with q elements. These orders were introduced by Roggenkamp in [27]. Recall that an R -order over an integral domain R is an R -algebra Λ , finitely generated projective as R -modules and so that $K \otimes_R \Lambda$ is a semisimple K -algebra, for K being the field of fractions of R .

Drozd conjectures at the end of [6] that $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$ should be equivalent to the module category of a particular Green order Λ_p over $\hat{\mathbb{Z}}_p$ for all primes p . Drozd proves the case $p = 2$ and $p = 3$ by explicitly associating the generators of the ring given by Baues or Baues, Dreckmann, Franjou and Pirashvili respectively to matrices in the corresponding matrix rings and computes the kernel and the image of the so-defined mapping. The relations in [2] are sufficiently involved so that going beyond $p = 3$ by this method seems to be not realistic.

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In this paper we prove Drozd's conjecture. Our method is conceptual. We develop a recollement diagram of $\mathcal{A}_{\mathbb{F}_p}^n$ by $\mathcal{A}_{\mathbb{F}_p}^{n-1}$ and the module category of the group ring $\mathbb{F}_p \mathfrak{S}_n$, analogous to the one described by Schwartz [31, Section 5.5] for functors $\mathbb{F}_p - \text{mod} \rightarrow \mathbb{F}_p - \text{mod}$. This recollement diagram for $\mathcal{A}_{\mathbb{F}_p}^n$ may be of independent interest since it is completely general. We do not know though if there is a recollement diagram for $\mathcal{A}_{\mathbb{Z}_p}^n$ directly. Comparison of these two diagrams gives many informations. A second ingredient then is the study of various *Ext*-groups between simple functors, using work of Franjou, Friedlander, Scorichenko and Suslin [11]. The third main ingredient is the explicit projective functor mapping to the reduction modulo p functor. It should be noted that we do not actually use the ring defined by Baues, Dreckmann, Franjou and Pirashvili in [2]. We just use that there is an algebra which is finitely generated, so that the Krull-Schmidt property, lifting of idempotents and similar properties are valid for $\mathcal{A}_{\mathbb{Z}_p}^n$. For this reason we do not give a Morita bimodule between the Baues, Dreckmann, Franjou and Pirashvili-ring and the order we get. As an application of our result, we count the number of 'torsion free' indecomposable polynomial functors in $\mathcal{A}_{\mathbb{Z}_p}^p$.

Our paper is organized as follows. In Section 1 we give the essential definitions and relate the different concepts. In Section 2 we recall some of the most important discoveries used in the sequel. Section 3 describes the classical recollement diagrams as well as the new one we have to use for $\mathcal{A}_{\mathbb{F}_p}$, and we derive first consequences. The first main result is proved in Section 4. We give the structure of $\mathcal{A}_{\mathbb{F}_p}^p$ there. Finally, in Section 5 we determine $\mathcal{A}_{\mathbb{Z}_p}^p$ and prove the second main theorem there.

Acknowledgement: This research was done during the years 2001, 2002 and 2003 as joint work with Steffen König. The result was proved in its present form in 2003, and we presented the result on various occasions, such as in Leicester in March 2003 "Workshop on Hochschild cohomology and applications", in the "Representation Theory Days" Jena October 2003, in Strasbourg in October 2003 in Valenciennes in February 2004, in Bern in February 2005, at the "Joint Meeting DMV-AMS" in Mainz in June 2005, and in October 2005 in the "séminaire Chevalley" Paris. Recently we received numerous encouragements to publish our manuscript. Steffen König¹ wrote to me that he will not find the time to finish the paper, and he gave me the autorisation to publish the paper alone. I wish to thank Steffen König for having shared his insight with me, and for allowing me to publish the paper.

1. GENERALITIES ON POLYNOMIAL FUNCTORS

1.1. Definitions. Let \mathcal{A} be a category with direct sums and \mathcal{B} be a category with direct sums and kernels. Then, following Eilenberg and MacLane [7] define the cross effect $F^{(1)}$ of a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to be the bifunctor $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ defined on objects by $F^{(1)}(V|W) := \ker(F(V \oplus W) \rightarrow F(V) \oplus F(W))$, and on morphism by the naturality of the construction. For $n \geq 1$, the n -th cross effect $F^{(n)}$ is the cross effect of the $n - 1$ -st cross effect of F , seen as functor in the first variable. Hence,

$$F^{(n)}(V|W|V_1|\dots|V_{n-1}) := \ker \left(F^{(n-1)}(V \oplus W|V_1|\dots|V_{n-1}) \rightarrow F^{(n-1)}(V|V_1|\dots|V_{n-1}) \oplus F^{(n-1)}(W|V_1|\dots|V_{n-1}) \right)$$

for objects $V, W, V_1, \dots, V_{n-1}$ in \mathcal{A} . Suppose in the sequel $F(0) = 0$. A functor F is said to be polynomial of degree at most n if $F^{(n)} = 0$ for an $n \in \mathbb{N}$ (see Pirashvili [21]). Given a commutative ring R , let $R - \text{mod}$ be the category of finitely generated R -modules, let $R - \text{Mod}$ be the category of all R -modules, and let $R - \text{free}$ be the category of finitely generated free R -modules. Further, call \mathcal{F}_R^n the category of polynomial functors of degree at most n from $R - \text{mod}$ to $R - \text{mod}$, and $\mathcal{F}_R := \varinjlim_n \mathcal{F}_R^n$. Moreover, let \mathcal{A}_R^n be the category

¹email to the author from April 12, 2013

of polynomial functors of degree at most n from $\mathbb{Z} - \text{free}$ to $R - \text{mod}$, and $\mathcal{A}_R := \varinjlim_n \mathcal{A}_R^n$. All these categories are abelian. Observe that additive functors are exactly the degree 1 polynomial functors. The only degree 0 polynomial functor is the trivial functor due to our hypothesis that $F(0) = 0$.

Friedlander and Suslin define in [12, Definition 2.1] the category of strict polynomial functors \mathcal{P}_k over a field k . A strict polynomial functor F is defined by associating to each finite dimensional k -vector space a k -vector space $F(V)$ and to associate for any two finite dimensional k -vector spaces V and W an element in

$$S^*(\text{Hom}_k(\text{Hom}_k(V, W), k)) \otimes \text{Hom}_k(F(V), F(W))$$

which in addition satisfy the usual compatibility relations for compositions and the identity. Each of these elements can be interpreted as mapping $\text{Hom}_k(V, W) \rightarrow \text{Hom}_k(F(V), F(W))$ by interpreting the formal polynomial $S^*(\text{Hom}_k(\text{Hom}_k(V, W), k)) \otimes \text{Hom}_k(F(V), F(W))$ as polynomial mapping, so that any strict polynomial functor induces a polynomial functor $k - \text{mod} \rightarrow k - \text{Mod}$. Hence, we have a forgetful functor $\mathcal{P}_k \rightarrow \mathcal{F}_k$. It is shown in [12] that the category of exact degree n strict polynomial functors \mathcal{P}_k^n from $k - \text{mod}$ to $k - \text{Mod}$ for a field k is equivalent to the category of modules over the Schur algebra $S_k(n, n)$. Moreover, $\mathcal{P}_k = \bigoplus_n \mathcal{P}_k^n$. Finally, the forgetful functor $\mathcal{P}_k \rightarrow \mathcal{F}_k$ sends a strict polynomial functor of degree at most d to a polynomial functor of degree at most d (cf [22, Remark 4.1]).

1.2. Functors with values in characteristic 0.

Lemma 1.1. *Let R be an integral domain of characteristic 0. If F is a polynomial functor $F : \mathbb{Z} - \text{free} \rightarrow R - \text{mod}$ of degree d , then $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^m) \xrightarrow{F} \text{Hom}_R(F(\mathbb{Z}^n), F(\mathbb{Z}^m))$ is polynomial of degree d in the $n \cdot m$ coordinate functions $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^m)$.*

Proof. We shall show by induction on $n+d$ that for any k homomorphisms $f_1, f_2, \dots, f_k \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^m)$ and integers $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{Z}$ one gets $F(\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_k f_k)$ is a degree d polynomial in the variables $\lambda_1, \lambda_2, \dots, \lambda_k$.

If the degree of F is 1, there is nothing to show since then the functor is linear.

Let $n > 1$. Now, we know that

$$F(\mathbb{Z}^{n-1} \oplus \mathbb{Z}) = F(\mathbb{Z}^{n-1}) \oplus F(\mathbb{Z}) \oplus F^{(1)}(\mathbb{Z}^{n-1} | \mathbb{Z}).$$

Hence, the restriction f'_i of each of the f_i to \mathbb{Z}^{n-1} and the restriction f''_i to the last component \mathbb{Z} define morphisms $F(\sum_{i=1}^k \lambda_i f'_i) : F(\mathbb{Z}^{n-1}) \rightarrow F(\mathbb{Z}^m)$, $F(\sum_{i=1}^k \lambda_i f''_i) : F(\mathbb{Z}) \rightarrow F(\mathbb{Z}^m)$ and $F(\sum_{i=1}^k \lambda_i f_i) : F^{(1)}(\mathbb{Z}^{n-1} | \mathbb{Z}) \rightarrow F(\mathbb{Z}^m)$. In the first two cases, the dimension of the source space is less than n , while the degree of F is unchanged, whereas in the third case the dimension of the source space is n , but the degree of the functor is $d-1$. So, in any of these cases by the induction hypothesis we can express $F(\sum_{i=1}^k \lambda_i f_i)$ a polynomial of degree n in the variables $\lambda_1, \lambda_2, \dots, \lambda_k$.

We are left with the case $n = 1$. The very same reduction applied to the image and induction on $m+d$ implies that one can suppose that $m = 1$. But then, Eilenberg and MacLane [7, (8.3)] show that for $\lambda \in \mathbb{Z}$ one has

$$F(\lambda \cdot) = F((\lambda - 1) \cdot) + F(1) + F^{(1)}(\lambda \cdot | 1),$$

where $F(1)$ is the identity. Now,

$$F(\lambda \cdot) - F((\lambda - 1) \cdot) = F(1) + F^{(1)}(\lambda \cdot | 1),$$

where by the induction hypothesis, since the degree of $F^{(1)}$ is less than the degree of F , the right hand side $F(1) + F^{(1)}(\lambda \cdot | 1)$ is polynomial of degree $d-1$ in λ .

We shall now adapt an argument of Kuhn [17, Lemma 4.8] to this slightly more general situation. We claim that a function $f : \mathbb{Z} \rightarrow R$ is a polynomial if and only if some derivative $f^{(r)}$ vanishes, where $f^{(r)}(n) = f^{(r-1)}(n) - f^{(r-1)}(n-1)$.

We assume for the moment that R contains \mathbb{Q} . Suppose f is a polynomial. Then, it is clear that $f^{(\deg(f)+1)} = 0$. Suppose to the contrary that $f^{(r)} = 0$. The polynomials

$\binom{X}{k} := \frac{X \cdot (X-1) \cdots (X-k+1)}{k!}$ for $k \in \{0, 1, \dots, d\}$ form an R -basis of the polynomials of degree at most d in $R[X]$, since $d!$ is invertible in R . Moreover, $\binom{X}{k} - \binom{X-1}{k} = \binom{X}{k-1}$. Now, by induction, $f^{(s)}$ is a polynomial, and hence a linear combination of polynomials $\binom{X}{k}$. The relation $\binom{X}{k} - \binom{X-1}{k} = \binom{X}{k-1}$ gives a polynomial $h^{(s-1)}$ so that $(h^{(s-1)})^{(1)} = f^{(s)}$. By induction on n , the values $g^{(1)}(n) = g(n) - g(n-1)$ determine the values $g(n)$ up to the value of $g(0)$. Therefore, up to this constant value, $f^{(s-1)} = h^{(s-1)}$.

Now, suppose R an integral domain of characteristic 0. Then, since $R \subseteq \text{frac}(R)$, f can be considered as being in values $\text{frac}(R)$ which contains \mathbb{Q} . This proves the claim.

Now, define $f(\lambda) := F(\lambda \cdot)$ and apply the claim to conclude that $F(\lambda \cdot)$ is polynomial of degree d . \blacksquare

Let K be a field. A priori the category \mathcal{A}_K is different from the category \mathcal{F}_K . Nevertheless, in some cases we get one inclusion.

Lemma 1.2. *Let K be either a prime field of finite characteristic or let K be a field of characteristic 0. Let $(\circ(K \otimes_{\mathbb{Z}} -))^* : \mathcal{F}_K \rightarrow \mathcal{A}_K$ be the functor defined by $(\circ(K \otimes_{\mathbb{Z}} -))^*(F) := F \circ (K \otimes_{\mathbb{Z}} -)$. Then, $(\circ(K \otimes_{\mathbb{Z}} -))^*$ induces a fully faithful embedding $\mathcal{F}_K \hookrightarrow \mathcal{A}_K$.*

Proof: The functor $\circ(- \otimes_{\mathbb{Z}} K) : \mathcal{F}_K \rightarrow \mathcal{A}_K$ induces for any two functors F and G in \mathcal{F}_K a mapping

$$\varphi : \text{Hom}_{\mathcal{F}_K}(F, G) \longrightarrow \text{Hom}_{\mathcal{A}_K}(F \circ (- \otimes_{\mathbb{Z}} K), G \circ (- \otimes_{\mathbb{Z}} K)).$$

We shall need to show that this mapping is an isomorphism.

Injectivity: Let η_1 and η_2 be two objects in $\text{Hom}_{\mathcal{F}_K}(F, G)$. Suppose $\varphi(\eta_1) = \varphi(\eta_2)$. Observe that for any $V \in \mathbb{Z} - \text{free}$ we have

$$(\varphi(\eta_1))(V) = \eta_1(K \otimes_{\mathbb{Z}} V) \in \text{Hom}_K(F(K \otimes_{\mathbb{Z}} V), G(K \otimes_{\mathbb{Z}} V))$$

and likewise for η_2 , satisfying that for any V and W and any $\rho \in \text{Hom}_{\mathbb{Z}}(V, W)$ one has $(\eta_1(K \otimes_{\mathbb{Z}} V)) \circ \hat{G}(\rho) = \hat{F}(\rho) \circ (\eta_1(K \otimes_{\mathbb{Z}} W))$.

