# Preperiodic dynatomic curves for $z \mapsto z^d + c$

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**Abstract**

The preperiodic dynatomic curve $\mathcal{X}_{n,p}$ is the closure in $\mathbb{C}^2$ of the set consisting of $(c, z)$ such that $z$ is a preperiodic point of the polynomial $z \mapsto z^d + c$ with preperiod $n$ and period $p$ $(n, p \geq 1)$. We prove that each $\mathcal{X}_{n,p}$ has exactly $d-1$ irreducible components, these components are all smooth and have pairwise transverse intersections at the singular points of $\mathcal{X}_{n,p}$. We also compute the genus of each component and the Galois group of the defining polynomial of $\mathcal{X}_{n,p}$.

**Keywords and phrases**: dynatomic curves, smooth, irreducible, Multibrot set, Galois group, complex dynamics.

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# 1 Introduction

Fix $d \geq 2$. For $c \in \mathbb{C}$, set $f_c(z) = z^d + c$. For $p \geq 1$, define

$$\check{\mathcal{X}}_{0,p} := \big\{(c, z) \in \mathbb{C}^2 \mid f_c^p(z) = z \text{ and for all } 0 < k < p, \quad f_c^k(z) \neq z\big\}.$$

$$\mathcal{X}_{0,p} := \text{the closure of } \check{\mathcal{X}}_{0,p} \text{ in } \mathbb{C}^2 \ .$$

It is known that all $\mathcal{X}_{0,p}$ are affine algebraic curves, called the *periodic dynatomic curves*. These curves have been the subject of several studies in algebraic and holomorphic dynamical systems. The known results for these curves mainly include the smoothness (Douady-Hubbard [DH1], Milnor [Mil1], Buff-Tan [BT]); irreducibility (Bousch [B], Buff-Tan [BT], Morton [Mo], Lau-Schleicher [LS], Schleicher [S]); the genus (Bousch [B]) and the associated Galois groups (Bousch [B], Morton [Mo], Lau-Schleicher [LS], Schleicher [S]).

In the present work, we study some topological and algebraic properties of *preperiodic dynatomic curves*.

**Definition 1.1.** *For $n \geq 0$, $p \geq 1$, a point $z$ is called a $p$-periodic point if $f_c^p(z) = z$ but $f_c^k(z) \neq z$ for $0 < k < p$, and an $(n, p)$-preperiodic point of $f_c$ if $f_c^n(z)$ is a $p$-periodic point of $f_c$ but $f_c^l(z)$ is not periodic for any $0 \leq l < n$.*

Now, for any $n \geq 1, p \geq 1$, define

$$\check{\mathcal{X}}_{n,p} = \big\{(c,z) \in \mathbb{C}^2 \big| z \text{ is a } (n,p)\text{-preperiodic point of } f_c\big\}$$
$$\mathcal{X}_{n,p} := \text{ the closure of } \check{\mathcal{X}}_{n,p} \text{ in } \mathbb{C}^2 \ .$$

In fact, as we shall see below, all $\mathcal{X}_{n,p}$ are also affine algebraic curves, called the *preperiodic dynatomic curves*. Limited work has been done for this kind of curves. The special case $d = 2$ has been previously studied by Bousch [B], who established in this case that for any integers $n, p \geq 1$, the curve $\mathcal{X}_{n,p}$ is also smooth and irreducible (as the periodic dynatomic curves), and computed its associated Galois group.

The main purpose of this work is to extend these results to arbitrary $d \geq 2$. An obvious difference with the previous case is that, for $d > 2$, the curve $\mathcal{X}_{n,p}$ is no longer irreducible: it consists of $d-1$ irreducible components. We may understand this by a simple observation. Consider the curve $\mathcal{X}_{1,p}$ of $(1,p)$-preperiodic points, that is, the points $z$ which are not p-periodic, but whose image $z_0 = f(z)$ is. The periodic point $z_{p-1} = f^{p-1}(z_0)$ is another preimage of $z_0$. Because $f_c(z) = z^d + c$, we have $z = \omega z_{p-1}$, where $\omega$ is a $d$-th root of unity. According to the value of $\omega$, we can partition the $(1,p)$-preperiodic points into $d-1$ classes, and this decomposition is of algebraic nature: it corresponds to a factorization of $f_c^{p+1}(z) - f_c(z)$.

We show that these $d-1$ components are smooth and irreducible. Our approach to smoothness is by using elementary calculations on quadratic differentials and Thurston's contraction principle, following the method of Buff-Tan ([BT]). The approach to irreducibility is based on the connectedness of periodic dynatomic curves and then by an induction on the preperiodic index $n$. Moreover, we study the features of the singular points of $\mathcal{X}_{n,p}$.

Following Bousch, we compute the genus of each irreducible component and the associated Galois group of the curve $\mathcal{X}_{n,p}$.

Here is a list our main results. They are to be compared with results of periodic dynatomic curves.

Denote by $\{\nu_d(p)\}_{p\geq 1}$ the unique sequence of positive integers satisfying the recursive relation

$$d^p = \sum_{k|p} \nu_d(k), \quad \text{integer } d \geq 2 \tag{1.1}$$

and let $\varphi(m)$ be the Euler totient function (i.e. the number of positive integers less than $m$ and co-prime to $m$). For $n, p \geq 1$, define the numbers

$$M_{n,p} := \nu_d(p) d^{n-2}(d-1)\big(n-1-\sum_{t=1}^{[\frac{n-1}{p}]} d^{-tp}\big),$$

where $[x]$ denotes the maximal integer less than or equal to $x$, and

$$K_{n,p} := \nu_d(p)(d^{p-1}-1)d^{n-1-p}\big(\sum_{t=1}^{[\frac{n-1}{p}]-1} d^{-t(p-1)} - \sum_{t=1}^{[\frac{n-1}{p}]-1} d^{-pt}\big) + (d^{[\frac{n-1}{p}]}-1)\nu_d(p) d^{n-2-[\frac{n-1}{p}]p}$$

(one can refer to (5.3) and (5.4) for the computation of them). For $n, p \geq 1$, set

$$g_p(d) = 1 + \frac{dp - d - p - 1}{2d}\nu_d(p) - \frac{d-1}{2d}\sum_{k|p,k<p}\varphi\Big(\frac{p}{k}\Big)k \cdot \nu_d(k),$$

$$g_{n,p} = 1 + \frac{1}{2}\nu_d(p)d^{n-2}(pd - d - p - 1) + \frac{1}{2}(M_{n,p} + K_{n,p}) - \frac{1}{2}d^{n-2}(d-1)\sum_{k|p,k<p}\varphi\Big(\frac{p}{k}\Big)k \cdot \nu_d(k).$$

**Theorem 1.2.** *For any $d \geq 2$, $n, p \geq 1$, the preperiodic dynatomic curve $\mathcal{X}_{n,p}$ has the following properties :*

1. *The set $\mathcal{X}_{n,p}$ is an affine algebraic curve. It has $d-1$ irreducible components and each one is smooth. Moreover, these components are pairwise intersecting at the singular points of $\mathcal{X}_{n,p}$. In particular, if $d = 2$, the curve $\mathcal{X}_{n,p}$ is smooth and irreducible.*
2. *The genus of every irreducible component of $\mathcal{X}_{n,p}$ (in some kind of compactification) is $g_{n,p}(d)$, and all irreducible components are mutually homeomorphic.*
3. *The Galois group associated with $\mathcal{X}_{n,p}$ is the same as that associated with $\mathcal{X}_{\leq n,p} := \cup_{l=0}^{n}\mathcal{X}_{l,p}$, which consists of all permutations on the roots of the defining polynomial of $\mathcal{X}_{\leq n,p}$ that commute with $f_c$ and the rotation of argument $1/d$.*

Here is a tableau comparing these various curves, where $\mathbf{S}_m$ denotes the group of permutations on $\{1, \ldots, m\}$ and $G_{n,p}(d)$ is the associated Galois group of $\mathcal{X}_{n,p}$.

| periodic $\mathcal{X}_{0,p}$ | $d = 2$ | $d > 2$ |
|---|---|---|
| | irreducible | irreducible |
| | smooth | smooth |
| genus | $g_p(2)$ | $g_p(d)$ |
| Galois group | $\mathbf{S}_{\nu_2(p)/p} \ltimes \mathbb{Z}_p^{\nu_2(p)/p}$ | $\mathbf{S}_{\nu_d(p)/p} \ltimes \mathbb{Z}_p^{\nu_d(p)/p}$ |

| preperiodic $\mathcal{X}_{n,p}$, $n \geq 1$ | $d = 2$ | $d > 2$ |
|---|---|---|
| | irreducible | $d-1$ irreducible components |
| | smooth | not smooth, but each component is smooth |
| component-wise genus | $g_{n,p}(2)$ | $g_{n,p}(d)$ |
| Galois group | $G_{n,p}(2)$ | $G_{n,p}(d)$ |
| pairwise intersection | empty | $C_{n,p}$(singular) : singularity set of $\mathcal{X}_{n,p}$ |

This manuscript is organized as follows:

In section 2, we summarize some preliminaries that will be used in this paper.

In section 3, we will prove that every $\mathcal{X}_{n,p}$ is an affine algebraic curve and find its defining polynomial.

In section 4, we give the irreducible factorization of $\mathcal{X}_{n,p}$, and prove that each irreducible factor is smooth and these irreducible components are pairwise intersecting at the singular points of $\mathcal{X}_{n,p}$.

In section 5, we calculate the genus of each irreducible component.

In section 6, we describe $\mathcal{X}_{n,p}$ from the algebraic point of view by calculating its associated Galois group.

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# 2 Preliminaries

**1. Filled in Julia set and Multibrot set.** These material can be found in [DH1, DH2] and [Eb].

For $c \in \mathbb{C}$, we denote by $K_c$ the filled-in Julia set of $f_c$, that is the set of points $z \in \mathbb{C}$ whose orbit under $f_c$ is bounded. We denote by $M_d$ the *Multibrot set* in the parameter plane, that is the set of parameters $c \in \mathbb{C}$ for which the critical point 0 belongs to $K_c$. It is known that $M_d$ is connected.

Assume $c \in M_d$. Then $K_c$ is connected. There is a conformal isomorphism $\phi_c : \mathbb{C} \setminus \overline{K}_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfying $\phi_c \circ f_c = \big(\phi_c\big)^d$ and $\phi_c'(\infty) = 1$ (i,e. $\dfrac{\phi_c(z)}{z} \longrightarrow_{z\to\infty} 1$). The dynamical ray of angle $\theta \in \mathbb{T}$ is defined by

$$R_c(\theta) := \big\{ z \in \mathbb{C} \setminus K_c \mid \arg\big(\phi_c(z)\big) = 2\pi\theta \big\}.$$

Assume $c \notin M_d$. Then $K_c$ is a Cantor set and all periodic points of $f_c$ are *repelling*, that is $|(f^p)'(z)| > 1$ for $p \geq 1$ and all $p$-periodic point $z$. There is a conformal isomorphism $\phi_c : U_c \to V_c$ between neighborhoods of $\infty$ in $\mathbb{C}$, which satisfies $\phi_c \circ f_c = \big(\phi_c\big)^d$ on $U_c$. We may choose $U_c$ so that $U_c$ contains the critical value $c$ and $V_c$ is the complement of a closed disk. For each $\theta \in \mathbb{T}$, there is an infimum $r_c(\theta) \geq 1$ such that $\phi_c^{-1}$ extends analytically along $R_0(\theta) \cap \big\{ z \in \mathbb{C} \mid r_c(\theta) < |z| \big\}$. We denote by $\psi_c$ this extension and by $R_c(\theta)$ the dynamical ray

$$R_c(\theta) := \psi_c\Big( R_0(\theta) \cap \big\{ z \in \mathbb{C} \mid r_c(\theta) < |z| \big\} \Big).$$

As $|z| \searrow r_c(\theta)$, the point $\psi_c(re^{2\pi i\theta})$ converges to a point $x \in \mathbb{C}$ ([DH2, Pro.8.3]). If $r_c(\theta) > 1$, then $x \in \mathbb{C} \setminus K_c$ is an iterated preimage of 0 and we say that $R_c(\theta)$ bifurcates at $x$. If $r_c(\theta) = 1$, then $x$ belongs to $K_c$ and we say that $R_c(\theta)$ lands at $x$.

There are three kinds of important parameters in $M_d$: super-attracting, parabolic, and Misiurewicz parameters. Recall that a point $z$ is said to be $p$-periodic if $f_c^p(z) = z$ but $f_c^k(z) \neq z$ for $0 < k < p$. We call $c \in \mathbb{C}$

- a *$p$-super-attracting parameter* if 0 is $p$-periodic by $f_c$;
- a *$p$-parabolic parameter* if $f_c$ has a $p$-periodic point $z_0$ with $(f^p)'(z_0) = 1$ or a $m$-periodic point $z_0$ such that $m \mid p$ and $(f^m)'(z_0)$ is a $\frac{p}{m}$-th root of unity;
- a *$(n,p)$-Misiurewicz parameter* if 0 is a $(n,p)$-preperiodic point of $f_c$.

A well-known result in complex dynamics says that any parabolic cycle of a rational map has a critical point in its basin, whose orbit eventually converges to , but is disjoint with the cycle (see [Mil2, Thm.10.15]). So for the family of unicritical polynomials $\{f_c \mid c \in \mathbb{C}\}$, the three classes of parameters above are pairwise disjoint. We write this point as a lemma, since it will be repeatedly used throughout the paper.

**Lemma 2.1.** *If the critical point 0 is (pre)periodic for $f_c$, then $c$ is not a parabolic parameter.*

**2. Affine algebraic curve and singularity.** These material can be found in [G].

A polynomial $f \in \mathbb{C}[x,y]$ is called *squarefree* if it is not divisible by $h(x,y)^2$ for any non-constant $h(x,y) \in \mathbb{C}[x,y]$. An *affine algebraic curve over* $\mathbb{C}$ is defined as

$$\mathcal{C} = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$$

where $f$ is a non-constant squarefree polynomial in $\mathbb{C}[x,y]$, called the *defining polynomial* of $\mathcal{C}$. If $f = \prod_{i=1}^m f_i$, where $f_i$ are the irreducible factor of $f$, we say that the affine curve defined by $f_i$ is a *irreducible component* of $\mathcal{C}$.

Let $f \in \mathbb{C}[x,y]$. The *total degree* of $f(x,y)$ as a multivariate polynomial is the highest degree of its terms, denoted by $\mathrm{Deg}(f)$. Correspondingly, we denote by $\deg_x(f)$ and $\deg_y(f)$ the degrees of $f$ when considered as a polynomial in the variable $x$ and $y$ respectively. The following lemma is repeatedly used in this paper.

**Lemma 2.2. (1)** *If $f = f_1 f_2$ with $f_1, f_2 \in \mathbb{C}[x,y]$, then $Deg(f) = Deg(f_1)+Deg(f_2)$, $deg_x(f) = deg_x(f_1)+deg_x(f_2)$ and $deg_y(f) = deg_y(f_1)+deg_y(f_2)$.*

**(2)** *For $f_1, f_2 \in \mathbb{C}[x,y]$, if $f(x,y) = f_1(x, f_2(x,y))$, then $deg_y(f) = deg_y(f_1) \cdot deg_y(f_2)$.*

**(3)** *For $f_1, f_2 \in \mathbb{C}[x,y]$, if $f(x,y) = f_1(x, f_2(x,y))$ and $Deg(f_1) = deg_y(f_1) \geq 1$, $Deg(f_2) > 1$, then $Deg(f) = Deg(f_1) \cdot Deg(f_2)$.*

*Proof.* (1). Refer to [F, section 1.1].

(2). It is straightforward by a simple computation.

