Central limit theorem for commutative semigroups of toral endomorphisms

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CENTRAL LIMIT THEOREM FOR COMMUTATIVE SEMIGROUPS OF TORAL ENDOMORPHISMS

GUY COHEN AND JEAN-PIERRE CONZE

ABSTRACT. Let $\mathcal{S}$ be an abelian finitely generated semigroup of endomorphisms of a probability space $(\Omega, \mathcal{A}, \mu)$, with $(T_1, ..., T_d)$ a system of generators in $\mathcal{S}$. Given an increasing sequence of domains $(D_n) \subset \mathbb{N}^d$, a question is the convergence in distribution of the normalized sequence $|D_n|^{-\frac{1}{2}} \sum_{\underline{k} \in D_n} f \circ T^{\underline{k}}$, for $f \in L^2_0(\mu)$, where $T^{\underline{k}} = T_1^{k_1} ... T_d^{k_d}$, $\underline{k} = (k_1, ..., k_d) \in \mathbb{N}^d$.

After a preliminary spectral study when the action of $\mathcal{S}$ has a Lebesgue spectrum, we consider $\mathbb{N}^d$- or $\mathbb{Z}^d$-actions given by commuting toral automorphisms or endomorphisms on $\mathbb{T}^\rho$, $\rho \geq 1$. For a totally ergodic action by automorphisms, we show a CLT for the above normalized sequence or other summation methods like barycenters, as well as a criterion of non-degeneracy of the variance, when f is regular on the torus. A CLT is also proved for some semigroups of endomorphisms. Classical results on the existence and the construction of such actions by automorphisms are recalled.

CONTENTS

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Introduction

Let $\mathcal{S}$ be an abelian finitely generated semigroup of endomorphisms of a probability space $(\Omega, \mathcal{A}, \mu)$. Each $T \in \mathcal{S}$ is a measurable map from Ω to Ω preserving the probability measure μ. For $f \in L^1(\mu)$ a random field is defined by $(f(T.)_{T\in\mathcal{S}})$ for which limit theorems can be investigated: law of large numbers, behavior in distribution.

By choosing a system $(T_1, ..., T_d)$ of generators in $\mathcal{S}$, every $T \in \mathcal{S}$ can be represented[1] as $T = T^{\underline{k}} = T_1^{k_1}...T_d^{k_d}$, for $\underline{k} = (k_1, ..., k_d) \in \mathbb{N}^d$. Given an increasing sequence of domains $(D_n) \subset \mathbb{N}^d$, there are cases where convergence in distribution toward a normal law can be shown for the standard normalized sequence and the "multidimensional periodogram" respectively defined by

$$(1) \qquad |D_n|^{-\frac{1}{2}} \sum_{\underline{k}\in D_n} T^{\underline{k}} f, \quad |D_n|^{-\frac{1}{2}} \sum_{\underline{k}\in D_n} e^{2\pi i\langle \underline{k},\theta\rangle}\, T^{\underline{k}} f, \ f \in L^2_0(\mu), \ \theta \in \mathbb{R}^d.$$

Let us take for $(\Omega, \mathcal{A}, \mu)$ a compact abelian group G endowed with its Borel σ-algebra and its Haar measure. In this framework, the first examples of dynamical systems satisfying a CLT in a class of regular functions are due to R. Fortet and M. Kac for endomorphims of $\mathbb{T}^1$. In 1960 V. Leonov ([17]) showed that, if T is an ergodic endomorphism of G, then the CLT holds for every function f on G under a certain regularity condition on f.

The d-dimensional extension of this situation leads to the question of validity of a CLT for algebraic actions on an abelian compact group G, i.e., when $T^{\underline{k}}$ in Formula (1) is given by an action of $\mathbb{N}^d$ on G by automorphisms or more generally endomorphisms.

By composition, one obtains an action by isometries on $\mathcal{H} = L^2_0(\mu)$, the space of square integrable functions f such that $\mu(f) = 0$. The spectral analysis of this action is the content of Section 1 where the methods of summation are also discussed.

In Section 2 we consider d-dimensional actions given by commuting toral automorphisms on $\mathbb{T}^\rho$, $\rho \geq 1$. For f with a certain regularity on the torus, a CLT is shown for the above normalized sequence (Theorem 2.28) and other summation methods like barycenters (Theorem 2.31), as well as a criterion of non-degeneracy of the variance. The barycenters yield a class of operators with a polynomial decay to zero of the iterates applied to regular functions. This contrasts with the non commutative case where a spectral gap can be expected.

We will focus on d-dimensional actions by automorphisms on tori. Some results are valid for semigroups of endomorphisms. The martingale property is a powerful tool, which can used in some cases. Here we use a different method, based on moments, valid in the

[1]We underline the elements of $\mathbb{N}^d$ or $\mathbb{Z}^d$ to distinguish them from the scalars and write $T^{\underline{k}} f$ for $f \circ T^{\underline{k}}$.

general case. The key of the method is the mixing property for $\mathbb{Z}^d$-actions given of toral automorphisms ([23]) which is a consequence of deep results of the theory of S-units. In an appendix, classical results on the construction of $\mathbb{Z}^d$-actions by automorphisms are recalled.

1. **Spectral analysis**

In this section we consider the general framework of the action of an abelian finitely generated semigroup $\mathcal{S}$ of isometries on a Hilbert space $\mathcal{H}$. We have in mind the example of a semigroup $\mathcal{S}$ of endomorphisms of a compact abelian group G acting on $\mathcal{H} = L^2_0(G,\mu)$, with μ the Haar measure of G.