Since $\circ(- \otimes_{\mathbb{Z}} K)$ is dense, η_1 and η_2 coincide on every object of $K - \text{mod}$, and so $\eta_1 = \eta_2$. As a consequence, without any further hypothesis,

$$\varphi : \text{Hom}_{\mathcal{F}_K}(F, G) \hookrightarrow \text{Hom}_{\mathcal{A}_K}(F \circ (- \otimes_{\mathbb{Z}} K), G \circ (- \otimes_{\mathbb{Z}} K)).$$

Surjectivity: Let $\eta \in \text{Hom}_{\mathcal{A}_K}(F \circ (- \otimes_{\mathbb{Z}} K), G \circ (- \otimes_{\mathbb{Z}} K))$ be a natural transformation. We need to show that there is a natural transformation $\eta' \in \text{Hom}_{\mathcal{F}_K}(F, G)$ so that $\varphi(\eta') = \eta$.

In the case K being a prime field of finite characteristic define for any $V \in K - \text{mod}$ the mapping $\eta'(V) := \eta(P_V)$, where P_V is a fixed chosen projective cover of V , so that $K \otimes_{\mathbb{Z}} P_V = V$.

In the case K being of characteristic 0, fix for any K -vector space V a free abelian subgroup P_V so that $K \otimes_{\mathbb{Z}} P_V = V$. Define $\eta'(V) := \eta(P_V)$.

We need to show that η' is a natural transformation.

Let $\varphi \in \text{Hom}_K(V, W)$.

Consider first the case of K being a prime field of finite characteristic. Since P_V and P_W are projective covers of V and W as abelian groups, there is a $\hat{\varphi} \in \text{Hom}_{\mathbb{Z}}(P_V, P_W)$ so that $K \otimes_{\mathbb{Z}} \hat{\varphi} = \varphi$ under the identification $K \otimes_{\mathbb{Z}} P_V = V$ and $K \otimes_{\mathbb{Z}} P_W = W$. Since η is a natural transformation, $\hat{G}(\hat{\varphi}) \circ \eta(P_V) = \eta(P_W) \circ \hat{F}(\hat{\varphi})$. But, by definition, $\hat{G}(\hat{\varphi}) = G(\varphi)$ and $\hat{F}(\hat{\varphi}) = F(\varphi)$, as well as $\eta(P_W) = \eta'(W)$ and $\eta(P_V) = \eta'(V)$. So, η' is a natural transformation.

Suppose now that K is a field of characteristic 0. Since $\mathcal{F}_K = \mathcal{P}_K$ in this case, we know that F (and G resp.) are polynomial laws transforming any linear mapping $V \rightarrow W$ into a linear mapping $F(V) \rightarrow F(W)$ (and $G(V) \rightarrow G(W)$ resp.) which depends polynomially

in the coefficients of any matrix representation with respect to any fixed bases. We know that for any \mathbb{Z} -linear mapping $\hat{\varphi} : P_V \rightarrow P_W$ that the equation

$$(\ddagger) : G(K \otimes_{\mathbb{Z}} \varphi) \circ \eta(P_V) = \hat{G}(\hat{\varphi}) \circ \eta(P_V) = \eta(P_W) \circ \hat{F}(\hat{\varphi}) = \eta(P_W) \circ F(K \otimes_{\mathbb{Z}} \varphi).$$

Since this equation holds evaluated in infinitely coefficients, since \mathbb{Z} and K are both infinite, this above equation (\ddagger) holds *as polynomial equation*.

Therefore, the equation hold as well for φ , since there the only difference is that the polynomials are evaluated not only on integer coefficients, but also on coefficients in K . Since the equation holds as polynomials, this equation holds true also evaluated on K .

Therefore again $\eta'_W \circ F(\varphi) = G(\varphi) \circ \eta'_V$. This proves that η' is a natural transformation. \blacksquare

We are now concerned with the question when a polynomial functor $\mathbb{Z}\text{-free} \rightarrow R\text{-mod}$ can be extended to a polynomial functor $R\text{-mod} \rightarrow R\text{-mod}$ by composing with the 'extending scalars' functor $\mathbb{Z}\text{-free} \xrightarrow{R \otimes_{\mathbb{Z}} -} R\text{-mod}$. In other words we study the question when $(\circ(K \otimes_{\mathbb{Z}} -))^*$ is an equivalence $\mathcal{A}_R \simeq \mathcal{F}_R$. In order to prove this, by Lemma 1.2, one needs to show that $(\circ(K \otimes_{\mathbb{Z}} -))^*$ is dense as well.

We have to deal with mainly two cases: the case of R being a field of characteristic 0 and the case of R being a field of characteristic p . We shall see that fields with characteristic 0 behave more like characteristic ∞ .

Lemma 1.3. *Let R be a field of characteristic 0 and let $F : \mathbb{Z}\text{-free} \rightarrow R\text{-mod}$ be a polynomial functor of degree d . Then, F extends to a polynomial functor $\hat{F} : R\text{-free} \rightarrow R\text{-mod}$ so that $\hat{F} \circ (R \otimes_{\mathbb{Z}} -) = F$. In particular, $(\circ(R \otimes_{\mathbb{Z}} -))^*$ induces an equivalence $\mathcal{A}_R \simeq \mathcal{F}_R$.*

Proof. By Lemma 1.2 we know that $\mathcal{F}_K^d \hookrightarrow \mathcal{A}_K^d$.

We have to show that this embedding is dense. Let F be a degree d polynomial functor in \mathcal{A}_K^d . By Lemma 1.1 we know that for any n and m the functor F induces a degree d polynomial mapping with coefficients in K in the coordinate functions of matrices in $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^m)$. Moreover, F is a functor, that is $F(\alpha\beta) = F(\alpha)F(\beta)$ and $F(id_A) = id_A$ for any free abelian group A and any two composable morphisms of abelian groups α and β .

Let $\alpha : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ and $\beta : \mathbb{Z}^k \rightarrow \mathbb{Z}^n$. The equation $F(\alpha\beta) = F(\alpha)F(\beta)$ translates into an equation between the evaluation of the corresponding polynomials in each degree. Since \mathbb{Z} and K are of characteristic 0, the polynomial equation holds if evaluated on infinitely many values, and so the polynomial equations actually holds *as polynomials*. Friedlander and Suslin remark in [12, remark after Definition 2.1] that this is actually equivalent to saying that F actually is a strict polynomial functor $\hat{G} \in \mathcal{P}_K$ of degree d . Now, for K infinite, $\mathcal{P}_K^{\leq d} \simeq \mathcal{F}_K^d$. ([12]). So, actually \hat{G} , and in turn F is a degree d polynomial functor in \mathcal{F}_K^d . This proves the lemma. \blacksquare

1.3. Functors with values in fields of finite characteristic.

Lemma 1.4. *Let \mathbb{F} be a field of characteristic p and let $F : \mathbb{Z}\text{-free} \rightarrow \mathbb{F}\text{-mod}$ be polynomial functor of degree at most $p-1$ which preserves the initial object, i.e. $F(0) = 0$. Then, for any homomorphism $\alpha : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ one gets $F(p \cdot \alpha) = 0$.*

Proof. Let $M = \mathbb{Z}^n$ and $N = \mathbb{Z}^m$. We write $p \cdot \alpha$ in the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\delta_M} & M \oplus M \oplus \dots \oplus M & \xrightarrow{\sigma_M} & M \\ \alpha \downarrow & & \downarrow \alpha \oplus \alpha \oplus \dots \oplus \alpha & & \downarrow \alpha \\ N & \xrightarrow{\delta_N} & N \oplus N \oplus \dots \oplus N & \xrightarrow{\sigma_N} & N \end{array}$$

where δ is the diagonal mapping and σ is the summation mapping, on M or on N respectively.

Denote by $F^{(i)}$ the i^{th} cross effect of the functor F . Since F is polynomial of degree at most p ,

$$F(A_1 \oplus A_2 \oplus \cdots \oplus A_n) = \bigoplus_{i=1}^p \bigoplus_{j_1 < \cdots < j_i} F^{(i)}(A_{j_1} | \dots | A_{j_i})$$

for p abelian groups A_1, \dots, A_p , and this decomposition is functorial with respect to these groups. Moreover, since $\deg(F) = p$, and since $F(0) = 0$, one gets $F^{(p)} = 0$.

Since $F(p \cdot \alpha) = F(\alpha \circ \sigma_M \circ \delta_M) = F(\alpha) \circ F(\sigma_M \circ \delta_M)$, it is necessary and sufficient to show that $F(\sigma_M \circ \delta_M) = 0$. Now,

$$F(M \oplus M \oplus \cdots \oplus M) = \bigoplus_{i=1}^p \bigoplus_{j_1 < \cdots < j_i} F^{(i)}(M_{j_1} | \dots | M_{j_i})$$

where $M_l = M$ is the copy of M in the l -th position of $\bigoplus_{i=1}^p M$, for all $l \leq p$. The mapping $F(\sigma_M \circ \delta_M) = F(\sigma_M) \circ F(\delta_M)$ factors as a sum $\sum_{i=1}^p F(\sigma_M) \circ \iota_i \circ \pi_i \circ F(\delta_M)$ where ι_i is the embedding of $\bigoplus_{j_1 < \cdots < j_i} F^{(i)}(M_{j_1} | \dots | M_{j_i})$ into $F(M \oplus M \oplus \cdots \oplus M)$ and π_i is the projection of $F(M \oplus M \oplus \cdots \oplus M)$ to this direct factor. But now, for $i < p$, $F(\sigma_M)|_{(\bigoplus_{j_1 < \cdots < j_i} F^{(i)}(M_{j_1} | \dots | M_{j_i}))} \circ \pi_i \circ F(\delta_M)$ is a sum of p identical mappings, which sum up to 0 in characteristic p . Hence, $F(\sigma_M \circ \delta_M) = F(\sigma_M) \circ F^{(p)}(\alpha | \alpha | \dots | \alpha) \circ F(\delta_M) = 0$ using that $F^{(p)} = 0$. \blacksquare

Corollary 1.5. *Let \mathbb{F} be a field of characteristic p and let $F : \mathbb{Z} - \text{free} \rightarrow \mathbb{F} - \text{mod}$ be polynomial functor of degree at most $p - 1$ which preserves the initial object, i.e. $F(0) = 0$. Then, for any two homomorphisms $\alpha : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ and $\beta : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ so that $\alpha - \beta \in p \cdot \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^m)$, one gets $F(\alpha) = F(\beta)$.*

Proof. This is a consequence of the previous lemma and [7, p. 76, formula 8.5] and [7, Theorem 9.3]. Indeed, $F(\sum_{n=0}^p \rho_n) = \sum_{n=0}^p \sum_{i_1 < \cdots < i_n} F^{(n)}(\rho_{i_1} | \dots | \rho_{i_n})$ implies

$$F(\alpha + p \cdot \gamma) = F(\alpha) + \underbrace{F^{(p)}(\gamma | \gamma | \dots | \gamma)}_{p \text{ factors}} + \underbrace{F^{(p+1)}(\alpha | \gamma | \gamma | \dots | \gamma)}_{p \text{ factors}} = F(\alpha) + F(p\gamma) = F(\alpha)$$

by Lemma 1.4. \blacksquare

Lemma 1.6. *Let \mathbb{F} be the prime field of characteristic p and let $F : \mathbb{Z} - \text{free} \rightarrow \mathbb{F} - \text{mod}$ be a polynomial functor of degree less or equal to $p - 1$. Then, F factors through the functor $\mathbb{F} \otimes_{\mathbb{Z}} - : \mathbb{Z} - \text{free} \rightarrow \mathbb{F} - \text{mod}$. Moreover, if $F = F' \circ (\mathbb{F} \otimes_{\mathbb{Z}} -)$, then F is polynomial of degree m if and only if F' is polynomial of degree m . Hence, $(\circ(\mathbb{F} \otimes_{\mathbb{Z}} -))^*$ induces an equivalence $\mathcal{A}_{\mathbb{F}}^{\leq p-1} \simeq \mathcal{F}_{\mathbb{F}}^{\leq p-1}$.*

Proof. Since $(\mathbb{F} \otimes_{\mathbb{Z}} -)^* : \mathcal{F}_{\mathbb{F}} \hookrightarrow \mathcal{A}_{\mathbb{F}}$ is a fully faithful embedding, we need to show that $(\mathbb{F} \otimes_{\mathbb{Z}} -)^*$ is dense. So, given a functor $F : \mathbb{Z} - \text{free} \rightarrow \mathbb{F} - \text{mod}$. One has to show that there is a functor $\hat{F} : \mathbb{F} - \text{mod} \rightarrow \mathbb{F} - \text{mod}$ with $\hat{F} \circ (\mathbb{F} \otimes_{\mathbb{Z}} -) = F$.