(3). Set $d_1 := \mathrm{Deg}(f_1)$ and $d_2 := \mathrm{Deg}(f_2)$. According to the conditions of the lemma, we have $\deg_y(f_1) = d_1 \geq 1$ and $d_2 > 1$. On one hand, since $\mathrm{Deg}(f_1) = \deg_y(f_1) = d_1$, then there is a unique term in $f_1$ with the form $a_1 y^{d_1}$, where $a_1$ is a non-zero constant. So, by (1) and $d_1 \geq 1$, it follows that $\mathrm{Deg}(a_1 f_2^{d_1}) = d_1 d_2$. On the other hand, any other term of $f_1$ has the form $ax^s y^t$, where $a$ is a non-zero constant and either $s + t < d_1$ or $s + t = d_1$ and $s \geq 1$. According to point (1) and $d_2 > 1$,

$$\mathrm{Deg}(x^s f_2^t(x, y)) = s + t d_2 < d_1 d_2.$$

So we get $\mathrm{Deg}(f) = d_1 d_2$. □

Let $\mathcal{C}$ be an affine algebraic curve for $\mathbb{C}$ defined by $f \in \mathbb{C}[x, y]$, and let $P = (a, b) \in \mathcal{C}$. The multiplicity of $\mathcal{C}$ at $P$, denoted by $mult_P(\mathcal{C})$, is defined as the order $s$ of the first non-vanishing term in the Taylor expansion of $f$ at $P$, i.e.

$$f(x, y) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{t=0}^{s} \binom{s}{t} (x - a)^t (y - b)^{s-t} \frac{\partial^s f}{\partial x^t \partial y^{s-t}}(a, b).$$

If $mult_P(\mathcal{C}) = 1$, the point $P$ is called a *smooth point* of $\mathcal{C}$. If $mult_P(\mathcal{C}) = r > 1$, then we say that $P$ is a *singular point of multiplicity* $r$. We say that $\mathcal{C}$ or $f$ is *smooth* if any point on $\mathcal{C}$ is smooth. Note that the first non-vanishing term is a homogeneous polynomial about $x - a$ and $y - b$, so all its irreducible factors are linear and they are called the *tangents* of $\mathcal{C}$ at $P$.

A singular point $P$ of multiplicity $r$ on an affine plane curve $\mathcal{C}$ is called *ordinary* if the $r$ tangents to $\mathcal{C}$ at $P$ are distinct.

The following result provides a topological interpretation of the irreducibility of polynomials.

**Lemma 2.3.** *A squarefree polynomial $f \in \mathbb{C}[x, y]$ is irreducible if and only if the set of smooth points of $f$ is connected.*

## 3. Periodic dynatomic curves.

In this paper, some of proofs and statements rely on the work of the periodic curves $\mathcal{X}_{0,p}$. We list the related results in the following lemma. Its proof can be found in [B], [BT], [Eb], [GO], [LS], [Mil1], [S].

By abuse of notation, we will identify polynomials in $\mathbb{C}[c, z]$ as polynomials in $\mathbf{C}[z]$ with $\mathbf{C} = \mathbb{C}[c]$. Denote by $\mathbf{K}$ a fixed algebraically closed field containing $\mathbf{C}$.

Let $f \in \mathbb{C}[c, z]$. By the *zeros of* $f \in \mathbb{C}[c, z]$, we mean the points $(c, z) \in \mathbb{C}^2$ with $f(c, z) = 0$. By the *roots of* $f \in \mathbf{C}[z]$, we mean the roots of $f$ in $\mathbf{K}$ when it is considered as a polynomial in $\mathbf{C}[z]$.

Recall that $\{\nu_d(p)\}_{p \geq 1}$ is a unique sequence of positive integers satisfying the recursive relation $d^p = \sum_{k|p} \nu_d(k)$, $\mathrm{Deg}(f)$ denotes the total degree of $f$ and $\deg_z(f)$ denotes the degree of $f$ as a polynomial in $\mathbf{C}[z]$.

**Lemma 2.4.** *Let $\mathcal{X}_{0,p}$ be a periodic dynatomic curve. Then*

1. *[B, BT] There exists a unique sequence of monic polynomials $\{Q_{0,p} \in \mathbf{C}[z]\}_{p\geq 1}$ such that for all $p \geq 1$,*

$$\Phi_{0,p}(c,z) := f_c^{\circ p}(z) - z = \prod_{k|p} Q_{0,k}(c,z).$$

   *Moreover, we have $Deg(Q_{0,p}) = deg_z(Q_{0,p}) = \nu_d(p)$.*

2. *[BT] Let $c_0$ be an arbitrary parameter. Then a point $z_0$ is a root of $Q_{0,p}(c_0,z) \in \mathbb{C}[z]$ if and only if one of the three exclusive conditions is satisfied:*

   **(1)** *$z_0$ is a $p$-periodic point of $f_{c_0}$ and $[f_{c_0}^{\circ p}]'(z_0) \neq 1$,*

   **(2)** *$z_0$ is a $p$-periodic point of $f_{c_0}$ and $[f_{c_0}^{\circ p}]'(z_0) = 1$,*

   **(3)** *$z_0$ is an $m$-periodic point of $f_{c_0}$, where $m$ is a proper factor of $p$, and $[f_{c_0}^{\circ m}]'(z_0)$ is a primitive $\frac{p}{m}$-th root of unity.*

3. *[B, BT, GO, LS, S] The polynomial $Q_{0,p}$ is smooth and irreducible for all $p \geq 1$ and*

$$\mathcal{X}_{0,p} = \{(c,z) \in \mathbb{C} \mid Q_{0,p}(c,z) = 0\}.$$

4. *[B, BT, GO] The projection $\pi_{0,p} : \mathcal{X}_{0,p} \longrightarrow \mathbb{C}$, defined by $\pi_{0,p}(c,z) = c$, is a degree $\nu_d(p)$ (given in (1.1)) branched covering with two kinds of critical points:*

   **(1)** *$C_{0,p}$(primitive) $= \{(c,z) \in \mathcal{X}_{0,p} \mid (c,z)$ satisfies condition (2) of point 2$\}$. In this case, $(c,z)$ is a simple critical point.*

   **(2)** *$C_{0,p}$(satellite) $= \{(c,z) \in \mathcal{X}_{0,p} \mid (c,z)$ satisfies condition (3) of point 2$\}$. In this case, the multiplicity of the critical point $(c,z)$ is $\frac{p}{m} - 1$.*

   *The critical value set of $\pi_{0,p}$ consists of the parabolic parameters of period $p$.*

5. *[Eb, Mil1] The projection $\varpi_{0,p} : \mathcal{X}_{0,p} \longrightarrow \mathbb{C}$, defined by $\varpi_{0,p}(c,z) = z$, is a degree $\nu_d(p)/d$ branched covering, which is injective near each point $(c_0, 0) \in \mathcal{X}_{0,p}$.*

6. *[B] The Galois group $G_{0,p}$ for the polynomial $Q_{0,p} \in \mathbf{C}[z]$ consists of the permutations on roots of $Q_{0,p} \in \mathbf{C}[z]$ that commute with $f_c$.*

# 3 The defining polynomial for $\mathcal{X}_{n,p}$

The objective of this section is to show that $\mathcal{X}_{n,p}$ is an affine algebraic curve and find its defining polynomial.

Recall that $\mathbf{C}$ denotes the ring $\mathbb{C}[c]$. For $n \geq 0, p \geq 1$, set $\Phi_{n,p}(c,z) = f_c^{\circ(n+p)}(z) - f_c^{\circ n}(z)$.

**Lemma 3.1.** *The polynomial $\Phi_{n,p} \in \mathbf{C}[z]$ has no repeated roots. Consequently, it is squarefree.*

*Proof.* To prove this lemma, it is enough to show that there exists $c_0 \in \mathbb{C}$ such that all roots of $\Phi_{n,p}(c_0, z)$ are simple. In fact, given $c_0 \in \mathbb{C} \setminus M_d$, a point $z_0$ is a root of $\Phi_{n,p}(c_0, z) \in \mathbb{C}[z]$ if and only if $z_0$ is a $(l, k)$-preperiodic point of $f_{c_0}$, where $0 \leq l \leq n$ and $k \mid p$. For such a $c_0$, the critical point 0 goes to infinity and all periodic points of $f_{c_0}$ are repelling. It follows that

$$(\partial \Phi_{n,p}/\partial z)(c_0, z_0) = [f_{c_0}^{\circ n}]'(z_0)\big([f_{c_0}^{\circ p}]'(z_0) - 1\big) \neq 0,$$

which completes the proof. □

**Lemma 3.2.** *There exists a unique double indexed sequence of squarefree, monic polynomials $\{Q_{n,p} \in \mathbf{C}[z]\}_{n,p\geq 1}$, such that for all $n, p \geq 1$,*

$$\Phi_{n,p}(c, z) = \Phi_{n-1,p}(c, z) \prod_{k|p} Q_{n,k}(c, z). \tag{3.1}$$

*Moreover, we have $Deg(Q_{n,p}) = deg_z(Q_{n,p}) = \nu_d(p)(d-1)d^{n-1}$.*

*Proof.* The definition of $\{Q_{n,p}\}_{n,p\geq 1}$ is based on the polynomials $\{Q_{0,p}\}_{p\geq 1}$ which appear in part 1 of Lemma 2.4. We firstly show that $Q_{0,p}(c, z)$ divides $Q_{0,p}(c, f_c(z))$ for any $p \geq 1$. Since the polynomials $Q_{0,p}(c, f_c(z)) \in \mathbf{C}[z]$ are monic, we may perform a Euclidean division to find a monic quotient $Q \in \mathbf{C}[z]$ and a remainder $R \in \mathbf{C}[z]$ with degree$(R)$ <degree$(Q_{0,p})$, such that $Q_{0,p}(c, f_c(z)) = Q_{0,p}Q + R$. We need to show that $R = 0$, which enable us to set $Q_{1,p}(c, z) := Q$.

Following Lemma 3.1 and part 1 of Lemma 2.4, the polynomial $Q_{0,p} \in \mathbf{C}[z]$ does not have repeated factors. So its discriminant $\Delta_{0,p} \in \mathbb{C}[c]$ does not identically vanish, and hence $\Delta_{0,p}(c) \neq 0$ outside a finite set. Fix $c_0 \in \mathbb{C}$ such that $\Delta_{0,p}(c_0) \neq 0$. Then any root $z_0$ of $Q_{0,p}(c_0, z)$ is simple. By part 2 of Lemma 2.4, the point $z_0$ is also a root of $Q_{0,p}(c_0, f_{c_0}(z))$. As a consequence, $R(c_0, z) = 0$ for all $z \in \mathbb{C}$. Since this is true for every $c_0$ outside a finite set, we have $R = 0$ as required.

For $n, p \geq 1$, we define $Q_{n,p}(c, z) := Q_{1,p}(c, f_c^{n-1}(z))$. It is clear that each $Q_{n,p} \in \mathbf{C}[z]$ is monic. Note that $\Phi_{n,p}(c, z) = \Phi_{0,p}(c, f_c^n(z))$ for any $n, p \geq 1$, then we have

$$\begin{aligned}
\Phi_{n,p}(c, z) &= \Phi_{0,p}(c, f_c^n(z)) \overset{Lem.2.4}{=} \prod_{k|p} Q_{0,k}(c, f_c^n(z)) = \prod_{k|p} Q_{0,k}(c, f_c^{n-1}(z)) Q_{1,k}(c, f_c^{n-1}(z)) \\
&= \prod_{k|p} Q_{0,k}(c, f_c^{n-1}(z)) \prod_{k|p} Q_{1,k}(c, f_c^{n-1}(z)) = \Phi_{0,p}(c, f_c^{n-1}(z)) \prod_{k|p} Q_{n,k}(c, z) \\
&= \Phi_{n-1,p}(c, z) \prod_{k|p} Q_{n,k}(c, z).
\end{aligned}$$

Since each $\Phi_{n,p}$ is squarefree (Lemma 3.1), so is each $Q_{n,p}$.

Repeatedly applying (2) and (3) of Lemma 2.2, we have $\mathrm{Deg}(f_c^k(z)) = \deg_z(f_c^k(z)) = d^k$ for $k \geq 1$. It follows that $\mathrm{Deg}(\Phi_{n,p}) = \deg_z(\Phi_{n,p}) = d^{n+p}$ for $n \geq 0, p \geq 1$. Then by the recursive formulas (3.1), (1.1) and (1) of Lemma 2.2, the degree conclusion in the lemma holds. □

By the definition of $Q_{n,p}$, we get the inductive formulas

$$\begin{cases} Q_{n-1,p}(c, f_c(z)) = Q_{n,p}(c,z), & n \geq 2; \\ Q_{0,p}(c, f_c(z)) = Q_{0,p}(c,z)Q_{1,p}(c,z), & n = 1. \end{cases} \tag{3.2}$$

for each $p \geq 1$. This equation implies that we can obtain the properties of $Q_{n,p}$ by induction on $n$.

In fact, $Q_{n,p}(c,z)$ is the defining polynomial of $\mathcal{X}_{n,p}$. To see this, we will now study the properties of the roots of $Q_{n,p}(c_0, z) \in \mathbb{C}[z]$ for an arbitrary parameter $c_0 \in \mathbb{C}$.

**Proposition 3.3.** *Let $n \geq 1$, $p \geq 1$ be any pair of integers and $c_0 \in \mathbb{C}$ be any parameter. Then $z_0 \in \mathbb{C}$ is a root of $Q_{n,p}(c_0, z)$ if and only if one of the following 5 mutually exclusive conditions holds:*

**(1)** *$z_0$ is a $(n,p)$-preperiodic point of $f_{c_0}$ such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and $[f_{c_0}^p]'(f_{c_0}^n(z_0)) \neq 1$.*

**(2)** *$z_0$ is a $(n,p)$-preperiodic point of $f_{c_0}$ such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and $[f_{c_0}^p]'(f_{c_0}^n(z_0)) = 1$.*

**(3)** *$z_0$ is a $(n,m)$-preperiodic point of $f_{c_0}$ such that $f_{c_0}^l(z_0) \neq 0$ for any $0 \leq l < n$ and $m$ is a proper factor of $p$ with $[f_{c_0}^m]'(f_{c_0}^n(z_0))$ a primitive $\frac{p}{m}$-th root of unity.*

**(4)** *$z_0$ is a $(n,p)$-preperiodic point of $f_{c_0}$ such that $f_{c_0}^l(z_0) = 0$ for some $0 \leq l < n$.*

**(5)** *$f_{c_0}^{(n-1)}(z_0) = 0$ and $0$ is a $p$-periodic point of $f_{c_0}$.*

We remark that in case (4), the case of $l = n-1$ never occurs.

*Proof.* Fix $c_0 \in \mathbb{C}$. The proof goes by induction on $n$. As $n = 1$, $Q_{0,p}(c, f_c(z)) = Q_{0,p}(c,z) \cdot Q_{1,p}(c,z)$. We claim that $z_0$ is a common root of $Q_{0,p}(c_0,z)$ and $Q_{1,p}(c_0,z)$ if and only if $z_0 = 0$ is a $p$ periodic point of $f_{c_0}$.

For sufficiency, we only need to note that, in this case, 0 is a multiple root of $Q_{0,p}(c_0, f_{c_0}(z))$, but a simple root of $Q_{0,p}(c_0,z)$ by part 2 of Lemma 2.4. For necessity, $z_0$ must be a multiple root of $Q_{0,p}(c_0, f_{c_0}(z))$. It follows that either $f_{c_0}(z_0)$ is a multiple root of $Q_{0,p}(c_0,z)$ or $z_0$ is a critical point of $f_{c_0}$. In the former case, by 4 of Lemma 2.4, $c_0$ is a parabolic parameter and $f_{c_0}(z_0)$ is a parabolic periodic point. It means that $Q_{0,p}(c_0, f_{c_0}(z))$ and $Q_{0,p}(c_0,z)$ have the same zero multiplicity at $z_0$. Then $Q_{1,p}(c_0,z_0) \neq 0$. In the latter case, we have $z_0 = 0$, and by 2 of Lemma 2.4 0 is a $p$ periodic point of $f_{c_0}$.

Such $c_0, z_0$ correspond to the condition (5). In any other case, $z_0$ is a root of $Q_{1,p}(c_0,z)$ if and only if $f_{c_0}(z_0)$ is a root of $Q_{0,p}(c_0,z)$ but $z_0$ is not periodic. In fact, if $z_0$ were periodic,

it would have the same period and multiplier as its first image. By part 2 of Lemma 2.4, we get that $Q_{0,p}(c_0, z_0)$ vanishes, which leads to a contradiction. Then part 2 of Theorem 2.4 implies that $z_0$ satisfies one of the conditions $(1), (2), (3), (4)$ in Proposition 3.3.