With the notations of the introduction, every $T \in \mathcal{S}$ is represented as $T = T^{\underline{\ell}} = T_1^{\ell_1}...T_d^{\ell_d}$, where $(T_1,...,T_d)$ is a system of generators in $\mathcal{S}$ and $\underline{\ell} = (\ell_1,...,\ell_d) \in \mathbb{N}^d$.

Given $f \in \mathcal{H}$, for $d > 1$, there are various choices of the sets of summation D_n for the field $(T^{\underline{\ell}} f, \underline{\ell} \in \mathbb{N}^d)$. We discuss this point, as well as the behavior of the associated (by discrete Fourier transform) kernels. The second subsection is devoted to the spectral analysis of the d-dimensional action.

1.1. **Summation and kernels.**

If $(D_n)_{n\geq 1}$ is a sequence of subsets of $\mathbb{N}^d$, the corresponding rotated sum and kernel are respectively: $\sum_{\underline{\ell}\in D_n} \mathrm{e}^{2\pi i\langle \underline{\ell},\theta\rangle} T^{\underline{\ell}} f$ and $\frac{1}{|D_n|}\big|\sum_{\underline{\ell}\in D_n} \mathrm{e}^{2\pi i\langle \underline{\ell},t\rangle}\big|^2$. The simplest choice for (D_n) is an increasing family of squares or rectangles.

Notation 1.1. More generally, we will call *summation sequence* a sequence (R_n) of functions from $\mathbb{N}^d$ to $\mathbb{R}^+$. It could be also defined on $\mathbb{Z}^d$, but for simplicity in this section we consider summation for $\underline{\ell} \in \mathbb{N}^d$. We will suppose that $\sup_n \|R_n\|_\infty < \infty$. If $T = (T^{\underline{\ell}})$ is a semigroup of isometries, the associated sequence of operators on $\mathcal{H}$ is

$$R_n(T) : f \in \mathcal{H} \to R_n(T)f := \sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell}) T^{\underline{\ell}} f.$$

For simplicity we just write R_n instead of $R_n(T)$. By introducing a rotation term, these operators extend to a family of operators R_n^θ, for $\theta \in \mathbb{R}^d$,

$$f \to R_n^\theta f := \sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell})\, e^{2\pi i\langle \underline{\ell},\theta\rangle}\, T^{\underline{\ell}} f.$$

We have $\|\sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell}) e^{2\pi i\langle \underline{\ell},.\rangle}\|^2_{L^2(\mathbb{T}^d,dt)} = \sum_{\underline{\ell}\in\mathbb{N}^d} |R_n(\underline{\ell})|^2$. Taking the discrete Fourier transform, we associate to R_n the normalized "kernel" $\tilde{R}_n$ defined on $\mathbb{T}^d$ by:

$$\tilde{R}_n(t) = \frac{|\sum_{\underline{\ell}\in\mathbb{N}^d} R_n(\underline{\ell}) e^{2\pi i\langle \underline{\ell},t\rangle}|^2}{\sum_{\underline{\ell}\in\mathbb{N}^d} |R_n(\underline{\ell})|^2}.$$

We say that $(R_n)_{n\geq 1}$ is *regular* if $(\tilde{R}_n)_{n\geq 1}$ weakly converges to a measure ζ on $\mathbb{T}^d$, i.e., $\int_{\mathbb{T}^d} \tilde{R}_n\, \varphi\, dt \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{T}^d} \varphi\, d\zeta$ for every continuous function φ on $\mathbb{T}^d$.

Lemma 1.2. *The kernel $(\tilde{R}_n)$ associated to sets $D_n \subset \mathbb{N}^d$ converges to the Dirac measure δ_0 if and only if (D_n) satisfies the Følner's type condition:*

$$\lim_{n\to\infty} |D_n|^{-1} |(D_n + \underline{p}) \cap D_n| = 1,\ \forall \underline{p} \in \mathbb{Z}^d. \tag{2}$$

Proof. We have for the characters $\chi_{\underline{p}}(t) = \mathrm{e}^{2\pi i \langle \underline{p}, t\rangle}$, $\underline{p} \in \mathbb{Z}^d$,

$$\int_{\mathbb{T}^d} \tilde{R}_n(t)\chi_{\underline{p}}(t)dt = \frac{1}{|D_n|}\int_{\mathbb{T}^d} \sum_{\underline{k},\underline{k}'\in D_n} \mathrm{e}^{2\pi i\langle(\underline{k}-\underline{k}'+\underline{p}),t\rangle} dt = \frac{|(D_n+\underline{p})\cap D_n|}{|D_n|}.$$

Therefore, since $\int_{\mathbb{T}^d} \tilde{R}_n(t)dt = 1$ and $\sup_n \|\tilde{R}_n * \varphi\|_\infty \leq \|\varphi\|_\infty \sup_n \|\tilde{R}_n\|_1 = \|\varphi\|_\infty$, by linearity and density, $(\tilde{R}_n)$ weakly converges to δ_0 if and only if (2) is satisfied. □

A family of examples satisfying (2) can be obtained as follows: take a non-empty domain $D \subset \mathbb{R}^d$ with *smooth* boundary and finite area and put $D_n = \lambda_n D \cap \mathbb{Z}^d$, where (λ_n) is an increasing sequence of real numbers tending to $+\infty$. Below we consider different examples.

Squares and rectangles. Using the usual one-dimensional Fejér kernel $K_N(t) = \frac{1}{N}(\frac{\sin \pi N t}{\sin \pi t})^2$, the d-dimensional Fejér kernels on $\mathbb{T}^d$ corresponding to rectangles are defined by

$$K_{N_1,\dots,N_d}(t_1,\dots,t_d) = K_{N_1}(t_1)\cdots K_{N_d}(t_d),\ \text{for } \underline{N} = (N_1,\dots,N_d) \in \mathbb{N}^d.$$

They are the kernels associated to $D_{\underline{N}} := \{\underline{k} \in \mathbb{N}^d :\ k_i \leq N_i, 1 \leq i \leq d\}$.