For any $V \in \mathbb{F} - \text{mod}$ choose P_V a projective cover as abelian group. Then, P_V is in \mathbb{Z} -free. By the universal property of projective covers one has for any $\alpha \in \text{Hom}_{\mathbb{F}}(V, W)$ a (non-unique) $\hat{\alpha} \in \text{Hom}_{\mathbb{Z}}(P_V, P_W)$ so that

$$\begin{array}{ccc} P_V & \xrightarrow{\hat{\alpha}} & P_W \\ \downarrow & & \downarrow \\ V & \xrightarrow{\alpha} & W \end{array}$$

is commutative. Put

$$\hat{F}(V) := F(P_V) \quad \text{and} \quad \hat{F}(\alpha) := F(\hat{\alpha}).$$

We need to show that this gives a functor $\hat{F} : \mathbb{F} - \text{mod} \rightarrow \mathbb{F} - \text{mod}$. Let $\hat{\alpha}$ and $\hat{\alpha}'$ be two different lifts of $\alpha : V \rightarrow W$, then $\hat{\alpha} - \hat{\alpha}'$ lifts the 0-mapping, and so $\hat{\alpha} - \hat{\alpha}' \in p \cdot \text{Hom}_{\mathbb{Z}}(P_V, P_W)$. Corollary 1.5 implies that $\hat{F}(\hat{\alpha}) = \hat{F}(\hat{\alpha}')$. Using Corollary 1.5 again

one gets that $\widehat{F}(id_V) = id_{\widehat{F}(V)}$ since id_{P_V} is a lift of id_V . Moreover, let $\alpha : U \rightarrow V$ and $\beta : V \rightarrow W$, then choosing lifts $\widehat{\alpha} : P_U \rightarrow P_V$ and $\widehat{\beta} : P_V \rightarrow P_W$, one gets $\widehat{\alpha\beta} - \widehat{\alpha}\widehat{\beta}$ lifts the 0-mapping. So, $\widehat{\alpha\beta} - \widehat{\alpha}\widehat{\beta} \in p \cdot Hom_{\mathbb{Z}}(P_U, P_W)$ and again by Corollary 1.5 one has $F(\widehat{\alpha\beta}) = F(\widehat{\alpha}\widehat{\beta})$. \blacksquare

2. A REVIEW ON POLYNOMIAL FUNCTORS AND FUNCTOR COHOMOLOGY

2.1. Polynomial functors are modules. Let R be a commutative ring. We know that by a result of Baues, Dreckmann, Franjou and Pirashvili [2] a polynomial functor of degree at most n from free abelian groups to R -modules is defined by giving R -modules $F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ for all $m \leq n$ and mappings

$$h_k^m : F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \rightarrow F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$$

for $k \leq m \leq n-1$ and

$$p_k^{m+1} : F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \rightarrow F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$$

for $k \leq m \leq n-1$ satisfying the relations :

$$(*) \quad h_j^m p_i^m = \left\{ \begin{array}{ll} p_i^{m+1} h_{j+1}^{m+1} & \text{for } j < i \\ p_{i+1}^{m+1} h_j^{m+1} & \text{for } j > i \\ \begin{array}{l} 1 + t_i^m + p_i^{m+1} h_{i+1}^{m+1} + p_{i+1}^{m+1} h_i^{m+1} + p_{i+1}^{m+1} t_i^{m+1} h_{i+1}^{m+1} \\ + p_i^{m+1} t_{i+1}^{m+1} h_i^{m+1} + p_{i+1}^{m+1} p_i^{m+2} t_{i+1}^{m+2} h_i^{m+2} h_{i+1}^{m+1} \end{array} & \text{for } j = i \end{array} \right\}$$

Define the algebra Γ_R^n over R by a quiver with n vertices $F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ for any m with $1 \leq m \leq n$ and arrows $h_k^m : F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \rightarrow F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ and $p_k^{m+1} : F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \rightarrow F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ subject to the relations (*). Observe that the relations above do not form a set of admissible relations. The third relation though should be read as the defining equation for the symbols t_i^n , and this way the relations (*) is a set of admissible relations. The result [2] of Baues et alii implies that $\Gamma_R^n - mod$ is equivalent to the category of polynomial functors of degree at most n .

Since the ring homomorphism $\widehat{\mathbb{Z}}_p \rightarrow \mathbb{F}_p$ induces an embedding $\mathbb{F}_p - mod \rightarrow \widehat{\mathbb{Z}}_p - mod$, we get an induced embedding $\Gamma_{\mathbb{F}_p}^n - mod \rightarrow \Gamma_{\widehat{\mathbb{Z}}_p}^n - mod$ which is also induced by the surjective ring homomorphism $\Gamma_{\widehat{\mathbb{Z}}_p}^n \rightarrow \Gamma_{\mathbb{F}_p}^n$.

2.2. Some facts on functor cohomology. We shall give some facts that we will need from Franjou, Friedlander, Scorichenko and Suslin [11]. Basically, these results reduce the computation of extension groups between polynomial functors to questions between extension groups between strict polynomial functors.

Let k be a commutative ring. Let $\mathcal{F}(k)$ be the category of functors $k - mod \rightarrow k - mod$ and let $\mathcal{P}(k)$ the category of strict polynomial functors between k -modules. Let $\mathcal{F}^n = \mathcal{F}^n(k)$ be the category of degree n polynomial functors from $k - mod$ to $k - mod$. If $k = \mathbb{F}_q$ for $q = p^s$, the field with q elements, we write $\mathcal{F}^n(k) = \mathcal{F}^n(q)$. In this case, for any strict polynomial functor P in $\mathcal{P}(k)$ let $P^{(m)}$ be the functor twisted by the Frobenius endomorphism defined by $\mathbb{F}_q \ni x \mapsto x^{p^m} \in \mathbb{F}_q$.

Theorem 1. [11] *Given any two homogeneous strict polynomial functors P and Q between \mathbb{F}_q -vector spaces for $q = p^s$. If the degrees of P and Q are different and strictly smaller than q , then $Ext_{\mathcal{F}(k)}^*(P, Q) = 0$. Moreover, if the degree of P and Q coincide, then*

$$\lim_{\rightarrow m} Ext_{\mathcal{P}(k)}^*(P^{(m)}, Q^{(m)}) \simeq Ext_{\mathcal{F}(k)}^*(P, Q)$$

Frobenius twisting decreases the 'degree of homological triviality' as is shown in a result of H.H. Andersen.

Proposition 2.1. (*H. H. Andersen; see [11, Corollary 1.3]*) *For two homogeneous strict polynomial functors P and Q between \mathbb{F}_q -vector spaces of the same degree, for $q = p^s$ and for $m \in \mathbb{N} \cup \{0\}$ we get*

$$\text{Ext}_{\mathcal{P}(k)}^*(P^{(m)}, Q^{(m)}) \leq \text{Ext}_{\mathcal{P}(k)}^*(P^{(m+1)}, Q^{(m+1)})$$

The first of the two statements in Theorem 1 actually is due to Kuhn:

Lemma 2.2. [17] *Any functor $F \in \mathcal{F}(q)$ decomposes into a direct sum $F = \bigoplus_{i=0}^{q-1} F_i$ where $F_i(V) := \{x \in F(V) \mid F(\lambda \cdot)(x) = \lambda^i \cdot x \forall \lambda \in \mathbb{F}_q\}$. This induces a decomposition of the category of functors between \mathbb{F}_q -vector spaces $\mathcal{F}(q) = \prod_{i=0}^{q-1} \mathcal{F}(q)_i$.*

Finally, a result due to Kuhn will be essential in the sequel.

Theorem 2. (*N. Kuhn [18, 19]*) *The injective envelope $I_{\mathbb{F}_p}$ of the trivial module in the category of analytic functors $\mathcal{F}^\omega(\mathbb{F}_p)$ from finite dimensional \mathbb{F}_p -vector spaces to \mathbb{F}_p -vector spaces is uniserial and the only composition factors in $\mathcal{F}^p(\mathbb{F}_p)$ are the two composition factors of $\text{soc}(\text{soc}(I_{\mathbb{F}_p}))$.*

The next result of Franjou, Lannes and Schwartz proves that the categories $\mathcal{A}_{\mathbb{F}_p}^p$ and $\mathcal{F}_{\mathbb{F}_p}^p$ are different.

Theorem 3. (*Franjou-Lannes-Schwartz [8]; Franjou-Pirashvili [9]*)

$$\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}(id, \mathbb{F}_p \otimes id) \simeq \mathbb{F}_p[e_1, e_2, \dots] / (e_h^p; h \geq 0) \otimes \Lambda(\xi_1)$$

where $\Lambda(\xi_1)$ is the exterior algebra in one variable with generator in degree $2p - 1$ and e_h are generators in degree $2p^h$. Moreover,

$$\text{Ext}_{\mathcal{F}_{\mathbb{F}_p}^p}(id, id) \simeq \mathbb{F}_p[e_0, e_1, \dots] / (e_h^p; h \geq 0).$$

Remark that $\mathbb{F}_p \otimes_{\mathbb{Z}} id = id$ as functors on the category $\mathbb{F}_p - \text{mod}$. As a consequence, $\text{Ext}_{\mathcal{F}_{\mathbb{F}_p}^p}^2(id, id) \neq 0$. Indeed, by [8, 7.3] the following four term sequence is a non zero element:

$$0 \longrightarrow id \longrightarrow S_p \longrightarrow S^p \longrightarrow id \longrightarrow 0;$$

where S_p is the degree p homogeneous part of the coinvariants under the \mathfrak{S}_p action on the tensor algebra, and where S^p is the degree p homogeneous part of the invariants of the tensor algebra.

3. ON RECOLLEMENT DIAGRAMS

We remind the reader to the notion of a recollement diagram. A recollement diagram is given by three categories \mathcal{A}, \mathcal{B} and \mathcal{C} with functors

$$\begin{array}{ccc} \xleftarrow{q} & & \xleftarrow{l} \\ \mathcal{A} & \xrightarrow{i} \mathcal{B} & \xrightarrow{e} \mathcal{C} \\ \xleftarrow{p} & & \xleftarrow{r} \end{array}$$

so that

- (1) (l, e) and (e, r) are adjoint pairs.
- (2) (q, i) and (i, p) are adjoint pairs.
- (3) i is a full embedding and $e(B') = 0 \Leftrightarrow B' \simeq i(A')$ for an $A' \in \mathcal{A}$.
- (4) the adjointness morphisms $e \circ r \longrightarrow id_{\mathcal{C}}$ and $id_{\mathcal{C}} \longrightarrow e \circ l$ are isomorphisms.

We denote a recollement diagram as above by $(\mathcal{A}, \mathcal{B}, \mathcal{C}, (e, l, r), (i, q, p))$.

We shall give a result which is a special case of a recent result of Chrysostomos Psaroudakis and Jorge Vitoria [25]. We shall give our original proof below, since in our special case the proof is much easier than the proof for the general statement from [25].

Proposition 3.1. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, (e, l, r), (i, q, p))$ be a recollement diagram. Suppose $\mathcal{B} \simeq B - \text{mod}$ and $\mathcal{C} = C - \text{mod}$ are module categories and that \mathcal{B} satisfies the Krull-Schmidt theorem on projective modules. Suppose that e is representable. Then, $\mathcal{A} = A - \text{mod}$ again is a module category, there is an idempotent e' in B so that C is Morita equivalent to $e'Be'$, and A is Morita equivalent to $B/Be'B$.*

Proof. Since e has a left and a right adjoint, e is exact. Therefore, $e = \text{Hom}_{\mathcal{B}}(P, -)$ where P is a projective object. Since l and r are left and right adjoints to e , we get that $l = P \otimes_{\text{End}_{\mathcal{B}}(P)} -$, that $r = \text{Hom}_{\text{End}_{\mathcal{B}}(P)}(\text{Hom}_{\mathcal{B}}(P, B), -)$ and that $C \simeq \text{End}_{\mathcal{B}}(P)$. Since \mathcal{B} is a Krull-Schmidt category, then up to Morita equivalence, one can choose $P = Be'$ for an idempotent $e'^2 = e' \in B$ and we get C is Morita equivalent to $e'Be'$.

The third condition in the definition of a recollement diagram implies that \mathcal{A} can be identified with those B -modules M for which $e'M = 0$. Hence,

$$\mathcal{A} \simeq \{M \in B - \text{mod} \mid e'M = 0\} \simeq B/Be'B - \text{mod}.$$

This proves the proposition. \blacksquare

Another important observation is that, by the adjointness properties, l maps projective object in \mathcal{C} to projective objects in \mathcal{B} , and that r maps injective objects in \mathcal{C} to injective objects in \mathcal{B} .

3.1. Analyzing Schwartz' recollement for polynomial functors. Let $q = p^s$ for a prime p . Then, using the notation of Section 2.2, following Kuhn [20, Theorem 1.3] or Schwartz [31, §5.5] we have a recollement diagram

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{n-1}(q) & \rightarrow & \mathcal{F}^n(q) & \rightarrow & \prod_{n(\lambda)=n} \mathbb{F}_q \mathfrak{S}_\lambda - \text{mod} \\ & \leftarrow & & \leftarrow & \end{array}$$

where $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_{s-1}}$ where \mathfrak{S}_k is the symmetric group on k elements and where $n(\lambda) := \lambda_0 + \cdots + \lambda_{s-1}$. Moreover, the functor $\mathcal{F}^n(q) \rightarrow \prod_{n(\lambda)=n} \mathbb{F}_q \mathfrak{S}_\lambda - \text{mod}$ is representable by id^λ and for a partition $\lambda = (\lambda_0 \geq \cdots \geq \lambda_{s-1})$, we set $id^\lambda := \bigoplus_{j=1}^{s-1} id^{\otimes \lambda_j}$.

Remark 3.2. Hence, in case $s = 1$ and $n < p$, the recollement becomes

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{n-1}(p) & \rightarrow & \mathcal{F}^n(p) & \rightarrow & \prod_{\text{partitions of } n} (\mathbb{F}_p - \text{mod}) \\ & \leftarrow & & \leftarrow & \end{array}$$

since $\mathbb{F}_p \mathfrak{S}_n$ is semisimple, since \mathbb{F}_p is a splitting field, and therefore its module category is equivalent to a direct product of copies $\mathbb{F}_p - \text{mod}$.

In case $s = 1$ and $n = p$, the recollement becomes

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{p-1}(p) & \rightarrow & \mathcal{F}^p(p) & \rightarrow & \mathbb{F}_p \mathfrak{S}_p - \text{mod} \\ & \leftarrow & & \leftarrow & \end{array}$$

We have an immediate consequence of the above result.

Lemma 3.3. *For any irreducible polynomial functor F there is a strict polynomial functor \hat{F} such that the forgetful functor, which assigns to every strict polynomial functor its polynomial functor by evaluating the polynomial as mapping, maps \hat{F} to F .*

Proof. This is done by induction on the degree. The simple objects in $\mathcal{F}^d(q)$ are in bijection with the union of the simple objects in $\mathcal{F}^{d-1}(q)$ and the simple objects in $\mathbb{F}_q \mathfrak{S}_d - \text{mod}$. Now, any simple $\mathbb{F}_q \mathfrak{S}_d$ -module is image of a simple module of the Schur algebra $S_{\mathbb{F}_q}(d, d)$ under the Schur functor as is a classical fact. Since the category of degree d strict polynomial functors is equivalent to the category of modules over the Schur algebra

$S(d, d)$, the simple objects in $\mathcal{F}^d(q)$ of degree d are images of a strict polynomial functor. By induction, the simple objects of degree less than d are images of strict polynomial functors. \blacksquare

3.2. Recollement for $\mathcal{A}_{\mathbb{F}_p}$. For the category $\mathcal{A}_{\mathbb{F}_p}$ we get a similar recollement diagram. Indeed,

Proposition 3.4.