Assume that the proposition is established for $1 \leq l < n$. At this time, $Q_{n,p}(c, z) = Q_{n-1,p}(c, f_c(z))$. So for any $c_0 \in \mathbb{C}$, $z_0$ is a root of $Q_{n,p}(c_0, z)$ if and only if $f_{c_0}(z_0)$ is a root of $Q_{n-1,p}(c_0, z)$. By Lemma 2.1, if $f_{c_0}(z_0)$ satisfies property (2) or (3), then the orbit of $z_0$ does not contain 0. Therefore by the inductive assumption, the point $z_0$ satisfies one of the 5 exclusive conditions in Proposition 3.3. □

In Proposition 3.3, the zeros of $Q_{n,p}(c, z)$ are divided into 5 classes. We give some notations to denote the sets consisting of zeros of most classes in the following table.

| The set | The points in the set |
|---|---|
| $C_{n,p}$(primitive) | $(c, z)$ satisfies the condition (2) in Proposition 3.3 |
| $C_{n,p}$(satellite) | $(c, z)$ satisfies the condition (3) in Proposition 3.3 |
| $C_{n,p}$(Misiurewicz) | $(c, z)$ satisfies the condition (4) in Proposition 3.3 |
| $C_{n,p}$(singular) | $(c, z)$ satisfies the condition (5) in Proposition 3.3 |

Recall that for any $n, p \geq 1$, the sets $\check{\mathcal{X}}_{n,p}$ and $\mathcal{X}_{n,p}$ are defined by

$$\check{\mathcal{X}}_{n,p} = \big\{(c, z) \in \mathbb{C}^2 \big| z \text{ is a } (n, p)\text{-preperiodic point of } f_c\big\}$$

and

$$\mathcal{X}_{n,p} := \text{ the closure of } \check{\mathcal{X}}_{n,p} \text{ in } \mathbb{C}^2 \ .$$

**Proposition 3.4.** *For $n, p \geq 1$, we have*

$$\mathcal{X}_{n,p} = \big\{(c, z) \big| Q_{n,p}(c, z) = 0\big\} \quad \textit{and} \quad \mathcal{X}_{n,p} \setminus \check{\mathcal{X}}_{n,p} = C_{n,p}(\textit{satellite}) \cup C_{n,p}(\textit{singular})$$

*Proof.* Set $X := \{(c, z) \mid Q_{n,p}(c, z) = 0\}$. Then $X$ is a closed, perfect set. By the definition of $\check{\mathcal{X}}_{n,p}$ and Proposition 3.3, we have

$$X \setminus \big( \ C_{n,p}(\text{satellite}) \cup C_{n,p}(\text{singular}) \ \big) \ = \check{\mathcal{X}}_{n,p} \subset X \tag{3.3}$$

We claim that the sets $C_{n,p}$(satellite) and $C_{n,p}$(singular) are both finite. If so, we get

$$X = \overline{X \setminus (C_{n,p}(\text{satellite}) \cup C_{n,p}(\text{singular}))} \ = \ \overline{\check{\mathcal{X}}_{n,p}} = \mathcal{X}_{n,p} \subset X.$$

Hence it remains to check the claim.

If $(c_0, z_0) \in C_{n,p}$(satellite), it satisfies that $f_{c_0}^{n+p}(z_0) - f_{c_0}^n(z_0) = 0$ and $[f_{c_0}^p]'(f_{c_0}^n(z_0)) = 1$. Hence $c_0$ is a root of the resultant $R \in \mathbb{C}[c]$ of the equations $f_c^{n+p}(z) - f_c^n(z) = 0$ and $(f_c^p)'(f_c^n(z)) = 1$. For a parameter $c$ outside the Multibrot set, all the periodic points of $f_c$ are repelling, so the polynomials $f_c^{n+p}(z) - f_c^n(z)$ and $(f_c^p)'(f_c^n(z)) - 1$ do not have a common root. It follows that $R$ is not identically zero, and hence, its roots form a finite set. If $(c_0, z_0) \in C_{n,p}$(singular), then $Q_{0,p}(c_0, 0) = 0$ by point (5) of Proposition 3.3 and point 2 of Lemma 2.4, whereas the roots of $Q_{0,p}(c, 0)$ form a finite set. □

# 4 The irreducible factorization of $Q_{n,p}$

In this section, we will show that the curve $\mathcal{X}_{n,p}$, $n \geq 1$, has $d-1$ smooth irreducible components and analyze the properties of its singular points. We always assume $n \geq 1$ without emphasizing.

## 4.1 Factorization of $Q_{n,p}$ and the features of its singular points

Recall that for $f \in \mathbb{C}[c,z]$, $\mathrm{Deg}(f)$ denotes the total degree of $f$ and $\deg_z(f)$ denotes the degree of the variable $z$ in $f$.

**Lemma 4.1. (Algebraic version)** *There exists a unique sequence of monic polynomials $\{q^j_{n,p} \in \mathbf{C}[z]\}_{1\leq j\leq d-1}$ such that*

$$Q_{n,p}(c,z) = \prod_{j=1}^{d-1} q^j_{n,p}(c,z).$$

*All points in $C_{n,p}(singular)$ are zeros of $q^j_{n,p} \in \mathbb{C}[c,z]$, and there are no other common zeros for $q^i_{n,p}$ and $q^j_{n,p}$ with $i \neq j$. Moreover, we have $Deg(q^j_{n,p}) = deg_z(q^j_{n,p}) = \nu_d(p)d^{n-1}$.*

**(Topological version)** *Let $\mathcal{V}^j_{n,p} = \{(c,z) \in \mathbb{C}^2 \big| q^j_{n,p}(c,z) = 0\}$ $(1 \leq j \leq d-1)$. Then $C_{n,p}(singular) \subset \mathcal{V}^j_{n,p}$ for each $j$ and $\left\{\mathcal{V}^j_{n,p} \setminus C_{n,p}(singular)\right\}_{1\leq j\leq d-1}$ are pairwise disjoint.*

*Proof.* Recall that $\mathbf{C} = \mathbb{C}[c]$ and $\mathbf{K}$ is a fixed algebraically closed field containing $\mathbf{C}$.

Let $\Delta$ be a root of $Q_{0,p} \in \mathbf{C}[z]$. Then by part 1 of Lemma 2.4,

$$\Phi_{0,p}(c,\Delta) = f^p_c(\Delta) - \Delta = 0.$$

We see from this equation that $\Delta$ is periodic under $f_c$ and $\Delta, \ldots, f^{p-1}_c(\Delta)$ are roots of $\Phi_{0,p}$. Note that $\Phi_{0,p}(c,0) = f^p_c(0)$ is a polynomial in the variable $c$ of the degree $d^{p-1}$, so $\Delta \neq 0$. Consequently, $\omega\Delta, \ldots, \omega^{d-1}\Delta$ are not roots of $Q_{0,p}$, where $\omega = e^{\frac{2\pi i}{d}}$, because they are not periodic under $f_c$. Then by the equation $Q_{0,p}(c, f_c(z)) = Q_{0,p}(c,z)Q_{1,p}(c,z)$ (see (3.2)), we get that $\omega\Delta, \ldots, \omega^{d-1}\Delta$ are roots of $Q_{1,p} \in \mathbf{C}[z]$.

Let us factorize $Q_{0,p}$ in $\mathbf{K}$ by

$$Q_{0,p}(c,z) = \prod_{i=1}^{\nu_d(p)} (z - \Delta_i)$$

($\Delta_{s_1} \neq \Delta_{s_2}$ for $s_1 \neq s_2$, because all roots of $\Phi_{0,p} \in \mathbf{C}[z]$ are simple (Lemma 3.1), and so are $Q_{0,p}$ (part 1 of Lemma 2.4)). Then $Q_{1,p}$ can be expressed as

$$Q_{1,p} = \prod_{i=1}^{\nu_d(p)} (z - \omega\Delta_i) \cdots (z - \omega^{d-1}\Delta_i) = \prod_{j=1}^{d-1} \prod_{i=1}^{\nu_d(p)} (z - \omega^j \Delta_i) \tag{4.1}$$

To see this, we first note that for any $s, t \in [1, d-1]$ and $i_1 \neq i_2 \in [1, \nu_d(p)]$, $\omega^s \Delta_{i_1} \neq \omega^t \Delta_{i_2}$. But it is impossible because both $\Delta_{i_1}$ and $\Delta_{i_2}$ are periodic. Thus $\{\omega\Delta_i, \ldots, \omega^{d-1}\Delta_i\}_{i=1}^{\nu_d(p)}$ are pairwise distinct roots of $Q_{1,p} \in \mathbf{C}[z]$ by the discussion above, then $\prod_{i=1}^{\nu_d(p)}(z - \omega\Delta_i)\cdots(z - \omega^{d-1}\Delta_i)$ is a divisor of $Q_{1,p}$. As its degree is $(d-1)\nu_d(p)$, equal to the degree of $Q_{1,p}$, and $Q_{1,p}$ is monic, we get (4.1). For $j \in [1, d-1]$, set

$$q_{1,p}^j(c,z) = \prod_{i=1}^{\nu_d(p)}(z - \omega^j\Delta_i) = (\omega^j)^{\nu_d(p)} \prod_{i=1}^{\nu_d(p)}(\omega^{-j}z - \Delta_i) = (\omega^j)^{\nu_d(p)} Q_{0,p}(c, \omega^{-j}z). \quad (4.2)$$

Note that $d \mid \nu_d(p)$, so $(\omega^j)^{\nu_d(p)} = 1$. Then $q_{1,p}^j(c,z)$ is a monic polynomial in $\mathbf{C}[z]$, satisfying

$$Q_{1,p}(c,z) = \prod_{j=1}^{d-1} q_{1,p}^j(c,z). \quad (4.3)$$

This gives a factorization of $Q_{1,p}$ in $\mathbf{C}[z]$. By formula (4.2) and the degree conclusion in point 1 of Lemma 2.4, the total degree $\mathrm{Deg}(q_{1,p}^j)$ and $\deg_z(q_{1,p}^j)$ are both $\nu_d(p)$.

For $n \geq 2$, we can define $q_{n,p}^j(c,z)$ inductively by $q_{n,p}^j(c,z) = q_{n-1,p}^j(c, f_c(z))$. Using induction, the degree conclusion in the lemma follows directly form (2) and (3) of Lemma 2.2. As $Q_{n,p}(c,z) = Q_{n-1,p}(c, f_c(z))$, we have

$$Q_{n,p}(c,z) = \prod_{j=1}^{d-1} q_{n,p}^j(c,z). \quad (4.4)$$

This is a factorization of $Q_{n,p}(c,z)$ in $\mathbf{C}[z]$.

We are left to prove that each $q_{n,p}^j(c,z)$ satisfies the remaining properties announced in the lemma. For $n = 1$, since $q_{1,p}^j(c,z) = Q_{0,p}(c, \omega^{-j}z)$, then $(c_0, z_0)$ is a common root of $q_{1,p}^i(c,z)$ and $q_{1,p}^j(c,z)$ for some $1 \leq i \neq j \leq d-1 \iff$ both $(c_0, \omega^{-i}z_0)$ and $(c_0, \omega^{-j}z_0)$ are zeros of $Q_{0,p}(c,z)$. It follows that $\omega^{-i}z_0$ and $\omega^{-j}z_0$ are both periodic point of $f_{c_0}$, hence $z_0 = 0$. Note that, in case (3) of Lemma 2.4 item 2, the critical point 0 is never periodic (Lemma 2.1), so 0 has period $p$. It follows that $(c_0, z_0) \in C_{1,p}(\text{singular})$. On the other hand, if $(c_0, z_0) \in C_{1,p}(\text{singular})$, then $(c_0, \omega^{-i}z_0) = (c_0, \omega^{-j}z_0) = (c_0, 0)$ is a zero of $Q_{0,p}(c,z)$. For $n \geq 2$, the conclusion can be deduced from the case of $n = 1$ and the definition of $q_{n,p}^j(c,z)$. □

For convenience, we summarize the definitions of $q_{1,p}^j$ in term of $Q_{0,p}$ and the inductive definitions of $q_{n,p}^j$ ($n \geq 2$) in terms of $q_{n-1,p}^j$ as a corollary.

**Corollary 4.2.** *For any $p \geq 1, 1 \leq j \leq d-1$, and $\omega = e^{\frac{2\pi i}{d}}$, we have*

$$\begin{cases} q_{1,p}^j(c,z) = Q_{0,p}(c, \omega^{-j}z), \\ q_{n,p}^j(c,z) = q_{n-1,p}^j(c, f_c(z)), \quad n \geq 2. \end{cases}$$

**Example:** Here are some examples of $Q_{n,p}$ and their decomposition. Let $d = 3$. Suppose $p = 1$, then we have $Q_{0,1}(c, z) = z^3 + c - z$,

$$\begin{aligned} Q_{1,1}(c,z) &= c^2 + cz + z^2 + 2cz^3 + z^4 + z^6 \\ &= (z^3 + c - e^{-\frac{2}{3}\pi i}z)(z^3 + c - e^{-\frac{4}{3}\pi i}z) \\ &= q^1_{1,1}(c,z) \cdot q^2_{1,1}(c,z). \end{aligned}$$

and

$$\begin{aligned} Q_{2,1}(c,z) &= 3c^2 + 3c^4 + (c^6 + 3c + 10c^3 + 6c^5)z^3 + (1 + 12c^2 + 15c^4)z^6 \\ &\quad +(6c + 20c^3)z^9 + (1 + 15c^2)z^{12} + 6cz^{15} + z^{18} \\ &= \big((1 - e^{-\frac{2}{3}\pi i})c + c^3 + (3c^2 - e^{-\frac{2}{3}\pi i})z^3 + 3cz^6 + z^9\big) \times \\ &\quad \big((1 - e^{-\frac{4}{3}\pi i})c + c^3 + (3c^2 - e^{-\frac{4}{3}\pi i})z^3 + 3cz^6 + z^9\big) \\ &= q^1_{2,1}(c,z) \cdot q^2_{2,1}(c,z). \end{aligned}$$

Suppose $p = 2$, then we have $Q_{0,2}(c, z) = 1 + c^2 + cz + z^2 + 2cz^3 + z^4 + z^6$ and

$$\begin{aligned} Q_{1,2}(c,z) &= 1 + 2c^2 + c^4 - (c + c^3) - z^2 + (3c + 4c^3)z^3 - 3c^2z^4 \\ &\quad +(1 + 6c^2)z^6 - 3cz^7 + 4cz^9 - z^{10} + z^{12} \\ &= \big(1 + c^2 + e^{-\frac{2}{3}\pi i}z + e^{-\frac{4}{3}\pi i}z^2 + 2cz^3 + e^{-\frac{2}{3}\pi i}z^4 + z^6\big) \times \\ &\quad \big(1 + c^2 + e^{-\frac{4}{3}\pi i}z + e^{-\frac{2}{3}\pi i}z^2 + 2cz^3 + e^{-\frac{4}{3}\pi i}z^4 + z^6\big) \\ &= q^1_{1,2}(c,z) \cdot q^2_{1,2}(c,z) \end{aligned}$$

From Lemma 4.1, we see that in the case $d \geq 3$, the polynomial $Q_{n,p}$ is both reducible and non-smooth, because $C_{n,p}$(singular), which is non-empty, belongs to the set of singular points of $Q_{n,p}$.

We now turn to the study of the components $q^j_{n,p}(c, z)$. The following theorem is the core of this section.

**Theorem 4.3.** *Given $d \geq 2$, for any $n, p \geq 1$, $j \in [1, d-1]$, the polynomial $q^j_{n,p}(c, z)$ is smooth and irreducible.*

The proof of this theorem is postponed to § 4.2.

By this theorem, all components $\mathcal{V}^j_{n,p}$ are Riemann surfaces. Together with Lemma 4.1, this implies that the singularity set of $\mathcal{X}_{n,p}$ is equal to $C_{n,p}$(singular). The next proposition characterizes the features of these singularities.