Other examples

Kernels with unbounded gaps

Analogously to Lemma 1.2, we have that if $(D_{\underline{n}})$ is a sequence of domains such that $\lim_{\underline{n}} \frac{|(D_{\underline{n}}+\underline{p})\cap D_{\underline{n}}|}{|D_{\underline{n}}|} = 0$ for every $\underline{p} \neq 0$ and if $(\tilde{R}_{\underline{n}})$ is the kernel associated to (D_n), then for every continuous function φ, $\lim_{\underline{n}}(\tilde{R}_{\underline{n}} * \varphi)(\theta) = \int_{\mathbb{T}^d} \varphi(t)dt$ for every $\theta \in \mathbb{T}^d$.

An example is the following. Let k_j be a sequence with $k_{j+1} - k_j \to \infty$ and put $D_n = \{k_j :\ 0 \leq j \leq n-1\}$. For $p \neq 0$ the number of solutions of $k_j - k_\ell = p$, for $j, \ell \geq 0$ is finite, so that $\lim_{n\to\infty} \frac{|(D_n+p)\cap D_n|}{|D_n|} = 0$ for $p \neq 0$.

Iteration of barycenter operators

Another method of summation, which is not of Følner type, is given by a barycenter of operators. Let $T_1, \dots, T_d$ be d commuting unitary operators on a Hilbert space $\mathcal{H}$ generating a group $\mathcal{S}$. If $(p_1, \dots, p_d)$ is a probability vector such that $p_j > 0, \forall j$, for

$\theta = (\theta_1, \theta_2, ..., \theta_d) \in \mathbb{T}^d$, we will consider the operators respectively defined on $\mathcal{H}$ by

$$P : f \to \sum_{j=1}^{d} p_j \, T_j f, \quad P_\theta : f \to \sum_{j=1}^{d} p_j \, e^{2\pi i \theta_j} \, T_j f. \tag{3}$$

1.2. **Lebesgue spectrum, variance.**

Let $\mathcal{S}$ be a finitely generated *torsion free* commutative group of unitary operators on a Hilbert space $\mathcal{H}$. Assume that $\mathcal{S}$ has dimension $d \geq 1$. Let $(T_1, ..., T_d)$ be a system of independent generators in $\mathcal{S}$. Each $T \in \mathcal{S}$ can be written in a unique way as $T = T^{\underline{\ell}} = T_1^{\ell_1} ... T_d^{\ell_d}$, with $\underline{\ell} = (\ell_1, ..., \ell_d) \in \mathbb{Z}^d$.

For every $f \in \mathcal{H}$, there is a positive finite measure ν_f on $\mathbb{T}^d$ such that, for every $\underline{\ell} \in \mathbb{Z}^d$, $\hat{\nu}_f(\underline{\ell}) = \langle T^{\ell_1} ... T^{\ell_d} f, f \rangle$.

Definition 1.3. Recall that the action of $\mathcal{S}$ on $\mathcal{H}$ has a *Lebesgue spectrum*, if there exists $\mathcal{K}_0$, a closed subspace of $\mathcal{H}$, such that the images by $T^{\underline{\ell}} \mathcal{K}_0$ are pairwise orthogonal and span a dense subspace in $\mathcal{H}$.

Any set of generators $(T_1, ..., T_d)$ in $\mathcal{S}$ defines a unitary representation of $\mathbb{Z}^d$ on $L^2_0(\Omega, \mu)$. With the Lebesgue spectrum property, for every $f \in \mathcal{H}$, the corresponding spectral measure ν_f of f on $\mathbb{T}^d$ has a density φ_f. A change of basis induces for the spectral density the composition by an automorphism acting on $\mathbb{T}^d$.

As to examples of totally ergodic $\mathbb{Z}^d$-actions by unitary operators, let us mention the natural $\mathbb{Z}^d$-action on the full $\mathbb{Z}^d$-shift or $\mathbb{Z}^d$-subshifts. Another family of examples is provided by the action of a group of commuting automorphisms on tori. In the present paper, we will focus mainly on this latter class of examples.

Notation 1.4. For any orthonormal basis $(\psi_j)_{j \in J}$ of $\mathcal{K}_0$, the family $(T^{\underline{\ell}} \psi_j)_{j \in J, \underline{\ell} \in \mathbb{Z}^d}$ is an orthonormal basis of $\mathcal{H}$. Let $\mathcal{H}_j$ be the closed subspace (invariant by the $\mathbb{Z}^d$-action) generated by $(T^{\underline{n}} \psi_j)_{\underline{n} \in \mathbb{Z}^d}$.

We set $a_{j,\underline{n}} := \langle f, T^{\underline{n}} \psi_j \rangle$, $j \in J$. Let f_j be the orthogonal projection of f on $\mathcal{H}_j$ and γ_j an everywhere finite square integrable function on $\mathbb{T}^d$ with Fourier coefficients $a_{j,\underline{n}}$.

The spectral measure is the sum of the spectral measures of f_j. For f_j, the density of the spectral measure is $|\gamma_j|^2$. Therefore, by orthogonality of the subspaces $\mathcal{H}_j$, the density of the spectral measure of f is $\varphi_f = \sum_{j \in J} |\gamma_j|^2$.

