The dynatomic curves for unimodel polynomials are smooth and irreducible

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The dynatomic periodic curves for polynomial $\mathbf{z \mapsto z^d + c}$ are smooth and irreducible

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Abstract

We prove here the smoothness and the irreducibility of the periodic dynatomic curves $(c, z) \in \mathbb{C}^2$ such that z is n-periodic for $z^d + c$, where $d \geq 2$.

We use the method provided by Xavier Buff and Tan Lei in [BT] where they prove the conclusion for $d = 2$. The proof for smoothness is based on elementary calculations on the pushforwards of specific quadratic differentials, following Thurston and Epstein, while the proof for irreducibility is a simplified version of Lau-Schleicher's proof by using elementary arithmetic properties of kneading sequence instead of internal addresses.

1 Introduction

For $c \in \mathbb{C}$, set $f_c(z) = z^d + c$, where $d \geq 2$. For $n \geq 1$, define

$$X_n := \big\{(c, z) \in \mathbb{C}^2 \mid f_c^n(z) = z,\ (f_c^n)'(z) \neq 1 \text{ and for all } 0 < m < n,\quad f_c^m(z) \neq z\big\}.$$

The objective of this note is to give an elementary proof of the following results:

Theorem 1.1. *For every $n \geq 1$, the closure of X_n in $\mathbb{C}^2$ is smooth.*

Theorem 1.2. *For every $n \geq 1$ the closure of X_n in $\mathbb{C}^2$ is irreducible.*

The first example is, as $d = 2$

$$X_1 = \big\{(c, z) \in \mathbb{C}^2 \mid z^2 + c = z\big\} \smallsetminus \big\{(\frac{1}{4}, \frac{1}{2})\big\} = \big\{(c, z) \in \mathbb{C}^2 \mid c = z - z^2\big\} \smallsetminus \big\{(\frac{1}{4}, \frac{1}{2})\big\}$$

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and

$$\overline{X_1} = \{(c,z) \in \mathbb{C}^2 \mid c = z - z^2\}.$$

In the case $d = 2$, Theorem 1.1 was proved by Douady-Hubbard and Buff-Tan in different methods; Theorem 1.2 was proved by Bousch, Morton, Lau-Schleicher and Buff-Tan with different approaches.

Our approach here to the two Theorems is a generalisation to that used by Xavier Buff and Tan Lei in [BT], where they prove the conclusion for $d = 2$. To prove Theorem 1.1, we use elementary calculations on quadratic differentials and Thurston's contraction principle. To prove Theorem 1.2, we use a dynamical method by a purely arithmetic argument on kneading sequences(Lemma 3.2 below).

Section 2 proves the smoothness and Section 3 proves the irreducibility. The two sections can be read independently.

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2 Smoothness of the periodic curves

For $n \geq 1$, and $(c,z) \in \mathbb{C}^2$, we say that z is periodic of *period* n for $f_c : z \mapsto z^d + c$, where $d \geq 2$, if $f_c^{\circ n}(z) = z$ and for all $0 < m < n$, $f_c^m(z) \neq z$. In this case the *multiplier* of z for f_c is defined to be $[f_c^n]'(z)$.

We define

$$X_n := \{(c,z) \in \mathbb{C}^2 \mid z \text{ is of period } n \text{ for } f_c \text{ and of multiplier distinct from } 1\}.$$

The objective of here is to give an elementary proof of the following result:

Theorem 2.1. *For every $n \geq 1$, the closure $\overline{X_n}$ of X_n in $\mathbb{C}^2$ is smooth. More precisely, the boundary ∂X_n is the finite set of $(c,z) \in \mathbb{C}^2$ such that z is of period $m \leq n$ dividing n for f_c whose multiplier is of the form $e^{2\pi i u/v}$ with $u, v \geq 1$ co-prime and $v = n/m$. In a neighborhood of a point $(c_0, z_0) \in \overline{X_n}$, the set $\overline{X_n}$ is locally the graph of a holomorphic map* $\begin{cases} c \mapsto z(c) \text{ with } z(c_0) = z_0 & \text{if } (c_0,z_0) \in X_n \\ z \mapsto c(z) \text{ with } c(z_0) = c_0 \text{ and } c'(z_0) = 0 & \text{if } (c_0,z_0) \in \partial X_n \end{cases}.$

The idea is to prove that some partial derivative of some defining function of $\overline{X_n}$ is non vanishing. Following A. Epstein, we will express this derivative as the coefficient of a quadratic differential of the form $(f_c)_*\mathcal{Q} - \mathcal{Q}$. Thurston's contraction principle gives $(f_c)_*\mathcal{Q} - \mathcal{Q} \neq 0$, therefore the non-nullness of our partial derivative.

2.1 Quadratic differentials and contraction principle

A meromorphic quadratic differential (or in short, a quadratic differential) $\mathcal{Q}$ on $\mathbb{C}$ takes the form $\mathcal{Q} = q\ \mathrm{dz}^2$ with q a meromorphic function on $\mathbb{C}$.

We use $\mathcal{Q}(\mathbb{C})$ to denote the set of meromorphic quadratic differentials on $\mathbb{C}$ whose poles (if any) are all simple. If $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ and U is a bounded open subset of $\mathbb{C}$, the norm

$$\|\mathcal{Q}\|_U := \iint_U |q|$$

is well defined and finite.

For example

$$\|\frac{\mathrm{d}z^2}{z}\|_{\{|z|<R\}} = \int_0^{2\pi}\int_0^R \frac{1}{r} r\, dr d\theta = 2\pi R\ .$$

For $f : \mathbb{C} \to \mathbb{C}$ a non-constant polynomial and $\mathcal{Q} = q\ \mathrm{d}z^2$ a meromorphic quadratic differential on $\mathbb{C}$, the pushforward $f_*\mathcal{Q}$ is defined by the quadratic differential

$$f_*\mathcal{Q} := Tq\ \mathrm{d}z^2 \quad \text{with} \quad Tq(z) := \sum_{f(w)=z} \frac{q(w)}{f'(w)^2}.$$

If $Q \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ also.

The following lemma is a weak version of Thurston's contraction principle.

Lemma 2.2 (contraction principle)**.** *For a non-constant polynomial f and a round disk V of radius large enough so that $U := f^{-1}(V)$ is relatively compact in V, we have*

$$\|f_*\mathcal{Q}\|_V \leq \|\mathcal{Q}\|_U < \|\mathcal{Q}\|_V, \quad \forall\, \mathcal{Q} \in \mathcal{Q}(\mathbb{C}).$$

Proof. The strict inequality on the right is a consequence of the fact that U is relatively compact in V. The inequality on the left comes from

$$\begin{aligned}
\|f_*\mathcal{Q}\|_V &= \iint_{z\in V} \left| \sum_{f(w)=z} \frac{q(w)}{f'(w)^2} \right| \ |\mathrm{d}z|^2 \\
&\leq \iint_{z\in V} \sum_{f(w)=z} \left| \frac{q(w)}{f'(w)^2} \right| \ |\mathrm{d}z|^2 \\
&= \iint_{w\in U} \big|q(w)\big|\ |\mathrm{d}w|^2 = \|\mathcal{Q}\|_U.
\end{aligned}$$

□

Corollary 2.3. *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial and if $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$, then $f_*\mathcal{Q} \neq \mathcal{Q}$.*

Remark 2.1. Thurston's contraction principal says that if $\mathcal{Q}$ is a meromorphic quadratic differential on $\mathbb{P}^1$ and $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function, if one requires $f_*\mathcal{Q} = \mathcal{Q}$ with $\mathcal{Q} \neq 0$, then f is necessarily a Lattès example.

The formulas below appeared in [L] chapter 2, we write them together as a lemma.

Lemma 2.4 (Levin)**.** *For $f = f_c$, we have*

$$\begin{cases} f_*\left(\dfrac{\mathrm{dz}^2}{z}\right) = 0 & \\ f_*\left(\dfrac{\mathrm{dz}^2}{z-a}\right) = \dfrac{1}{f'(a)}\left(\dfrac{\mathrm{dz}^2}{z-f(a)} - \dfrac{\mathrm{dz}^2}{z-c}\right) & \textit{if } a \neq 0 \\ f_*\left(\dfrac{\mathrm{dz}^2}{(z-a)^2}\right) = \dfrac{\mathrm{dz}^2}{(z-f(a))^2} - \dfrac{d-1}{af'(a)}\left(\dfrac{\mathrm{dz}^2}{z-f(a)} - \dfrac{\mathrm{dz}^2}{z-c}\right) & \textit{if } a \neq 0. \end{cases} \tag{2.1}$$

2.2 Proof of Theorem 2.1

Lemma 2.5 (compare with [Mil])**.** *Given $z \in \mathbb{C}$, for $n \geq 0$ and $d \geq 2$, define $z_n : c \mapsto f_c^{\circ n}(z)$ and $\delta_n = f_c'(z_n) = dz_n^{d-1}$. Then*

$$\frac{\mathrm{dz}_n}{dc} = 1 + \delta_{n-1} + \delta_{n-1}\delta_{n-2} + \ldots + \delta_{n-1}\delta_{n-2}\cdots\delta_1.$$

Proof. From $z_n = z_{n-1}^d + c$, $d \geq 2$, we obtain

$$\frac{\mathrm{dz}_n}{dc} = 1 + \delta_{n-1}\frac{\mathrm{dz}_{n-1}}{dc} \quad \text{with} \quad \frac{\mathrm{dz}_0}{dc} = 0.$$

The result follows by induction. □

Proof of Theorem 2.1.

Let $P_n(c,z) := f_c^{\circ n}(z) - z$ and consider the algebraic curve

$$Y_n := \{(c,z) \in \mathbb{C}^2 \mid P_n(c,z) = 0\}.$$

If $(c,z) \in Y_n$, the point z is periodic for f_c of period $m \leq n$. Then m divides n[1]. Therefore Y_n is the set of (c,z) such that z is periodic for f_c of period $m \leq n$ and m dividing n.

As Y_n is a closed subset of $\mathbb{C}^2$, we have $\overline{X_n} \subset Y_n$.

We decompose Y_n into

$$\begin{aligned} Y_n \;=\;& X_n \\ & \sqcup \{(c,z) \mid \; z \text{ is of period } n \text{ for } f_c \text{ with multiplier } 1\} \\ & \sqcup \{(c,z) \mid \; z \text{ is of period } m \text{ for } f_c \text{ with } m < n \text{ and } m \text{ dividing } n\} \end{aligned}$$

We will examine case by case points in Y_n, determine points in $\overline{X_n}$ and establish the smoothness of $\overline{X_n}$ at each of these points.

Case 1. Consider a point $(c_0, z_0) \in X_n \subset Y_n$.

[1] use the formula $0 = f_c^{\circ n}(z) - z = f_c^{\circ km+\ell}(z) - z = f_c^{\circ \ell}(f_c^{\circ km}(z)) - z = f_c^{\circ \ell}(z) - z$ and the minimality of m to conclude that m divides n.

If $(c, z) \in Y_n$ is close to $(c_0, z_0) \in X_n$, the points of the orbit of z are close to points of the orbit of z_0 and there are therefore at least n distinct points in the orbit of z. It follows that the period of z is equal to n. This shows that in a neighborhood of (c_0, z_0), the curves X_n and Y_n coincide. It suffices to show that Y_n is smooth in a neighborhood of (c_0, z_0). As $[f_{c_0}^{\circ n}]'(z_0) \neq 1$, we have

$$\frac{\partial P_n}{\partial z}(c_0, z_0) \neq 0.$$

The implicit function theorem implies that Y_n, therefore X_n, is smooth in a neighborhood of (c_0, z_0).

Case 2. Now consider a point $(c_0, z_0) \in Y_n$ such that z_0 is of period equal to n for f_{c_0} with multiplier 1.

Fix any $\ell \geq n$ that is a multiple of n. And consider P_ℓ and Y_ℓ. We know that

$$(c_0, z_0) \in Y_\ell \quad \text{and} \quad [f_{c_0}^{\ell}]'(z_0) = 1 \ . \tag{2.2}$$

Claim. For any triple (c_0, z_0, ℓ) satisfying (2.2), we have $\dfrac{\partial P_\ell}{\partial c}(c_0, z_0) \neq 0$.