**Proposition 4.4.** *Given $d \geq 2$, for $n, p \geq 1$, each singularity $(c_0, z_0)$ of $\mathcal{X}_{n,p}$ has multiplicity $d - 1$. Furthermore, if $f^l_{c_0}(z_0) = 0$ for some $0 \leq l \leq n - 2$, then $\mathcal{X}_{n,p}$ has one tangent of multiplicity $d - 1$ at $(c_0, z_0)$; otherwise, the singularity $(c_0, z_0)$ is ordinary.*

*Proof.* Let $(c_0, z_0)$ be a singular point of $\mathcal{X}_{n,p}$. Since each component of $\mathcal{X}_{n,p}$ is smooth and they are pairwise intersecting at $(c_0, z_0)$, then the first non-vanishing term of $Q_{n,p}(c, z)$ at $(c_0, z_0)$ is $d - 1$. Hence the multiplicity of the singularity $(c_0, z_0)$ is $d - 1$.

If $n = 1$, by (5) of Proposition 3.3, the fact that $(c_0, z_0) \in C_{1,p}$(singular) implies that $z_0 = 0$ and $(c_0, 0) \in \mathcal{X}_{0,p}$. According to Lemma 2.1 $c_0$ is not a parabolic parameter. Then it follows from part 4 of Lemma 2.4 that $(c_0, 0)$ is not a critical point of $\pi_{0,p}$, and hence $\big(\partial Q_{0,p}/\partial z\big)(c_0, 0) \neq 0$. Meanwhile, according to part 5 of Lemma 2.4, $\big(\partial Q_{0,p}/\partial c\big)(c_0, 0) \neq 0$. Thus $Q_{0,p}(c, z)$ has a local expression

$$Q_{0,p}(c, z) = a_{0,p}(c - c_0) + b_{0,p}z + \text{ higher order terms}$$

around $(c_0, 0)$ with $a_{0,p}, b_{0,p} \neq 0$. It follows that

$$q^j_{1,p}(c, z) = Q_{0,p}(c, \omega^{-j}z) = a_{0,p}(c - c_0) + b_{0,p}\omega^{-j}z + \text{ higher order terms}$$

Therefore the tangents of $\mathcal{V}^j_{1,p}$ $(1 \leq j \leq d - 1)$ at $(c_0, 0)$ are pairwise distinct.

For $n \geq 2$, we denote by $a^j_{n,p}(c - c_0) + b^j_{n,p}(z - z_0)$ the equation of the tangent of $\mathcal{V}^j_{n,p}$ at $(c_0, z_0)$. By the formula $q^j_{n,p}(c, z) = q_{1,p}(c, f^{n-1}_c(z))$ (Corollary 4.2), we have that

$$\begin{cases} a^j_{n,p} &= \dfrac{\partial q^j_{n,p}}{\partial c}(c_0, z_0) = \dfrac{\partial q^j_{1,p}}{\partial c}(c_0, 0) + \dfrac{\partial q^j_{1,p}}{\partial z}(c_0, 0)\dfrac{\partial f^{n-1}_c}{\partial c}(c_0, z_0) = a_{0,p} + b_{0,p}\omega^{-j}\dfrac{\partial f^{n-1}_c}{\partial c}(c_0, z_0) \\ b^j_{n,p} &= \dfrac{\partial q^j_{n,p}}{\partial z}(c_0, z_0) = \dfrac{\partial q^j_{1,p}}{\partial z}(c_0, 0)(f^{n-1}_{c_0})'(z_0) = b_{0,p}\omega^{-j}(f^{n-1}_{c_0})'(z_0) \end{cases}$$

If there exists $0 \leq l \leq n - 2$ such that $f^l_{c_0}(z_0) = 0$, then $(f^{n-1}_{c_0})'(z_0) = 0$, and hence $b^j_{n,p} = 0$. It follows that the first non-vanishing term of $Q_{n,p}$ at $(c_0, z_0)$ is $a(c - c_0)^{d-1}$ where $a$ is a non-zero constant, i.e., $\mathcal{X}_{n,p}$ has the tangent $c = c_0$ of multiplicity $d - 1$ at $(c_0, z_0)$. In the other cases, we get $(f^{n-1}_{c_0})'(z_0) \neq 0$. Combining this point and the fact that $a_{0,p}, b_{0,p} \neq 0$, it is not difficult to check that the pairs $(a^j_{n,p}, b^j_{n,p})(1 \leq j \leq d - 1)$ are pairwise non-colinear. Hence the tangents of $\mathcal{V}^j_{n,p}$ $(1 \leq j \leq d - 1)$ at $(c_0, z_0)$ are pairwise distinct, that is, $(c_0, z_0)$ is ordinary. □

## 4.2 Proof of the smoothness and irreducibility of $q^j_{n,p}$

The objective here is to prove Theorem 4.3.

The approach to prove the smoothness is similar to that in [BT]. The idea is to prove that some partial derivative of $q^j_{n,p}$ is non vanishing. Following A. Epstein, we will express this derivative as the coefficient of a quadratic differential of the form $(f_c)_\star\mathcal{Q} - \mathcal{Q}$. Thurston's contraction principle gives $(f_c)_\star\mathcal{Q} - \mathcal{Q} \neq 0$, whence our partial derivative is non-zero.

The approach to the irreducibility is based on the connectedness of periodic curve $\mathcal{X}_{0,p}$. Then we will show the connectivity of $\mathcal{V}^j_{n,p}$ using a branched covering by induction on the preperiodic index $n$.

Here we list some definitions and results about quadratic differentials and Thurston's contraction principle. All their proofs can be found in [BT] and [Le].

We use $\mathcal{Q}(\mathbb{C})$ to denote the set of meromorphic quadratic differentials on $\mathbb{C}$ whose poles (if any) are all simple. If $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ and $U$ is a bounded open subset of $\mathbb{C}$, the norm

$$\|\mathcal{Q}\|_U := \iint_U |q|$$

is well defined and finite.

For $f : \mathbb{C} \to \mathbb{C}$ a non-constant polynomial and $\mathcal{Q} = q\ \mathrm{dz}^2$ a meromorphic quadratic differential on $\mathbb{C}$, the pushforward $f_*\mathcal{Q}$ is defined by the quadratic differential

$$f_*\mathcal{Q} := Tq\ \mathrm{dz}^2 \quad \text{with} \quad Tq(z) := \sum_{f(w)=z} \frac{q(w)}{f'(w)^2}.$$

If $Q \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ also. The following lemma is a weak version of Thurston's contraction principle.

**Lemma 4.5.** *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial and if $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \neq \mathcal{Q}$.*

The formulas below appeared in [Le] chapter 2, we write them together as a lemma.

**Lemma 4.6** (Levin)**.** *For $f = f_c$, we have*

$$\begin{cases} f_*\left(\dfrac{\mathrm{dz}^2}{z}\right) = 0 \\ f_*\left(\dfrac{\mathrm{dz}^2}{z-a}\right) = \dfrac{1}{f'(a)}\left(\dfrac{\mathrm{dz}^2}{z-f(a)} - \dfrac{\mathrm{dz}^2}{z-c}\right) \quad \text{if } a \neq 0 \end{cases} \tag{4.5}$$

To prove the irreducibility of $q^j_{n,p}$, we need the following Lemma.

**Lemma 4.7.** *For each $n, p \geq 1$, $1 \leq j \leq d-1$, the polynomial $q^j_{n,p}(c, 0)$ (in the variable $c$) has degree $\nu_d(p)d^{n-2}$.*

*Proof.* For $n = 1$, we see that $q^j_{1,p}(c, 0) = Q_{0,p}(c, 0)$ from Corollary 4.2. Then the result follows directly from point 5 of Lemma 2.4.

For $n \geq 2$, $q^j_{n,p}(c, 0) = q^j_{1,p}(c, f^{n-1}(0))$. Since $\mathrm{Deg}(q^j_{1,p}) = \deg_z(q^j_{1,p}) = \nu_d(p)$ (see Lemma 4.1) and $\mathrm{Deg}(f_c^{n-1}(0)) = d^{n-2}$ (which is easily checked), we have

$$\mathrm{Deg}(q^j_{n,p}(c, 0)) = \mathrm{Deg}(q^j_{1,p}(c, z)) \cdot \mathrm{Deg}(f_c^{n-1}(0)) = \nu_d(p)d^{n-2}$$

by (2) and (3) of Lemma 2.2. Then the proof is completed. □

**Proof of Theorem 4.3**. The proof goes by induction on $n$.

For $n = 1$, as $q^j_{1,p}(c, z) = Q_{0,p}(c, \omega^{-j}z)$ and $Q_{0,p}(c, z)$ is smooth and irreducible, we know that $q^j_{1,p}(c, z)$ are smooth and irreducible. Assume that for $1 \leq l < n$, the polynomial

$q^j_{l,p}(c,z)$ are smooth and irreducible. Then we will show that $q^j_{n,p}(c,z)$ are smooth and irreducible. Now fix any $j_0 \in [1, d-1]$.

**Smoothness of $q^{j_0}_{n,p}$:** As $q^{j_0}_{n,p}(c,z) = q^{j_0}_{n-1,p}(c, f_c(z))$, for any $(c_0,z_0)$ a zero of $q^{j_0}_{n,p}(c,z)$, we have

$$\begin{cases} \dfrac{\partial q^{j_0}_{n,p}}{\partial c}(c_0,z_0) = \dfrac{\partial q^{j_0}_{n-1,p}}{\partial c}(c_0,w_0) + \dfrac{\partial q^{j_0}_{n-1,p}}{\partial z}(c_0,w_0) \\ \dfrac{\partial q^{j_0}_{n,p}}{\partial z}(c_0,z_0) = \dfrac{\partial q^{j_0}_{n-1,p}}{\partial z}(c_0,w_0)\cdot f'_{c_0}(z_0) \end{cases} \tag{4.6}$$

where $w_0 = f_{c_0}(z_0)$. Then if $z_0 \neq 0$, by the smoothness of $\mathcal{V}^{j_0}_{n,p}$ (assumption of induction), $[\partial q^{j_0}_{n,p}/\partial c](c_0,z_0)$ and $[\partial q^{j_0}_{n,p}/\partial z](c_0,z_0)$ can not be equal to 0 simultaneously, it follows that $q^{j_0}_{n,p}(c,z)$ is smooth at $(c_0,z_0)$. So we are left to prove that $q^{j_0}_{n,p}(c,z)$ is smooth at $(c_0,0) \in \mathcal{V}^{j_0}_{n,p}$. In this situation, $c_0$ is either a $p$-periodic super-attracting parameter or a $(n,p)$-Misiurewicz parameter, and $[\partial q^{j_0}_{n,p}/\partial z](c_0,0) = 0$. So we have to show $[\partial q^{j_0}_{n,p}/\partial c](c_0,0) \neq 0$.

In the former case $f^{n-1}_{c_0}(0) = 0$, then $p|n-1$. Since $q^{j_0}_{n,p}(c,z) = q^{j_0}_{1,p}(c, f^{n-1}_c(z))$, we have

$$\frac{\partial q^{j_0}_{n,p}}{\partial c}(c_0,0) = \frac{\partial q^{j_0}_{1,p}}{\partial c}(c_0,0) + \frac{\partial q^{j_0}_{1,p}}{\partial z}(c_0,0)\frac{\partial f^{n-1}_c}{\partial c}(c_0,0) \tag{4.7}$$

Note that $Q_{0,p}(c_0,0) = 0$ and $p|n-1$, then differentiating both sides of the equation

$$f^{n-1}_c(z) - z = \prod_{k|n-1} Q_{0,k}(c,z),$$

which is raised in point 1 of Lemma 2.4, with respect to $c$ and $z$ respectively at the point $(c_0,0)$, we have that

$$\begin{cases} \dfrac{\partial f^{n-1}_c}{\partial c}(c_0,0) &= \dfrac{\partial Q_{0,p}}{\partial c}(c_0,0) \displaystyle\prod_{\substack{k|n-1 \\ k\neq p}} Q_{0,k}(c_0,0) \\ -1 &= \dfrac{\partial Q_{0,p}}{\partial z}(c_0,0) \displaystyle\prod_{\substack{k|n-1 \\ k\neq p}} Q_{0,k}(c_0,0) \end{cases} \tag{4.8}$$

Since $q^{j_0}_{1,p}(c,z) = Q_{0,p}(c, \omega^{-j_0} z)$, then

$$\frac{\partial q^{j_0}_{1,p}}{\partial c}(c_0,0) = \frac{\partial Q_{0,p}}{\partial c}(c_0,0), \quad \frac{\partial q^{j_0}_{1,p}}{\partial z}(c_0,0) = \omega^{-j_0}\frac{\partial Q_{0,p}}{\partial z}(c_0,0).$$

By substituting these two formulas into equation (4.7) and applying equation (4.8), we find

$$\begin{aligned} \frac{\partial q^{j_0}_{n,p}}{\partial c}(c_0,0) &= \frac{\partial Q_{0,p}}{\partial c}(c_0,0) + \omega^{-j_0}\frac{\partial Q_{0,p}}{\partial z}(c_0,0)\frac{\partial Q_{0,p}}{\partial c}(c_0,0) \prod_{\substack{k|n-1 \\ k\neq p}} Q_{0,k}(c_0,0) \\ &= \frac{\partial Q_{0,p}}{\partial c}(c_0,0)\big(1 + \omega^{-j_0}\frac{\partial Q_{0,p}}{\partial z}(c_0,0) \prod_{\substack{k|n-1 \\ k\neq p}} Q_{0,k}(c_0,0)\big) \\ &= \frac{\partial Q_{0,p}}{\partial c}(c_0,0)(1-\omega^{-j_0}). \end{aligned} \tag{4.9}$$

By point 5 of Lemma 2.4, $[\partial Q_{0,p}/\partial c](c_0,0) \neq 0$, then so is $[\partial q_{n,p}^{j_0}/\partial c](c_0,0)$.

In the latter case, since

$$\frac{\partial Q_{n,p}}{\partial c}(c_0,0) = \prod_{1\leq j\neq j_0\leq d-1} q_{n,p}^{j}(c_0,0)\cdot\frac{\partial q_{n,p}^{j_0}}{\partial c}(c_0,0)$$

and the point $(c_0,0)$ is not a zero of $\prod_{j\neq j_0} q_{n,p}^{j}(c,z)$ by Lemma 4.1, we only have to show $[\partial Q_{n,p}/\partial c](c_0,0)\neq 0$. Furthermore, since

$$\frac{\partial \Phi_{n,p}}{\partial c}(c_0,0) = \Phi_{n-1,p}(c_0,0)\cdot\prod_{k|p,k<p} Q_{n,k}(c_0,0)\cdot\frac{\partial Q_{n,p}}{\partial c}(c_0,0)$$

and $\Phi_{n-1,p}(c_0,0)\cdot\prod_{k|p,k<p} Q_{n,k}(c_0,0)\neq 0$, it is equivalent to show $[\partial\Phi_{n,p}/\partial c](c_0,0)\neq 0$. We shall choose a meromorphic quadratic differential with simple poles such that

$$(f_{c_0})_*\mathcal{Q} = \mathcal{Q} + \frac{\partial\Phi_{n,p}}{\partial c}(c_0,0)\cdot\frac{dz^2}{z-c_0}.$$

Then by Lemma 4.5, we obtain $[\partial\Phi_{n,p}/\partial c](c_0,0)\neq 0$.