Since $\int_{\mathbb{T}^d} \sum_{j \in J} |\gamma_j(\theta)|^2 \, d\theta = \sum_{j \in J} \sum_{\underline{n} \in \mathbb{Z}^d} |a_{j,\underline{n}}|^2 = \int_{\mathbb{T}^d} \varphi_f(\theta) \, d\theta = \|f\|^2 < \infty$, the set $\Lambda_0 := \{\theta \in \mathbb{T}^d : \sum_{j \in J} |\gamma_j(\theta)|^2 < \infty\}$ has full measure.

Using the basis $(T^{\underline{\ell}} \psi_j)$ we associate to $f \in L^2_0(\mu)$ and $\theta \in \mathbb{T}^d$ an element $M_\theta f$ in $\mathcal{K}_0$ with orthogonal "increments" and such that the rotated sums corresponding to $M_\theta f$

approximate in a sense (cf. (34)) the rotated sums of f:

$$M_\theta f := \sum_j \gamma_j(\theta)\, \psi_j. \tag{4}$$

One easily see that $M_\theta f$ is defined for θ in a set Λ_0 of full measure in $\mathbb{T}^d$. It is defined for every θ, $\theta \to \|M_\theta\|_2^2$ is continuous and is equal everywhere to φ_f under assumption

$$\sum_{j\in J} \big(\sum_n |a_{j,\underline{n}}|\big)^2 < +\infty. \tag{5}$$

Remark that the choice of the system (ψ_j) generating the orthonormal basis $(T^{\underline{n}}\psi_j)$ is not unique, so that the definition of $M_\theta f$ is not canonical. But for algebraic automorphisms of a compact abelian group G, Fourier analysis gives a natural choice for the basis.

Mixing: Recall that $\mathcal{S}$ is mixing (or more precisely *mixing of order 2*) if

$$\lim_{\|\underline{n}\|\to\infty} |\langle T^{\underline{n}} f, g\rangle| = 0,\ \forall f, g \in \mathcal{H}. \tag{6}$$

The Lebesgue spectrum property implies mixing of order 2.

Variance for summation sequences

Let $(D_n) \subset \mathbb{N}^d$ be an increasing sequence of subsets. For $f \in L^2_0(\mu)$, the asymptotic variance at θ along (D_n) is, when it exists, the limit

$$\sigma^2_\theta(f) = \lim_n \frac{\| \sum_{\underline{\ell}\in D_n} e^{2\pi i\langle \underline{\ell},\theta\rangle} T^{\underline{\ell}} f\|_2^2}{|D_n|}. \tag{7}$$

By the spectral theorem, if $\varphi_f \in L^1(\mathbb{T}^d)$ is the spectral density of f and $\tilde{R}_n$ the kernel associated to (D_n), then

$$|D_n|^{-1} \| \sum_{\underline{\ell}\in D_n} e^{2\pi i\langle \underline{\ell},\theta\rangle} T^{\underline{\ell}} f\|_2^2 = (\tilde{R}_n * \varphi_f)(\theta). \tag{8}$$

If (D_n) is a sequence of d-cubes, we obtain, when it exists, the usual asymptotic variance at θ. By the Fejér-Lebesgue theorem, for every f in $\mathcal{H}$, for of cubes, it exists for a.e. θ.

More generally, if (R_n) is a summation sequence and $(\tilde{R}_n)$ the associated kernel, when φ_f is continuous and $(\tilde{R}_n)$ is weakly converging to a measure ζ on $\mathbb{T}^d$, we have

$$\frac{\|R^\theta_n f\|_2^2}{\sum_{\underline{\ell}} |R_n(\underline{\ell})|^2} = \int_{\mathbb{T}^d} \tilde{R}_n(\theta - t)\varphi_f(t)\, dt \ \to \int_{\mathbb{T}^d} \varphi_f(\theta - t)\, d\zeta(t).$$

For Følner sequences (Condition (2)), ζ is the Dirac measure at 0 and when φ_f is continuous the corresponding asymptotic variance $\sigma^2_\theta(f)$ is equal to $\varphi_f(\theta)$.

Variance for barycenter

Let P and P_θ be defined by (3) for d commuting unitary operators $T_1, ..., T_d$ on a Hilbert space $\mathcal{H}$ generating a group $\mathcal{S}$ with the Lebesgue spectrum property and let $(p_1, ..., p_d)$

be a probability vector such that $p_j > 0, \forall j$. If φ_f is the spectral density of f in $\mathcal{H}$ with respect to the action of $\mathcal{S}$, we have:

$$\|P_\theta^n f\|_2^2 = \int_{\mathbb{T}^d} |\sum_{j=1}^d p_j\, e^{2\pi i t_j}|^{2n}\, \varphi_f(\theta_1 - t_1, ..., \theta_d - t_d)\; dt_1 \dots dt_d.$$

In order to find the normalization of $P^n f$ for $f \in \mathcal{H}$, we need an estimation, when $n \to \infty$, of the integral $I_n := \int_{\mathbb{T}^d} |\sum_j p_j e^{2\pi i t_j}|^{2n}\; dt_1 ... dt_d$.

Proposition 1.5. *If $(p_1, ..., p_d)$ is a probability vector such that $p_j > 0, \forall j$, we have*

$$\lim_n n^{\frac{d-1}{2}} \int_{\mathbb{T}^d} |\sum_j p_j e^{2\pi i t_j}|^{2n}\; dt_1 ... dt_d = (4\pi)^{-\frac{d-1}{2}}\, (p_1 ... p_d)^{-\frac{1}{2}}. \tag{9}$$

Lemma 1.6. *Let r be an integer ≥ 1 and let $(q_1, ..., q_r)$ be a vector such that $q_j > 0, \forall j$ and $\sum_j q_j \leq 1$. Then the quadratic form Q on $\mathbb{R}^r$ defined by*

$$Q(\underline{t}) = \sum_{j=1}^r q_j t_j^2 - (\sum_{j=1}^r q_j t_j)^2 \tag{10}$$

is positive definite with determinant $(1 - \sum_j q_j)\, q_1 ... q_r$.