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{A}^{n-1}(p) & \rightarrow & \mathcal{A}^n(p) & \xrightarrow{e} & \mathbb{F}_p\mathfrak{S}_n - \text{mod} \\ & \leftarrow & & \leftarrow & \end{array}$$

is a recollement diagram with

$$e := \text{Hom}_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}, -) : \mathcal{A}^n(\mathbb{F}_p) \longrightarrow \mathbb{F}_p\mathfrak{S}_n - \text{mod}.$$

Proof. Following Franjou and Pirashvili [10, Section 3.2] we define for a functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ from an additive category to an abelian category and any integer d a mapping $\rho_{d,F} : F^{(d+1)}(X|\dots|X) \longrightarrow F(X)$ induced by the codiagonal mapping $X^{d+1} \longrightarrow X$, applying F to this codiagonal, and restriction to the $d+1$ cross effect direct summand. Set

$$t_d(F) := \text{coker}(\rho_{d,F}).$$

By Franjou and Pirashvili [10, Section 3.2] this defines a left adjoint functor to the embedding of the category of degree d polynomial functors $F : \mathcal{C} \longrightarrow \mathcal{A}$ into the category of functors $F : \mathcal{C} \longrightarrow \mathcal{A}$. Similarly, we may apply F to the diagonal $X \longrightarrow X^{d+1}$ and compose with the natural projection $F(X^{d+1}) \longrightarrow F^{(d+1)}(X|\dots|X)$ to obtain a natural transformation

$$\theta_{d,F} : F(X) \longrightarrow F^{(d+1)}(X|\dots|X).$$

Then the morphism t^d from functors $\mathcal{C} \longrightarrow \mathcal{A}$ to degree d functors $\mathcal{C} \longrightarrow \mathcal{A}$ defined by

$$t^d F := \ker(\theta_{d,F})$$

is right adjoint to the embedding of the category of degree d functors $\mathcal{C} \longrightarrow \mathcal{A}$ into the category of all functors $\mathcal{C} \longrightarrow \mathcal{A}$.

We see that $t^d F$ is the maximal degree d subfunctor of F , and t_d is the maximal degree d quotient functor of F .

We shall show the following auxiliary lemma needed for the proof of the proposition.

Lemma 3.5. $\mathbb{F}_p \otimes id^{\otimes n}$ is projective and injective in $\mathcal{A}_{\mathbb{F}_p}^n$.

Proof. We shall imitate the arguments in L. Piriou's thesis [23].

Since

$$P_W := (V \mapsto \mathbb{F}_p[\text{Hom}_{\mathbb{Z}}(W, V)])$$

is projective by Yoneda's lemma, using that the duality D on functors $\mathbb{Z}\text{-free} \longrightarrow \mathbb{F}_p\text{-Mod}$ given by

$$(DF)(V) = \text{Hom}_{\mathbb{F}_p}(F(\text{Hom}_{\mathbb{Z}}(V, \mathbb{Z})), \mathbb{F}_p),$$

one sees that for any $W \in \mathbb{Z}\text{-free}$ the functor I_W defined by

$$I_W(V) := \mathbb{F}_p^{\text{Hom}_{\mathbb{Z}}(V, W)}$$

is injective in this category. Now, if $W = \mathbb{Z}^m$, then

$$I_W = \bigotimes_{j=1}^m I_{\mathbb{Z}},$$

and the latter is an injective as well. The injective object $I_{\mathbb{Z}}$ has as direct factor the constant functor induced by $0 \longrightarrow \mathbb{Z} \longrightarrow 0$. Let $\bar{I}_{\mathbb{Z}}$ be the quotient of $I_{\mathbb{Z}}$ by this constant summand.

$\bar{I}_{\mathbb{Z}}$ is again injective, since the constant functors are a direct factor in the category of functors $\mathcal{A}_{\mathbb{F}_p}$. Now, $\mathbb{F}_p \otimes id$ is in the socle of $\bar{I}_{\mathbb{Z}}$ since

$$Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes id, I_{\mathbb{Z}}) = Hom_{\mathbb{F}_p}(Hom_{\mathcal{A}_{\mathbb{F}_p}}(P_{\mathbb{Z}}, \mathbb{F}_p \otimes id), \mathbb{F}_p) = Hom_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z}, \mathbb{F}_p) = \mathbb{F}_p,$$

using that $\mathbb{F}_p \otimes id$ is self-dual.

Since I_W is injective, its biggest degree n polynomial subfunctor $t^n(I_W)$ is injective in $\mathcal{A}_{\mathbb{F}_p}^n$. Moreover,

$$(\dagger) : \quad t^n(F \otimes G) = \sum_{k+l=n} t^k(F) \otimes t^l(G).$$

Piriou's proof in [23, Proposition 1.3.1] for polynomial functors for \mathbb{F}_p -vector spaces carries through literally. Hence, $t^n(I_{\mathbb{Z}^n}) = \bigotimes_{j=1}^n (t^1(I_{\mathbb{Z}}))$. Since $\mathbb{F}_p \otimes id$ is a direct factor of $t^1(I_{\mathbb{Z}})$, the functor $\mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}$ is a direct factor of $t^n(I_{\mathbb{Z}^n})$ and is therefore injective in $\mathcal{A}_{\mathbb{F}_p}^n$. Now, $D(\mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}) = \mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}$ and hence $\mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}$ is projective as well. \blacksquare

We need to prove that

$$\mathcal{A}_{\mathbb{F}_p}^{n-1} = \{F \in \mathcal{A}_{\mathbb{F}_p}^n \mid Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}, F) = 0\}.$$

We show first the inclusion " \subseteq ":

Let G be in $\mathcal{A}_{\mathbb{F}_p}^{n-1}$. Then,

$$Hom_{\mathcal{A}_{\mathbb{F}_p}^{n-1}}(t_{n-1}((\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes n}), G) \simeq Hom_{\mathcal{A}_{\mathbb{F}_p}}((\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes n}, G).$$

Now, for any functor F in $\mathcal{A}_{\mathbb{F}_p}$, one has $t_n((\mathbb{F}_p \otimes_{\mathbb{Z}} -) \otimes_{\mathbb{F}_p} F) = (\mathbb{F}_p \otimes_{\mathbb{Z}} -) \otimes_{\mathbb{F}_p} t_{n-1}(F)$, as is easily verified by definition. Then, one has $t_{n-1}((\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes n}) = 0$; indeed, $t_0(\mathbb{F}_p \otimes_{\mathbb{Z}} id) = 0$ and therefore in the sum decomposition obtained as the dual of (\dagger) one factor in each summand is t_0 , hence 0. Therefore,

$$\mathcal{A}_{\mathbb{F}_p}^{n-1} \subseteq \{F \in \mathcal{A}_{\mathbb{F}_p}^n \mid Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}, F) = 0\}.$$

We have to show the inclusion " \supseteq ": Let $G \in \mathcal{A}_{\mathbb{F}_p}^n \setminus \mathcal{A}_{\mathbb{F}_p}^{n-1}$. We have to show that $Hom_{\mathcal{A}_{\mathbb{F}_p}^n}((\mathbb{F}_p \otimes_{\mathbb{Z}} -)^{\otimes n}, G) \neq 0$. But

$$Hom_{\mathcal{A}_{\mathbb{F}_p}^n}((\mathbb{F}_p \otimes_{\mathbb{Z}} -)^{\otimes n}, G) = Hom_{\mathcal{A}_{\mathbb{F}_p}^n}(t_n(P_{\mathbb{Z}^n}), G) = Hom_{\mathcal{A}_{\mathbb{F}_p}}(P_{\mathbb{Z}^n}, G) = G(\mathbb{Z}^n)$$

by the adjointness property and Yoneda's lemma (completely analogous to [23, Proposition 1.1.4]). Now, for functors G which are of degree n and not of degree $n-1$ we get that the evaluation on \mathbb{Z}^n is not zero.

Since $DP_{\mathbb{Z}^n} = I_{\mathbb{Z}^n}$ for the projective object $P_{\mathbb{Z}^n} = \mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}^n, -)]$, we get that

$$Dt_n P_{\mathbb{Z}^n} = t^n DP_{\mathbb{Z}^n} = t^n I_{\mathbb{Z}^n} = \mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}.$$

Hence,

$$t_n P_{\mathbb{Z}^n} = D(\mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}) = \mathbb{F}_p \otimes_{\mathbb{Z}} id^{\otimes n}.$$

Since $\mathbb{F}_p \otimes id^{\otimes n}$ is projective there is a right adjoint and a left adjoint to

$$Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes id^{\otimes n}, -),$$

namely the functor $M \mapsto (\mathbb{F}_p \otimes id^{\otimes n} \otimes M)^{\mathfrak{S}_n}$ is the right adjoint, and the functor $M \mapsto (\mathbb{F}_p \otimes id^{\otimes n} \otimes M)_{\mathfrak{S}_n}$ is the left adjoint.

Moreover, the unit and the counit of the adjunctions induce the identity on $\mathbb{F}_p \mathfrak{S}_n$ -mod. This again is done literally as in Piriou [23, Proposition 2.2.2].

This shows Proposition 3.4. \blacksquare

As a consequence we show the following lemma.

Lemma 3.6. *Any simple object in $\mathcal{A}_{\mathbb{F}_p}^n$ is in the image of $\circ(\mathbb{F}_p \otimes -) : \mathcal{F}_{\mathbb{F}_p}^n \hookrightarrow \mathcal{A}_{\mathbb{F}_p}^n$ and any simple object of $\mathcal{A}_{\mathbb{F}_p}^n$ gives a simple object in $\mathcal{F}_{\mathbb{F}_p}^n$ this way.*

Proof. We shall use induction on n . There is a morphism of recollement diagrams as follows, where the vertical functors are fully faithful embeddings of categories by Lemma 1.2.

$$\begin{array}{ccccc}
& \leftarrow & & \leftarrow & \\
\mathcal{F}_{\mathbb{F}_p}^{n-1} & \rightarrow & \mathcal{F}_{\mathbb{F}_p}^n & \xrightarrow{e_{\mathcal{F}^n}} & \mathbb{F}_p \mathfrak{S}_n - \text{mod} \\
& \leftarrow & & \leftarrow & \\
\downarrow i_{n-1} & & \downarrow i_n & & \parallel \\
\mathcal{A}_{\mathbb{F}_p}^{n-1} & \rightarrow & \mathcal{A}_{\mathbb{F}_p}^n & \xrightarrow{e_{\mathcal{A}^n}} & \mathbb{F}_p \mathfrak{S}_n - \text{mod} \\
& \leftarrow & & \leftarrow &
\end{array}$$

Therefore the number of simple objects in $\mathcal{A}_{\mathbb{F}_p}^n$ and in $\mathcal{F}_{\mathbb{F}_p}^n$ coincides.

The statement is clear for $n \leq p-1$ by Lemma 1.6. Let $n \geq p$ and let S be a simple object in $\mathcal{F}_{\mathbb{F}_p}^n$. We may suppose, using the induction hypothesis that $e_{\mathcal{F}^n}(S) \neq 0$. Then, suppose X is a simple and proper subobject of $i_n S$. Since $e_{\mathcal{A}^n}$ is exact, $e_{\mathcal{A}^n}(X)$ is a subobject of $e_{\mathcal{A}^n}(i_n S) = e_{\mathcal{F}^n}(S)$. Since $e_{\mathcal{F}^n}$ is exact, $e_{\mathcal{F}^n}(S)$ is simple. So, $e_{\mathcal{A}^n}(X)$ is either 0 or isomorphic to $e_{\mathcal{F}^n}(S)$. Since X is a proper subobject of $i_n S$ we see that $e_{\mathcal{A}^n}(X) = 0$, and hence $X \in \mathcal{A}^{n-1}$. Since the simple objects of \mathcal{F}^{n-1} and of \mathcal{A}^{n-1} coincide by the induction hypothesis, X is a proper non zero subobject of S . This gives the contradiction. \blacksquare

We have seen in Lemma 1.6 and Remark 3.2 that

$$\mathcal{F}_{\mathbb{F}_p}^{p-1} \simeq \mathcal{A}_{\mathbb{F}_p}^{p-1} \simeq \prod_{n < p} \prod_{\lambda \vdash n} \mathbb{F}_p - \text{mod}$$

4. THE STRUCTURE OF POLYNOMIAL FUNCTORS MODULO p

The situation of polynomial functors of degree p is different from those of degree $n < p$. We are going to describe in this section their structure completely. From now on we assume that $p \geq 5$ since the representation theory of $\mathbb{F}_3 \mathfrak{S}_2$ and of $\mathbb{F}_3 \mathfrak{S}_3$ is slightly different from the case $p \geq 5$.

Let us recall the relation between the Schur algebra $S_{\mathbb{F}_p}(p, p)$ and the group algebra $\mathbb{F}_p \mathfrak{S}_p$. For general informations on Schur algebras and Brauer tree algebras see [13] and [16]. The algebra $S_{\mathbb{F}_p}(p, p)$ admits p simple modules S_1, \dots, S_{p-1}, S_p whereas the group algebra $\mathbb{F}_p \mathfrak{S}_p$ admits $p-1$ simple modules S'_1, \dots, S'_{p-1} . The projective indecomposable $S_{\mathbb{F}_p}(p, p)$ -modules have composition series

$$\begin{array}{cccccccc}
S_1 & S_2 & S_3 & & S_{p-2} & & S_{p-1} & S_p \\
S_2, S_1 & S_3, S_2 & S_4, \dots, S_{p-3} & & S_{p-1}, S_{p-2} & & S_p, S_{p-1} & \\
S_1 & S_2 & S_3 & & S_{p-2} & & S_{p-1} & S_{p-1}
\end{array}$$

whereas the projective indecomposable $\mathbb{F}_p \mathfrak{S}_p$ -modules have composition series

$$\begin{array}{cccccccc}
S'_1 & S'_2 & S'_3 & & S'_{p-2} & & S'_{p-1} & \\
S'_2, S'_1 & S'_3, S'_2 & S'_4, \dots, S'_{p-3} & & S'_{p-1}, S'_{p-2} & & S'_{p-2} & \\
S'_1 & S'_2 & S'_3 & & S'_{p-2} & & S'_{p-1} & S'_{p-1}
\end{array}$$

Lemma 4.1. *Let L be the simple polynomial functor in $\mathcal{F}_{\mathbb{F}_p}^p$ so that L is of degree p and so that L corresponds to the trivial representation of $\mathbb{F}_p \mathfrak{S}_p$. Then, $\text{Ext}_{\mathcal{F}_{\mathbb{F}_p}}^1(\text{id}, L) \neq 0 \neq \text{Ext}_{\mathcal{F}_{\mathbb{F}_p}}^1(L, \text{id})$. Moreover, if $\text{Ext}_{\mathcal{F}_{\mathbb{F}_p}}^1(\text{id}, S) \neq 0$ or $0 \neq \text{Ext}_{\mathcal{F}_{\mathbb{F}_p}}^1(S, \text{id})$ for a simple degree p -functor S , then $L \simeq S$.*

Proof. Let S_1 be the simple polynomial functor of degree 1. The identity functor id is trivially of degree 1 and simple, which implies $S_1 = id$. Then, $Ext_{\mathcal{F}_p}^1(L, id)$ is not necessarily zero for L being an irreducible polynomial functor of degree p . Now, since we are working over \mathbb{F}_p , we get that $id^{(1)} \simeq id$ as polynomial functor, but not as strict polynomial functor. As strict polynomial functor, $I^{(1)}$ is of degree p .