We shall use the following notations:

$$\begin{aligned} z_k &:= f_{c_0}^{\circ n+k}(0), \quad \delta_k := f'_{c_0}(z_k) = dz_k^{d-1}, \qquad 0\leq k\leq p-1\\ y_l &:= f_{c_0}^{l}(0), \qquad\quad \varepsilon_l := f'_{c_0}(y_l) = dy_l^{d-1}, \qquad 1\leq l\leq n-1 \end{aligned}$$

With these notations and a bit of calculations, we get

$$\begin{aligned} \frac{\partial\Phi_{n,p}}{\partial c}(c_0,0) &= \frac{\partial f_c^{\circ(n+p)}}{\partial c}(c_0,0) - \frac{\partial f_c^{\circ n}}{\partial c}(c_0,0)\\ &= (\delta_0\cdots\delta_{p-1}-1)(\varepsilon_{n-1}\cdots\varepsilon_1+\cdots+\varepsilon_{n-1}\varepsilon_{n-2}+\varepsilon_{n-1}+1)\\ &\quad +\delta_{p-1}\cdots\delta_1+\cdots+\delta_{p-1}+1 \end{aligned}$$

Denote $(\delta_0\cdots\delta_{p-1}-1)(\varepsilon_{n-1}\cdots\varepsilon_1+\cdots+\varepsilon_{n-1}\varepsilon_{n-2}+\varepsilon_{n-1}+1)$ by $\alpha$. Let

$$\mathcal{Q} = \sum_{k=0}^{p-1}\frac{\rho_k}{z-z_k}dz^2 + \sum_{l=1}^{n-1}\frac{\lambda_l}{z-y_l}dz^2$$

be a quadratic differential in $\mathcal{Q}(\mathbb{C})$. Here $\rho_k$ $(0\leq k\leq p-1)$, $\lambda_l$ $(1\leq l\leq n-1)$ are undetermined coefficients (note that $y_1=c_0$). Applying Lemma 4.6 and writing $f$ for $f_{c_0}$, we have

$$\begin{aligned} f_*\mathcal{Q} &= \sum_{k=0}^{p-1}\frac{\rho_k}{\delta_k}\left(\frac{dz^2}{z-z_{k+1}}-\frac{dz^2}{z-c_0}\right) + \sum_{l=1}^{n-2}\frac{\lambda_l}{\varepsilon_l}\left(\frac{dz^2}{z-y_{l+1}}-\frac{dz^2}{z-c_0}\right) + \frac{\lambda_{n-1}}{\varepsilon_{n-1}}\left(\frac{dz^2}{z-z_0}-\frac{dz^2}{z-c_0}\right)\\ &= \left(\frac{\rho_{p-1}}{\delta_{p-1}}+\frac{\lambda_{n-1}}{\varepsilon_{n-1}}\right)\frac{dz^2}{z-z_0} + \frac{\rho_0}{\delta_0}\frac{dz^2}{z-z_1}+\cdots+\frac{\rho_{p-2}}{\delta_{p-2}}\frac{dz^2}{z-z_{p-1}}\\ &\quad +\left(\alpha-\sum_{l=1}^{n-1}\frac{\lambda_l}{\varepsilon_l}\right)\frac{dz^2}{z-y_1}+\frac{\lambda_1}{\varepsilon_1}\frac{dz^2}{z-y_2}+\cdots+\frac{\lambda_{n-2}}{\varepsilon_{n-2}}\frac{dz^2}{z-y_{n-1}}-\left(\alpha+\sum_{k=0}^{p-1}\frac{\rho_k}{\delta_k}\right)\frac{dz^2}{z-c_0} \end{aligned}$$

We want to choose $\mathcal{Q}$ so that

$$f_*\mathcal{Q} - \mathcal{Q} = -\left(\alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k}\right) \frac{dz^2}{z - c_0}$$

It amounts then to solve the following linear system on the unknown coefficient vector $(\rho_0, \ldots, \rho_{p-1}, \lambda_1, \ldots, \lambda_{n-1})$ :

$$\begin{pmatrix}
\frac{1}{\delta_0} & -1 & & & & & & & & & & \\
 & \ddots & \ddots & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & \\
 & & & \ddots & \ddots & & & & & & & \\
 & & & & \frac{1}{\delta_{p-2}} & -1 & & & & & & \\
-1 & & & & & \frac{1}{\delta_{p-1}} & & & & & \frac{1}{\varepsilon_{n-1}} & \\
 & & & & & & 1+\frac{1}{\varepsilon_1} & \frac{1}{\varepsilon_2} & \frac{1}{\varepsilon_3} & \cdots & \frac{1}{\varepsilon_{n-2}} & \frac{1}{\varepsilon_{n-1}} \\
 & & & & & & \frac{1}{\varepsilon_1} & -1 & & & & \\
 & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & \frac{1}{\varepsilon_{n-2}} & -1
\end{pmatrix}
\begin{pmatrix} \rho_0 \\ \cdot \\ \cdot \\ \cdot \\ \rho_{p-2} \\ \rho_{p-1} \\ \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{n-1} \end{pmatrix}
=
\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ \alpha \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

Denote by $A$ the coefficient matrix, we have

$$\det(A) = \frac{(-1)^{n-1}\alpha}{\delta_0 \cdots \delta_{p-1} \cdot \varepsilon_1 \cdots \varepsilon_{n-1}}$$

Then whether $\alpha = 0$ or not, this linear system has non-zero solutions, and one of its solutions is

$$\begin{cases}
\rho_0 &= \delta_0 \cdots \delta_{p-1} \\
\rho_1 &= \delta_1 \cdots \delta_{p-1} \\
\vdots & \\
\rho_{p-1} &= \delta_{p-1} \\
\lambda_1 &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1} \cdots \varepsilon_1 \\
\vdots & \\
\lambda_{n-2} &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1}\varepsilon_{n-2} \\
\lambda_{n-1} &= (\delta_0 \cdots \delta_{p-1} - 1) \cdot \varepsilon_{n-1}
\end{cases} \tag{4.10}$$

Therefore, for $(\rho_0, \ldots, \rho_{p-1}, \lambda_1, \ldots, \lambda_{n-1})$ satisfies (4.10), we have

$$f_*\mathcal{Q} - \mathcal{Q} \;=\; -\left(\alpha + \sum_{k=0}^{p-1} \frac{\rho_k}{\delta_k}\right) \frac{dz^2}{z - c_0} = -\frac{\partial \Phi_{n,p}}{\partial c}(c_0, 0) \cdot \frac{dz^2}{z - c_0}$$

As a consequence $[\partial \Phi_{n,p}/\partial c](c_0, 0) \neq 0$.

**Irreducibility of $q_{n,p}^{j_0}$:** For $n \geq 2$, $q_{n,p}^j(c, z)$ is defined by $q_{n,p}^j(c, z) = q_{n-1,p}^j(c, f_c(z))$. Interpreting these equations from a topological view, we obtain a sequence of maps

$$\Big\{\wp_{n,p}^j : \mathcal{V}_{n,p}^j \longrightarrow \mathcal{V}_{n-1,p}^j,\ (c, z) \mapsto (c, f_c(z)) \mid n \geq 2,\ p \geq 1,\ 1 \leq j \leq d-1\Big\}.$$

Note that for $n = 1$, we can also define a map $\wp^j_{1,p} : \mathcal{V}^j_{1,p} \to \mathcal{X}_{0,p}$ by $\wp^j_{1,p}(c, z) = (c, f_c(z))$. By the smoothness of $\mathcal{V}^j_{n,p}$, we can check the following results.

- The map $\wp^j_{1,p} : \mathcal{V}^j_{1,p} \to \mathcal{X}_{0,p}$ is a homeomorphism. To see this, notice that $q^j_{1,p}(c, z) = Q_{0,p}(c, \omega^{-j} z)$ (Corollary 4.2), so we can define a map $\phi^j_{1,p}$ from $\mathcal{X}_{0,p}$ to $\mathcal{V}^j_{1,p}$ by mapping a point $(c_0, w_0) \in \mathcal{X}_{0,p}$ to $(c_0, \omega^j z_0) \in \mathcal{V}^j_{1,p}$, where $z_0$ is the point in the orbit of $w_0$ under $f_{c_0}$ with $f_{c_0}(z_0) = w_0$. By a simple computation, we can see that $\phi^j_{1,p} \circ \wp^j_{1,p} = id_{\mathcal{V}^j_{1,p}}$ and $\wp^j_{1,p} \circ \phi^j_{1,p} = id_{\mathcal{X}_{0,p}}$. Hence $\wp^j_{1,p}$ is a homeomorphism.

- For $n \geq 2$, the map $\wp^j_{n,p} : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{n-1,p}$ is a degree $d$ branched covering with critical set
$$D^j_{n,p} = \big\{(c, 0) \mid q^j_{n,p}(c, 0) = 0\big\}.$$
and each critical point has multiplicity $d - 1$.

  In fact, a point $(c_0, w_0) \in \mathcal{V}^j_{n-1,p} \setminus \wp(D^j_{n,p})$ has $d$ preimages $(c_0, z_1), \ldots, (c_0, z_d)$ under $\wp^j_{n,p}$, where $z_1, \ldots, z_d$ are preimages of $w_0$ under $f_{c_0}$. Fix $i \in [1, d]$. If $[\partial q^j_{n,p}/\partial z](c_0, z_i) \neq 0$, then by equation (4.6), $[\partial q^j_{n-1,p}/\partial z](c_0, w_0) \neq 0$. It implies that some neighborhoods of $(c_0, z_i)$ and $(c_0, w_0)$ can be parameterized by $c$ respectively. Using such two local coordinates, the map $\wp^j_{n,p}$ has a local expression $c \mapsto c$ near $(c_0, z_i)$, which means that $\wp^j_{n,p}$ is a local homeomorphism near $(c_0, z_i)$. If $[\partial q^j_{n,p}/\partial z](c_0, z_i) = 0$, then by equation (4.6), the fact of $z_i \neq 0$ and the smoothness of $q^j_{n,p}$, we have that $[\partial q^j_{n-1,p}/\partial z](c_0, w_0) = 0$ and
$$\frac{\partial q^j_{n,p}}{\partial c}(c_0, z_i) = \frac{\partial q^j_{n-1,p}}{\partial c}(c_0, w_0) \neq 0$$
  It implies that some neighborhoods of $(c_0, z_i)$ and $(c_0, w_0)$ can be parameterized by $z$ respectively, and $c'(z_i) = 0$. Using such two local coordinates, the map $\wp^j_{n,p}$ has a local expression $z \mapsto f_{c(z)}(z)$ near $(c_0, z_i)$. Since $z_i \neq 0$, then $\dfrac{df_{c(z)}(z)}{dz}\Big|_{z=z_i} = dz_i \neq 0$, which still means that $\wp^j_{n,p}$ is a local homeomorphism near $(c_0, z_i)$.

  By the discussion above, we can see that
$$\wp^j_{n,p} : \mathcal{V}^j_{n,p} \setminus (\wp^j_{n,p})^{-1}(\wp(D^j_{n,p})) \to \mathcal{V}^j_{n-1,p} \setminus \wp(D^j_{n,p})$$
  is a degree $d$ covering. On the other hand, for any point in $\wp^j_{n,p}(D^j_{n,p})$, it has only one preimage, which belongs to $D^j_{n,p}$. Hence we have that $\wp : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{n-1,p}$ is a degree $d$ branched covering (because $(\wp^j_{n,p})^{-1}(\wp(D^j_{n,p})) = D^j_{n,p}$ and $D^j_{n,p}$ is finite) and the local degree of $\wp^j_{n,p}$ at each point of $D^j_{n,p}$ is $d$.

By the smoothness of $q^{j_0}_{n,p}(c, z)$ and the inductive assumption of irreducibility, we know that $\mathcal{V}^{j_0}_{n-1,p}$ and each connected component of $\mathcal{V}^{j_0}_{n,p}$ is a Riemann surface. Then the restriction of $\wp^{j_0}_{n,p}$ on any connected component of $\mathcal{V}^{j_0}_{n,p}$ is also a branched covering. Lemma 4.7 implies that the critical set $D^{j_0}_{n,p}$ of $\wp^{j_0}_{n,p}$ is non-empty. Since each critical point has multiplicity $d - 1$, the set $\mathcal{V}^{j_0}_{n,p}$ must be connected. By Lemma 2.3 and the smoothness of $q^{j_0}_{n,p}$, we conclude that $q^{j_0}_{n,p}(c, z)$ is irreducible in $\mathbb{C}[c, z]$.

□

# 5 Genus of the compactification of $\mathcal{V}^j_{n,p}$

In the previous section, we have seen that $\mathcal{X}_{n,p}$ consists of $d-1$ Riemann surfaces $\mathcal{V}^j_{n,p}$ which are pairwise intersecting at the singular points of $\mathcal{X}_{n,p}$. In order to give a complete topological description of $\mathcal{X}_{n,p}$, we also need the topological characterization of each $\mathcal{V}^j_{n,p}$.

In fact, by adding an ideal boundary point at each end of $\mathcal{V}^j_{n,p}$, we obtain a compactification of $\mathcal{V}^j_{n,p}$, denoted by $\widehat{\mathcal{V}}^j_{n,p}$, such that $\widehat{\mathcal{V}}^j_{n,p}$ is a compact Riemann surface (in § 5.1 ). The genus of $\widehat{\mathcal{V}}^j_{n,p}$ is calculated (in § 5.2). Topologically, $\mathcal{X}_{n,p}$ is completely determined by the number of its singular points, the genus of $\widehat{\mathcal{V}}^j_{n,p}$ and the number of ideal boundary points added on $\mathcal{V}^j_{n,p}$ (or the number of ends of $\mathcal{V}^j_{n,p}$).

## 5.1 Compactification of $\mathcal{V}^j_{n,p}$

Denote by $\pi^j_{n,p}\colon \mathcal{V}^j_{n,p} \to \mathbb{C}$ the projection from $\mathcal{V}^j_{n,p}$ to the parameter plane, i.e., $\pi^j_{n,p}(c,z) = c$. It is easy to see

$$\pi^j_{n,p} = \pi_{0,p} \circ \wp^j_{1,p} \circ \cdots \circ \wp^j_{n-1,p} \circ \wp^j_{n,p} \tag{5.1}$$

where $\pi_{0,p}$ is the projection from $\mathcal{X}_{0,p}$ to the parameter plane and $\wp^j_{n,p}$ is defined in the proof of irreducibility. It follows that $\pi^j_{n,p}$ is a degree $\nu_d(p)d^{n-1}$ branched covering. To study the critical points of $\pi^j_{n,p}$, we define a subset $C^{\rm crit}_{n,p}(\text{singular})$ of $C_{n,p}(\text{singular})$ by

$$C^{\rm crit}_{n,p}(\text{singular}) = \{(c,z) \in C_{n,p}(\text{singular}) \mid f^l_c(z) = 0 \text{ for some } 0 \le l \le n-2\} \tag{5.2}$$

**Lemma 5.1.** *For any $l, p \ge 1$, the critical set of $\pi^j_{l,p}$ is the union of $C^j_{l,p}$(primitive), $C^j_{l,p}$(satellite), $C^j_{l,p}$(Misiurewicz) and $C^{crit}_{l,p}$(singular), where $C^j_{l,p}(M) := C_{l,p}(M) \cap \mathcal{V}^j_{l,p}$ and $M$ indicates different properties.*

*Proof.* We first note that $(c_0, z_0)$ is a critical point of $\pi^j_{l,p}$ if and only if $[\partial q^j_{l,p}/\partial z](c_0,z_0) = 0$. By part 4 of Lemma 2.4 and the fact that $\wp^j_{1,p}$ is homeomorphic (which is shown in the proof of irreducibility of $q^j_{l,p}$), the critical set of $\pi^j_{1,p}$ is $C^j_{1,p}(\text{primitive}) \cup C^j_{1,p}(\text{satellite})$. In the case $l = 1$, $C_{l,p}(\text{Misiurewicz})$ and $C^{\rm crit}_{l,p}(\text{singular})$ are empty.