Proof. We give a proof by induction on r. Let us consider the polynomial of second order with respect to the variable t_1:

$$q_1 t_1^2 + \sum_2^r q_j t_j^2 - (q_1 t_1 + \sum_2^r q_j t_j)^2 = (q_1 - q_1^2) t_1^2 - 2q_1 (\sum_2^r q_j t_j) t_1 + \sum_2^r q_j t_j^2 - (\sum_2^r q_j t_j)^2.$$

It is always ≥ 0, since its discriminant

$$q_1^2 (\sum_2^r q_j t_j)^2 - q_1(1-q_1)(\sum_2^r q_j t_j^2 - (\sum_2^r q_j t_j)^2) = q_1 (1-q_1)^2 [(\sum_2^r \frac{q_j}{1-q_1} t_j)^2 - \sum_2^r \frac{q_j}{1-q_1} t_j^2],$$

is < 0 for $\sum_{j=2}^r t_j^2 \neq 0$ by induction hypothesis, since $\frac{q_j}{1-q_1} > 0$ and $\sum_{j=2}^r \frac{q_j}{1-q_1} \leq 1$.

The quadratic form is given by the symmetric matrix: $A = \operatorname{diag}(q_1, ..., q_r)\; B$, where

$$B \quad = \begin{pmatrix} 1-q_1 & -q_2 & . & -q_r \\ -q_1 & 1-q_2 & . & -q_r \\ . & . & . & . \\ -q_1 & -q_2 & . & 1-q_r \end{pmatrix}.$$

The determinant of B is of the form $\alpha + \sum_j \beta_j q_j$, where the coefficients $\alpha, \beta_1, ..., \beta_r$ are constant. Giving to $q_1, ..., q_r$ the values 0 except for one of them, we find $\alpha = 1$, $\beta_1 = \beta_2 = ... = \beta_r = -1$. Hence $\det A = (1 - \sum_j q_j)\, q_1 ... q_r$. □

Remark that the positive definiteness follows also from the properties of F since Q gives the approximation of F defined below at order 2.

Proof of Proposition 1.5 Observe that by strict convexity $|\sum_j p_j e^{2\pi i t_j}| \leq 1$ and $|\sum_j p_j e^{2\pi i t_j}| = 1$ if and only if $\underline{t} = \underline{0}$ modulo $\mathbb{Z}^d$. Since $|\sum_j p_j e^{2\pi i t_j}|^{2n} = |p_1 + \sum_{j=2}^d p_j e^{2\pi i (t_j - t_1)}|^{2n}$ and using the invariance of the integral by translation, we have $I_n = \int_{\mathbb{T}^{d-1}} |p_1 + \sum_{j=2}^d p_j e^{2\pi i t_j}|^{2n}\, dt_2 ... dt_d$.

Putting $q_j := p_{j+1}$, $j = 1, ..., d-1$, $r = d-1$, we have $q_j > 0$, $\sum_1^r q_j < 1$, and the computation reduces to estimate:

$$I_n := \int_{\mathbb{T}^r} [|1 + \sum_{j=1}^r q_j (e^{2\pi i t_j} - 1)|^2]^n \, dt_1 ... dt_r = \int_{\mathbb{T}^r} [1 - F(\underline{t})]^n \, dt_1 ... dt_r,$$

with $F(\underline{t}) := 1 - |1 + \sum_{j=1}^r q_j (e^{2\pi i t_j} - 1)|^2$.

A point $\underline{t} = (t_1, ... t_r)$ of the torus by coordinates such that: $-\frac{1}{2} < t_j < \frac{1}{2}$. We have $F(\underline{t}) \geq 0$ and $F(\underline{t}) = 0$ if and only if $\underline{t} = 0$. A stronger property is the existence of $c > 0$ such that

$$F(\underline{t}) \geq c \|\underline{t}\|^2, \ \forall \underline{t} : -\frac{1}{2} < t_j < \frac{1}{2}. \tag{11}$$

Indeed Inequality (11) is clear, outside a small open neighborhood V of 0, since $F(\underline{t})$ is bounded away from 0 for $\underline{t}$ in V. On V, we can replace F by a positive definite form as we will see below. This shows the result on V.

From the convergence

$$\lim_n n^{\frac{r}{2}} \int_{\{\underline{t} \in \mathbb{T}^r : \|\underline{t}\| > \frac{\ln n}{\sqrt{n}}\}} (1 - F(\underline{t}))^n \, d\underline{t} \leq \lim_n n^{\frac{d-1}{2}} (1 - c \frac{(\ln n)^2}{n})^n = 0,$$

it follows:

$$\lim_n n^{\frac{r}{2}} \int_{\underline{t} \in \mathbb{T}^{d-1}} (1 - F(\underline{t}))^n \, d\underline{t} = \lim_n n^{\frac{r}{2}} J_n,$$

where $J_n := \int_{\{\underline{t} \in \mathbb{T}^r : \|\underline{t}\| \leq \frac{\ln n}{\sqrt{n}}\}} (1 - F(\underline{t}))^n \, d\underline{t}$.