Theorem 2 implies that there is only one simple functor L of degree p with $Ext_{\mathcal{F}_p}^1(L, id) \neq 0$. Proposition 2.1 in connection with Theorem 1, imply that L is the simple functor corresponding to the trivial $\mathbb{F}_p\mathfrak{S}_p$ -module, since this is the module which has an extension with the unique simple module of the Schur algebra $S_{\mathbb{F}_p}(p, p)$ which is not a simple $\mathbb{F}_p\mathfrak{S}_p$ -module (cf the discussion preceding the statement of the lemma).

This implies that

$$Ext_{\mathcal{P}(p)}^1(L, id^{(1)}) \simeq Ext_{\mathcal{F}_{\mathbb{F}_p}}^1(L, id^{(1)}) \simeq Ext_{\mathcal{F}_{\mathbb{F}_p}}^1(L, id).$$

Since the category of degree p strict polynomial functors is equivalent to the category of modules over the Schur algebra $S_{\mathbb{F}_p}(p, p)$, one sees that

$$Ext_{\mathcal{P}_{\mathbb{F}_p}}^1(L, id^{(1)}) \simeq Ext_{S_{\mathbb{F}_p}(p, p)}^1(V, I_0)$$

for I_0 being the simple $S_{\mathbb{F}_p}(p, p)$ -module corresponding to the p singular partition of p and V being the simple $S_{\mathbb{F}_p}(p, p)$ -module corresponding to L . Finally, it is a classical fact (cf e.g. [15]) that $Ext_{S_{\mathbb{F}_p}(p, p)}^1(I_0, V) \neq 0$ or $Ext_{S_{\mathbb{F}_p}(p, p)}^1(V, I_0) \neq 0$ if and only if V corresponds to the trivial module of the symmetric group, and in this case, the dimension of $Ext_{S_{\mathbb{F}_p}(p, p)}^1(I_0, V)$ and of $Ext_{S_{\mathbb{F}_p}(p, p)}^1(V, I_0)$ is 1. This proves the statement. \blacksquare

We get as a corollary the following statement.

Corollary 4.2. $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(\mathbb{F}_p \otimes_{\mathbb{Z}} id, L) \neq 0 \neq Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(L, \mathbb{F}_p \otimes_{\mathbb{Z}} id)$.

Proof. We know that $Ext_{\mathcal{F}_{\mathbb{F}_p}}^1(id, L) \neq 0$. So, there is a non split exact sequence

$$0 \longrightarrow L \longrightarrow X \longrightarrow id \longrightarrow 0$$

for some functor X in $\mathcal{F}_{\mathbb{F}_p}^p$. Since $\mathcal{F}_{\mathbb{F}_p}^p \hookrightarrow \mathcal{A}_{\mathbb{F}_p}^p$ by Lemma 1.2. This induces an exact sequence

$$0 \longrightarrow L(\mathbb{F}_p \otimes_{\mathbb{Z}} id) \longrightarrow X(\mathbb{F}_p \otimes_{\mathbb{Z}} id) \longrightarrow \mathbb{F}_p \otimes_{\mathbb{Z}} id \longrightarrow 0$$

in $\mathcal{A}_{\mathbb{F}_p}^p$ where $L(\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is simple by Lemma 3.6. This sequence is non split since the functor pre-composing with $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ is a fully faithful embedding. Hence $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(\mathbb{F}_p \otimes_{\mathbb{Z}} id, L) \neq 0$. Therefore, L is a direct factor of the top of the radical of the projective cover of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$. Similarly, $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(L, \mathbb{F}_p \otimes_{\mathbb{Z}} id) \neq 0$. \blacksquare

Actually, the argument of Lemma 4.1 gives another slightly different statement.

Lemma 4.3. *Let X and Y be two simple functors of degree at most p . Then,*

$$Ext_{\mathcal{F}_{\mathbb{F}_p}}^1(X, Y) \neq 0 \Rightarrow deg(X) - deg(Y) \in \{0, p - 1\}.$$

Proof. By Lemma 3.6 we know that the simple functors X and Y can be considered to lie in $\mathcal{F}_{\mathbb{F}_p}^p$. Proposition 2.1 in connection with Theorem 1 imply this result. \blacksquare

Remark 4.4. At the present stage it might happen that $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(X, Y) \neq 0$ even though $Ext_{\mathcal{F}_{\mathbb{F}_p}^p}^1(X, Y) = 0$.

Denoting by $\rho_p(i)$ the number of p -regular partitions of i , the algebra $\Gamma_{\mathbb{F}_p}^p$ is Morita equivalent to a direct product of $\left(\sum_{i=1}^{p-1} \rho_p(i)\right) - 1$ copies of \mathbb{F}_p and of an indecomposable ring $\Gamma_{\mathbb{F}_p, 0}^p$. By the recollement diagram preceding Lemma 3.3 this ring $\Gamma_{\mathbb{F}_p, 0}^p$ has a projective

module $P = \Gamma_{\mathbb{F}_p,0}^p \cdot e$ so that the endomorphism ring of P is Morita equivalent to the Brauer tree algebra corresponding to $\mathbb{F}_p \mathfrak{S}_p$. There is a projective indecomposable $\Gamma_{\mathbb{F}_p,0}^p$ -module P_0 so that $P_0 \oplus P$ is a progenerator of $\Gamma_{\mathbb{F}_p,0}^p$ and the endomorphism ring of $P_0 \oplus P$ is basic and Morita equivalent to $\Gamma_{\mathbb{F}_p,0}^p$. Moreover, $\Gamma_{\mathbb{F}_p,0}^p / (\Gamma_{\mathbb{F}_p,0}^p \cdot e \cdot \Gamma_{\mathbb{F}_p,0}^p)$ is Morita equivalent to \mathbb{F}_p .

The next remark constructs appropriate projective objects.

Let R and S be commutative rings. The functor $R[-]/(I^{n+1})$ which assigns to an S -module V the quotient of the semi-group ring $R[V]$ on V by the $n+1$ -st power of the augmentation ideal. This functor $S\text{-free} \rightarrow R\text{-mod}$ is polynomial of degree n . Define $proj_n^m := \mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}^m, -)]/(I^{n+1})$ and $proj_{\infty}^m := \mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}^m, -)]$.

Lemma 4.5. *The functor $proj_n^m$ in $\mathcal{A}_{\mathbb{F}_p}^n$ is projective and contains a projective cover of the reduction modulo p functor $\mathbb{F}_p \otimes_{\mathbb{Z}} id$.*

Proof. For any degree n polynomial functor F one gets $Hom_{\mathcal{A}_{\mathbb{F}_p}}(proj_n^m, F) \simeq F(\mathbb{Z}^m)$. This is true if one does not factorizes the power of the augmentation ideal, and since all functors are of degree at most n , each natural transformation from $proj_{\infty}^m$ to F is zero on I^{n+1} (see [8, Section 1]).

So, $Hom_{\mathcal{A}}(proj_n^m, -)$ is exact, as evaluation on exact sequences of functors is exact. Hence, $proj_n^m$ is a projective object in $\mathcal{A}_{\mathbb{F}_p}^n$ and since $Hom_{\mathcal{A}}(proj_n^m, id) = id(\mathbb{Z}^m) = \mathbb{Z}^m \neq 0$, the projective cover of the reduction modulo p functor is a direct summand in $proj_n^m$. ■

Remark 4.6. The situation is different for $\mathcal{F}_{\mathbb{F}_p}$. Indeed, the functor $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p^m, -)]/I^{n+1}$ is projective in $\mathcal{F}_{\mathbb{F}_p}^n$ again since $Hom_{\mathcal{F}_{\mathbb{F}_p}}(\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p^m, -)]/I^{n+1}, F) \simeq F(\mathbb{F}_p^m)$. But, automatically $I^p = 0$ for $m = 1$ and the evaluation at \mathbb{F}_p in this case, so that the endomorphism ring of $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p, -)]/I^{n+1}$ is an ℓ -dimensional vector space where $\ell = \min(p, n)$.

Recall the embedding $\mathcal{F}_{\mathbb{F}_p}^n \rightarrow \mathcal{A}_{\mathbb{F}_p}^n$ given by pre-composing with $\mathbb{F} \otimes_{\mathbb{Z}} id$. The image of $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p, -)]/I^{n+1}$ under this embedding is $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p \otimes_{\mathbb{Z}} id)]/I^{n+1}$ which is different from $\mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}, -)]/I^{n+1}$. As we will see, the projective indecomposable cover of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ in $\mathcal{A}_{\mathbb{F}_p}^p$ is a direct factor of the functor $\mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}, -)]/I^{p+1}$ and this is the only indecomposable functor which is not in the image of the embedding $\mathcal{F}_{\mathbb{F}_p}^p \rightarrow \mathcal{A}_{\mathbb{F}_p}^p$.

Lemma 4.7. *The projective functor $proj_n^1$ has one composition factor of degree d for each $0 \leq d \leq n$ for all $n \leq p-1$. In particular $proj_n^1$ contains a simple constant functor as a direct summand.*

Proof. First, $proj_n^1(0) = \mathbb{F}_p$, and so the simple functor of degree 0 is a direct factor of $proj_n^1$.

Furthermore, $proj_n^1 \twoheadrightarrow proj_{n-1}^1$ for trivial reasons. Moreover, $proj_n^1$ is of degree n and not of degree $n-1$. We compute that $End_{\mathcal{A}_{\mathbb{F}_p}^n}(proj_n^1) = proj_n^1(\mathbb{Z}) = \mathbb{F}_p[\mathbb{Z}]/I^{n+1}$ is an $n+1$ -dimensional vector space. Moreover, by Lemma 1.6 and Remark 3.2 the projective module $proj_{p-1}^1$ is semisimple since $\mathcal{A}_{\mathbb{F}_p}^{p-1}$ is a semisimple category. Observe that $proj_n^1$ has exactly one composition factor more than $proj_{n-1}^1$. This composition factor is of degree n since

$$Hom_{\mathcal{A}_{\mathbb{F}_p}^n}(proj_n^1, \mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}) = \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{F}_p$$

and since $\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}$ is the projective object corresponding to degree n polynomial functors in the recollement diagram. ■

Recall that L denotes the simple functor in $\mathcal{A}_{\mathbb{F}_p}^p$ mapping to the trivial $\mathbb{F}_p \mathfrak{S}_p$ -module in the recollement diagram.

Proposition 4.8. *Suppose $p \geq 5$. The projective cover $P_{\mathbb{F}_p \otimes_{\mathbb{Z}} id}$ of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ in $\mathcal{A}_{\mathbb{F}_p}^p$ is uniserial with top and socle being $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ and with $\text{rad}(P_{\mathbb{F}_p \otimes_{\mathbb{Z}} id})/\text{soc}(P_{\mathbb{F}_p \otimes_{\mathbb{Z}} id}) \simeq L$. Moreover,*

$$\text{proj}_p^1 \simeq S_0 \oplus P_{\mathbb{F}_p \otimes_{\mathbb{Z}} id} \oplus S_2 \oplus \cdots \oplus S_{p-1}$$

for simple functors S_i of degree i .

Proof. We shall divide the proof into various claims.

Claim 4.9. *No direct summand of the top of proj_p^1 is of degree p .*

Proof: We have $\text{Hom}_{\mathcal{A}_{\mathbb{F}_p}^p}(\text{proj}_p^1, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}) = \mathbb{F}_p$. On the other hand, we know from the recollement diagram that $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$ is the projective cover of the simple modules coming from $\mathbb{F}_p \mathfrak{S}_p$, that is those of degree p . Now, each projective indecomposable module of $\mathbb{F}_p \mathfrak{S}_p$ has the property that the top of this module is isomorphic to the socle of this module and that the top and the socle of this projective indecomposable module are different. Hence, suppose a simple polynomial functor of degree p would be in the top of proj_p^1 , then let Q be its projective cover in $\mathcal{A}_{\mathbb{F}_p}^p$. Further, Q is a direct summand of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$. Since the top of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$ is isomorphic to the socle of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$, the above homomorphism space would be at least 2-dimensional, corresponding to the mapping of proj_p^1 on the top and on the socle of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$. ■

Claim 4.10. *Let A and B be two polynomial functors of degree at most n and let*

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

be an exact sequence of functors. Then, the degree of C is at most n as well.

Proof: We get a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{(2)}(U|V) & \longrightarrow & C^{(2)}(U|V) & \longrightarrow & B^{(2)}(U|V) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A(U \oplus V) & \longrightarrow & C(U \oplus V) & \longrightarrow & B(U \oplus V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A(U) \oplus A(V) & \longrightarrow & C(U) \oplus C(V) & \longrightarrow & B(U) \oplus B(V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

and the snake lemma implies that $0 \longrightarrow A^{(2)}(U|V) \longrightarrow C^{(2)}(U|V) \longrightarrow B^{(2)}(U|V) \longrightarrow 0$ is exact. Induction on the degree gives the result. ■

Corollary 4.11. *If $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^1(S, T) \neq 0$ for two simple functors S and T , then $\text{deg}(S) - \text{deg}(T) \in \{0, p-1\}$ and if $\text{deg}(S) = \text{deg}(T)$, then $\text{deg}(S) = p$.*

Proof. We know by Lemma 1.6 that $\mathcal{A}_{\mathbb{F}_p}^{p-1}$ is semisimple. Moreover, the category of constant functors is a direct factor in the category of polynomial functors. Using Claim 4.10 this shows the statement. ■

We denote by P_V the projective cover of the functor V in $\mathcal{A}_{\mathbb{F}_p}^p$.