Proof. For $k \geq 0$, define inductively $z_{k+1} = f_{c_0}(z_k)$ and define $\delta_k := f'_{c_0}(z_k)$. We have, by Lemma 2.5

$$\frac{\partial P_\ell}{\partial c}(c_0, z_0) = \frac{d}{dc}(f_c^{\circ \ell}(z_0) - z_0)\Big|_{c_0} = 1 + \delta_{\ell-1} + \delta_{\ell-1}\delta_{\ell-2} + \ldots + \delta_{\ell-1}\delta_{\ell-2}\cdots\delta_1.$$

Now consider the quadratic differential $\mathcal{Q} \in \mathcal{Q}(\mathbb{C})$ defined by

$$\mathcal{Q}(z) := \sum_{k=0}^{\ell-1} \frac{\rho_k}{z - z_k}\,\mathrm{d}z^2, \quad \text{with } \rho_k = \delta_{\ell-1}\delta_{\ell-2}\cdots\delta_k.$$

Applying Lemma 2.4, and writing f for f_{c_0}, we obtain

$$f_*\mathcal{Q}(z) = \sum_{k=0}^{\ell-1} \frac{\rho_k}{\delta_k}\left(\frac{\mathrm{d}z^2}{z - z_{k+1}} - \frac{\mathrm{d}z^2}{z - c_0}\right) = \mathcal{Q}(z) - \frac{\partial P_\ell}{\partial c}(c_0, z_0) \cdot \frac{\mathrm{d}z^2}{z - c_0}.$$

By Corollary 2.3, we can not have $f_*\mathcal{Q} = \mathcal{Q}$. It follows that

$$\frac{\partial P_\ell}{\partial c}(c_0, z_0) \neq 0.$$

This ends the proof of the claim.

Now let $\ell = n$, by implicit function theorem,there exists unique locally holomorphic function $c(z)$ with $f_{c(z)}^n(z) = z, c(z_0) = z_0$ and $c'(z_0) = 0$(for $\frac{\partial P_\ell}{\partial z}(c_0, z_0) = 0$). Then there is neighborhood U of (c_0, z_0) in $\mathbb{C}^2$ such that

$$Y_n \cap U = \{(c(z), z) | |z - z_0| < \varepsilon\}.$$

As z_0 is a n periodic point of f_{c_0} and the map $z \mapsto [f_{c(z)}^{\circ n}]'(z)$ is holomorphic and **can not be**[2] constantly 1, we can choose ε small enough such that z is n periodic point of $f_{c(z)}$ with multiplier $\neq 1$ for $|z - z_0| < \varepsilon$. Then

$$U \cap Y_n \setminus \{(c_0, z_0)\} \subset U \cap X_n \subset U \cap Y_n.$$

It follows $(c_0, z_0) \in \partial X_n$ and $U \cap Y_n$ is a neighborhood of (c_0, z_0) on $\overline{X_n}$. Then $\overline{X_n}$ is smooth at (c_0, z_0) and parametered locally by z.

Case 3. Finally consider $(c_0, z_0) \in Y_n$ so that z_0 is of period $m < n$ for f_{c_0} with m dividing n.

Note that $Y_m \subset Y_n$.

If $[f_{c_0}^n]'(z_0) \neq 1$ then $[f_{c_0}^m]'(z_0) \neq 1$. By the existence and the unicity of the implicit function theorem the local solutions of $f_c^n(z) - z = 0$ and $f_c^m(z) = z$ coincide, that is, Y_m and Y_n coincide locally. So at point (c_0, z_0), Y_n is locally the graph of a holomorphic function $z(c)$ with $z(c_0) = z_0$ and $z(c)$ is m periodic point of f_c. It follows that $(c_0, z_0) \notin \overline{X_n}$.

If $[f_{c_0}^n]'(z_0) = 1$ and $[f_{c_0}^m]'(z_0) = 1$, then both triples (c_0, z_0, m) and (c_0, z_0, n) satisfy (2.2) . The claim in Case 2 and implicit function theorem imply that Y_m and Y_n again coincide in a neighborhood of (c_0, z_0). For the same reason as above, $(c_0, z_0) \notin \overline{X_n}$.

Set $\rho := [f_{c_0}^m]'(z_0)$. We consider now the only remaining case $\rho \neq 1$ and $\rho^{n/m} = [f_{c_0}^n]'(z_0) = 1$.

Fix any integer $s \geq 2$ such that $\rho^s = 1$. Let c_* be any point outside Mandelbrot set, then each zero point of $f_{c_*}^m(z) - z$ is simple. It follows that $f_{c_*}^m(z) - z$ divides $f_{c_*}^{\circ ms}(z) - z$. Since c_* is any point outside Mandelbrot set, the polynomial $f_c^m(z) - z$ must divides $f_c^{ms}(z) - z$. Let $P(c, z)$ be the polynomial defined by the equation:

$$f_c^{\circ ms}(z) - z = \left(f_c^{\circ m}(z) - z\right) \cdot P(c, z). \tag{2.3}$$

Claim. Let $Z_s := \{(c, z) \mid P(c, z) = 0\}$. Then $(c_0, z_0) \in Z_s$ and there is a neighborhood V of (c_0, z_0) in $\mathbb{C}^2$ such that

$$Z_s \cap V = \{(c(z), z) || z - z_0| < \varepsilon_0, c(z) \text{ is holomorphic with } c(z_0) = c_0 \text{ and } c'(z_0) = 0\}.$$

Proof. We will prove at first that the map $z \mapsto f_{c_0}^{ms}(z) - z$ has a zero of order at least 3 at z_0. Define $F(z) = f^m(z + z_0) - z_0$, then it is equivalent to show the function $F^s(z) := f_{c_0}^{ms}(z + z_0) - z_0$ has a local expansion $z + O(z^3)$ at 0. We have $F(z) = \rho z + az^2 + O(z^3)$ in a neighborhood of 0. One checks by induction

$$\forall\, k \geq 1, \quad F^{\circ k}(z) = \rho^k z + a\rho^{k-1}(1 + \rho + \rho^2 + \cdots + \rho^{k-1})z^2 + O(z^3)\ .$$

[2]One can prove that $D := \{(c, z) \mid f_c^{\circ n}(z) = z, [f_c^n]'(z) = 1\}$ is finite as follows: Denote by $X(c)$, resp. $Y(z)$ the resultant of the two polynomials $f_c^{\circ n}(z) - z$ and $[f_c^n]'(z) - 1$ considered as polynomials of z, resp. of c. Then $X(c)$ is a polynomial of c, resp. $Y(z)$ is a polynomial of z. The projection of D to each coordinate equals the zeros of X, resp. of Y. As no point of the form $(0, z)$, $(c, 0)$ is in D, we have $X(0) \neq 0 \neq Y(0)$ so X, Y each has finite many roots. As $D \subset (X^{-1}(0) \times \mathbb{C}) \cap (\mathbb{C} \times Y^{-1}(0))$ we know that D is finite.

But $\rho \neq 1$ and $\rho^s = 1$, it follows that $1 + \rho + \rho^2 + \cdots + \rho^{s-1} = 0$ and $F^{\circ s}(z) = z + O(z^3)$.

Since $z \mapsto f_{c_0}^{\circ m}(z) - z$ has a simple zero, we see from (2.3) that $z \mapsto P(c_0, z)$ has a zero of order at least 2 at z_0. Therefore $(c_0, z_0) \in Z_s$ and

$$\frac{\partial P}{\partial z}(c_0, z_0) = 0. \tag{2.4}$$

We proceed now to prove

$$\frac{\partial P}{\partial c}(c_0, z_0) \neq 0. \tag{2.5}$$

This will be down in two steps:

Step 1. Let $Q(c, z) := f_c^{\circ m}(z) - z$. We have

$$Q(c_0, z_0) = 0 \quad \text{and} \quad \frac{\partial Q}{\partial z}(c_0, z_0) = \rho - 1 \neq 0.$$

According to the implicit function theorem, there is a germ of a holomorphic function $\zeta : (\mathbb{C}, c_0) \to (\mathbb{C}, z_0)$ with $Q\big(c, \zeta(c)\big) = 0$. In other words, $\zeta(c)$ is a periodic point of period m for f_c. Let ρ_c denote the multiplier of $\zeta(c)$ for f_c and set

$$\dot{\rho} := \frac{d\rho_c}{dc}\Big|_{c_0}.$$

Lemma 2.6. *We have*

$$\frac{\partial P}{\partial c}(c_0, z_0) = \frac{s \cdot \dot{\rho}}{\rho(\rho - 1)}.$$

Proof. Differentiating the equation (2.3) with respect to z, and then evaluating at $\big(c, \zeta(c)\big)$, we get:

$$\rho_c^s - 1 = (\rho_c - 1) \cdot P\big(c, \zeta(c)\big) + \underbrace{\big(f_c^m\big(\zeta(c)\big) - \zeta(c)\big)}_{=0} \cdot \frac{\partial P}{\partial z}\big(c, \zeta(c)\big) = (\rho_c - 1) \cdot P\big(c, \zeta(c)\big).$$

Setting

$$R(c) := P\big(c, \zeta(c)\big) = \frac{\rho_c^s - 1}{\rho_c - 1},$$

we have

$$R'(c_0) = \frac{\partial P}{\partial c}(c_0, z_0) + \underbrace{\frac{\partial P}{\partial z}(c_0, z_0)}_{=0} \cdot \zeta'(c_0) = \frac{\partial P}{\partial c}(c_0, z_0).$$

Using $\rho^s = 1$ and $\rho^{s-1} = 1/\rho$, we deduce that

$$\frac{\partial P}{\partial c}(c_0, z_0) = \frac{d}{dc}\left(\frac{\rho_c^s - 1}{\rho_c - 1}\right)\Big|_{c_0} = \left(\frac{s\rho^{s-1}}{\rho - 1} - \frac{\rho^s - 1}{(\rho - 1)^2}\right)\frac{d\rho_c}{dc}\Big|_{c_0} = \frac{s \cdot \dot{\rho}}{\rho(\rho - 1)}. \qquad \square$$

Step 2. $\dot{\rho} \neq 0$. The proof of this fact will be postponed to the following section 2.3 using quadratic differential with double poles (see also [DH] for a parabolic implosion approach).

This ends the proof of (2.5) , as well as the proof of the claim by combining (2.5) and the implicit function theorem plus the observation that $(c_0, z_0) \in Z_s$.

Write now $\rho = e^{2\pi i u/v}$ with u, v co-prime and $v > 0$. Then any s satisfying $\rho^s = 1$ takes the form $s = kv$ for some integer $k \geq 1$. With the same reason as that of existence of polynomial $P(c, z)$ in (2.3), there are polynomials g, h such that

$$f_c^{\circ ms}(z) - z = f_c^{\circ mkv}(z) - z = (f_c^{\circ mv}(z) - z)g(c, z) = (f_c^{\circ m}(z) - z)h(c, z)g(c, z) \ .$$

By definition we have $Z_s = \{(c, z) \mid g(c, z)h(c, z) = 0\} \supset \{(c, z) \mid h(c, z) = 0\} = Z_v$. By the claim in Case 3, we conclude that Z_s and Z_v coincide in a neighborhood of (c_0, z_0) as the graph of a single holomorphic function $c(z)$ with vanishing derivative at z_0.

Remark:(1) If necessary, we can decrease ε_0 in claim of case 3 such that $f^m_{c(z)}(z) - z \neq 0$ for $0 < |z - z_0| < \varepsilon_0$. Otherwise, there exist a sequence $\{z_k\}$ with $z_k \to z_0$ (correspondingly, $c_k := c(z_k) \to c_0$) such that $f^m_{c_k}(z_k) - z_k = 0$ and $h(c_k, z_k)g(c_k, z_k) = 0$. It follows that $[f^{\circ ms}_{c_k}]'(z_k) - 1 = 0$, that is, $\{c_k\}$ is a sequence of parabolic parameter with period of parabolic orbit less than m converging to c_0. It is impossible.