By taking the Taylor approximation of order 2 at 0 of the exponential function $e^{it_j} = 1 + it_j - \frac{t_j^2}{2} + i\gamma_1(t_j) + \gamma_2(t_j)$, with $|\gamma_1(t_j) + |\gamma_2(t_j)| = o(|t_j|^2)$, we obtain:

$$F(t) = Q(2\pi t) + \gamma(t), \quad \text{with } Q(\underline{t}) = \sum q_j t_j^2 - (\sum q_j t_j)^2 \ \text{ and } \gamma(t) = o(\|t\|^2).$$

The quadratic form Q is the form defined by (10). Therefore, it is positive definite by Lemma 1.6 and there is $c > 0$ such that $Q(\underline{t}) \geq c \|\underline{t}\|^2, \forall \underline{t} \in \mathbb{R}^r$.

We have $\lim_{\delta \downarrow 0} \sup_{\|t\| \leq \delta} F(t)/Q(2\pi t) = 1$. With the notation $\underline{u} = (u_1, ..., u_r)$, $\underline{t} = (t_1, ..., t_r)$ and the change of variable $\underline{u} = \sqrt{n}\, \underline{t}$, we get:

$$n^{\frac{r}{2}} J_n \sim \int_{\{\|\underline{u}\| \leq \ln n\}} (1 - Q(\frac{2\pi \underline{u}}{\sqrt{n}}))^n \, d\underline{u} \to \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} e^{-Q(u)} \, d\underline{u}.$$

We have $\int_{\mathbb{R}^r} e^{-Q(u)} \, d\underline{u} = \pi^{\frac{r}{2}} \det(A)^{-\frac{1}{2}} = \pi^{\frac{r}{2}} (p_1 ... p_d)^{-\frac{1}{2}}$. Therefore we obtain:

$$\lim_n n^{\frac{d-1}{2}} \int_{\mathbb{T}^d} |\sum_j p_j e^{2\pi i t_j}|^{2n} \, dt_1 ... dt_d = (4\pi)^{-\frac{r}{2}} (p_1 ... p_d)^{-\frac{1}{2}}.$$

As expected the final result is symmetric into the parameters $p_1, ..., p_d$, although the variables $t_1, ..., t_d$ play a dissymmetric role in the calculation. □

Example: With $K_n(t_1, t_2) := \sqrt{\pi n}|(\frac{e^{2\pi i t_1}+e^{2\pi i t_2}}{2})|^{2n}$, we have $\int_{\mathbb{T}^2} K_n(t_1, t_2)dt_1dt_2 \to 1$. This can be shown also using Stirling's approximation:

$$\begin{aligned}\int_{\mathbb{T}^2} K_n(t_1, t_2)dt_1dt_2 &= \frac{\sqrt{\pi n}}{4^n}\int_{\mathbb{T}^2} |\sum_{k=0}^{n}\binom{n}{k}\mathrm{e}^{2\pi i k t_1}\mathrm{e}^{2\pi i(n-k)t_2}\big|^2 dt_1dt_2\\ &= \frac{\sqrt{\pi n}}{4^n}\sum_{k=0}^{n}\binom{n}{k}^2 = \frac{\sqrt{\pi n}}{4^n}\binom{2n}{n} \underset{n\to\infty}{\to} 1.\end{aligned}$$

Proposition 1.5 gives the normalization for the iterates of P or P_θ:

Proposition 1.7. *If φ_f is continuous, then for every $\theta \in \mathbb{T}^d$ we have*

$$\lim_{n\to\infty} (4\pi)^{\frac{d-1}{2}}\,(p_1...p_d)^{\frac{1}{2}}\,n^{\frac{d-1}{2}}\,\|P_\theta^n f\|_2^2 = \int_{\mathbb{T}} \varphi_f(\theta_1 + u, ..., \theta_d + u)\,du. \tag{12}$$

Proof. Let us put $c_n := (4\pi)^{\frac{d-1}{2}}\,(p_1...p_d)^{\frac{1}{2}}\,n^{\frac{d-1}{2}}$ for the normalization coefficient and

$$K_n(t_1, ..., t_d) := c_n\,|\sum_{j=1}^{d} p_j\,e^{2\pi i t_j}|^{2n}.$$

We have $c_n\,\|P_\theta^n f\|_2^2 = (K_n * \varphi_f)(\theta_1, ..., \theta_d)$ and, by (9) $\int_{\mathbb{T}^d} K_n(t_1, ..., t_d)\,dt_1...dt_d \to 1$.

Let us show that for φ continuous on $\mathbb{T}^d$, $\lim_n \int_{\mathbb{T}^d} K_n\,\varphi\,dt_1...dt_d = \int_{\mathbb{T}} \varphi(u, ..., u)\,du$. Using the density of trigonometric polynomials for the uniform norm, it is enough to prove it for characters $\chi_{\underline{k}}(\underline{t}) = \mathrm{e}^{2\pi i\sum_j k_j t_j}$, i.e., to prove that for $\varphi = \chi_{\underline{k}}$ the limit is 0 if $\sum_\ell k_\ell \neq 0$, and 1 if $\sum_\ell k_\ell = 0$. We have

$$\int_{\mathbb{T}^d} K_n(t_1, ..., t_d)\,e^{2\pi i\sum_\ell k_\ell t_\ell}\,dt_1...dt_d = c_n\int_{\mathbb{T}^d} |\sum_{j=1}^{d} p_j\,e^{2\pi i t_j}\big|^{2n}\,e^{2\pi i\sum_\ell k_\ell t_\ell}\,dt_1...dt_d$$

$$= (c_n\int_{\mathbb{T}^{d-1}} |p_1 + \sum_{j=2}^{d} p_j\,e^{2\pi i(t_j - t_1)}\big|^{2n}\,e^{2\pi i\sum_{\ell=2}^{d} k_\ell(t_\ell - t_1)}\,dt_2...dt_d)\int_{\mathbb{T}} e^{2\pi i(\sum_\ell k_\ell)t_1}dt_1.$$