Claim 4.12. *$\text{proj}_p^1 \simeq S_0 \oplus S_2 \oplus S_3 \oplus \cdots \oplus S_{p-1} \oplus M$ for simple projective functors S_i of degree i and the projective cover M of the functor $\mathbb{F}_p \otimes_{\mathbb{Z}} id$.*

Proof. $\text{End}_{\mathcal{A}_{\mathbb{F}_p}^p}(\text{proj}_p^1) = \text{proj}_p^1(\mathbb{Z}) = \mathbb{F}_p[\mathbb{Z}]/I^{p+1}$ is a $p+1$ -dimensional \mathbb{F}_p -vector space. We know already that proj_{p-1}^1 is a quotient of proj_p^1 and that this is a semisimple functor with $p-1$ direct factors. So, every direct summand of the semisimple functor proj_{p-1}^1 is a direct factor of the head of proj_p^1 . Denote by $S_0, S_1, S_2, \dots, S_{p-1}$ the simple direct factors of

$proj_{p-1}^1$ and let S_i be of degree i . Then, since the degree 0 functors split off in any case, $P_{S_0} = S_0$. Moreover, $S_0 \oplus P_{S_0} \oplus P_{S_1} \oplus \cdots \oplus P_{S_{p-1}}$ is a direct factor of $proj_p^1$.

We need to study the functors P_{S_i} . If all the composition factors of P_{S_i} for an $i \leq p-1$ are of degree $p-1$ at most, then by Claim 4.10 we get that P_{S_i} is of degree at most $p-1$ as well. Since the category $\mathcal{A}_{\mathbb{F}_p}^{p-1}$ is semisimple, we get $P_{S_i} = S_i$.

Suppose that a degree p simple polynomial functor S is a composition factor of $P_{S_{i_0}}$. Then, since $\mathcal{A}_{\mathbb{F}_p}^{p-1}$ is semisimple, again by Claim 4.10, $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(S_{i_0}, S) \neq 0$. Now, simple functors are self-dual under the duality $(DF)(V) := F(V^*)^*$ (cf [18] for functors in $\mathcal{F}_{\mathbb{F}_p}$ and by Lemma 3.6 for simple functors in $\mathcal{A}_{\mathbb{F}_p}$). So, $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(S, S_{i_0}) \neq 0$. Since $Hom_{\mathcal{A}_{\mathbb{F}_p}^p}(proj_p^1, \mathbb{F}_p \otimes_{\mathbb{Z}} -)$ is a one-dimensional \mathbb{F}_p -vector space, this happens for precisely one $i_0 \in \{1, 2, \dots, p-1\}$. We already know by Claim 4.2 that $i_0 = 1$ and S is the simple module corresponding to the trivial representation of $\mathbb{F}_p \mathfrak{S}_p$. This proves the claim. \blacksquare

Claim 4.13. *For the projective cover P_L of L we get $rad(P_L)/rad^2(P_L) \simeq (\mathbb{F}_p \otimes id) \oplus L_2$, where L_2 is simple of degree p , $L_2 \not\cong L$ and $rad^2(P_L) \simeq L$ as well as $rad^3(P_L) = 0$.*

Remark 4.14. We do not claim here that $soc(P_L)$ is simple. However, the radical layer structure of P_L can be described by

$$\begin{array}{ccc} & L & \\ (\mathbb{F}_p \otimes id) & & L_2 \\ & L & \end{array}$$

and where it is not clear if $soc(P_L)$ is simple and isomorphic to L or if the socle is isomorphic to $L \oplus L_2$ or to $L \oplus \mathbb{F}_p \otimes id$. But $Hom_{\mathcal{A}_{\mathbb{F}_p}^p}(\mathbb{F}_p \otimes id^{\otimes p}, P_L)$ is the projective cover of the trivial $\mathbb{F}_p \mathfrak{S}_p$ -module. This projective cover is uniserial with composition series

$$\begin{array}{c} L \\ L_2 \cdot \\ L \end{array}$$

Since L_2 is simple of degree p , its image in $\mathbb{F}_p \mathfrak{S}_p$ is given by the known module structure of $\mathbb{F}_p \mathfrak{S}_p$. In particular, $L \not\cong L_2$. This shows that the only uniserial module of length 3 which is a quotient of P_L , if there is any, can have composition series

$$\begin{array}{c} L \\ L_2 \cdot \\ L \end{array}$$

In particular, $soc(P_L) \not\cong L \oplus L_2$.

Proof of Claim 4.13. By Claim 4.2 we know that $(\mathbb{F}_p \otimes id)$ is composition factor of $top(rad(P_L))$ and since $Hom_{\mathcal{A}_{\mathbb{F}_p}^p}(proj_p^1, \mathbb{F}_p \otimes_{\mathbb{Z}} -)$ is a one-dimensional \mathbb{F}_p -vector space we know that it has multiplicity 1. Since the image of P_L in $\mathbb{F}_p \mathfrak{S}_p - mod$ is uniserial with top and socle L and simple $rad(P_L)/soc(P_L) \simeq L_2$, we have the above structure. \blacksquare

Claim 4.15. *The projective cover $M = P_{\mathbb{F}_p \otimes id}$ of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ has $rad(M)/rad^2(M) \simeq L$.*

Proof. By Claim 4.2 we know that L is a direct factor of $top(rad(M))$. Since $\mathcal{A}_{\mathbb{F}_p}^{p-1}$ is semisimple, using Claim 4.10 we see that no simple functor of degree $p-1$ at most can be a direct factor of $top(rad(M))$. Suppose $top(rad(M))$ has a second simple direct factor T of degree p . Then, $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(\mathbb{F}_p \otimes_{\mathbb{Z}} id, T) \neq 0$. Simple functors are self-dual (cf as above [18] for functors in $\mathcal{F}_{\mathbb{F}_p}$ and by Lemma 3.6 for simple functors in $\mathcal{A}_{\mathbb{F}_p}$). So, $Ext_{\mathcal{A}_{\mathbb{F}_p}^p}^1(T, \mathbb{F}_p \otimes_{\mathbb{Z}} id) \neq 0$. But, we have seen in Corollary 4.2 that there is one simple functor of degree p with a non trivial extension group with $\mathbb{F}_p \otimes_{\mathbb{Z}} id$, namely L . Moreover, since $proj_p^1$ contains the projective cover of the simple degree 1-functor as a direct factor (see Claim 4.12), and since

by Corollary 4.11 this is the only degree where non trivial first extension groups can occur, we see that $T \simeq L$.

Suppose $L \oplus L | \text{rad}(M)$. Then, $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^1(\mathbb{F}_p \otimes_{\mathbb{Z}} id, L)$ is two-dimensional at least, and again by the self-duality of the simple functors, $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^1(L, \mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is at least two-dimensional. Therefore, $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ occurs twice in $\text{top}(\text{rad}(P_L))$. Since M is a direct factor of proj_p^1 , the space $\text{Hom}_{\mathcal{A}_{\mathbb{F}_p}^p}(\text{proj}_p^1, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p})$ would be two-dimensional at least. This contradiction shows that $\text{rad}(M)$ has simple top L . Hence, $\text{top}(\text{rad}(M)) \simeq L$. \blacksquare

Remark 4.16. The radical layer structure of M is therefore given by

$$\begin{array}{c} (\mathbb{F}_p \otimes id) \\ L \\ \text{rad}^2(M) \end{array}$$

and by Claim 4.13 we get $\text{rad}^3(P_L) = 0$, and therefore we obtain $\text{rad}^4(M) = 0$.

Claim 4.17. For the projective cover M of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ we get that $\text{top}(\text{rad}^2(M)) \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} id$.

Proof. Degree p functors can only have extensions with degree p -functors or degree 1-functors by Corollary 4.11. Moreover, the structure of P_L implies that we get that $\text{top}(\text{rad}^2(M))$ is a direct summand of $\text{top}(\text{rad}(P_L))$, whence is isomorphic to either 0, or to L_2 (which is defined in Claim 4.13), or to $\mathbb{F}_p \otimes_{\mathbb{Z}} id$, or to $L_2 \oplus (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$.

Suppose $L_2 \oplus (\mathbb{F}_p \otimes_{\mathbb{Z}} id) \simeq \text{top}(\text{rad}^2(M))$. We shall use the fact that by Claim 4.13 we know the structure of P_L .

We get two possibilities for the projective resolution of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$. Either

$$P_L \hookrightarrow M \twoheadrightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$$

is exact, or

$$L \hookrightarrow P_L \longrightarrow M \twoheadrightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$$

is exact.

In the second case, $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^2((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L) \neq 0$. By the self-duality of the simple functors, $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^2(L, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) \neq 0$. Our information is sufficient for being able to write down the first terms of the projective resolution of L ,

$$0 \longleftarrow L \longleftarrow P_L \longleftarrow M \oplus P_{L_2} \longleftarrow P_L \oplus P_{L_3} \longleftarrow \dots$$

for some projective P_{L_3} , for some simple object L_3 of degree p , given by the known projective resolution of the trivial $\mathbb{F}_p \mathfrak{S}_p$ -module. Since $p \neq 2$, we get $L_3 \not\cong L$. In an case $\text{Hom}(P_L \oplus P_{L_3}, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) = 0$, and therefore $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^2((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L) = 0$. This contradiction excludes the case

$$L \hookrightarrow P_L \longrightarrow M \twoheadrightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$$

is exact.

If $P_L \hookrightarrow M \twoheadrightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is exact, the projective dimension of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is 1. But, we know by Theorem 3 and the example following it, that $\text{Ext}_{\mathcal{F}_{\mathbb{F}_p}^p}^2(id, id) \neq 0$. By consequence, also $\text{Ext}_{\mathcal{A}_{\mathbb{F}_p}^p}^2((\mathbb{F}_p \otimes_{\mathbb{Z}} id), (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) \neq 0$ and therefore the projective dimension of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is at least 2.

These two observations exclude $L_2 \oplus (\mathbb{F}_p \otimes_{\mathbb{Z}} id) \simeq \text{top}(\text{rad}^2(M))$.

Suppose $L_2 \simeq \text{top}(\text{rad}^2(M))$. Then, using the structure of P_L , either $\text{rad}^3(M) = 0$ or $\text{rad}^3(M) = L$, and then $\text{rad}^4(M) = 0$.

If $\text{rad}^3(M) = 0$, then we get a non split exact sequence

$$0 \longleftarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id) \longleftarrow M \longleftarrow P_L \longleftarrow M \longleftarrow L_2 \longleftarrow 0$$

and therefore $Ext_{\mathcal{F}_{\mathbb{F}_p}}^3(L_2, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) \neq 0$. Dualizing, $Ext_{\mathcal{F}_{\mathbb{F}_p}}^3((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L_2) \neq 0$. Our knowledge of the various projective covers of simples is sufficient to write down the first terms of the projective resolution of L_2 . We get

$$0 \longleftarrow L_2 \longleftarrow P_{L_2} \longleftarrow P_L \oplus P_{L_3} \longleftarrow M \oplus P_{L_2} \oplus P_{L_4} \longleftarrow P_L \oplus P_{L_3} \oplus P_{L_5} \longleftarrow \dots$$

for projective objects P_{L_4} and P_{L_5} corresponding to degree p simple functors L_4 and L_5 , given by the known projective resolution of the trivial $\mathbb{F}_p \mathfrak{S}_p$ -module. Moreover, L_3 and L_5 are both different from L_2 , since $p \geq 5$. This implies $Ext_{\mathcal{F}_{\mathbb{F}_p}}^3((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L_2) = 0$. This contradiction excludes this case as well.

So, assume $rad^3(M) = L$ and $rad^4(M) = 0$. This is impossible since then the endomorphism ring of M would be one-dimensional. This contradicts the fact that $End_{\mathcal{A}_{\mathbb{F}_p}^p}(proj_p^1)$ is $p+1$ -dimensional.

Hence, $L_2 \not\cong top(rad^2(M))$.

We still have the possibility that $rad^2(M) = 0$. But again, this would imply that $End_{\mathcal{A}_{\mathbb{F}_p}^p}(M)$ would be one-dimensional and therefore $End_{\mathcal{A}_{\mathbb{F}_p}^p}(proj_p^1)$ is p -dimensional. Contradiction.

This proves the claim. ■

Claim 4.18. $rad^3(M) = 0$.

Proof. We know by Claim 4.13 that $rad^3(P_L) = 0$. Since by Claim 4.15 we have $top(rad(M)) \simeq L$, one sees that $rad^4(M) = 0$. Moreover, $rad^3(M)$ is either 0 or L , since $top(rad^2(M)) \simeq (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ by Claim 4.17 and $top(rad(M)) \simeq L$ by Claim 4.15.

Suppose $rad^3(M) \simeq L$. Then M is uniserial with composition length 4, and $top(rad(M)) = L$. Therefore P_L maps onto $rad(M)$ with image being a uniserial module N of length 3 with $rad(N)/soc(N) = \mathbb{F}_p \otimes id$.

But this contradicts the structure of P_L as described in Claim 4.13 and in particular Remark 4.14.

This proves the claim. ■

Examining what we showed implies that $proj_p^1$ is as stated in Proposition 4.8. ■

We now come to our first main result in describing the structure of $\Gamma_{\mathbb{F}_p, 0}$.

Theorem 4. $\Gamma_{\mathbb{F}_p, 0}$ is a Brauer tree algebra over \mathbb{F}_p without exceptional vertex and associated to a stem with p edges.

$$\bullet_1 - \bullet_2 - \bullet_3 - \dots - \bullet_{p+1}$$

Proof. The case $p \leq 3$ is a consequence of Drozd's results. By Proposition 4.8 we know that the projective cover of the functor $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ is uniserial with top and socle $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ and with second layer V , where V is the simple functor corresponding to the trivial $\mathbb{F}_p \mathfrak{S}_p$ -module.

We know furthermore that except the projective cover of $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ only the projective indecomposable functor P_L has a composition factor $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ and that this composition factor is a direct summand of $top(rad(P_L))$.