(2) $\qquad Z_s = (Y_n \setminus Y_m) \cup \{(c_0, z_0)\}, \ X_n \subset Y_n \setminus Y_m$

Lemma 2.7. *There exists $0 < \varepsilon_1 < \varepsilon_0$ such that z is mv periodic point of $f_{c(z)}$ with multiplier$\neq 1$ for $0 < |z - z_0| < \varepsilon_1$. $c(z)$ is defined in the claim of case 3.*

Proof. Note that $P(c(z), z) = 0$ implies z is periodic point of $f_{c(z)}$ with period less than ms. As $(f^m_{c_0})'(z_0) = \rho = e^{2\pi i u/v}$, by lemma 3.9 below, when c is close enough to c_0, the orbit of f_{c_0} containing z_0 splits into two periodic orbit of f_c with period m and mv. Then we can choose $\varepsilon_1 < \varepsilon_0$ such that z belongs to one of the two splitted orbits of $f_{c(z)}$ for $0 < |z - z_0| < \varepsilon_1$. By remark (1), the period of z under $f_{c(z)}$ must be mv. The parabolic parameter in M_d with period of parabolic point less than a fixed number are finite, so we can decrease ε_1 if necessary, such that $c(z)$ is not parabolic parameter for $0 < |z - z_0| < \varepsilon_1$. □

Now let V_1 be a neighborhood of (c_0, z_0) in $\mathbb{C}^2$ with property that

$$V_1 \cap Z_s = V_1 \cap Z_v = \{(c(z), z) || z - z_0| < \varepsilon_1\}.$$

If $n = mv$, by lemma 2.7 and remark (2), we have

$$(V_1 \cap Z_v) \setminus \{(c_0, z_0)\} \subset V_1 \cap X_n \subset V_1 \cap (Y_n \setminus Y_m) = (V_1 \cap Z_v) \setminus \{(c_0, z_0)\}.$$

It follows $(c_0, z_0) \in \partial X_n$ and $\overline{X_n}$ coincides with Z_v at neighborhood of (c_0, z_0). Then $\overline{X_n}$ is smooth at point (c_0, z_0)

If $n = mvk$ for some $k > 1$

$$(V_1 \cap Z_s) \setminus \{(c_0, z_0)\} \subset V_1 \cap (Y_n \setminus Y_m) = (V_1 \cap Z_s) \setminus \{(c_0, z_0)\}.$$

Then $V_1 \cap Z_s$ is the neighborhood of (c_0, z_0) in $(Y_n \setminus Y_m) \cup \{(c_0, z_0)\}$. For $X_n \subset Y_n \setminus Y_m$ and $X_n \cap (V_1 \cap Z_s) = \emptyset$, we have $(c_0, z_0) \notin \overline{X_n}$ □

2.3 Quadratic differentials with double poles

Set $f := f_{c_0}$,

$$z_k := f^k(z_0), \quad \delta_k := dz_k^{d-1} = f'(z_k), \quad \zeta_k(c) := f_c^{\circ k}\big(\zeta(c)\big) \quad \text{and} \quad \dot{\zeta}_k := \zeta_k'(c_0).$$

Then

$$\zeta_{k+1}(c) = f_c\big(\zeta_k(c)\big) \quad \text{and} \quad \zeta_m = \zeta_0.$$

Since

$$\delta_0\delta_1 \cdots \delta_{m-1} = \rho \neq 0,$$

there is a unique m-tuple $(\mu_0, \ldots, \mu_{m-1})$ such that

$$\mu_{k+1} = \frac{\mu_k}{dz_k^{d-1}} - \frac{d-1}{dz_k^d},$$

where the indices are considered to be modulo m.

Now consider the quadratic differential $\mathcal{Q}$ (with double poles) defined by

$$\mathcal{Q} := \sum_{k=0}^{m-1} \left(\frac{1}{(z - z_k)^2} + \frac{\mu_k}{z - z_k} \right) \mathrm{dz}^2.$$

Lemma 2.8 (Compare with [L])**.** *We have*

$$f_*\mathcal{Q} = \mathcal{Q} - \frac{\dot{\rho}}{\rho} \cdot \frac{\mathrm{dz}^2}{z - c_0}.$$

Proof. By construction of $\mathcal{Q}$ and the calculation of $f_*\mathcal{Q}$ in Lemma 2.4, the polar parts of $\mathcal{Q}$ and $f_*\mathcal{Q}$ along the cycle of z_0 are identical. But $f_*\mathcal{Q}$ has an extra simple pole at the critical value c_0 with coefficient

$$\sum_{k=0}^{m-1} \left(-\frac{\mu_k}{dz_k^{d-1}} + \frac{d-1}{dz_k^d} \right) = -\sum_{k=0}^{m-1} \mu_{k+1}.$$

We need to show that this coefficient is equal to $-\frac{\dot{\rho}}{\rho}$.

Using $\zeta_{k+1}(c) = \zeta_k(c)^d + c$, we get

$$\dot{\zeta}_{k+1} = dz_k^{d-1}\dot{\zeta}_k + 1.$$

It follows that

$$\dot\zeta_{k+1}\mu_{k+1} - \mu_{k+1} = dz_k^{d-1}\dot\zeta_k\mu_{k+1} = \dot\zeta_k\mu_k - \frac{(d-1)\dot\zeta_k}{z_k}.$$

Therefore

$$\sum_{k=0}^{m-1}\mu_{k+1} = \sum_{k=0}^{m-1}\left(\dot\zeta_{k+1}\mu_{k+1} - \dot\zeta_k\mu_k + \frac{(d-1)\dot\zeta_k}{z_k}\right) = (d-1)\sum_{k=0}^{m-1}\frac{\dot\zeta_k}{z_k} = \frac{\dot\rho}{\rho},$$

where last equality is obtained by evaluating at c_0 of the logarithmic derivative of

$$\rho_c := \prod_{k=0}^{m-1} d\zeta_k^{d-1}(c).$$

□

Lemma 2.9 (Epstein[E]). *We have $f_*\mathcal{Q} \neq \mathcal{Q}$.*

Proof. The proof rests again on the contraction principle, but we can not apply directly Lemma 2.2 since $\mathcal{Q}$ is not integrable near the cycle $\langle z_0, \ldots, z_{m-1}\rangle$. Consider a sufficiently large round disk V so that $U := f^{-1}(V)$ is relatively compact in V. Given $\varepsilon > 0$, we set

$$V_\varepsilon := \bigcup_{k=1}^{m} f^k\big(D(z_0,\varepsilon)\big) \quad \text{and} \quad U_\varepsilon := f^{-1}(V_\varepsilon).$$

When ε tends to 0, we have

$$\|f_*\mathcal{Q}\|_{V-V_\varepsilon} \le \|\mathcal{Q}\|_{U-U_\varepsilon} = \|\mathcal{Q}\|_{V-V_\varepsilon} - \|\mathcal{Q}\|_{V-U} + \|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon} - \|\mathcal{Q}\|_{U_\varepsilon-V_\varepsilon}.$$

If we had $f_*\mathcal{Q} = \mathcal{Q}$, we would have

$$0 < \|\mathcal{Q}\|_{V-U} \le \|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon}.$$

However, $\|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon}$ tends to 0 as ε tends to 0, which is a contradiction. Indeed, $\mathcal{Q} = q(z)dz^2$, the meromorphic function q is equivalent to $\dfrac{1}{(z-z_0)}$ as z tends to z_0. In addition, since the multiplier of z_0 has modulus 1,

$$D(z_0,\varepsilon) \subset U_\varepsilon - V_\varepsilon \subset D(z_0,\varepsilon') \quad \text{with} \quad \frac{\varepsilon'}{\varepsilon} \xrightarrow{\varepsilon\to 0} 1.$$

Therefor,

$$\|\mathcal{Q}\|_{V_\varepsilon-U_\varepsilon} \le \int_0^{2\pi}\int_\varepsilon^{\varepsilon'} \frac{1+o(1)}{r^2} r\,dr\,d\theta = 2\pi(1+o(1))\log\frac{\varepsilon'}{\varepsilon} \xrightarrow{\varepsilon\to 0} 0$$

□

The fact $\dot\rho \neq 0$ follows from the above two lemmas.

3 The irreducibility of the periodic curves

Recall that f_c denote the polynomial $z \mapsto z^d + c$, where $d \geq 2$, and we have defined

$$X_n := \left\{(c, z) \in \mathbb{C}^2 \mid f_c^n(z) = z,\ [f_c^n]'(z) \neq 1 \text{ and for all } 0 < m < n,\quad f_c^m(z) \neq z\right\}.$$

The objective here is to prove:

Theorem 3.1. *For every $n \geq 1$, the set X_n is connected.*

It follows immediately that the closure of X_n in $\mathbb{C}^2$ is irreducible.

3.1 Kneading sequences

Set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $\tau : \mathbb{T} \to \mathbb{T}$ be the angle map

$$\tau : \mathbb{T} \ni \theta \mapsto d\theta \in \mathbb{T},\ d \geq 2.$$

We shall often make the confusion between an angle $\theta \in \mathbb{T}$ and its representative in $[0, 1[$. In particular, the angle $\theta/d \in \mathbb{T}$ is the element of $\tau^{-1}(\theta)$ with representative in $[0, 1/d[$ and the angle $(\theta + (d-1))/d$ is the element of $\tau^{-1}(\theta)$ with representative in $[(d-1)/d, 1[$.

Every angle $\theta \in \mathbb{T}$ has an associated kneading sequence $\nu(\theta) = \nu_1\nu_2\nu_3\ldots$ defined by

$$\nu_k = \begin{cases} 1 & \text{if } \tau^{k-1}(\theta) \in \left]\frac{\theta}{d}, \frac{\theta+1}{d}\right[, \\ 2 & \text{if } \tau^{k-1}(\theta) \in \left]\frac{\theta+1}{d}, \frac{\theta+2}{d}\right[, \\ . & \\ . & \\ . & \\ d-1 & \text{if } \tau^{k-1}(\theta) \in \left]\frac{\theta+(d-2)}{d}, \frac{\theta+(d-1)}{d}\right[, \\ 0 & \text{if } \tau^{k-1}(\theta) \in \mathbb{T} \setminus \left[\frac{\theta}{d}, \frac{\theta+(d-1)}{d}\right], \\ \star & \text{if } \tau^{k-1}(\theta) \in \left\{\frac{\theta}{d}, \frac{\theta+1}{d}, \ldots, \frac{\theta+(d-2)}{d}, \frac{\theta+(d-1)}{d}\right\}. \end{cases}$$

For example,

- as $d = 3$, $\nu(\frac{1}{7}) = \overline{12102\star}$ and $\nu(\frac{27}{28}) = \overline{22200\star}$;

We shall say that an angle $\theta \in \mathbb{T}$, periodic under τ, is *maximal in its orbit* if its representative in $[0, 1)$ is maximal among the representatives of $\tau^j(\theta)$ in $[0, 1)$ for all $j \geq 1$. If the period is n and the d-expansion $(d \geq 2)$ of θ is $.\overline{\varepsilon_1 \ldots \varepsilon_n}$, then θ is maximal

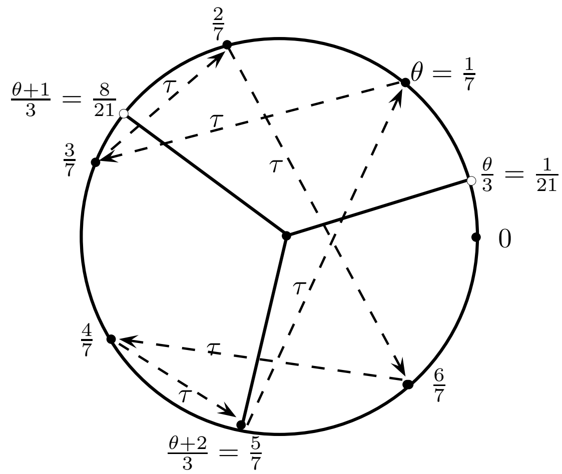


Figure 1: As $d = 3$, the kneading sequence of $\theta = 1/7$ is $\nu(1/7) = \overline{12102\star}$

in its orbit if and only if the periodic sequence $\overline{\varepsilon_1 \dots \varepsilon_n}$ is maximal (in the lexicographic order) among its shifts. For example, as $d = 4$, $\dfrac{5}{31} = .\overline{02211}$ is not maximal in its orbit but $\dfrac{20}{31} = .\overline{22110}$ is maximal in the same orbit.

The following lemma indicates cases where the $d-$expansion ($d \geq 2$) and the kneading sequence coincide.