Therefore it remains to show that the limit of the first factor when $n \to \infty$ is 1. Using the proof and the result of Proposition 1.5, we find that this factor is equivalent to

$$(4\pi)^{\frac{d-1}{2}}\,(p_1...p_d)^{\frac{1}{2}}\int_{\{\underline{u}:\|\underline{u}\|\le\frac{\ln n}{\sqrt{n}}\}} (1 - Q(\frac{2\pi\underline{u}}{\sqrt{n}}))^n e^{2\pi i\sum_1^r k_{\ell+1}\frac{u_\ell}{\sqrt{n}}}\,d\underline{u},$$

which tends to 1. □

1.3. **Nullity of variance and coboundaries.**

Let $\mathcal{H}$ be a Hilbert space and let T_1 and T_2 be two commuting unitary operators acting on $\mathcal{H}$. Assuming the Lebesgue spectrum property for the $\mathbb{Z}^2$-action generated by T_1 and T_2, we study in this subsection the degeneracy of the variance. Here we consider, for simplicity, the case of two unitary commuting operators, but the results are valid for any finite family of commuting unitary operators.

Single Lebesgue spectrum

At first, let us assume that there is $\psi \in \mathcal{H}$ such that the family of vectors $T_1^k T_2^r \psi$ for $(k,r) \in \mathbb{Z}^2$ is an orthonormal basis of $\mathcal{H}$ (simplicity of the spectrum).

Lemma 1.8. *Let f be in $\mathcal{H}$ and $f = \sum_{(k,r)\in\mathbb{Z}^2} a_{k,r}\, T_1^k T_2^r \psi$ be the representation of f in the orthonormal basis $(T_1^k T_2^r \psi,\ (k,r) \in \mathbb{Z}^2)$. If*

$$A := \sum_{k,r\in\mathbb{Z}^2} (1 + |k| + |r|)\, |a_{k,r}| < +\infty, \tag{13}$$

there exists $u, v \in \mathcal{H}$ with $\|u\|, \|v\| \le A$ such that

$$f = \Big(\sum_{(k,r)\in\mathbb{Z}^2} a_{k,r} \Big)\, \psi + (I - T_1)u + (I - T_2)v.$$

If $\sum_{(k,r)\in\mathbb{Z}^2} a_{k,r} = 0$, then f is sum of two coboundaries respectively for T_1 and T_2:

$$f = (I - T_1)u + (I - T_2)v.$$

Proof. 1) We start with a formal computation. Let us decompose f into vectors whose coefficients are supported on disjoint quadrants of increasing dimensions. If $f = \sum_{k,r\in\mathbb{Z}^2} a_{k,r} T_1^k T_2^r \psi$, we write

$$f = f_{0,0} + f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1} + f_{1,1} + f_{-1,1} + f_{1,-1} + f_{-1,-1}, \tag{14}$$

with

$$f_{0,0} = a_{0,0}\psi,\ f_{1,0} = \sum_{k>0} a_{k,0}\, T_1^k \psi,\ f_{0,1} = \sum_{r>0} a_{0,r}\, T_2^r \psi,$$
$$f_{-1,0} = \sum_{k>0} a_{-k,0}\, T_1^{-k} \psi,\ f_{0,-1} = \sum_{r>0} a_{0,-r}\, T_2^{-r} \psi,$$
$$f_{1,1} = \sum_{k,r>0} a_{k,r}\, T_1^k T_2^r \psi,\ f_{-1,1} = \sum_{k,r>0} a_{-k,r}\, T_1^{-k} T_2^r \psi,$$
$$f_{1,-1} = \sum_{k,r>0} a_{k,-r}\, T_1^k T_2^{-r} \psi,\ f_{-1,-1} = \sum_{k,r>0} a_{-k,-r}\, T_1^{-k} T_2^{-r} \psi.$$

For each component given by a quadrant, we solve the corresponding coboundary equation up to *constant* $\times\, \psi$.

With f decomposed as in (14), the components can be formally written in the following way, with $\varepsilon_i, \varepsilon_i' \in \{0, +1, -1\}$, for $i = 1, 2$:

$$\begin{aligned} f_{0,0} &= u_{0,0} = a_{0,0}\,\psi, \quad f_{\varepsilon_1,0} = u^0_{\varepsilon_1,0} + (T_1^{\varepsilon_1} - I)u^{\varepsilon_1,\,0}_{\varepsilon_1,\,0}, \quad f_{0,\varepsilon_2} = u^0_{0,\varepsilon_2} + (T_2^{\varepsilon_2} - I)u^{0,\,\varepsilon_2}_{0,\,\varepsilon_2}, \\ f_{\varepsilon_1,\varepsilon_2} &= u^0_{\varepsilon_1,\varepsilon_2} + (T_1^{\varepsilon_1} - I)u^{\varepsilon_1,0}_{\varepsilon_1,\varepsilon_2} + (T_2^{\varepsilon_2} - I)u^{0,\varepsilon_2}_{\varepsilon_1,\varepsilon_2} - (T_1^{\varepsilon_1} - I)(T_2^{\varepsilon_2} - I)u^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1,\varepsilon_2}, \end{aligned}$$