For $l \ge 2$, by Corollary 4.2, we have $q^j_{l,p}(c,z) = q^j_{1,p}(c, f^{l-1}_c(z))$. Then a point $(c_0,z_0)$ is critical for $\pi^j_{l,p}$ if and only if

$$\frac{\partial q^j_{l,p}}{\partial z}(c_0,z_0) = \frac{\partial q^j_{1,p}}{\partial z}(c_0, f^{l-1}_{c_0}(z_0)) \cdot (f^{l-1}_{c_0})'(z_0) = 0$$

It is equivalent that either $(c_0, f^{l-1}_{c_0}(z_0))$ is a critical point of $\wp^j_{1,p}$ or $f^l_{c_0}(z_0) = 0$ for some $0 \le q \le n-2$. By Proposition 3.3, the former case happens if and only if $(c_0,z_0) \in C^j_{l,p}(\text{primitive}) \cup C^j_{l,p}(\text{satellite})$, and the latter case happens if and only if $(c_0,z_0) \in C^j_{l,p}(\text{Misiurewicz}) \cup C^{\rm crit}_{l,p}(\text{singular})$. □

From this Lemma, we see that the critical value set of $\pi^j_{n,p}$ is contained in the union of parabolic, super-attracting and Misiurewicz parameters. Hence $\mathbb{C} \setminus M_d$ contains no critical values. It follows that each connected component of $(\wp^j_{n,p})^{-1}(\mathbb{C} \setminus M_d)$, called an *end* of $\mathcal{V}^j_{n,p}$, is conformal to $\mathbb{C} \setminus \overline{\mathbb{D}}$. By adding an ideal boundary point at the infinitely far boundary, each end of $\mathcal{V}^j_{n,p}$ is conformal to the unit disk, and then $\mathcal{V}^j_{n,p}$ becomes a compact Riemann surface. This gives a kind of compactification of $\mathcal{V}^j_{n,p}$ and we will calculate in the next subsection the genus of this compact Riemann surface.

More precisely, set $\{\ \mathcal{E}^j_{n,p,i}\}$ $(1 \le i \le m^j_{n,p})$ the ends of $\mathcal{V}^j_{n,p}$. Denote by $\infty^j_{n,p,i}$ the point added at the infinitely far boundary of $\mathcal{E}^j_{n,p,i}$. Then the surface $\widehat{\mathcal{V}}^j_{n,p} := \mathcal{V}^j_{n,p} \cup \{\infty^j_{n,p,i}\}_{i=1}^{m^j_{n,p}}$ is a compactification of $\mathcal{V}^j_{n,p}$ and $\widehat{\mathcal{E}}^j_{n,p,i} := \mathcal{E}^j_{n,p,i} \cup \{\infty^j_{n,p,i}\}$ is called an *end* of $\widehat{\mathcal{V}}^j_{n,p}$. In this case, the map $\pi^j_{n,p}$ can be extended to

$$\widehat{\pi}^j_{n,p} : \widehat{\mathcal{V}}^j_{n,p} \longrightarrow \widehat{\mathbb{C}}$$

by setting $\widehat{\pi}^j_{n,p}(\infty^j_{n,p,i}) = \infty$.

## 5.2 Calculation of the genus of $\widehat{\mathcal{V}}^j_{n,p}$

Now, for any $n, p \ge 1,\ j \in [1, d-1]$, we have obtained a branched covering $\widehat{\pi}^j_{n,p} \colon \widehat{\mathcal{V}}^j_{n,p} \to \widehat{\mathbb{C}}$ of degree $\nu_d(p)d^{n-1}$ between two compact Riemann surface. By the Riemann-Hurwitz formula, we have

$$2 - 2g^j_{n,p} + \text{ total number of critical points of } \widehat{\pi}^j_{n,p} = 2\nu_d(p)d^{n-1}.$$

where $g^j_{n,p}$ denotes the genus of $\widehat{\mathcal{V}}^j_{n,p}$. So in order to calculate the genus of $\widehat{\mathcal{V}}^j_{n,p}$, we only need to count the number of critical points of $\widehat{\pi}^j_{n,p}$ counting with multiplicity. By Lemma 5.1, we know that the critical points of $\widehat{\pi}^j_{n,p}$ consists of the points of $C^j_{n,p}$(primitive), $C^j_{n,p}$(satellite), $C^j_{n,p}$(Misiurewicz), $C^{\text{crit}}_{n,p}$(singular) and maybe some added ideal boundary points. So we will count them class by class.

- Counting the points of $C^j_{n,p}$(primitive) and $C^j_{n,p}$(satellite).

  In [B], Bousch counts the number of critical points in $C_{0,p}$(primitive) and $C_{0,p}$(satellite). His argument can be directly extended to our case (see also [Sil, Thm. 4.17]). so we only list the result without the counting process. The number of critical points counted with multiplicity of $\widehat{\pi}^j_{n,p}$ in $C^j_{n,p}$(primitive) and $C^j_{n,p}$(satellite) are

  $$d^{n-1}p\big[(d-1)\nu_d(p)/d - \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)\big]$$

  and

  $$d^{n-1} \sum_{k|p,k<p} (\nu_d(k)/d)(d-1)\varphi(p/k)k(p/k-1).$$

- Counting the points of $C^j_{n,p}$(Misiurewicz).

  Recall that $D^j_{s,p} = \{(c,0) \in \mathbb{C}^2 \big| q^j_{s,p}(c,0) = 0\}, s \geq 2$, is the set of critical points of $\wp^j_{s,p}$. By Proposition 3.3, if $(c,0) \in D^j_{s,p}$, then $c$ is either a $(s,p)$-Misiurewicz parameter or a $p$-super-attracting parameter. So we divide the set $D^j_{s,p}$ into two sets

  $$D^j_{s,p}(\text{Misiurewicz}) = \{(c,0) \in D^j_{s,p} \mid \text{c is a Misiurewicz parameter}\}$$

  and

  $$D^j_{s,p}(\text{period}) = \{(c,0) \in D^j_{s,p} \mid \text{c is a super-attracting parameter}\}$$

  By the definition of $C^j_{n,p}$(Misiurewicz), we have

  $$C^j_{n,p}(\text{Misiurewicz}) = \bigcup_{s=2}^{n} (h^j_{n,s,p})^{-1}(D^j_{s,p}(\text{Misiurewicz})),$$

  where $h^j_{n,s,p} := \wp^j_{s+1,p} \circ \cdots \circ \wp^j_{n,p} : \mathcal{V}^j_{n,p} \longrightarrow \mathcal{V}^j_{s,p}$

  Fix any $s \in [2,n]$. Since the degree of $q^j_{s,p}(c,0)$ is $\nu_d(p)d^{s-2}$ (Lemma 4.7) and $[\partial q^j_{s,p}/\partial c](c,0) \neq 0$ at each $(c,0) \in D^j_{s,p}$ (this point is shown in the proof of smoothness of $q^j_{s,p}(c,z)$), we get $\#D^j_{s,p} = \nu_d(p)d^{s-2}$. By point (5) of Proposition 3.3, the set $D^j_{s,p}$(period) is non-empty if and only if $p|s-1$. In this case, we also see that $D^j_{s,p}$(period) $= \{(c,0) \mid Q_{0,p}(c,0) = 0\}$, then $\#D^j_{s,p}$(period) equals to $\nu_d(p)/d$ if $p|s-1$ and 0 otherwise. It follows that

  $$\#D^j_{s,p}(\text{Misiurewicz}) = \begin{cases} \nu_d(p)d^{s-2}, & \text{if } p \nmid s-1; \\ \nu_d(p)d^{s-2} - \nu_d(p)/d, & \text{if } p|s-1. \end{cases}$$

  Note that the critical value set of $h^j_{n,s,p}$ is disjoint from $D^j_{s,p}$(Misiurewicz), so

  $$\#(h^j_{n,s,p})^{-1}(D^j_{s,p}(\text{Misiurewicz})) = \#D^j_{s,p}(\text{Misiurewicz}) \cdot d^{n-s}$$

  and each point in $(h^j_{n,s,p})^{-1}(D^j_{s,p}(\text{Misiurewicz}))$ is a critical point of $\widehat{\pi}^j_{n,p}$ with multiplicity $d-1$. Therefore the number of critical points counting with multiplicity of $\widehat{\pi}^j_{n,p}$ in $C^j_{n,p}$(Misiurewicz), denoted by $M_{n,p}$, is equal to

  $$\begin{aligned} M_{n,p} : &= \sum_{s=2}^{n} \#D^j_{s,p}(\text{Misiurewicz}) \cdot d^{n-s} \cdot (d-1) \\ &= \nu_d(p)d^{n-2}(d-1)\big(n-1-\sum_{t=1}^{[\frac{n-1}{p}]} d^{-tp}\big). \end{aligned} \tag{5.3}$$

- Counting the points of $C^{\text{crit}}_{n,p}$(singular).

  Recall that $C^{\text{crit}}_{n,p}$(singular) consists of points of the form $(c,z)$ with $f^{n-1}_c(z) = 0$ and such that there exists $l$ between 0 and $n-2$ (both included) with $f^l(z) = 0$ and s.t. 0 is $p$-periodic.

We divide the set $C_{n,p}$(singular) into several subsets $C^t_{n,p}$(singular) which consists of points $(c,z) \in C_{n,p}$(singular) such that

$$n-1-tp = \min\{\ l \mid f_c^l(z) = 0\ \}$$

The index $t$ can take the values $0, \ldots, [\frac{n-1}{p}]$, where $[x]$ denotes the maximal integer less than or equal to $x$, and the sets $C^t_{n,p}$ are pairwise disjoint and form a partition of $C_{n,p}$(singular). From (5.2), we see that $C^{\rm crit}_{n,p}$(singular) is the union of $C^t_{n,p}$(singular), $t \geq 1$. Then we get $\#C^{\rm crit}_{n,p}$(singular) $= 0$ if $n-1 < p$. So in the following discussion, we only concern the case of $n-1 \geq p$, i,e., $[\frac{n-1}{p}] \geq 1$.

Let $t \geq 1$. A point $(c,z) \in C^t_{n,p}$(sigular) if and only if $(c,0) \in D^j_{tp+1,p}$(period), $f_c^{n-1-tp}(z) = 0$ and $(f_c^{n-1-tp})'(z) \neq 0$. Hence

$$C^t_{n,p}(\text{sigular}) = (h^j_{n,tp+1,p})^{-1}(D^j_{tp+1,p}(\text{period})) \setminus (h^j_{n,(t+1)p+1,p})^{-1}(D^j_{(t+1)p+1,p}(\text{period}))$$

if $(t+1)p+1 \leq n$, and $C^t_{n,p}(\text{sigular}) = (h^j_{n,tp+1,p})^{-1}(D^j_{tp+1,p}(\text{period}))$ otherwise. So we have

$$\#C^t_{n,p}(\text{sigular}) = \begin{cases} d^{n-1-tp}\cdot \nu_d(p)/d, & \text{if } t = [\frac{n-1}{p}]; \\ d^{n-1-tp}\cdot \nu_d(p)/d - d\cdot d^{n-1-(t+1)p}\cdot \nu_d(p)/d, & \text{if } 1 \leq t < [\frac{n-1}{p}]. \end{cases}$$

On the other hand, the map $h^j_{n,tp+1,p} : \mathcal{V}^j_{n,p} \to \mathcal{V}^j_{tp+1,p}$ is injective in a neighborhood of any point $(c,z) \in C^t_{n,p}$(sigular), and the map $\pi^j_{kp+1,p} : \mathcal{V}^j_{tp+1} \to \mathbb{C}$ has the local degree $d^t$ at the point $(c,0)$, so the number of critical points counting with multiplicity in $C^t_{n,p}$(sigular) is $(d^t-1)\#C^t_{n,p}$(sigular). Then the total number of critical points counting with multiplicity in $C_{n,p}$(sigular), in the case of $[\frac{n-1}{p}] \geq 1$, is

$$\begin{aligned} K_{n,p} : &= \sum_{t=1}^{[\frac{n-1}{p}]} (d^t-1)\#C^t_{n,p}(\text{sigular}) \\ &= \nu_d(p)(d^{p-1}-1)d^{n-1-p}(\xi_{n,p}-\zeta_{n,p}) + (d^{[\frac{n-1}{p}]}-1)\nu_d(p)d^{n-2-[\frac{n-1}{p}]p} \quad (5.4) \end{aligned}$$

where $\xi_{n,p} := \sum_{t=1}^{[\frac{n-1}{p}]-1} d^{-t(p-1)}$ and $\zeta_{n,p} := \sum_{t=1}^{[\frac{n-1}{p}]-1} d^{-pt}$.

Note that when $[\frac{n-1}{p}] = 0$, the number computed by formula (5.4) is 0, which is still equal to the number of $C^{\rm crit}_{n,p}$(singular). So the number $K_{n,p}$, defined by (5.4), is equal to the number of critical points counting with multiplicity in $C^{\rm crit}_{n,p}$(singular) in all cases.

- Counting the ideal boundary points.

  In [B], [Mil1], Bousch and Milnor show that the local degree of $\pi_{0,p}$ at each ideal boundary point is 2 (in the case of $d=2$) by analysing the asymptotic behavior of $f_c(z)$ as $(c,z)$ goes to an ideal boundary point on $\mathcal{X}_{0,p}$. Their argument can be easily generalized to our case with degree $d \geq 2$. Just to be self-contained we give an alternative proof using the monodromy action (Lemma 5.3 below). By Lemma 5.3,

the local degree of $\widehat{\mathcal{V}}^j_{n,p}$ at each ideal boundary point is $d$. It follows that the number of ideal boundary points is $\nu_d(p)d^{n-2}$ because $\widehat{\pi}^j_{n,p}$ is a degree $\nu_d(p)d^{p-1}$ branched covering. So the number of critical points counting with multiplicity is equal to $\nu_d(p)d^{n-2}(d-1)$.

By the Riemann-Hurwitz formula, we have

$$g^j_{n,p} = 1+\frac{1}{2}\nu_d(p)d^{n-2}(pd-p-1-d)+\frac{1}{2}(M_{n,p}+K_{n,p})-\frac{1}{2}d^{n-2}(d-1)\sum_{k|p,k<p} k\nu_d(k)\varphi(p/k).$$

Here is a genus computation of some examples.

| $d$ | $n$ | $p$ | $\nu_d(p)$ | $M_{n,p}$ | $K_{n,p}$ | $g_{n,p}$ |
|---|---|---|---|---|---|---|
| 3 | 1 | 1 | 3 | 0 | 0 | 0 |
| 3 | 2 | 1 | 3 | 4 | 2 | 1 |
| 2 | 2 | 2 | 2 | 2 | 0 | 0 |
| 2 | 3 | 2 | 2 | 7 | 1 | 1 |
| 2 | 2 | 3 | 6 | 6 | 0 | 2 |

**Corollary 5.2.** *Fix $n, p \geq 1$, the surfaces $\mathcal{V}^j_{n,p}, 1 \leq j \leq d-1$ are pairwise homeomorphic.*

*Proof.* Topologically, the surface $\mathcal{V}^j_{n,p}$ is completely determined by the genus and the number of ideal boundary points of $\widehat{\mathcal{V}}^j_{n,p}$, whereas these two numbers are independent of $j$. □

**Lemma 5.3.** *All ideal boundary points are critical points of $\widehat{\pi}^j_{n,p}$ with multiplicity $d-1$.*

*Proof.* To prove this lemma, we first give a symbolic description of the dynamics on the filled-in Julia set for a parameter outside the Multibrot set.

If $c \in \mathbb{C}\setminus M_d$, the Julia set of $f_c$ is a Cantor set. If $c \in R_{M_d}(\theta)$ with $\theta \neq 0$ not necessarily periodic, the dynamical rays $R_c(\theta/d)\dots R_c\big((\theta+d-1)/d\big)$ bifurcate on the critical point. The set $R_c(\theta/d)\cup\ldots\cup R_c\big((\theta+d-1)/d\big)\cup\{0\}$ decomposes the complex plane into $d$ connected components. We denote by $U_0$ the component containing the dynamical ray $R_c(0)$ and by $U_1,\dots,U_{d-1}$ the other components in counterclockwise order.

The orbit of a point $x \in K_c$ has an itinerary with respect to this partition. In other words, to each $x \in K_c$, we can associate a sequence in $\mathbb{Z}_d^{\mathbb{N}}$ whose $j$-th entry is equal to $k$ if $f_c^{\circ j-1}(x) \in U_k$ . This gives a map $\iota_c : K_c \to \mathbb{Z}_d^{\mathbb{N}}$, which is bijective for any $c \in \mathbb{C}\setminus M_d$. Moreover, the dynamic of $f_c$ on $K_c$ is conjugate to shift on $\mathbb{Z}_d^{\mathbb{N}}$ via the map $\iota_c$.