Lemma 3.2 (Realization of kneading sequences)**.** *Let $\theta \in \mathbb{T}$ be a periodic angle which is maximal in its orbit and let $.\overline{\varepsilon_1 \dots \varepsilon_n}$ be its d-expansion $(d \geq 2)$. Then, $\varepsilon_n \in \{0, 1, 2, \dots, d-2\}$ and the kneading sequence $\nu(\theta)$ is equal to $\overline{\varepsilon_1 \dots \varepsilon_{n-1}\star}$.*

For example,

- as $d = 3$ $\qquad \dfrac{13}{14} = .\overline{221001}$ and $\nu(\theta) = \overline{22100\star}$.
- as $d = 4$ $\qquad \dfrac{28}{31} = .\overline{32130}$ and $\nu(\theta) = \overline{3213\star}$.

Proof. Since θ is maximal in its orbit under τ, the orbit of θ is disjoint from $]\dfrac{\theta}{d}, \dfrac{1}{d}] \bigcup]\dfrac{\theta+1}{d}, \dfrac{2}{d}] \bigcup \dots \bigcup]\dfrac{\theta+(d-2)}{d}, \dfrac{d-1}{d}] \bigcup]\theta, 1]$. It follows that the orbit $\tau^j(\theta)$, $j = 0, 1, \dots, n-2$ have the same itinerary relative to the two partitions $\mathbb{T} - \{0, \dfrac{1}{d}, \dfrac{2}{d}, \dots, \dfrac{d-2}{d}, \dfrac{d-1}{d}\}$ and $\mathbb{T} - \{\dfrac{\theta}{d}, \dfrac{\theta+1}{d}, \dots, \dfrac{\theta+(d-2)}{d}, \dfrac{\theta+(d-1)}{d}\}$ (see Figure 2). The first one gives the d-expansion ($d \geq 2$) whereas the second gives the kneading sequence. Therefore, the kneading sequence of θ is $\overline{\varepsilon_1 \dots \varepsilon_{n-1}\star}$. Since $\tau^{n-1}(\theta) \in \tau^{-1}(\theta) = \{\dfrac{\theta}{d}, \dfrac{\theta+1}{d}, \dots, \dfrac{\theta+(d-1)}{d}\}$ and

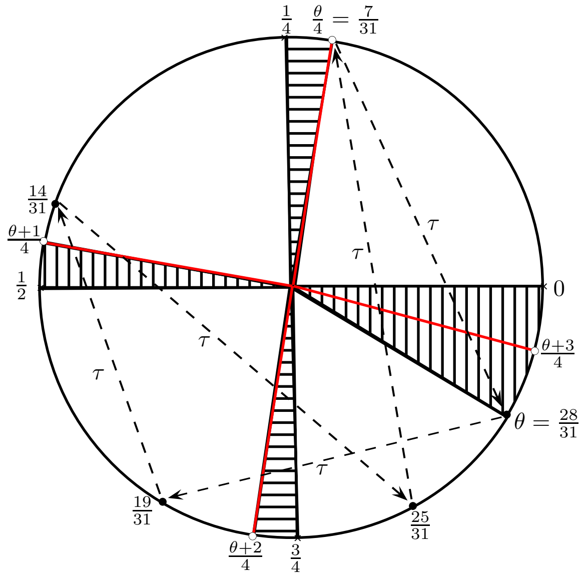


Figure 2: As $d = 4$, the kneading sequence of $\theta = 28/31$ is $\nu(28/31) = \overline{3213\star}$

since $\dfrac{\theta + (d-1)}{d} \in \,]\theta, 1]$, we must have $\tau^{n-1}(\theta) = \{\dfrac{\theta}{d}, \dfrac{\theta+1}{d}, \dots, \dfrac{\theta+(d-2)}{d}\} < \dfrac{d-1}{d}$. So ε_n, as the first digit of $\tau^{n-1}(\theta)$, must be in $\{0, 1, 2, \dots, d-2\}$. $\square$

3.2 Cyclic expression of kneading sequence

$X = \{0, 1, \dots, d-1\} (d \geq 2)$ is an alphabet. $X^\star$ is the set of all sequence of symbols from X with finite length, that is,

$$X^\star = \{\nu_1 \dots \nu_t | \nu_i \in X, t \in \mathbb{N}^\star\}.$$

The element of $X^\star$ is called word, its length is denoted by $|\cdot|$. For any $w \in X^\star$, w can be written as $u^n := \underbrace{u \dots u}_{n}$ with $u \in X^\star$ and $n \geq 1$.

For example: $\quad 121212 = 12^3, \quad 1234 = 1234.$

Definition 3.3. *A word is called primitive if it is not the form u^n for any $n > 1, u \in X^\star$.*

The following lemma is a basic result about primitive words due to F.W.Levi. One can refer to [KM] for the proof.

Lemma 3.4 (F.W.Levi)**.** *For each $w \in X^\star$, there exists an unique primitive word $a(w)$ such that $w = a(w)^n$ for some $n \geq 1$.*

$a(w)$ is called the primitive root of w, this lemma means the primitive root of a word is unique. Let w be a word, we denote by L_w the set of all words different from w only at the last digit.

Lemma 3.5. *If w is a non-primitive word, then any word in L_w is primitive.*

Proof. As w is not primitive, then $w = a^m$ where a is the primitive root of w and $m > 1$. w' is any element of L_w, then $w' = a^{m-1}a'$ for some $a' \in L_a$. Now assume w' is not primitive, then $w' = z^n$ where z is the primitive root of w' and $n > 1$. Obviously $|z| \neq |a|$.

If $|z| < |a|$, then $n > m \geq 2$ and $a = zb$ for some $b \in X^\star$.

$$a^{m-1}a' = z^n \Longrightarrow za^{m-1}a' = a^{m-1}a'z \Longrightarrow za^{m-1}a' = zba^{m-2}a'z \Longrightarrow$$

$$\exists v \in X^\star, s.t\ a = bv, |v| = |z| \Longrightarrow a^{m-1}bv' = ba^{m-2}a'z(a' = bv') \Longrightarrow$$

$$v' = z \text{ and } a^{m-1}b = ba^{m-2}a' \Longrightarrow a^{m-2}bvb = ba^{m-2}a' \Longrightarrow a' = vb.$$

It is a contradiction to $a = zb$.

If $|z| > |a|$, then there exists $z' \in L_z$ such that $z^{n-1}z' = a^m = w$ with $m > n \geq 2$. It reduces to the case above.

□

Now, let θ be a periodic angle with period $n \geq 2$. $\nu(\theta)$ is the kneading sequence of θ.

Definition 3.6. *If there is a word $w = \nu_1 \dots \nu_t$ such that $\nu(\theta) = \overline{w^{s-1}w_\star} := \overline{\underbrace{w \dots w}_{s-1} w_\star}$, where $w_\star = \nu_1 \dots \nu_{t-1}\star$ and t is a proper factor of n with $ts = n$, then $\nu(\theta)$ is called cyclic, otherwise $\nu(\theta)$ is called acyclic.*

Definition 3.7. $\nu(\theta) = \overline{w^{s-1}w_\star}$ *is cyclic. If w is a primitive word, we call $\overline{w^{s-1}w_\star}$ a cyclic expression of $\nu(\theta)$.*

The following proposition is a corollary of Lemma 3.4 and 3.5.

Proposition 3.8. *If $\nu(\theta)$ is cyclic, then its cyclic expression is unique.*

Proof. Assume $\overline{w^{s-1}w_\star}$ and $\overline{u^{l-1}u_\star}$ are two cyclic expression of $\nu(\theta)$ where $w = \nu_1 \dots \nu_t$ and $u = \epsilon_1 \dots \epsilon_m$. If $\nu_t = \epsilon_m$, then $w^s = u^l$. By Lemma 3.4, we have $w = u$. If $\nu_t \neq \epsilon_m$, then $w^s = u^{l-1}u'$ with some $u' \in L_u$, but this is a contradiction to Lemma 3.5. □

3.3 Filled-in Julia sets and the Multibrot set

Let us recall some results about filled-in Julia set and Multibrot set that will be used following. These can be found in [DH], [Mil] and [DE].

For $c \in \mathbb{C}$, we denote by K_c the filled-in Julia set of f_c, that is the set of points $z \in \mathbb{C}$ whose orbit under f_c is bounded. We denote by M_d the Multibrot set for $f_c(z) = z^d + c$, that is the set of parameters $c \in \mathbb{C}$ for which the critical point 0 belongs to K_c.

If $c \in M_d$, then K_c is connected. There is a conformal isomorphism $\phi_c : \mathbb{C}\smallsetminus\overline{K}_c \to \mathbb{C}\smallsetminus\overline{\mathbb{D}}$ which satisfies $\phi_c \circ f_c = \big(\phi_c\big)^d$ and $\phi_c'(\infty) = 1$. The dynamical ray of angle $\theta \in \mathbb{T}$ is

$$R_c(\theta) := \big\{z \in \mathbb{C}\smallsetminus K_c \mid \arg\big(\phi_c(z)\big) = 2\pi\theta\big\}.$$

If θ is rational, then as r tends to 1 from above, $\phi_c^{-1}(r\mathrm{e}^{2\pi i\theta})$ converges to a point $\gamma_c(\theta) \in K_c$. We say that $R_c(\theta)$ lands at $\gamma_c(\theta)$. We have $f_c \circ \gamma_c = \gamma_c \circ \tau$ on $\mathbb{Q}/\mathbb{Z}$. In particular, if θ is periodic under τ, then $\gamma_c(\theta)$ is periodic under f_c. In addition, $\gamma_c(\theta)$ is either repelling (its multiplier has modulus > 1) or parabolic (its multiplier is a root of unity).

If $c \notin M_d$, then K_c is a Cantor set. There is a conformal isomorphism $\phi_c : U_c \to V_c$ between neighborhoods of ∞ in $\mathbb{C}$, which satisfies $\phi_c \circ f_c = \big(\phi_c\big)^d$ on U_c. We may choose U_c so that U_c contains the critical value c and V_c is the complement of a closed disk. For each $\theta \in \mathbb{T}$, there is an infimum $r_c(\theta) \geq 1$ such that ϕ_c^{-1} extends analytically along $R_0(\theta)\cap\big\{z \in \mathbb{C} \mid r_c(\theta) < |z|\big\}$. We denote by ψ_c this extension and by $R_c(\theta)$ the dynamical ray

$$R_c(\theta) := \psi_c\Big(R_0(\theta) \cap \big\{z \in \mathbb{C} \mid r_c(\theta) < |z|\big\}\Big).$$

As r tends to $r_c(\theta)$ from above, $\psi_c(r\mathrm{e}^{2\pi i\theta})$ converges to a point $x \in \mathbb{C}$. If $r_c(\theta) > 1$, then $x \in \mathbb{C}\smallsetminus K_c$ is an iterated preimage of 0 and we say that $R_c(\theta)$ bifucates at x. If $r_c(\theta) = 1$, then $\gamma_c(\theta) := x$ belongs to K_c and we say that $R_c(\theta)$ lands at $\gamma_c(\theta)$. Again, $f_c \circ \gamma_c = \gamma_c \circ \tau$ on the set of θ such that $R_c(\theta)$ does not bifurcate. In particular, if θ is periodic under τ and $R_c(\theta)$ does not bifurcate, then $\gamma_c(\theta)$ is periodic under f_c.

The Multibrot set is connected. The map

$$\phi_{M_d} : \mathbb{C}\smallsetminus M_d \ni c \mapsto \phi_c(c) \in \mathbb{C}\smallsetminus\overline{\mathbb{D}}$$

is a conformal isomorphism. For $\theta \in \mathbb{T}$, the parameter ray $R_{M_d}(\theta)$ is

$$R_{M_d}(\theta) := \big\{c \in \mathbb{C}\smallsetminus M_d \mid \arg\big(\phi_{M_d}(c)\big) = 2\pi\theta\big\}.$$

It is known that if θ is rational, then as r tends to 1 from above, $\phi_{M_d}^{-1}(r\mathrm{e}^{2\pi i\theta})$ converges to a point $\gamma_{M_d}(\theta) \in M_d$. We say that $R_{M_d}(\theta)$ lands at $\gamma_{M_d}(\theta)$.