Since we know that the principal block of $\mathbb{F}_p \mathfrak{S}_p$ is a Brauer tree algebra without exceptional vertex associated to a stem with p vertices, this means that we only need to show that $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ is not in the socle of P_L , since the only basic algebra with the composition series as a Brauer tree algebra associated to a stem is actually a Brauer tree algebra associated to a stem.

For this we use the duality D on the category of polynomial functors. The projective indecomposable functor P_L is a direct factor of $\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} id}_{p \text{ factors}}$, since this is the

projective cover of all the degree p simple functors.

It is clear that $\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{p \text{ factors}}$ is self dual. Since all the simple functors are self-dual,

also $DP_L \simeq P_L$. If $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ is in the socle of P_L , the simple functor $D(\mathbb{F}_p \otimes_{\mathbb{Z}} -) \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} -$ is in the top of $DP_L \simeq P_L$, but the top of P_L is L by definition.

This proves the Theorem. \blacksquare

5. LIFTING TO CHARACTERISTIC 0

5.1. Lifting Brauer tree algebras to orders.

Proposition 5.1. *Let R be a complete discrete valuation ring with residue field k and field of fractions K . Let B be a Brauer tree algebra over k associated to a Brauer tree which is a stem without exceptional vertex. Let Λ be an R -order. Then, for any proper two-sided ideal $I \neq 0$ of B we get that*

$$\Lambda \otimes_R k \simeq B/I \implies \text{rank}_{\mathbb{Z}}(K_0(K \otimes_R \Lambda)) \leq \text{rank}_{\mathbb{Z}}(K_0(B/I))$$

Proof. We shall first suppose that Λ is indecomposable and that $I \leq \text{rad}(B)$.

Let S_1, S_2, \dots, S_n be representatives of the simple B -modules. The projective cover P_i of S_i has then a composition series where $\text{soc}(P_i) \simeq S_i$ and $\text{rad}(P_i)/\text{soc}(P_i) \simeq S_{i-1} \oplus S_{i+1}$ for all $i \in \{2, 3, \dots, n-1\}$, $\text{rad}(P_1)/\text{soc}(P_1) \simeq S_2$ and $\text{rad}(P_n)/\text{soc}(P_n) \simeq S_{n-1}$.

Denote $\overline{B} := B/I$. Since $I \leq \text{rad}(B)$, we get \overline{B} has the same number of simple modules, and moreover, the simple \overline{B} -modules and the simple B -modules coincide by the epimorphism $B \rightarrow \overline{B}$. Therefore the projective indecomposable \overline{B} -modules are $\overline{P}_i := \overline{B} \otimes_B P_i$ for $i \in \{1, 2, \dots, n\}$. Moreover, \overline{P}_i is the projective cover of S_i as \overline{B} -module.

Extending k if necessary, we may assume that the field of fractions K of R is a splitting field for Λ , since extending K does not decrease the rank of the Grothendieck group, using the Noether-Deuring theorem. Since k is a splitting field for B and for B/I , and since $k \otimes_R \Lambda \simeq \overline{B}$, the Cartan matrix of \overline{B} is symmetric (cf e.g. [16, Proposition 4.2.11]). Since the Cartan matrix of B equals

$$C := \begin{pmatrix} 2 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 2 & 1 \\ 0 & \dots & \dots & 0 & 1 & 2 \end{pmatrix},$$

we see that the composition length of \overline{P}_i differs from the composition length of P_i by at most 1. Indeed, if this would not be the case, then the composition matrix of \overline{B} would be decomposable into at least two blocks and \overline{B} would be decomposable as algebra. But, since Λ is indecomposable, so is $k \otimes_R \Lambda \simeq \overline{B}$. So, $I \leq \text{soc}(B)$.

Since R is complete, we may assume that B and Λ are both basic algebras.

Let Q_i be the projective cover of \overline{P}_i as Λ -module. Hence, $k \otimes_R Q_i \simeq \overline{P}_i$. Since

$$\dim_K \text{Hom}_{K\Lambda}(KQ_i, KQ_j) = \dim_R \text{Hom}_{\Lambda}(Q_i, Q_j) = \dim_k \text{Hom}_{\overline{B}}(\overline{P}_i, \overline{P}_j)$$

we know that KQ_i and KQ_j do not have a character in common if $|i - j| > 1$ and do have one character in common if $|i - j| = 1$. Since $\dim_k \text{Hom}_{\overline{B}}(\overline{P}_i, \overline{P}_i) \in \{1, 2\}$, the character of KQ_i for all $i \in \{1, \dots, n\}$ is a sum of at most two irreducible characters, and in case of two characters these are non isomorphic. Now, since $\text{Hom}_{K\Lambda}(KQ_i, KQ_j) = 0$ if $|i - j| > 1$, it follows that if $\dim_K \text{Hom}_{K\Lambda}(KQ_{i_0}, KQ_{i_0}) = 1$, then $i_0 \in \{1, n\}$. Otherwise, the character of KQ_{i_0} would be a constituent of KQ_{i_0+1} , of KQ_{i_0} and of KQ_{i_0-1} , which implies then $\text{Hom}_{K\Lambda}(KQ_{i_0+1}, KQ_{i_0-1}) \neq 0$. This would give a contradiction. This gives that I equals either S_1 or S_n or $S_1 \oplus S_n$.

Suppose now that S_1 is a direct factor of I (as left module) and suppose $\text{rank}_{\mathbb{Z}}(K_0(K \otimes_R \Lambda)) > \text{rank}_{\mathbb{Z}}(K_0(B))$. We shall prove that S_1 is not a direct factor of I . By symmetry, then neither S_n is a direct factor of I , and therefore, $I = 0$.

Under these hypotheses, $\dim(\overline{P}_1) = 2$ and as a consequence also $\dim_R(Q_1) = 2$. So, for the Wedderburn components corresponding to $K \otimes_R Q_1$ in $K \otimes_R \Lambda$ we have two possibilities. Either $K \otimes_R Q_1$ is a sum of two one-dimensional characters or $K \otimes_R Q_1$ is isomorphic to one two-dimensional character. Since $K \otimes_R \Lambda$ admits at least $n+1$ irreducible characters, $K \otimes_R Q_1$ must have two constituents. So, $K \otimes_R Q_1$ is a sum of two one-dimensional characters.

But now, let $\{e_i | i \in \{1, \dots, n\}\}$ be an orthogonal set of primitive idempotents with $\Lambda e_i \simeq Q_i$. Then, since e_1 and e_2 must be non zero on the common Wedderburn component of KQ_1 and KQ_2 , we get $e_2 e_1 \neq 0$. This is a contradiction to the fact that e_1 and e_2 are orthogonal.

We have to deal with the case Λ being decomposable. The structure of B implies that in this case, B/I is a direct product of algebras we have dealt with in the earlier case, and copies of k . By induction on the number of simple modules of each indecomposable factor the result holds for each of the pieces as well. Summing up for all of these pieces, we get the desired result.

Finally, we have to deal with the case that I is not contained in $\text{rad}(B)$. The same argument as for Λ decomposable applies here as well.

This proves the Proposition. ■

5.2. Proving that the Baues-Dreckmann-Franjou-Pirashvili ring is an order. Since $\hat{\mathbb{Z}}_p$ is a complete discrete valuation ring, we may lift idempotents from $\Gamma_{\mathbb{F}_p}^p$ to $\Gamma_{\hat{\mathbb{Z}}_p}^p$. Hence, there is an indecomposable direct factor $\Gamma_{\hat{\mathbb{Z}}_p,0}^p$ of the rank one free module $\Gamma_{\hat{\mathbb{Z}}_p}^p$ which maps surjectively to $\Gamma_{\mathbb{F}_p,0}^p$. Let $T_0^p := t(\Gamma_{\hat{\mathbb{Z}}_p,0}^p)$ be the torsion ideal in $\Gamma_{\hat{\mathbb{Z}}_p,0}^p$ and define $\Lambda_{\hat{\mathbb{Z}}_p,0}^p := \Gamma_{\hat{\mathbb{Z}}_p,0}^p / T_0^p$.

Proposition 5.2. $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is an order. Moreover, $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is a direct product of $p+1$ matrix rings over $\hat{\mathbb{Q}}_p$ and up to isomorphism there are at most p simple $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ -modules.

Proof. In fact, $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p}^p = \hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Gamma_{\hat{\mathbb{Z}}_p}^p$ and their common module categories are equivalent to the category of polynomial functors $\hat{\mathbb{Q}}_p\text{-mod} \rightarrow \hat{\mathbb{Q}}_p\text{-mod}$ of degree at most p by Lemma 1.3.

By Friedlander-Suslin [12], the category of exact degree n polynomial functors $\hat{\mathbb{Q}}_p\text{-mod} \rightarrow \hat{\mathbb{Q}}_p\text{-mod}$ is equivalent to the category of strict polynomial functors $\hat{\mathbb{Q}}_p\text{-mod} \rightarrow \hat{\mathbb{Q}}_p\text{-mod}$ and this category is equivalent to the category of modules over the Schur algebra $S_{\hat{\mathbb{Q}}_p}(n, n)$. Moreover, the category of strict polynomial functors of degree at most n is equivalent to the direct sum of the category of strict polynomial functors of exact degree m for each $m \in \{0, 1, \dots, n\}$. The Schur algebra $S_{\hat{\mathbb{Q}}_p}(p, p)$ is split semisimple (cf Green [13]) with exactly $p+1$ simple modules. This shows that $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is a direct product of $p+1$ full matrix rings over $\hat{\mathbb{Q}}_p$. Moreover, this shows also that $\Lambda_{\hat{\mathbb{Z}}_p}^p$ is an order since it is by definition torsion free and contains a basis of the semisimple algebra $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p}^p$ (which is Morita equivalent to $\prod_{i=0}^p S_{\hat{\mathbb{Q}}_p}(i, i)$).

In order to prove the second statement we just observe that the number of simple objects in $\mathcal{A}_{\mathbb{F}_p}^p$ equals the number of simple objects in $\mathcal{F}_{\mathbb{F}_p}^p$ by Lemma 3.6. Moreover, since $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is a quotient of $\Gamma_{\hat{\mathbb{Z}}_p,0}^p$, every simple $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ -module induces a simple $\Gamma_{\hat{\mathbb{Z}}_p,0}^p$ -module. We know that $\Gamma_{\mathbb{F}_p,0}^p$ is a Brauer tree algebra with p simple modules. Moreover, $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \Gamma_{\hat{\mathbb{Z}}_p,0}^p \simeq \Gamma_{\mathbb{F}_p,0}^p$

and so, $\Gamma_{\hat{\mathbb{Z}}_p,0}^p$ admits p simple modules. As a consequence $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ admits at most p simple modules. This proves the proposition. \blacksquare

Proposition 5.3. $t(\Gamma_{\mathbb{F}_p,0}^p) = 0$ and therefore $\Lambda_{\hat{\mathbb{Z}}_p,0}^p = \Gamma_{\hat{\mathbb{Z}}_p,0}^p$.

Proof. This is a consequence of Proposition 5.2, Theorem 4 and Proposition 5.1.

Indeed, since $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} -$ is right exact, the epimorphism

$$\Gamma_{\hat{\mathbb{Z}}_p,0}^p \longrightarrow \Lambda_{\hat{\mathbb{Z}}_p,0}^p$$

induces an epimorphism

$$B = \Gamma_{\mathbb{F}_p,0}^p \longrightarrow \Lambda_{\hat{\mathbb{Z}}_p,0}^p \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p$$

with kernel I , for B being a Brauer tree algebra associated to a stem with p edges and without exceptional vertex (Theorem 4). Since $t(\Gamma_{\hat{\mathbb{Z}}_p,0}^p) \subseteq \text{rad}(\Gamma_{\mathbb{F}_p,0}^p)$ by Proposition 5.2, $I \leq \text{rad}(B)$. Since $B/I \simeq \Lambda_{\hat{\mathbb{Z}}_p,0}^p \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p$ for an order $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$, Proposition 5.1 implies that in this case $I = 0$. Hence, $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} t(\Gamma_{\hat{\mathbb{Z}}_p,0}^p) = 0$ and therefore, $t(\Gamma_{\hat{\mathbb{Z}}_p,0}^p) = 0$. This proves the proposition. \blacksquare

5.3. Describing the order; the main result. We shall describe $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ and prove our main result. For this purpose we introduce some notation (cf [16, Section 4.4]). Let

$$\hat{\mathbb{Z}}_p \xrightarrow{p^i} \hat{\mathbb{Z}}_p \quad := \quad \{(a, b) \in \hat{\mathbb{Z}}_p \times \hat{\mathbb{Z}}_p \mid a - b \in p^i \hat{\mathbb{Z}}_p\}$$

and

$$\hat{\mathbb{Z}}_p \xrightarrow{p} \hat{\mathbb{Z}}_p \quad := \quad \hat{\mathbb{Z}}_p \xrightarrow{p} \hat{\mathbb{Z}}_p .$$

The following is the main result of our paper.

Theorem 5. *Let $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$ be the category of at polynomial functors from free abelian groups to $\hat{\mathbb{Z}}_p$ -modules and of degree at most p . Then, $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$ is equivalent to $\Gamma_{\hat{\mathbb{Z}}_p}^p$ -mod, where*

$$\Gamma_{\hat{\mathbb{Z}}_p}^p := \left(\prod_{1 < n < p} \hat{\mathbb{Z}}_p \right) \times \left(\prod_{\lambda \vdash p \text{ and } \lambda \text{ not a hook}} \hat{\mathbb{Z}}_p \right) \times \Lambda_{\hat{\mathbb{Z}}_p,0}^p$$

and where

$$\begin{aligned} \Lambda_{\hat{\mathbb{Z}}_p,0}^p &\simeq \hat{\mathbb{Z}}_p \oplus \left(\hat{\mathbb{Z}}_p \oplus \hat{\mathbb{Z}}_p \right) \oplus \left(\hat{\mathbb{Z}}_p \oplus \hat{\mathbb{Z}}_p \right) \oplus \dots \oplus \left(\hat{\mathbb{Z}}_p \oplus \hat{\mathbb{Z}}_p \right) \oplus \hat{\mathbb{Z}}_p \\ &= \left\{ (d_0) \times \left(\prod_{j=1}^{p-1} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \right) \times (a_p) \mid \forall j : a_j, b_j, c_j, d_j \in \hat{\mathbb{Z}}_p; p \mid c_j; p \mid (d_j - a_{j-1}) \right\} \end{aligned}$$

is a Green order with p isomorphism classes of indecomposable projective modules.