Now let $\pi := \pi^j_{n,p}\big|_{\mathcal{E}^j_{n,p,i}}$. The map $\pi : \mathcal{E}^j_{n,p,i} \longrightarrow \mathbb{C}\setminus M_d$ is a covering of degree $d^j_{n,p,i}$. Fix $c_0 \in \mathbb{C}\setminus(M_d\cup R_{M_d}(0))$, $d^j_{n,p,i} = \#(\pi^{-1}(c_0))$. Since $\mathcal{E}^j_{n,p,i}$ is connected, the monodromy group induced by $\pi$, denoted by $\mathrm{Mon}(\pi)$, acts on $\pi^{-1}(c_0)$ transitively. Then fixing any point $(c_0, z_0) \in \pi^{-1}(c_0)$, the set $\pi^{-1}(c_0)$ is exactly the orbit of $(c_0, z_0)$ under $\mathrm{Mon}(\pi)$.

Let $c_t : [0,1] \to \mathbb{C}\setminus M_d$ be a oriented simple closed curve based at $c_0$ such that $c_t$ intersects $R_{M_d}(0)$ at only one point $c_{t_0}$. Let $z_t$ be the $(n,p)$-preperiodic point of $f_{c_t}$

obtained from the analytic continuation of $z_0$ along $c_t$. Note that as $c$ varies in $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, the $(n,p)$-preperiodic points of $f_c$, the dynamical rays $R_c(0)$ and $R_c\big((\theta_c + s)/d\big)$ $(s \in \mathbb{Z}_d)$ move continuously. Consequently, we have

$$\begin{cases} \iota_{c_t}(z_t) &= \iota_{c_0}(z_0) \qquad \text{for } t \in [0,\ t_0) \\ \iota_{c_t}(z_t) &= \iota_{c_0}(z_1) \qquad \text{for } t \in (t_0,\ 1] \end{cases}$$

Furthermore, on one hand, $z_t$ and $R_{c_t}(0)$ move continuously for $t \in [0,1]$. On the other hand, when $c_t$ passes through $R_{M_d}(0)$, the dynamical rays $R_{c_t}\big((\theta_t + s)/d\big)$ $(s \in \mathbb{Z}_d)$ move discontinuously and jump from $R_{c_{t_-}}\big((\theta_{t_-} + s)/d\big)$ to $R_{c_{t_+}}\big((\theta_{t_+} + s + 1)/d\big)$, $t_- < t_0 < t_+$. So if $\iota_{c_0}(z_0) = \beta_n \dots \beta_1 \overline{\epsilon_1 \dots \epsilon_p}$, then

$$\iota_{c_0}(z_1) = (\beta_n + 1) \dots (\beta_1 + 1)\overline{(\epsilon_1 + 1) \dots (\epsilon_p + 1)} \tag{5.5}$$

Hence the map $\sigma_{c_t}$, an element of $\mathrm{Mon}(\pi)$ induced by $c_t$, maps $(c_0, z_0)$ to $(c_0, z_1)$ with $z_1$ satisfying (5.5). Since $\pi_1(\mathbb{C} \setminus M_d, c_0) = \langle c_t \rangle$, then we have

$$\Big(\pi^j_{n,p}\big|_{\mathcal{E}^j_{n,p,i}}\Big)^{-1}(c_0) = \big\{(c_0, z)\big|\iota_{c_0}(z) = (\beta_n + s) \dots (\beta_1 + s)\overline{(\epsilon_1 + s) \dots (\epsilon_p + s)},\ s \in \mathbb{Z}_d\big\}$$

As a consequence, $d^j_{n,p,i} = d$. □

# 6 The Galois group of $Q_{n,p}(c,z)$

The objective here is to study $\mathcal{X}_{n,p}$ from the algebraic point of view by calculating its associated Galois group.

Recall that $\mathbf{C}$ denotes the ring $\mathbb{C}[c]$ and $\mathbf{K}$ is a fixed algebraically closed field containing $\mathbf{C}$. Since the characteristic of $\mathbb{C}(c)$ is 0, any polynomial $f \in \mathbf{C}[z]$ induces a finite Galois extension $\mathbb{C}(c)(f)$ over $\mathbb{C}(c)$ (see [W, Thm. 3.2.6, 2.7.14]), where $\mathbb{C}(c)(f)$ is the splitting field of $f$, and hence a Galois group $G(f) := \mathrm{Gal}(\mathbb{C}(c)(f)/\mathbb{C}(c))$. In particular, we denote the Galois group of $Q_{n,p}$ by $G_{n,p}$.

For each $n \geq 0, p \geq 1$, denote $\mathfrak{R}_{n,p}$ the set of roots of $Q_{n,p} \in \mathbf{C}[z]$. By (3.2), we have $f_c(\mathfrak{R}_{n,p}) = \mathfrak{R}_{n-1,p}$ if $n \geq 1$ and $f_c(\mathfrak{R}_{0,p}) = \mathfrak{R}_{0,p}$. Let us consider

$$\mathfrak{R}_{\leq n,p} := \bigcup_{0 \leq l \leq n} \mathfrak{R}_{l,p}.$$

Then $f_c(\mathfrak{R}_{\leq n,p}) \subset \mathfrak{R}_{\leq n,p}$ and the action of $f_c$ induces a directed graph structure consisting of a certain number of disjoint cycles of order $p$, on each vertex of which is attached a tree of height $n$. More precisely, for each $0 \leq l \leq n$, we consider the roots in $\mathfrak{R}_{l,p}$ as the vertices of level $l$, and two vertices $\Delta_1, \Delta_2 \in \mathfrak{R}_{\leq n,p}$ are connected by an oriented edge from $\Delta_1$ to $\Delta_2$ if $f_c(\Delta_1) = \Delta_2$. Thus $\mathfrak{R}_{\leq n,p}$ has a graph structure, and we denote this graph by $\mathfrak{R}^T_{\leq n,p}$ (see Figure 1).

**Example.** For $d = 3, p = 4, n = 2$, the directed graph $\mathfrak{R}^T_{\leq 2,4}$ has 18 connected component, which are pairwise isomorphic. We draw one in the following.

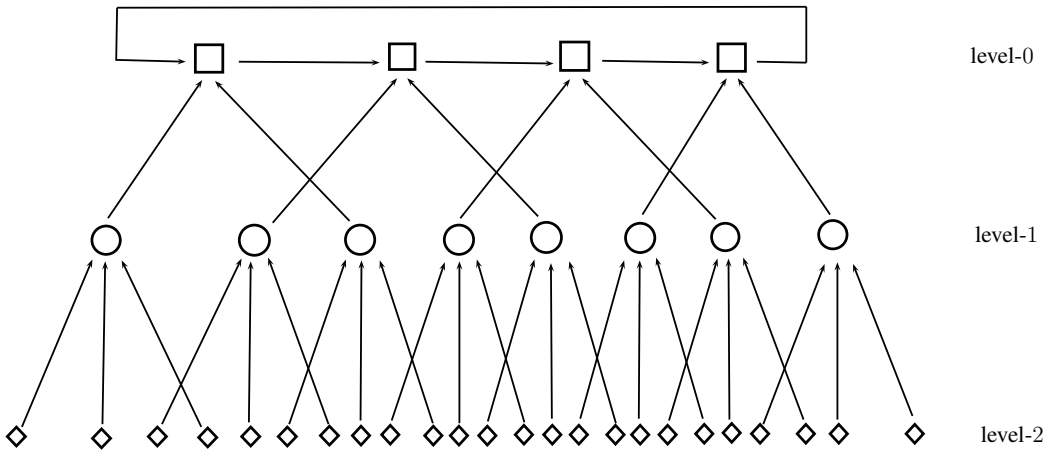


Figure 1. A connected component of $\mathfrak{R}^T_{\leq 2,4}$.

On the other hand, note that $\mathfrak{R}_{\leq n,p}$ is the set of roots of

$$Q_{\leq n,p} := \prod_{l=0}^{n} Q_{l,p} \in \mathbf{C}[z].$$

So, correspondingly, we consider the Galois group $G_{\leq n,p}$ of $Q_{\leq n,p}$. Firstly, we have the following simple result.

**Proposition 6.1.** *For each $n \geq 0, p \geq 1$, we have $G_{n,p} = G_{\leq n,p}$.*

*Proof.* By (3.2), any root of $Q_{l,p} \in \mathbf{C}[z]$ $(0 \leq l \leq n)$ can be written as a polynomial with coefficients in $\mathbf{C}$ of roots of $Q_{n,p}$. It follows that the splitting field of $Q_{\leq n,p} = \Pi_{l=0}^{n} Q_{l,p}$ over $\mathbb{C}(c)$ is the same as that of $Q_{n,p}$ over $\mathbb{C}(c)$. Hence $G_{\leq n,p} = G_{n,p}$. □

By this proposition, computing the Galois group $G_{n,p}$ is equivalent to computing the group $G_{\leq n,p}$. Let $\sigma$ be an element in $G_{\leq n,p}$. Since it fixes the base field $\mathbb{C}(c)$ pointwise, we have $\sigma(\mathfrak{R}_{l,p}) = \mathfrak{R}_{l,p}$ and $f_c \circ \sigma = \sigma \circ f_c$. Hence $\sigma$ induces an *automorphism* of the graph $\mathfrak{R}^T_{\leq n,p}$, i.e., $\sigma$ is a permutation on each $l$-level vertices of $\mathfrak{R}^T_{\leq n,p}$, and $\Delta_1, \Delta_2 \in \mathfrak{R}_{\leq n,p}$ are connected by an edge from $\Delta_1$ to $\Delta_2$ if and only if $\sigma(\Delta_1), \sigma(\Delta_2)$ are connected by an edge from $\sigma(\Delta_1)$ to $\sigma(\Delta_2)$. Clearly, different elements of $G_{\leq n,p}$ induce different automorphisms of $\mathfrak{R}^T_{\leq n,p}$. So $G_{\leq n,p}$ can be seen as a subgroup of $\mathrm{Aut}(\mathfrak{R}^T_{\leq n,p})$, the automorphic group of the graph $\mathfrak{R}^T_{\leq n,p}$.

In the case $d = 2$, Bousch [B] proved that

$$G_{\leq n,p} \simeq \mathrm{Aut}(\mathfrak{R}^T_{\leq n,p}) \simeq H_{\leq n,p}(f_c),$$

where $H_{\leq n,p}(f_c)$ denotes the set of all permutations on $\mathfrak{R}_{\leq n,p}$ that commute with $f_c$. In the general case, the result is similar but needs a small modification. We will exhibit this point in the following.

Let $\sigma$ be an element of the Galois group $G_{\leq n,p}$. As $\sigma$ fixes the field $\mathbb{C}(c)$ pointwise, it must satisfy the following two conditions:

**(P1)** $\sigma$ commutes with $f_c$, i.e., $\sigma \circ f_c = f_c \circ \sigma$.

**(P2)** $\sigma$ commutes with the rotation of argument $1/d$. That is, if $\sigma(\Delta) = \widetilde{\Delta}$ for $\Delta, \widetilde{\Delta} \in \mathfrak{R}_{\leq n,p}$, then $\sigma(\omega^j \Delta) = \omega^j \widetilde{\Delta}$, where $\omega = e^{\frac{2\pi i}{d}}$ and $1 \leq j \leq d-1$

Therefore, if a permutation on $\mathfrak{R}_{\leq n,p}$ wants to be a candidate of elements in the Galois group $G_{\leq n,p}$, it should satisfy the conditions (P1) and (P2).

In fact, in the case of $d = 2$, the condition (P1) implies (P2). To see this, let $\Delta_{n-1}$ be a root of $Q_{n-1,p}$ ($n \geq 1$) and $\Delta_n, -\Delta_n$ be the preimages of $\Delta_{n-1}$ under $f_c$. Let $\sigma$ be an element of $G_{\leq n,p}$ and set $\widetilde{\Delta}_n := \sigma(\Delta_n)$. By condition (P1) $\sigma$ must map $-\Delta_n$ to $-\widetilde{\Delta}_n$, then the condition (P2) holds. Therefore, it is possible for (P1) to be a sufficient condition for a permutation on $\mathfrak{R}_{\leq n,p}$ to be an element of $G_{\leq n,p}$, and Bousch [B] proved this point.

However, the situation has a little difference in the case of $d \geq 3$. Following the notations $\Delta_{n-1}, \Delta_n, \widetilde{\Delta}_n$ and $\sigma$ as above. In this case, $\Delta_{n-1}$ has at least 3 preimages, which are $\Delta_n, \omega\Delta_n, \ldots, \omega^{d-1}\Delta_n$. By condition (P1), we only know that $\sigma$ maps $\{\omega\Delta_n, \ldots, \omega^{d-1}\Delta_n\}$ bijectively to $\{\omega\widetilde{\Delta}_n, \ldots, \omega^{d-1}\widetilde{\Delta}_n\}$, but can not get $\sigma(\omega^j\Delta_n) = \omega^j(\widetilde{\Delta}_n)$ for $1 \leq j \leq d-1$. So, in case of $d \geq 3$, the condition (P2) can not be omitted.

What we would like to prove is that, except (P1) and (P2), no other restrictions are imposed on $G_{\leq n,p}$. The proof is similar to that of Theorem 4 in Chapter III of [B].

**Theorem 6.2.** *The Galois group $G_{\leq n,p}$ consists of all permutations on $\mathfrak{R}_{\leq n,p}$ which commute with $f_c$ and the rotation of argument $1/d$.*

*Proof.* We denote $r_d$ the rotation of argument $1/d$, and $H_{\leq n,p}(f_c, r_d)$ the set of permutations on $\mathfrak{R}_{\leq n,p}$ which commute with $f_c$ and $r_d$. By the definition, it is not difficult to check that $H_{\leq n,p}(f_c, r_d)$ leaves each $\mathfrak{R}_{l,p}$, and hence $\mathfrak{R}_{\leq l,p}$ invariant for $0 \leq l \leq n$.

Define a group homomorphism

$$\phi_n : G_{\leq n,p} \to H_{\leq n,p}(f_c, r_d)$$

such that $\phi_n(\sigma)$ is the restriction of $\sigma$ to $\mathfrak{R}_{\leq n,p}$. According to the discussion above, we just need to prove the surjectivity of $\phi_n$.

Note first that the result is true for $n = 0$ following 6 of Lemma 2.4.

For $n = 1$, since $H_{\leq 1,p}(f_c, r_d)$ leaves $\mathfrak{R}_{0,p}$ invariant, there is a natural homomorphism from $H_{\leq 1,p}(f_c, r_d)$ to $H_{\leq 0,p}(f_c, r_d)$ with $\widetilde{\tau} \mapsto \widetilde{\tau}|_{\mathfrak{R}_{0,p}}$. This homomorphism has an inversion which maps $\tau \in H_{\leq 0,p}(f_c, r_d)$ to $\widetilde{\tau} \in H_{\leq 1,p}(f_c, r_d)$ such that $\widetilde{\tau}|_{\mathfrak{R}_{0,p}} = \tau$ and $\widetilde{\tau}(\omega^j\Delta) = \omega^j\tau(\Delta)$ for each $\Delta \in \mathfrak{R}_{0,p}, j \in [1, d-1]$. Thus $H_{\leq 1,p}(f_c, r_d) \simeq H_{\leq 0,p}(f_c, r_d)$. Note that $G_{1,p} = G_{0,p}$ (because the splitting fields of $Q_{0,p}$ and $Q_{1,p}$ over $\mathbb{C}(c)$ coincide), then the result is true for $n = 1$.

Now we argue by induction on $n$. Assume that $\phi_{n-1} : G_{\leq n-1,p} \to H_{\leq n-1,p}(f_c, r_d)$ is surjective ($n \geq 2$).