If θ is periodic for τ of exact period n and if $c_0 := \gamma_{M_d}(\theta)$, then the point $\gamma_{c_0}(\theta)$ is periodic for f_{c_0} with period p dividing n ($ps = n,\ s \geq 1$) and multiplier a s-*th* root of unity. If the period of $\gamma_{c_0}(\theta)$ for f_{c_0} is exactly n then the multiplier is 1, c_0 is called primitive parabolic parameter, otherwise c_0 is called satellite parabolic parameter.

Lemma 3.9 (near parabolic map)**.** *c_0 is defined as above. When we make a small perturbation to c_0 in parameter space, If c_0 is a primitive parabolic parameter, then the parabolic orbit of f_{c_0} is splitted into a pair of nearby periodic orbits of f_c, both have length n; If c_0 is a satellite parabolic parameter, then the parabolic orbit of f_{c_0} is splitted into a pair of nearby periodic orbits of f_c, one has length p and the other has length $sp = n$.*

This lemma was proved by Milnor in [Mil] lemma 4.2 for the case $d = 2$, but we can translate the proof word by word to the general case.

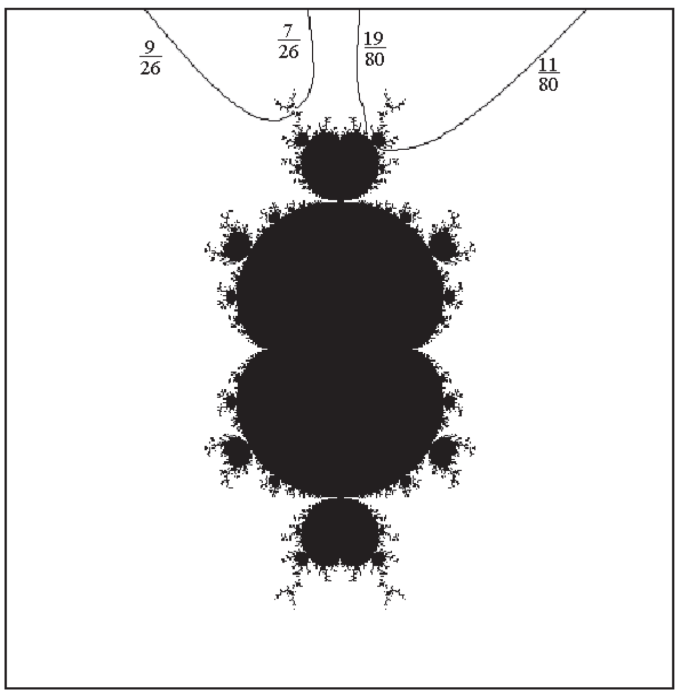


Figure 3: The parameter rays $R_{M_3}(7/26)$ and $R_{M_3}(9/26)$ land on a common root of a primitive hyperbolic component while $R_{M_3}(19/80)$ and $R_{M_3}(11/80)$ land on a common root of a satellite hyperbolic component. Only angles of rays are labelled in the graph.

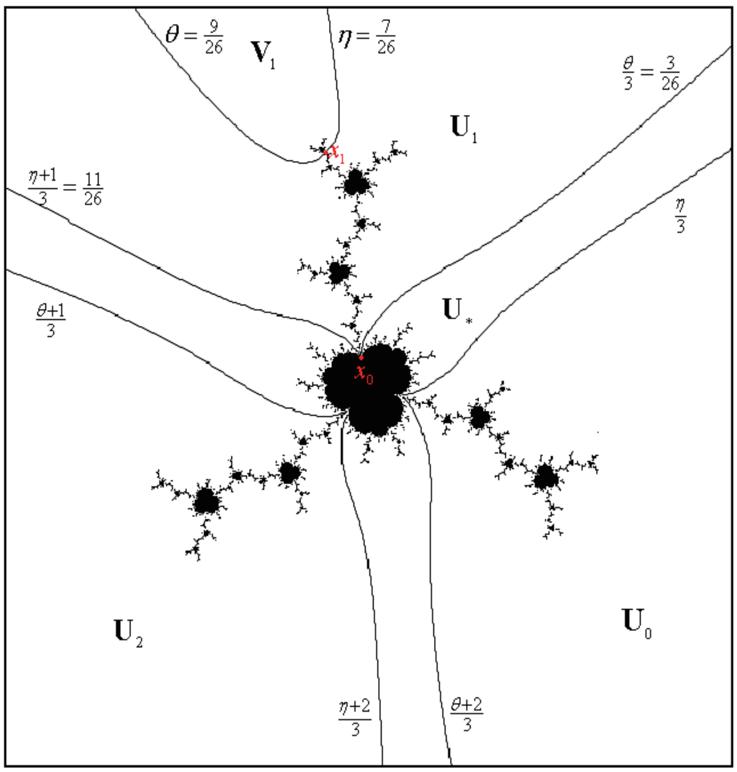


Figure 4: The dynamical plane of f_{c_0}. $c_0 := \gamma_{M_3}(7/26) = \gamma_{M_3}(9/26)$ is the root of some primitive hyperbolic component as illustrated in Figure 3. The dynamical rays $R_{c_0}(7/26)$ and $R_{c_0}(9/26)$ land on a common parabolic point of f_{c_0} with period 3.

Let H be periodic $n(n>1)$ hyperbolic component of M_d. For every parameter $c \in H$, f_c has an attracting periodic orbit $\{\ z(c), \dots, f_c^{n-1}(z(c))\ \}$. Its multiplier define a map

$$\mu_H : H \to \mathbb{D},\ c \mapsto \frac{\partial}{\partial z} f_c^n(z(c))$$

then $\mu_H : H \to \mathbb{D}$ is $d-1$ covering map with only one branched point .It extends continuously to a neighborhood of $\overline{H}$. Considering parameter $c \in \partial H$ such that $\mu_H(c) = 1$, Eberlein proved that among these points, there is exactly one c which is the landing point of two parameter rays of period n, this point is called root of H (see Figure 3); the other $d-2$ points are landing points of only one parameter ray of period n each, they are called co-root of H (see Figure 6). H is called primitive or satellite hyperbolic component according to whether its root is primitive or satellite parabolic parameter.

If c is the root of some hyperbolic component and $c \neq \gamma_{M_d}(0)$, then two periodic parameter rays $R_{M_d}(\theta)$ and $R_{M_d}(\eta)$ land on c, we say θ and η are companion angles, and θ, η have the same period under τ. c is primitive if and only if the orbit of $R_{M_d}(\theta)$ and $R_{M_d}(\eta)$ under τ are distinct. In dynamical plane, the dynamic rays $R_c(\theta)$ and $R_c(\eta)$ land at a common point $x_1 := \gamma_c(\theta) = \gamma_c(\eta)$. This point is on the parabolic orbit of f_c with its immediate basin containing the critical value. $R_c(\theta)$ and $R_c(\eta)$ are adjacent to the Fatou component containing c and the curve $R_c(\theta) \cup R_c(\eta) \cup \{x_1\}$ is a Jordan curve that cuts the plane into two connected components: one component, denoted by V_1, contains the critical value c; the other component, denoted by V_0, contains $R_c(0)$ and all points of parabolic cycle except x_1. Since V_1 contains the critical value, its preimage $U_\star = f_c^{-1}(V_1)$ is connected and contains the critical point 0. It is bounded by the dynamical rays $R_c(\theta/d), \dots, R_c\big((\theta+d-1)/d\big)$; $R_c(\eta/d), \dots, R_c\big((\eta+d-1)/d\big)$. Suppose $\theta > \eta$, and since each component of $\mathbb{C} \setminus \overline{U_\star}$ is conformally mapped to V_0 which is bounded by $R_c(\theta)$ and $R_c(\eta)$, it is easy to see that $R_c\big((\theta+k-1)/d\big)$ and $R_c\big((\eta+k)/d\big)$ land on a common point which is one of the preimage of x_1 for $k \in \mathbb{Z}_d$. Denote U_k the component of $\mathbb{C} \setminus R_c\big((\theta+k-1)/d\big) \cup \{\gamma_c\big((\eta+k)/d\big)\} \cup R_c\big((\eta+k)/d\big)$ disjoint with $U_\star$. See Figure 4 (primitive case) and Figure 5 (satellite case). Note that $f_c : U_k \to V_0$ is conformal.

If c is a co-root of some hyperbolic component, then exactly one period parameter ray $R_{M_d}(\beta)$ land on it (see Figure 6). In dynamical plane, $R_c(\beta)$ is the unique dynamical ray landing on a parabolic periodic point $\gamma_c(\beta) := x_1$, whose immediate basin contains the critical value c. The parameter c is a primitive parabolic parameter. Denote V_1 the union of Fatou component containing c and external ray $R_c(\beta)$, $V_0 = \mathbb{C} \setminus \overline{V_1}$, $U_\star = f_c^{-1}(V_1)$. U_k is the component of $f_c^{-1}(V_0)$ adjacent with $R_c\big((\beta+k-1)/d\big)$ and $R_c\big((\beta+k)/d\big)$, $k \in \mathbb{Z}_d$.(see Figure 7).

Remark: in our paper, if c is a parabolic parameter, then f_c has unique parabolic orbit, denoted by $\{x_0, x_1, \dots, x_{p-1}\}$. x_1 is the point whose immediate basin contains critical value c.

The following lemma provides a criterion for θ such that $\gamma_{M_d}(\theta)$ is a primitive parabolic parameter.

Definition 3.10. *Let θ be a periodic angle of period n and the d-expansion of θ be $.\overline{\epsilon_1 \dots \epsilon_n}$. We call $\epsilon_1 \dots \epsilon_n$ the periodic part of the d-expansion of θ.*

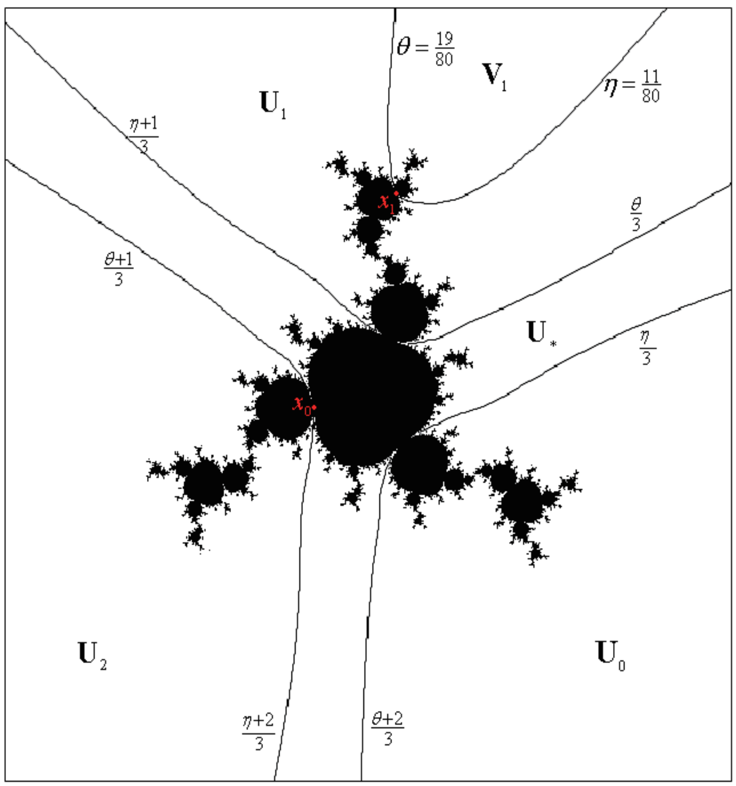


Figure 5: The dynamical plane of f_{c_1}. $c_1 := \gamma_{M_3}(11/80) = \gamma_{M_3}(19/80)$ is the root of some satellite hyperbolic component as illustrated in Figure 3. The dynamical rays $R_{c_1}(11/80)$ and $R_{c_1}(19/80)$ land on a common parabolic point of f_{c_1} with period 2.