Remark 5.4. Roggenkamp described the orders Λ which admit a set of lattices with periodic projective resolutions encoded by a Brauer tree ([27], see also [16]). Roggenkamp called these orders Green-orders and he described their structure in great detail.

Proof of the theorem. The case $p \leq 3$ was done by Drozd. Hence we may suppose that $p \geq 5$. Since $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is a Brauer tree algebra, there is a set of $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p,0}^p$ -modules having a periodic projective resolution given by the Brauer tree of $\Lambda_{\mathbb{F}_p,0}^p$. Lifting these projective resolutions to the order $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ gives a periodic projective resolution of certain $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ -modules M_i . These periodic resolutions are encoded by the same Brauer tree. It remains to show that the modules M_i are lattices. Actually, this is automatic. Indeed, since the resolution is periodic, each module M_i is also a kernel of a differential, after a complete

period of the periodic projective resolution. Hence, $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is a Green order with Brauer tree being a stem with p edges and without exceptional vertex.

We have to show that the maximal overorder of the Green order $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ is a direct product of matrix rings over $\hat{\mathbb{Z}}_p$ and that the image of $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ in each of the matrix rings is a hereditary order.

The first part is clear since $\hat{\mathbb{Q}}_p$ is a splitting field of $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$, and $\Lambda_{\hat{\mathbb{Z}}_p,0}^p$ can be embedded into a direct product of matrix rings over the ring of integers in $\hat{\mathbb{Q}}_p$ (see e.g. [26]). Let e_1, e_2, \dots, e_{p+1} be a complete set of primitive pairwise orthogonal idempotents of the center of $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda_{\hat{\mathbb{Z}}_p,0}^p$. Then,

$$\prod_{j=1}^{p-1} (\Lambda_{\hat{\mathbb{Z}}_p,0}^p \cdot e_j) \simeq \hat{\mathbb{Z}}_p \times \prod_{j=1}^{p-1} \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p^{x_j}) & \hat{\mathbb{Z}}_p \end{pmatrix} \times \hat{\mathbb{Z}}_p$$

for some $x_j \in \mathbb{N} \setminus \{0\}$. Moreover, since $\Lambda_{\hat{\mathbb{Z}}_p,0}^p \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p$ is a Brauer tree algebra without exceptional vertex, $x_1 = x_2 = \dots = x_{p-1}$ and as a consequence, if one of the matrix rings is hereditary, all of them are hereditary. The structure theory of Green orders (cf Roggenkamp [27]; see also [16, Section 4.4]) and of hereditary orders, (cf e.g. Reiner [26]) then gives the statement.

Define a functor

$$\text{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p} \left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \left(\bigotimes_{j=1}^p id \right), - \right) : \mathcal{A}_{\hat{\mathbb{Z}}_p}^p \longrightarrow \hat{\mathbb{Z}}_p \mathfrak{S}_p - \text{mod}$$

where we use again that the functor $\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \left(\bigotimes_{j=1}^p id \right) : \mathbb{Z} - \text{free} \longrightarrow \hat{\mathbb{Z}}_p - \text{mod}$ carries a natural $\hat{\mathbb{Z}}_p$ -linear \mathfrak{S}_p action. Again, by the very same formal reasons, this functor $E := \text{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p} \left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \left(\bigotimes_{j=1}^p id \right), - \right)$ has a left adjoint $E_*(M) := (M \otimes_{\mathbb{Z}} id^{\otimes p})_{\mathfrak{S}_p}$. Indeed, for any right $\hat{\mathbb{Z}}_p \mathfrak{S}_p$ -module M we get that the functor defined in the variable M by

$$\begin{aligned} (M \otimes_{\mathbb{Z}} id^{\otimes p})_{\mathfrak{S}_p} &= (M \otimes_{\mathbb{Z} \mathfrak{S}_p} id^{\otimes p}) \\ &= (M \otimes_{\hat{\mathbb{Z}}_p \mathfrak{S}_p} (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} id^{\otimes p})) \end{aligned}$$

is left adjoint to $\text{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p} \left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \left(\bigotimes_{j=1}^p id \right), - \right)$ by the usual Hom - \otimes adjunction.

Let $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p \text{ (lat)}$ be the category of at most degree p polynomial functors $\mathbb{Z} - \text{free} \longrightarrow \hat{\mathbb{Z}}_p - \text{free}$, i.e. having values in free $\hat{\mathbb{Z}}_p$ -modules. This category is equivalent to the category of $\Lambda_{\hat{\mathbb{Z}}_p}$ -lattices, i.e. finitely generated $\hat{\mathbb{Z}}_p$ -free $\Lambda_{\hat{\mathbb{Z}}_p}$ -modules. We shall prove that the functor $E_l : \hat{\mathbb{Z}}_p \mathfrak{S}_p - \text{lat} \longrightarrow \mathcal{A}_{\hat{\mathbb{Z}}_p}^p \text{ (lat)}$ defined by

$$E_l(M) := (M \otimes_{\mathbb{Z}} id^{\otimes p})_{\mathfrak{S}_p}$$

is right adjoint to the restriction of $\text{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p} \left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \left(\bigotimes_{j=1}^p id \right), - \right)$ to $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p \text{ (lat)}$.

Indeed, we get a duality defined for any F in $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p \text{ (lat)}$ as

$$DF := \text{Hom}_{\hat{\mathbb{Z}}_p} (F(\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}), \hat{\mathbb{Z}}_p).$$

Since we are dealing with torsion free modules, we get $D \circ D = id$ and also

$$\text{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p \text{ (lat)}}(F, G) \simeq \text{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p \text{ (lat)}}(DG, DF).$$

But then

$$\mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}(F, DG) \simeq \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}(G, DF).$$

Since

$$D(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p}) \simeq (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p}),$$

we get

$$D((\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p})_{\mathfrak{S}_p}) \simeq (D(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p}))_{\mathfrak{S}_p} \simeq (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p})_{\mathfrak{S}_p},$$

and hence, denoting $M^* := \mathrm{Hom}_{\hat{\mathbb{Z}}_p}(M, \hat{\mathbb{Z}}_p)$, we compute

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}(-, (M \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p})_{\mathfrak{S}_p}) &= \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}(-, (M \otimes_{\hat{\mathbb{Z}}_p} (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p}))_{\mathfrak{S}_p}) \\ &= \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}(-, (M \otimes_{\hat{\mathbb{Z}}_p} D(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p}))_{\mathfrak{S}_p}) \\ &= \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}(-, (D(M^* \otimes_{\hat{\mathbb{Z}}_p} (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p})))_{\mathfrak{S}_p}) \\ &= \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}^{\mathrm{lat}}((M^* \otimes_{\hat{\mathbb{Z}}_p} (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}^{\otimes p}))_{\mathfrak{S}_p}, D-) \\ &\simeq \mathrm{Hom}_{\hat{\mathbb{Z}}_p \mathfrak{S}_p}(M^*, \mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p \mathrm{id}), D-)) \\ &\simeq \mathrm{Hom}_{\hat{\mathbb{Z}}_p \mathfrak{S}_p}(\mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p \mathrm{id}), -), M), \end{aligned}$$

functorial in M .

Therefore E_l is right adjoint to the restriction of $\mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p \mathrm{id}), -)$ to $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$.

Moreover, the unit and the counit are both isomorphic to the identity functors on $\hat{\mathbb{Z}}_p \mathfrak{S}_p$ –*lat*. Therefore, E is exact, and since by definition E is represented by $P := \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p \mathrm{id})$, this object P is projective. Let e be an idempotent in $\Gamma_{\hat{\mathbb{Z}}_p}^p$ which corresponds to the projective indecomposable $\Gamma_{\hat{\mathbb{Z}}_p}^p$ –modules which occur in P . Then, replacing $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$ by $\Gamma_{\hat{\mathbb{Z}}_p}^p$ –*mod* the functor $\mathrm{Hom}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p \mathrm{id}), -)$ becomes the functor $E : \Gamma_{\hat{\mathbb{Z}}_p}^p$ –*mod* \rightarrow $\hat{\mathbb{Z}}_p \mathfrak{S}_p$ –*mod* and E is just multiplication by e .

We need to show that $\mathrm{End}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(P) \simeq \hat{\mathbb{Z}}_p \mathfrak{S}_p$, where the action is given by permutation of components in the tensor product. Once this is done, we know that $\mathrm{End}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(P) \simeq e \cdot \Gamma_{\hat{\mathbb{Z}}_p}^p \cdot e \simeq \hat{\mathbb{Z}}_p \mathfrak{S}_p$ and we observe that for all idempotents in $\Gamma_{\hat{\mathbb{Z}}_p}^p$ we get that this product $e \cdot \Gamma_{\hat{\mathbb{Z}}_p}^p \cdot e$ is again a product of Green orders with the same order of congruences. Since $\hat{\mathbb{Z}}_p \mathfrak{S}_p$ is a Green order with congruences modulo p only, we get that $x = 1$.

Claim 5.5. $\mathrm{End}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p \mathrm{id})) \simeq \hat{\mathbb{Z}}_p \mathfrak{S}_p$

Proof. The proof given by Piriou-Schwartz [24, Lemma 1.9] of the corresponding statement for $\mathcal{F}_{\mathbb{F}_p}$ carries over literally. For the reader's convenience we recall the (short) arguments.

Given an $x = \sum_{\sigma \in \mathfrak{S}_p} x_{\sigma} \sigma \in \hat{\mathbb{Z}}_p \mathfrak{S}_p$, then associate to this x the natural transformation η_x in $\mathrm{End}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\bigotimes_{j=1}^p (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}))$ given by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in \mathfrak{S}_p} x_{\sigma} (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}).$$

Inversely, a natural transformation η in $\mathrm{End}_{\mathcal{A}_{\hat{\mathbb{Z}}_p}^p}(\bigotimes_{j=1}^p (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathrm{id}))$ is determined by its value on \mathbb{Z}^n . Fix a basis $\{e_1, \dots, e_n\}$ of \mathbb{Z}^n . The image of $e_1 \otimes e_2 \otimes \cdots \otimes e_n$ under $\eta_{\mathbb{Z}^n}$ can

be uniquely written as $x_\eta \cdot (e_1 \otimes e_2 \otimes \cdots \otimes e_n)$ for an $x_\eta \in \hat{\mathbb{Z}}_p \mathfrak{S}_n$. The two mappings $x \mapsto \eta_x$ and $\eta \mapsto x_\eta$ are mutually inverse and obviously ring homomorphisms. ■

This proves the theorem. ■

Remark 5.6. The Schur algebra $S_{\hat{\mathbb{Z}}_p}(p, p)$ is a classical order which was completely described by König in [15].

$$S_{\hat{\mathbb{Z}}_p}(p, p)' \simeq \left(\begin{array}{c} \hat{\mathbb{Z}}_p \\ (p) \end{array} \begin{array}{c} \hat{\mathbb{Z}}_p \\ \hat{\mathbb{Z}}_p \end{array} \right) \oplus \left(\begin{array}{c} \hat{\mathbb{Z}}_p \\ (p) \end{array} \begin{array}{c} \hat{\mathbb{Z}}_p \\ \hat{\mathbb{Z}}_p \end{array} \right) \oplus \cdots \oplus \left(\begin{array}{c} \hat{\mathbb{Z}}_p \\ (p) \end{array} \begin{array}{c} \hat{\mathbb{Z}}_p \\ \hat{\mathbb{Z}}_p \end{array} \right) \oplus \hat{\mathbb{Z}}_p$$

where we denote by $S_{\hat{\mathbb{Z}}_p}(p, p)'$ the basic algebra of the Schur algebra $S_{\hat{\mathbb{Z}}_p}(p, p)$.

Now, any strict polynomial functor induces a polynomial functor. So, composing further to the Green order lifting the principal block of the group ring of the symmetric group, we get an induced functor

$$S_{\hat{\mathbb{Z}}_p}(p, p)' - \text{mod} \longrightarrow \mathcal{A}_{\hat{\mathbb{Z}}_p}^p \xrightarrow{\phi} e \cdot \Lambda \cdot e - \text{mod}.$$

Since the functor ϕ is induced by the Schur functor, this composed map is induced by the natural embedding of $e \cdot \Lambda \cdot e \hookrightarrow S_{\hat{\mathbb{Z}}_p}(p, p)'$.

6. IDENTIFYING THE LATTICES AS FUNCTORS

We shall identify the indecomposable functors of $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$ which correspond to indecomposable $\Gamma_{\hat{\mathbb{Z}}_p}$ -lattices. We call such polynomial functors 'polynomial lattices'.

The structure of $\Gamma_{\hat{\mathbb{Z}}_p, 0}$ implies that there are exactly $3p - 2$ indecomposable $\Gamma_{\hat{\mathbb{Z}}_p, 0}$ lattices. Indeed, the indecomposable lattices are the p projective indecomposable modules P_1, P_2, \dots, P_p , the $p - 1$ kernels of any fixed non zero homomorphism $P_i \rightarrow P_{i+1}$ for $i = 1, 2, \dots, p - 1$, as well as the $p - 1$ kernels of any fixed non zero homomorphism $P_i \rightarrow P_{i-1}$ for $i = 2, 3, \dots, p$. Therefore, there are exactly $3p - 2$ indecomposable 'lattices' of exactly degree p polynomial functors in $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$.

Moreover, the proof of Theorem 5 shows that exactly the projective indecomposable $\Gamma_{\hat{\mathbb{Z}}_p}$ -modules of degree $d \in \{2, \dots, p - 1\}$ will give rise to indecomposable lattices. Denote by $\rho(k)$ the number of partitions of k into non zero integers, we get the following corollary to Theorem 5.

Corollary 6.1. *Up to isomorphism there are exactly $3p - 2 + \sum_{k=2}^{p-1} \rho(k)$ indecomposable polynomial lattices in $\mathcal{A}_{\hat{\mathbb{Z}}_p}^p$ and $p + \sum_{k=2}^{p-1} \rho(k)$ of them are projective, while $2(p - 1)$ of them are not projective. The non projective polynomial lattices are kernels of mappings between projective indecomposable polynomial functors.*

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