Let $\tau \in H_{\leq n,p}(f_c, r_d)$. As $\tau$ commutes with $f_c$, it leaves $\mathfrak{R}_{\leq n-1,p}$ invariant. Then $\tau|_{n-1}$, the restriction of $\tau$ on $\mathfrak{R}_{\leq n-1,p}$, belongs to $H_{\leq n-1,p}(f_c, r_d)$. By the assumption

of induction, there is an element $\sigma_{n-1}$ of $G_{\leq n-1,p}$ with $\phi_{n-1}(\sigma_{n-1}) = \tau|_{n-1}$. According to the Galois theory (see [W, Thm. 2.88]), we can find an element $\sigma$ of $G_{\leq n,p}$ such that its restriction on the splitting field of $Q_{\leq n-1,p}$ over $\mathbb{C}(c)$ coincides with $\sigma_{n-1}$. Set $\tau' := \tau \cdot \phi_n(\sigma)^{-1}$, then $\tau' \in H_{\leq n,p}(f_c, r_d)$ and it fixes $\mathfrak{R}_{\leq n-1,p}$ pointwise. Now it remains to prove that $G_{\leq n,p}$ contains $\tau'$, i.e., there exists $\sigma' \in G_{\leq n,p}$ with $\phi_n(\sigma') = \tau'$, because if so, $\tau = \phi_n(\sigma')\phi_n(\sigma) = \phi_n(\sigma'\sigma)$, which implies $\phi_n$ is surjective.

Set $\kappa_l := \nu_d(p)(d-1)d^{l-1}$ for each $l \geq 1$ (which is the number of roots of $Q_{l,p}$), and denote

$$\mathfrak{R}_{n,p} = \{\Delta_n^i, \omega\Delta_n^i, \ldots, \omega^{d-1}\Delta_n^i;\}_{i=1}^{\kappa_{n-1}}$$

Since $\tau'$ fixes $\mathfrak{R}_{\leq n-1,p}$ pointwise and commutes with both $f_c, r_d$, it can be expressed as a product

$$\tau' = \prod_{i=1}^{\kappa_{n-1}} \big(s_i, s_i+1, \cdots, d-1, 0, \cdots, s_i-1\big),$$

where $(s_i, s_i+1, \cdots, d-1, 0, \cdots, s_i-1)$ is the cyclic permutation on $(\Delta_n^i, \ldots, \omega^{d-1}\Delta_n^i)$ mapping $\Delta_n^i$ to $\omega^{s_i}\Delta_n^i$ and so on. Notice that all these cyclic permutations $(s_i, \ldots, s_i-1)$ are commutable.

The argument in this paragraph is a classical correspondence between the Galois theory and the covering theory; see [Z] for the detail. Let $V_{n,p}$ be the set of singular values of the projection $\pi : \mathcal{X}_{n,p} \to \mathbb{C}$. Then $\pi_{n,p}$ restricts to a cover from the complement of the preimage of $V_{n,p}$ in $\mathcal{X}_{n,p}$ to the complement of $V_{n,p}$ in $\mathbb{C}$. For all $c_0$ not in $V_{n,p}$, there is thus an action of $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ on the roots

$$Z_{n,p} = \{z_n^i, \ldots, \omega^{d-1}z_n^i\}_{i=1}^{\kappa_{n-1}}$$

of $Q_{n,p}(c_0, z)$, which is seen as a polynomial in the variable $z$ and with complex coefficients. By the correspondence between the Galois theory and the covering theory (see [Z, Thm. 1]), there is a (non-unique) choice of bijection between the roots of $Q_{n,p} \in \mathbf{C}[z]$ and the roots of $Q_{n,p}(c_0, z) \in \mathbb{C}[z]$ such that the set of permutations induced by the Galois group on the set $\mathfrak{R}_{\leq n,p}$ is conjugated by this bijection to the set of permutations on $Z_{n,p}$ induced by $\pi_1(\mathbb{C}\setminus V_{n,p}, c_0)$. Thus we get a surjective (not injective, usually) morphism from $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ to the Galois group. Moreover, this bijection is such that any polynomial relation between the $\Delta_n^i$ with coefficient in $\mathbb{C}(c)$ will give a relation between the $z_n^i$, with $c_0$ being substituted to $c$. It implies that the action of $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ on $Z_{n,p}$ preserves commutation with multiplication by $\omega$.

Therefore, by the discussion above, to obtain the required permutation $\tau'$, we only need to find a path in the basic group $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ such that its monodromy action on $\{(z_n^i, \ldots, \omega^{d-1}z_n^i)\}_{i=1}^{\kappa_{n-1}}$ induces the same permutation as $\tau'$. We now show the following result, which is sufficient: for any $i \in [1, \kappa_{n-1}]$, there exists a path in $\pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ such that its monodromy action induces the permutation $(s_i, s_i+1, \cdots, s_i-1)$.

Fix any $i \in [1, \kappa_{n-1}]$. Suppose that $\{(c_0, z_n^i),\ (c_0, \omega z_n^i), \ldots, (c_0, \omega^{d-1}z_n^i)\}$ belong to $\mathcal{V}_{n,p}^t$. Let $\hat{c}$ be an $(n,p)$-Misiurewicz parameter with $(\hat{c}, 0) \in \mathcal{V}_{n,p}^t$. Such $\hat{c}$ exists because the set $D_{n,p}^t$(Misiurewicz) is non-empty (see section 5.2 item 2). By (3.2), $(\hat{c}, \hat{c}) \in \mathcal{X}_{n-1,p}$.

Since $\hat{c}$ is a Misiurewicz parameter and the orbit of $\hat{c}$ does not contain 0, then the point $(\hat{c}, \hat{c})$ belongs to no sets in Lemma 5.1 in the case $l = n-1$. Hence $w = \hat{c}$ is a simple root of the equation $Q_{n-1,p}(\hat{c}, w) = 0$ (in $w$). So by the Implicit Function Theorem, the equation $Q_{n-1,p}(c, w) = 0$ admits a unique solution $w = w(c)$ close to $\hat{c}$ fullfilling $w(\hat{c}) = \hat{c}$. Thus, a neighborhood of $(\hat{c}, 0)$ in $\mathcal{X}_{n,p}$ can be written as

$$\big\{(c, z_c) \cup (c, \omega z_c) \cup \cdots \cup (c, \omega^{d-1} z_c) \mid |c - \hat{c}| < \epsilon\big\}$$

where $z_c$ is one of the preimages of $w(c)$ under $f_c$ nearby 0.

When $c$ makes a small turn around $\hat{c}$, the set $\{z_c, \ldots, \omega^{d-1} z_c\}$ gets a cyclic permutation with $\omega^j z_c$ mapped to $\omega^{j+1} z_c$, because $\pi_{n,p}$ is a degree $d$ covering in a punctured neighborhood of $(\hat{c}, 0)$ (which is shown in §5.2 item 2), and the other $(n,p)$-preperiodic points of $f_c$ remain fixed, since $\pi_{n,p}$ is injective around each point $(\hat{c}, \xi)$ with $\xi$ a non-zero $(n,p)$-preperiodic point of $f_{\hat{c}}$. So if we choose a path $\gamma \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ homotopic to $\hat{c}$, the induced permutation by its monodromy action gives the cyclic permutation $(2, \ldots, d, 1)$ on $(z_n^*, \ldots, \omega^{d-1} z_n^*)$ for a $(n,p)$-preperiodic point $z_n^*$ of $f_{c_0}$ fullfilling $(c_0, z_n^*) \in \mathcal{V}_{n,p}^t$, and keeps the other $(n,p)$-preperiodic points of $f_{c_0}$ fixed. Now we join $(c_0, z_n^i)$ and $(c_0, z_n^*)$ by a curve from $(c_0, z_n^i)$ to $(c_0, z_n^*)$ in $\mathcal{V}_{n,p}^t \setminus \pi_{n,p}^{-1}(V_{n,p})$, and denote its projection under $\pi_{n,p}$ by $\beta$. Then $\beta \in \pi_1(\mathbb{C} \setminus V_{n,p}, c_0)$ and the path $\beta \cdot \gamma^{s_i} \cdot \beta^{-1}$ is what we expect. □

Applying this theorem, we can also characterize the Galois group $G_{\le n,p}$ by the automorphisms of the directed graph $\mathfrak{R}_{\le n,p}^T$, as in the $d = 2$ case. For $d \ge 3$, denote by $\mathrm{Aut}(\mathfrak{R}_{n,p}^T, r_d)$ the set of automorphisms of $\mathfrak{R}_{\le n,p}^T$ that commute with the rotation of argument $1/d$, and by $H_{\le n,p}(f_c, r_d)$ the set of permutations on $\mathfrak{R}_{\le n,p}$ that commute with $f_c$ and the rotation of argument $1/d$.

**Corollary 6.3.** *For $n \ge 0, p \ge 1$, $G_{\le n,p} \simeq \mathrm{Aut}(\mathfrak{R}_{n,p}^T, r_d) \simeq H_{\le n,p}(f_c, r_d)$.*

Following Bousch [B, Chap. 3, III] and Silverman ([Sil, § 3.9]), we express the Galois group $G_{n,p}$ ($n \ge 2$) as a wreath product.

**Definition 6.4.** *Let $G$ be a group and $\Sigma$ be a subgroup of $\mathbf{S}_m$, where $\mathbf{S}_m$ denotes the set of permutations on $\{1, \ldots, m\}$. Denote by $\Sigma \ltimes G^m$ the wreath product of $G$ and $\Sigma$. As a set, it consists of $g = \sigma(g_1, \cdots, g_m)$ where $g_i \in G$ and $\sigma \in \Sigma$. The multiplication is defined by*

$$g \cdot h = \sigma_g(g_1, \cdots, g_m) \cdot \sigma_h(h_1, \cdots, h_m) = \sigma_g \circ \sigma_h(g_{\sigma_h(1)} \cdot h_1, \cdots, g_{\sigma_h(m)} \cdot h_m).$$

*Under this multiplication law, $\Sigma \ltimes G^m$ is a group with $g^{-1} = \sigma_g^{-1}\big(g_{\sigma_g^{-1}(1)}^{-1}, \cdots, g_{\sigma_g^{-1}(1)}^{-1}\big)$ and unit element $(1, \ldots, 1)$.*

In [B], Bousch showed that the Galois group $G_{0,p}$ is isomorphic to the wreath product $\mathbf{S}_{\nu_d(p)/p} \ltimes (\mathbb{Z}/p\mathbb{Z})^{\nu_d(p)/p}$ (see also [Sil, § 3.9]). From the proof of Theorem 6.2, we have seen that $G_{1,p} = G_{0,p}$, so

$$G_{1,p} \simeq \mathbf{S}_{\nu_d(p)/p} \ltimes (\mathbb{Z}/p\mathbb{Z})^{\nu_d(p)/p}.$$

For $n \ge 2$, we can give inductively an isomorphic model of $G_{n,p}$ by a wreath product. Recall that $\kappa_n = \nu_d(p)(d-1)d^{n-1}$ ($n \ge 1$) is the number of roots of $Q_{n,p}$.

**Proposition 6.5.** *For $n \geq 2$, we have $G_{n,p} \cong G_{n-1,p} \ltimes (\mathbb{Z}/d\mathbb{Z})^{\kappa_{n-1}}$, where the action of $G_{n-1,p}$ on $(1, 2, ..., \kappa_{n-1})$ comes from the action of $G_{n-1,p}$ on the roots of $Q_{n-1,p}$, of which there are exactly $\kappa_{n-1}$.*

*Proof.* For $n \geq 2$, we denote $(\Delta_{n-1}^i)_{i=1}^{\kappa_{n-1}}$ the roots of $Q_{n-1,p} \in \mathbf{C}[z]$, and denote

$$( \ \{\Delta_n^i, \ldots, \omega^{d-1}\Delta_n^i\} \ )_{i=1}^{\kappa_{n-1}}$$

the roots of $Q_{n,p}$ such that $f_c(\Delta_n^i) = \Delta_{n-1}^i$.

| $\Delta_{n-1}^1$ | $\Delta_{n-1}^2$ | $\cdots\cdots$ | $\Delta_{n-1}^{\kappa_{n-1}}$ |
|---|---|---|---|
| $\Delta_n^1$ | $\Delta_n^2$ | $\cdots\cdots$ | $\Delta_n^{\kappa_{n-1}}$ |
| $\omega\Delta_n^1$ | $\omega\Delta_n^2$ | $\cdots\cdots$ | $\omega\Delta_n^{\kappa_{n-1}}$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $\omega^{d-1}\Delta_n^1$ | $\omega^{d-1}\Delta_n^2$ | $\cdots\cdots$ | $\omega^{d-1}\Delta_n^{\kappa_{n-1}}$ |

We define a group homomorphism

$$W : G_{n,p} \longrightarrow G_{n-1,p} \ltimes (\mathbb{Z}/d\mathbb{Z})^{\kappa_{n-1}}$$

such that $W(\sigma) = \sigma|_{n-1}(s_1, \ldots, s_{\kappa_{n-1}})$, where $\sigma|_{n-1}$ is the restriction of $\sigma$ on the splitting field of $Q_{n-1,p}$ over $\mathbb{C}(c)$, and the $i$-th digit in $(s_1, \ldots, s_{\kappa_{n-1}})$ is $s_i$ if and only if once $\sigma(\Delta_{n-1}^i) = \Delta_{n-1}^t$ for some $1 \leq t \leq \kappa_{n-1}$, then $\sigma(\Delta_n^i) = \omega^{s_i}\Delta_n^t$. The injectivity of $W$ is straightforward by the action of $G_{n,p}$ on $\mathfrak{R}_{\leq n,p}$ and the subjectivity of $W$ is due to Theorem 6.2. □

Before ending this section, we give a computation of $G_{n,p}$ for some small $n, p$. Note that although $G_{1,p}$ is isomorphic to a subgroup $\mathbf{S}_{\nu_d(p)/p} \ltimes (\mathbb{Z}/p\mathbb{Z})^{\nu_d(p)/p}$ of $\mathbf{S}_{\nu_d(p)}$, it is indeed a subgroup of $\mathbf{S}_{\nu_d(p)(d-1)}$. So mimicking the action of $G_{1,p}$ on

$$\{\omega\Delta_1^1, \ldots, \omega\Delta_1^{\nu_d(p)}; \ \ldots; \ \omega^{d-1}\Delta_1^1, \ldots, \omega^{d-1}\Delta_1^{\nu_d(p)}\},$$

we define a subgroup $\mathbf{P}_{\nu_d(p)(d-1),d}$ of $\mathbf{S}_{\nu_d(p)(d-1)}$ such that $\tau \in \mathbf{P}_{\nu_d(p)(d-1),d}$ if and only if

$$\begin{aligned} & \tau\big(1, \ldots, \nu_d(p); \ \ldots \ ; (d-2)\nu_d(p)+1, \ldots, (d-2)\nu_d(p)+\nu_d(p)\big) \\ = \ & \big(\sigma(1), \ldots, \sigma(\nu_d(p)); \ \ldots \ ; (d-2)\nu_d(p)+\sigma(1), \ldots, (d-2)\nu_d(p)+\sigma(\nu_d(p))\big) \end{aligned}$$

for a $\sigma \in \mathbf{S}_{\nu_d(p)}$. Then $\mathbf{P}_{\nu_d(p)(d-1),d} \simeq \mathbf{S}_{\nu_d(p)} \simeq G_{1,p}$, and $\mathbf{P}_{\nu_d(p)(d-1),d} = \mathbf{S}_{\nu_d(p)}$ in the case $d = 2$. The results of computation are listed in the following table.

| $d$ | $n$ | $p$ | $\nu_d(p)$ | $\kappa_{n-1}$ | $G_{n,p}(d)$ |
|---|---|---|---|---|---|
| 3 | 1 | 1 | 3 | – | $\mathbf{S}_3 \simeq \mathbf{P}_{6,3}$ |
| 3 | 2 | 1 | 3 | 6 | $\mathbf{P}_{6,3} \ltimes (\mathbb{Z}/3\mathbb{Z})^6$ |
| 3 | 3 | 1 | 3 | 18 | $(\mathbf{P}_{6,3} \ltimes (\mathbb{Z}/3\mathbb{Z})^6) \ltimes (\mathbb{Z}/3\mathbb{Z})^{18}$ |
| 2 | 3 | 2 | 2 | 4 | $((\mathbb{Z}/2\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z})^2) \ltimes (\mathbb{Z}/2\mathbb{Z})^4$ |
| 2 | 2 | 3 | 6 | 6 | $((\mathbb{Z}/2\mathbb{Z}) \ltimes (\mathbb{Z}/3\mathbb{Z})^2) \ltimes (\mathbb{Z}/2\mathbb{Z})^6$ |

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