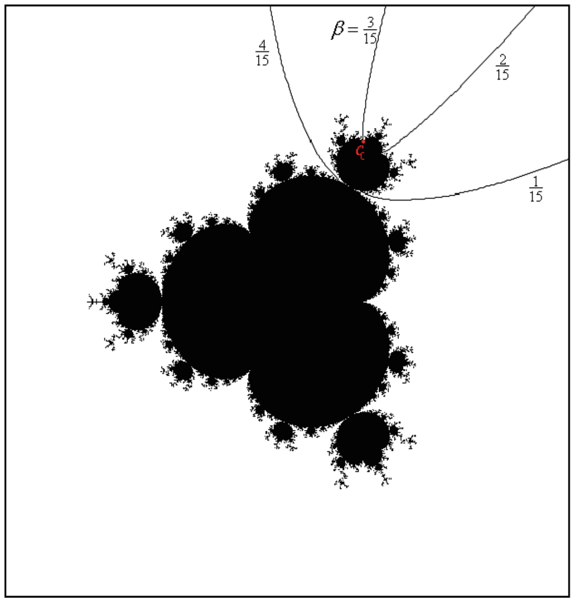


Figure 6: Multibrot set M_4. The parameter rays $R_{M_4}(1/15)$ and $R_{M_4}(4/15)$ land on the root of some hyperbolic component. $R_{M_4}(2/15)$ and $R_{M_4}(1/5)$ land on two co-root of this hyperbolic component respectively.

Lemma 3.11. *θ is periodic under τ with period $n \geq 2$. If $c_0 := \gamma_{M_d}(\theta)$ is the root of some satellite hyperbolic component, then θ satisfies the following properties:*

(1) *$\nu(\theta)$ is cyclic.*

(2) *Denote by $\overline{w^{s-1}w_\star}$ the cyclic expression of $\nu(\theta)$ where $w = \nu_1 \ldots \nu_t$, t is a proper factor of n and $ts = n$. Then the last digit of the period part of the d-expansion of θ is ν_t or $\nu_t - 1$.*

Moreover, if θ is maximal in its orbit, then $\nu(\theta)$ also satisfies

(3) *t is the length of parabolic orbit and the last digit of the period part of the d-expansion of θ must be $\nu_t - 1 \in [0, d-2]$.*

Proof. Let η be the companion angle of θ, then in dynamical plane of f_{c_0}, $R_{c_0}(\theta)$ and $R_{c_0}(\eta)$ land on x_1 (see Figure 5). As V_1 contains no points and external rays of the parabolic orbit, then $\{x_0, x_1, \ldots, x_{p-1}\}$ together with their external rays belong to $\bigcup_{k=0}^{d-1} \overline{U}_k$.

For c_0 is satellite parabolic parameter, the length p of parabolic orbit is a proper factor of n and f_{c_0} acts on the rays of the orbit transitively. Then we have, in $\nu(\theta) = \overline{\nu_1 \ldots \nu_{n-1}\star}$, $\nu_j = \nu_{j(\mathrm{mod})p}$ for $1 \leq j \leq n-1$, that is, $\nu(\theta) = \overline{u^{l-1}u_\star}$ where $u = \nu_1 \ldots \nu_p$. By definition of kneading sequence, we can see $\tau^{\circ(p-1)}(\theta) \in \big((\theta + \nu_p - 1)/d,\ (\theta + \nu_p)/d\big)$. It follows x_0

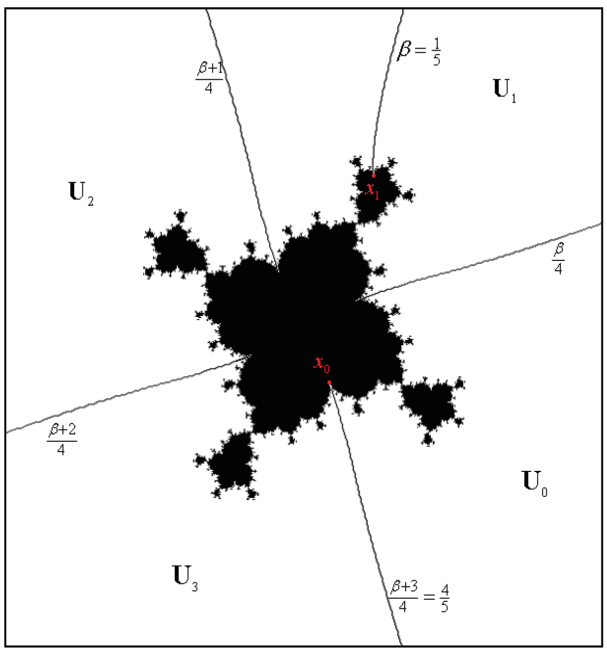
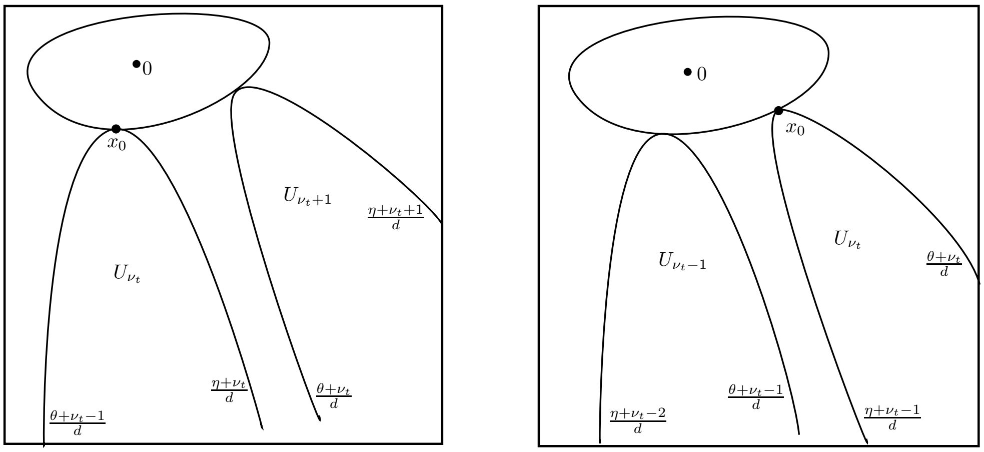


Figure 7: The dynamical plane of f_{c_0}. $c_0 := \gamma_{M_4}(1/5)$ is a co-root of the hyperbolic component illustrated in Figure 6. $R_{c_0}(1/5)$ is the unique dynamical ray landing on $\gamma_{c_0}(1/5)$ which is the parabolic point of f_{c_0} with period 2.

together with its external rays belong to $\overline{U}_{\nu_p}$. Then $\tau^{n-1}(\theta)$ is either $(\theta+\nu_p-1)/d$ $(\theta > \eta)$ or $(\theta+\nu_p)/d$ $(\theta < \eta)$ (see Figure 8). So the last digit of d-expansion of θ is either $\nu_p - 1$ $(\theta > \eta)$ or ν_p $(\theta < \eta)$. Let $w = \nu_1 \dots \nu_t$ be the primitive root of u, then $u = w^{p/t}$. We have $\overline{w^{s-1}w_\star}$ is the cyclic expression of $\nu(\theta)$ (proposition 3.8) and $\nu_t = \nu_p$, so θ satisfies property (1) and (2).



$\theta > \eta$ $\qquad$ $\theta < \eta$

Figure 8:

Furthermore, if θ is maximal in its orbit, then $\theta > \eta$, so the last digit of the period part of the d-expansion of θ must be $\nu_t - 1$. By lemma 3.2, $\theta = .\overline{w^{s-1}\nu_1 \dots \nu_{t-1}(\nu_t - 1)}$ and $0 \le \nu_t - 1 \le d - 2$. Note that the angles of external rays belonging to x_1 are θ, $\tau^p(\theta), \dots, \tau^{(s-1)p}(\theta)$ with the order $\theta > \tau^p(\theta) > \dots > \tau^{(s-1)p}(\theta)$. The maximum of θ implies η is the second largest angle in orbit of θ, then $\eta = \tau^p(\theta) = .\overline{u^{l-2}\nu_1 \dots \nu_{p-1}(\nu_p - 1)u}$. If u is not primitive, then $p/t > 1$. It follows $\tau^t(\theta) > \tau^p(\theta) = \eta$, a contradiction to that η is the second largest angle in orbit of θ. So u is a primitive word and hence $t = p$ is length of parabolic orbit.

□

Then once θ doesn't satisfy the property in this lemma, we have $\gamma_{M_d}(\theta)$ is a primitive parabolic parameter. The lemma below can be seen as a application of lemma 3.11.

Lemma 3.12. *Assume $\theta = .\overline{w^{s-1}\nu_1 \dots \nu_{t-1}(\nu_t - 1)}$ is maximal in its orbit, where $w = \nu_1 \dots \nu_t$ is primitive with $\nu_t \in [1, d-1]$ and t is a proper factor of n with $ts = n$. Let*

$$\beta_{v_t - i} = .\overline{w^{s-1}\nu_1 \dots \nu_{t-1}(\nu_t - i)} \quad \textit{for } 2 \le i \le \nu_t$$

$$\beta_{-1} = \begin{cases} .\overline{w^{s-1}\nu_1 \dots (\nu_{t-1} - 1)(d-1)} & \textit{as } t \ge 2 \\ .\overline{k \dots k(k-1)(d-1)} & \textit{as } t = 1 \end{cases}$$

Then $\gamma_{M_d}(\beta_{\nu_t - i})$ is a primitive parabolic parameter for any $2 \le i \le \nu_t$. $\gamma_{M_d}(\beta_{-1})$ is a satellite parabolic parameter for $\theta = .\overline{(d-1)\cdots(d-1)(d-2)}$ and a primitive parabolic parameter for any other case.

Proof. Let $\beta = .\overline{w^{s-1}\nu_1 \dots \nu_{t-1} j}$ be any angle among $\{\beta_{\nu_t - i}\}_{2 \le i \le \nu_t}$, then $0 \le j \le \nu_t - 2$. The maximum of θ implies the maximum of β in its orbit. Since w is primitive, by lemma 3.2, we have $\overline{w^{s-1}w_\star}$ is the cyclic expression of $\nu(\beta)$. As $j \le \nu_t - 2 < \nu_t - 1$, with the maximum of β, the property (3) in lemma 3.11 is not satisfied. So $\gamma_{M_d}(\beta)$ is a primitive parabolic parameter.

For β_{-1}, the maximum of θ implies β_{-1} is greater than $\tau(\beta_{-1}),\ \tau^2(\beta_{-1}), \dots, \tau^{n-2}(\beta_{-1})$ but less than $\tau^{n-1}(\beta_{-1})$. It follows $\nu(\beta) = \begin{cases} \overline{w^{s-1}\nu_1 \dots \nu_{t-1}\star} & \text{as } t \ge 2 \\ \overline{k \dots kk\star} & \text{as } t = 1 \end{cases} = \overline{w^{s-1}w_\star}$. It is the cyclic expression of $\nu(\beta)$, then if β satisfies the property in lemma 3.11, ν_t is either 0 or $d-1$. Since $1 \le \nu_t \le d-1$, we have ν_t must be $d-1$, then the maximum of θ implies $\theta = .\overline{(d-1)\cdots(d-1)(d-2)}$. So $\gamma_{M_d}(\beta_{-1})$ is a primitive parabolic parameter as long as $\theta \neq .\overline{(d-1)\cdots(d-1)(d-2)}$. In the case of $\theta = .\overline{(d-1)\cdots(d-1)(d-2)}$, we will see in lemma 3.14 that $\gamma_{M_d}(\theta)$ is the root of a hyperbolic component attached to the main cardioid and β_{-1} is the companion angle of θ. In this case, $\gamma_{M_d}(\beta_{-1})$ is a satellite parabolic parameter.

□

Remark. In this lemma, we distinguish β_{-1} according to whether $t \ge 2$ or $t = 1$. It is because that we don't find a uniform expression of β_{-1} for the two cases rather than the case of $t = 1$ is special.

3.4 Itineraries outside the Multibrot set

If $c \in \mathbb{C} \setminus M_d$, the Julia set of f_c is a Cantor set. If $c \in R_{M_d}(\theta)$ with $\theta \neq 0$ not necessarily periodic, then the dynamical rays $R_c(\theta/d) \dots R_c\big((\theta + d - 1)/d\big)$ bifurcate on the critical point. The set $R_c(\theta/d) \cup \dots \cup R_c\big((\theta + d - 1)/d\big) \cup \{0\}$ separates the complex plane in d connected components. We denote by U_0 the component containing the dynamical ray $R_c(0)$ and by $U_1, \dots, U_{d-1}$ the other component in counterclockwise (see Figure 9).

The orbit of a point $x \in K_c$ has an itinerary with respect to this partition. In other words, to each $x \in K_c$, we can associate a sequence $\iota_c(x) \in \{0, 1, \dots d-1\}^{\mathbb{N}}$ whose j-th term is equal to k if $f_c^{\circ j-1}(x) \in U_k$. A point $x \in K_c$ is periodic for f_c if and only if the itinerary $\iota_c(x)$ is periodic for the shift with the same period.

The map $\iota_c : K_c \to \{0, 1, \dots d-1\}^{\mathbb{N}}$ is a bijection. In particular, for each itinerary $\iota \in \{0, \dots, d-1\}^{\mathbb{N}}$ and each $c \in \mathbb{C} \setminus \big(M_d \cup R_{M_d}(0)\big)$, there is a unique point $x(\iota, c) \in K_c$ whose itinerary is ι. For a given $\iota \in \{0, \dots, d-1\}^{\mathbb{N}}$, the map $\mathbb{C} \setminus \big(M_d \cup R_{M_d}(0)\big) \longrightarrow \mathbb{C} \quad c \mapsto x(\iota, c) \in \mathbb{C}$ is continuous, and even holomorphic (as can be seen by applying the Implicit Function Theorem).

Proposition 3.13. *Let $\overline{\varepsilon_1 \dots \varepsilon_{n-1}\star}$ be the kneading sequence of a periodic angle θ with period $n \ge 2$. If $c_0 := \gamma_{M_d}(\theta)$ is a primitive parabolic parameter and if one follows continuously the periodic points of period n of f_c as c makes a small turn around c_0, then*

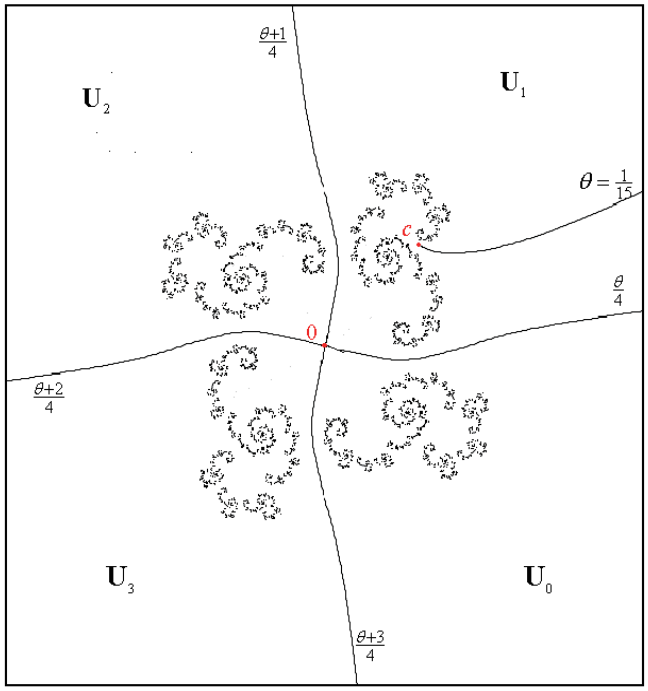


Figure 9: The regions $U_0,\ U_1,\ U_2,\ U_3$ for a parameter c belonging to $R_{M_4}(1/15)$.

the periodic points with itineraries $\overline{\varepsilon_1 \ldots \varepsilon_{n-1} k}$ and $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}(k+1)}$ get exchanged where $k \in \mathbb{Z}_d$ is the last digit of the period part of the d-expansion of θ.

Proof. Since c_0 is a primitive parabolic parameter, then the periodic point $x_1 := \gamma_{c_0}(\theta)$ has period n and multiplier 1. According to Case 2 in the proof of smoothness and lemma 3.9, the projection from a small neighborhood of (c_0, x_1) in X_n to the first coordinate is a degree 2 covering. So the neighborhood of (c_0, x_1) in $\overline{X_n}$ can be written as

$$\big\{(c_0 + \delta^2, x(\delta)), (c_0 + \delta^2, x(-\delta)) \;\big|\; |\delta| < \varepsilon\big\}$$

where $x : (\mathbb{C}, 0) \to (\mathbb{C}, x_1)$ is a holomorphic germ with $x'(0) \neq 0$. In particular, the pair of periodic points for f_c which are splitted from x_1 get exchanged when c makes a small turn around c_0. So, using analytic continuation on $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, it is enough to show that there exists a $c \in \mathbb{C} \setminus M_d$ close to c_0 such that $x(\pm\sqrt{c - c_0})$ have itineraries $\overline{\varepsilon_1 \ldots \varepsilon_{n-1} k}$ and $\overline{\varepsilon_1 \ldots \varepsilon_{n-1}(k+1)}$ where $k \in \mathbb{Z}_d$ is the last digit of the period part of the d-expansion of θ.

Let us denote by $V_0(c_0), V_1(c_0), U_0(c_0), \ldots, U_{d-1}(c_0)$ and $U_\star(c_0)$ the sets defined in the previous section. For $j \geq 0$, set $x_j := f_{c_0}^j(x_0)$ and observe that for $j \in [1, n-1]$, we have $x_j \in U_{\varepsilon_j}(c_0)$.

For $c \in R_{M_d}(\theta)$, consider the following compact subsets of the Riemann sphere :

$$R(c) := R_c(\theta) \cup \{c, \infty\} \quad \text{and} \quad S(c) := R_c(\theta/d) \cup \ldots \cup R_c\big((\theta + d - 1)/d\big) \cup \{0, \infty\}.$$

Denote by $U_0(c)$ the component of $\mathbb{C}\setminus S(c)$ containing $R_c(0)$ and by $U_1(c),\ldots,U_{d-1}(c)$ the other component in counterclockwise. From any sequence $\{c_m\}\subset R_{M_d}(\theta)$ converging to c_0, by extracting a subsequence if necessary, we can assume $R(c_m)$ and $S(c_m)$ converge respectively, for the Hausdorff topology on compact subsets of $\mathbb{C}\cup\{\infty\}$, to connected compact sets R and S. Since $S(c)=f_c^{-1}\big(R(c)\big)$, we have $S=f_{c_0}^{-1}(R)$. According to [PR, Section 2 and 3], $R\cap(\mathbb{C}\setminus K_{c_0})=R_{c_0}(\theta)$, the intersection of R with the boundary of K_{c_0} is reduced to $\{x_1\}$ and the intersection of R with the interior of K_{c_0} is contained in the immediate basin of x_1, whence in V_1. It follows $R\subset\overline{V_1}(c_0)$ and $S\subset\overline{U}_\star(c_0)$, that means any compact subset of $\mathbb{C}\setminus\overline{U}_\star(c_0)$ is contained in $\mathbb{C}\setminus S(c_m)$ for m sufficiently large .

For $j\in[1,n-1]$ and let D_j be a sufficiently small disk around x_j so that

$$\overline{D}_j\subset U_{\varepsilon_j}(c_0)\subset\mathbb{C}\setminus\overline{U}_\star(c_0).$$

According to the previous discussion, if m is sufficiently large, we have

$$f_{c_m}^{j-1}\big(x(\pm\sqrt{c_m-c_0})\big)\subset D_j\subset U_{\varepsilon_j}(c_m).$$

So the first $n-1$ symbols of the itineraries of $x(\pm\sqrt{c_m-c_0})$ are all $\epsilon_1,\ldots,\epsilon_{n-1}$. As $x(\sqrt{c_m-c_0})$ and $x(-\sqrt{c_m-c_0})$ are different n periodic points of f_{c_m}, their itineraries must be different. It follows $f_{c_m}^{n-1}\big(x(\pm\sqrt{c_m-c_0})\big)$, which are splitted from x_0, lie in different component of $\mathbb{C}\setminus S(c_m)$. Combining with the fact that $R_{c_0}\big((\theta+k)/d\big)$ lands on x_0 (k is the last digit of the period part of the d-expansion of θ), we have $f_{c_m}^{n-1}\big(x(\pm\sqrt{c_m-c_0})\big)$ belong to $U_k(c_m)$ and $U_{k+1}(c_m)$ respectively, then $x(\pm\sqrt{c_m-c_0})$ have itineraries $\overline{\varepsilon_1\ldots\varepsilon_{n-1}k}$ and $\overline{\varepsilon_1\ldots\varepsilon_{n-1}(k+1)}$ respectively.

□

Lemma 3.14. *For $\theta=1-1/(d^n-1)=.\overline{(d-1)\cdots(d-1)(d-2)}$ $(n\geq 2)$, we have $\gamma_{M_d}(\theta)$ is the root of some periodic n hyperbolic component attached to the main cardioid. If η is denoted the companion angle of θ, then $\eta=d\theta-d+1$.*

Proof. Let $c_0:=\gamma_{M_d}(\theta)$, then $x_1:=\gamma_{c_0}(\theta)$ is the parabolic periodic point of f_{c_0} as previous. By lemma 3.2, $\nu(\theta)=\overline{(d-1)\cdots(d-1)\star}$, so $\overline{(d-1)\cdots(d-1)\star}$ is the cyclic expression of $\nu(\theta)$. If $x_0\neq x_1$, then the length of parabolic orbit is greater than 1. It implies the property (3) in lemma 3.11 is not satisfied, so c_0 is a primitive parabolic parameter. According to proposition3.13, when $c\in\mathbb{C}\setminus M_d$ is close to c_0, x_1 splits into two n periodic point y,z of f_c with itineraries $\overline{(d-1)\cdots(d-1)(d-2)}$ and $\overline{(d-1)\cdots(d-1)(d-1)}$. It leads to a contradiction to the period n of y and z. So $x_0=x_1$ and then c_0 is the root of some periodic n satellite hyperbolic component attached to the main cardioid.

By the maximum of θ, we have U_{d-1} is bounded by $R_{c_0}\big((\theta+d-2)/d\big)$ and $R_{c_0}\big((\eta+d-1)/d\big)$. $\nu(\theta)=\overline{(d-1)\cdots(d-1)\star}$ implies $R_{c_0}(\theta)\subset\overline{U}_{d-1}$, then $\theta\leq(\eta+d-1)/d$ and x_0 is on the boundary of U_{d-1}. On the other hand, $(\eta+d-1)/d$ is in the orbit of θ, so $\theta\geq(\eta+d-1)/d$. Then we have $\eta=d\theta-d+1$.

□

Remark. The dynamical rays $R_{c_0}(\theta)$ and $R_{c_0}(\eta)$ are consecutive among the rays landing at x_0. Lemma 3.14 implies $R_{c_0}(\theta)$ is mapped to $R_{c_0}(\eta)$. It follows that each dynamical ray landing at x_0 is mapped to the one which is once further clockwise.

Proposition 3.15. *Let $\theta = 1 - 1/(d^n - 1) = .\overline{(d-1)\cdots(d-1)(d-2)}$ be periodic with period $n \geq 2$. If one follows continuously the periodic points of period n of f_c as c makes a small turn around $\gamma_{M_d}(\theta)$, then the periodic points in the cycle of $\iota_c^{-1}(\overline{(d-1)\cdots(d-1)(d-2)})$ get permuted cyclically.*

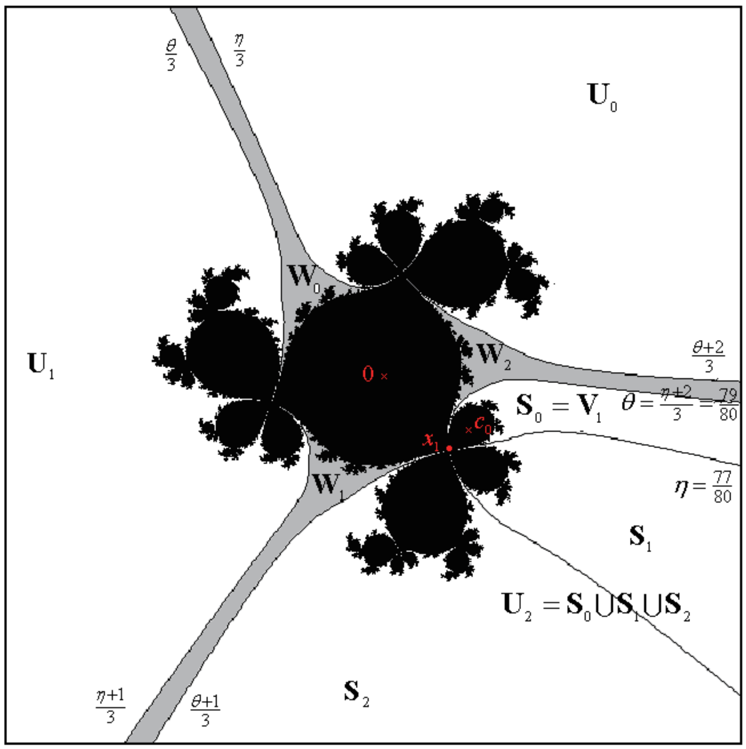


Figure 10: The dynamical plane of f_{c_0}. $c_0 := \gamma_{M_3}(\theta)$ with $\theta = .\overline{2221}$

Proof. Set $c_0 := \gamma_{M_d}(\theta)$. By Lemma 3.14, all the dynamical rays $R_{c_0}\big(\tau^j(\theta)\big)$ land on a common fixed point x_0. This fixed point is parabolic and the companion angle of θ, denoted by η, equals to $d\theta - (d-1) \equiv d\theta (\text{mod } \mathbb{Z})$. $V_1(c_0) \subset U_{d-1}(c_0)$ which is bounded by $R_{c_0}\big((\theta + d - 2)/d\big)$ and $R_{c_0}(\theta)$.

According to Case 3 in the proof of smoothness and lemma 3.9, we have the projection from a small neighborhood of (c_0, x_0) in X_n to the parameter plane is a degree n covering. Then the neighborhood of (c_0, x_0) in $\overline{X_n}$ can be written as

$$\big\{(c_0 + \delta^n, x(\delta)), (c_0 + \delta^n, x(\omega\delta)), \ldots, (c_0 + \delta^n, x(\omega^{n-1}\delta)) \;\big|\; |\delta| < \varepsilon\big\}$$

where $x : (\mathbb{C}, 0) \to (\mathbb{C}, x_0)$ is a holomorphic germ satisfying $x'(0) \neq 0$. So, for c close to c_0, the set $x\{\sqrt[n]{c - c_0})\}$ is a cycle of period n of f_c, and when c makes a small turn around c_0, the periodic points in the cycle $x\{\sqrt[n]{c - c_0})\}$ get permuted cyclically. So, combining with analytic continuation on $\mathbb{C} \setminus (M_d \cup R_{M_d}(0))$, it is enough to show there exists a $c \in \mathbb{C} \setminus M_d$ close enough to c_0 such that the point $\iota_c^{-1}(\overline{(d-1) \cdots (d-1)(d-2)})$ belongs to $x\{\sqrt[n]{c - c_0}\}$. Equivalently, we must show that there is a sequence $\{c_j\} \subset \mathbb{C} \setminus M_d$ converging to c_0, such that the periodic point $y_j := \iota_{c_j}^{-1}(\overline{(d-1) \cdots (d-1)(d-2)})$ converges to x_0.

Let $\{c_j\} \subset R_{M_d}(\theta)$ converge to c_0 as $j \to \infty$. Without loss of generality, we may assume that the sequence y_j converges to a point z, $R(c_j)$ converges to R and $S(c_j)$ converges to S in Hausdoff topology. The definition of $R(c)$, $S(c)$, $U_0(c), \dots, U_{d-1}(c)$ are in the proof of proposition 3.13. As (c_0, z) is on $\overline{X_n}$, then z is either the parabolic fixed point or repelling n periodic point of f_{c_0}.

Suppose z is a repelling n periodic point, set $z_i := f_{c_0}^i(z)$. Now we will define a new sequence of open domain $\{W_k(c_0)\}$. $W_k(c_0)$ is the connected component of $U_\star(c_0) \setminus$ the closure of Fatou component containing 0, adjacent with $U_k(c_0), U_{k+1}(c_0)$ (see Figure 10). According to [PR, Section 2 and 3], $R \cap (\mathbb{C} \setminus K_{c_0}) = R_{c_0}(\theta)$, the intersection of R with the boundary of K_{c_0} is reduced to $\{x_0\}$ and the intersection of R with the interior of K_{c_0} is contained in the immediate basin of x_0. It follows $\{z_0, \dots, z_{n-1}\} \bigcap S = \emptyset$. Then for j sufficiently large, $\{z_0, \dots, z_{n-1}\} \subset \mathbb{C} \setminus S_{c_j}$. As y_j has itineraries $\overline{(d-1) \cdots (d-1)(d-2)}$, we have $\{z_0, \dots z_{n-2}\} \subset U_{d-1}(c_0) \bigcup \overline{W}_{d-1}(c_0)$, $z_{n-1} \in U_{d-2}(c_0) \bigcup \overline{W}_{d-2}(c_0)$.

Claim 1. $z_{n-1} \notin \overline{W}_{d-2}(c_0)$.

Proof. In $J(f_{c_0})$, x_0 is the unique periodic point with more than one external rays landing on it (refer to [Poi, proposition 3.3]). So there is exactly one external ray landing on z_{n-1} with period n. Its angle is denoted by $\dfrac{a}{d^n - 1}$, a is a integer. If $z_{n-1} \in \overline{W}_{d-2}$, the angle of external ray belonging to z_{n-1} satisfy

$$\frac{\eta + d - 2}{d} < \frac{a}{d^n - 1} < \frac{\theta + d - 2}{d} \quad \left(\theta = 1 - \frac{1}{d^n - 1},\ \eta = d\theta - d + 1 \right).$$

by simple computation, we have

$$\frac{k(d^n - 1)}{d - 1} - d^{n-1} - 1 + \frac{1}{d} < a < \frac{k(d^n - 1)}{d - 1} - d^{n-1},$$

a contradiction to a is an integer. This ends the proof of claim 1.

Claim 2. $z_{n-1} \notin U_{d-2}(c_0)$.

Proof. If $z_{n-1} \in U_{d-2}(c_0)$, we label the sectors at x_0 by $S_i (0 \leq i \leq n-1)$ clockwise with $S_0 = V_1(c_0)$. The dynamics between these sectors satisfy

$$V_1(c_0) = S_0 \xrightarrow{f_{c_0}} S_1 \xrightarrow{f_{c_0}} \cdots \xrightarrow{f_{c_0}} S_{n-2} \xrightarrow{f_{c_0}} S_{n-1} = \mathbb{C} \setminus \overline{U}_{d-1}(c_0)$$

As $\{z_0, \dots z_{n-2}\} \subset U_{d-1}(c_0) \bigcup \overline{W}_{d-1}(c_0)$, we have $z_0 = f_{c_0}(z_{n-1})$ belongs to the union of $\overline{W_{d-1}}(c_0)$ and $\bigcup_{i=1}^{n-2} S_i$. If $z_0 \in S_{i_0}$ $(1 \leq i_0 \leq n-2)$, then $f_{c_0}^{(n-2-i_0)}(z_0) = z_{n-2-i_0} \in$

$f_{c_0}^{(n-2-i_0)}(S_{i_0}) = S_{n-2}$. It follows $f_{c_0}(z_{n-2-i_0}) = z_{n-1-i_0}$ must belong to $\overline{W}_{d-1}(c_0)$. So $z_{n-i_0} \in S_0$ and $f_{c_0}^{(i_0-1)}(z_{n-i_0}) = z_{n-1} \in f_{c_0}^{(i_0-1)}(S_0) = S_{i_0-1}$, contradiction to $z_{n-1} \in U_{d-2}$. If $z_0 \in \overline{W}_{d-1}(c_0)$, then $z_1 \in S_0$. We have $f_{c_0}^{(n-2)}(z_1) = z_{n-1} \in f_{c_0}^{(n-2)}(S_0) = S_{n-2}$, also a contradiction to $z_{n-1} \in U_{d-2}(c_0)$. This ends the proof of claim 2.

The two claim imply the assumption that z is repelling n periodic point is false and then z must be a parabolic fixed point of f_{c_0}, that is $z = x_0$. $\square$

3.5 Proof of Theorem 3.1

Fix $n > 1$ (the case $n = 1$ has been treated directly at the beginning). We proceed to show that X_n is connected.

Set $X := \mathbb{C} \setminus \big(M_d \cup R_{M_d}(0)\big)$ and $F_n := \mathbb{C}\setminus$ all the landing points of periodic n parameter rays. Take any pair of points $(a, w), (a', w')$ in X_n. By analytic continuation, we may assume $a, a' \in X$. Again by analytic continuation on simply connected open set X, we may assume $a = a'$. Thus it is enough to show that there exists a loop in F_n based on a such that the analytic continuation along the loop connects w and w'. We will give a algorithm to find such a loop.

Let z be any n periodic point of f_a.

step 1 In the orbit of z, there is a point with maximal itineraries among the shift of $\iota_a(z)$ in the lexicograph order, denoted by $\overline{\epsilon_1 \dots \epsilon_n}$. Set $\theta = .\overline{\epsilon_1 \dots \epsilon_n}$ (θ is maximal in its orbit). If θ satisfies the properties in lemma 3.11, do step 2 below. Otherwise, $\gamma_{M_d}(\theta)$ is a primitive parabolic parameter. According to lemma 3.2 and proposition 3.13, when a makes a turn around $\gamma_{M_d}(\theta)$, the periodic point of f_a with itineraries $\overline{\epsilon_1 \dots \epsilon_n}$ and $\overline{\epsilon_1 \dots (\epsilon_n + 1)}$ get changed. Then z is connected to a new orbit containing $\iota_a^{-1}(\overline{\epsilon_1 \dots (\epsilon_n + 1)})$. For this new orbit, repeat doing step 1.

step 2 $\theta = .\overline{\epsilon_1 \dots \epsilon_n}$ is maximal in its orbit and satisfies the properties in lemma 3.11. If $\theta = .\overline{(d-1) \cdots (d-1)(d-2)}$, step 2 ends. Otherwise, let $\overline{w^{s-1} w_\star}$ be the cyclic expression of $\nu(\theta)$ where $w = \nu_1 \dots \nu_t$, $\nu_t \in [1, d-1]$. As in lemma 3.12, we obtain a sequence of angles $\{\beta_{\nu_t - 2}, \dots, \beta_0,\ \beta_{-1}\}$ and know that $\gamma_{M_d}(\beta_{\nu_t - i})$ is a primitive parabolic parameter with $\nu(\theta) = \overline{\epsilon_1 \dots \epsilon_{n-1} \star}$ for any $i \in [2, \nu_t + 1]$. Then by proposition 3.13 again, as a makes a turn around $\gamma_{M_d}(\beta_{\nu_t - i})$ $(2 \le i \le \nu_t + 1)$, the periodic points of f_a with itineraries $\overline{\epsilon_1 \dots \epsilon_{n-1}(\nu_t - i)}$ and $\overline{\epsilon_1 \dots \epsilon_{n-1}(\nu_t - i + 1)}$ get changed. Then let a makes turns around from $\gamma_{M_d}(\beta_{\nu_t - 2})$ to $\gamma_{M_d}(\beta_{-1})$ one by one, we have $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{n-1} \epsilon_n})$ are connected with $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{n-1}(d-1)})$ by analytic continuation through the points $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{n-1}(\epsilon_n - 1)}), \dots, \iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{n-1} 0})$. For the new periodic point $\iota_a^{-1}(\overline{\epsilon_1 \dots \epsilon_{n-1}(d-1)})$, do step 1.

Every time a n periodic point of f_a passes though step 1 or step 2, the sum of all digits in the itineraries of the output periodic point is greater than that of the input one. For fixed n, this sum is bounded $\big($the bound is $(d-1)n - 1\big)$, then each n periodic point z can be connected to the orbit containing $\iota_a^{-1}\big(\overline{(d-1) \cdots (d-1)(d-2)}\big)$.

In our case, applying the procedure above to w and w', we have w and w' are connected to two points of the periodic orbit containing $\iota_a^{-1}(\overline{(d-1)\cdots(d-1)(d-2)})$. Proposition 3.15 tells us, by analytic continuation, any two point in this orbit can be connected as long as a makes the appropriate number of turns around $\gamma_{M_d}(1-\frac{1}{d^n-1})$. Thus w and w' are connected. $\square$

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