where

$$\begin{aligned} u^0_{\varepsilon_1,0} &= (\sum_{t\geq 1} a_{\varepsilon_1 t,\,0})\,\psi, \quad u^{\varepsilon_1,\,0}_{\varepsilon_1,\,0} = \sum_{k\geq 0}(\sum_{t\geq k+1} a_{\varepsilon_1 t,\,0})T_1^{\varepsilon_1 k}\,\psi, \\ u^0_{0,\varepsilon_2} &= (\sum_{s\geq 1} a_{0,\,\varepsilon_2 s})\,\psi, \quad u^{0,\,\varepsilon_2}_{0,\,\varepsilon_2} = \sum_{r\geq 0}(\sum_{s\geq r+1} a_{0,\,\varepsilon_2 s})T_2^{\varepsilon_2 r}\,\psi, \\ u^0_{\varepsilon_1,\varepsilon_2} &= (\sum_{t,s\geq 1} a_{\varepsilon_1 t,\,\varepsilon_2 s})\,\psi, \;\; u^{\varepsilon_1,0}_{\varepsilon_1,\varepsilon_2} = \sum_{k0,r\geq 1}(\sum_{t\geq k+1} a_{\varepsilon_1 t,\,\varepsilon_2 r})\,T_1^{\varepsilon_1 k}T_2^{\varepsilon_2 r}\,\psi, \\ u^{0,\varepsilon_2}_{\varepsilon_1,\varepsilon_2} &= \sum_{k\geq 1,r\geq 0}(\sum_{s\geq r+1} a_{\varepsilon_1 k,\,\varepsilon_2 s})\,T_1^{\varepsilon_1 k}T_2^{\varepsilon_2 r}\,\psi, \\ u^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1,\varepsilon_2} &= \sum_{k,r\geq 0}(\sum_{t\geq k+1,s\geq r+1} a_{\varepsilon_1 t,\,\varepsilon_2 s})\,T_1^{\varepsilon_1 k}T_2^{\varepsilon_2 r}\,\psi. \end{aligned}$$

More explicitly we have, for instance,

$$\begin{aligned} &f_{1,0} = u^0_{1,0} + (T_1 - I)u^{1,0}_{1,0} = (\sum_{t\geq 1} a_{t,0})\,\psi + (T_1 - I)\,[\sum_{k\geq 0}(\sum_{t\geq k+1} a_{t,0})T_1^k\,\psi], \\ &f_{1,1} = u^0_{1,1} + (T_1 - I)u^{1,0}_{1,1} + (T_2 - I)u^{0,1}_{1,1} + (T_1 - I)(T_2 - I)u^{1,1}_{1,1} \\ &= (\sum_{t,s\geq 1} a_{t,s})\,\psi + (T_1 - I)\,[\sum_{k\geq 0,r\geq 1}(\sum_{t\geq k+1} a_{t,r})\,T_1^k T_2^r\,\psi] \\ &+(T_2 - I)\,[\sum_{k\geq 1,r\geq 0}(\sum_{s\geq r+1} a_{k,s})\,T_1^k T_2^r\,\psi] - (T_1 - I)(T_2 - I)\,[\sum_{k,r\geq 0}(\sum_{t\geq k+1,s\geq r+1} a_{t,s})\,T_1^k T_2^r\,\psi]. \end{aligned}$$

By summing the previous expressions, we obtain the following representation of f:

$$\begin{aligned} f = (\sum a_{t,s})\,\psi \quad &+(T_1 - I)(u^1_1 - T_1^{-1}u^1_{-1} + u^{1,2}_1 - T_1^{-1}u^{-1,2}_{-1} + u^{1,-2}_1 - T_1^{-1}u^{-1,-2}_{-1}) \\ &+(T_2 - I)(u^1_2 - T_2^{-1}u^1_{-2} + u^{1,2}_2 - T_2^{-1}u^{1,-2}_{-2} + u^{-1,2}_2 - T_2^{-1}u^{-1,-2}_{-2}) \\ &+(T_1 - I)(T_2 - I)(u^{1,2}_{1,2} - T_1^{-1}u^{-1,2}_{-1,2} - T_2^{-1}u^{1,-2}_{1,-2} + T_1^{-1}T_2^{-1}u^{-1,-2}_{-1,-2}). \end{aligned}$$

The first term is the vector $a(f)\,\psi$, where $a(f)$ is the constant $\sum_{k,r\in\mathbb{Z}^2} a_{k,r}$ obtained as the sum $u^0_0 + u^0_1 + u^0_2 + u^0_{-1} + u^0_{-2} + u^{1,2}_0 + u^{-1,2}_0 + u^{1,-2}_0 + u^{-1,-2}_0$. The second term is a sum of coboundaries. If $a(f) = 0$, then f reduces to a sum of coboundaries.

2) Now we examine the question of convergence in the previous computation. We need the convergence of the following series (for $\varepsilon_1, \varepsilon_2 = \pm 1$):

$$\sum_{t,s\geq 1} a_{\varepsilon_1 t,\,\varepsilon_2 s},\ \sum_{t\geq k+1} a_{\varepsilon_1 t,\,\varepsilon_2 r},\ \sum_{s\geq r+1} a_{\varepsilon_1 k,\,\varepsilon_2 s},\ \sum_{t\geq k+1,s\geq r+1} a_{\varepsilon_1 t,\,\varepsilon_2 s},$$
$$\sum_{k\geq 0,r\geq 1} |\sum_{t\geq k+1} a_{\varepsilon_1 t,\,\varepsilon_2 r}|^2,\ \sum_{k\geq 1,r\geq 0} |\sum_{s\geq r+1} a_{\varepsilon_1 k,\,\varepsilon_2 s}|^2,\ \sum_{k,r\geq 0} |\sum_{t\geq k+1,s\geq r+1} a_{\varepsilon_1 t,\,\varepsilon_2 s}|^2.$$

Sufficient conditions for the convergence are:

$$\sum_{k,r\in\mathbb{Z}^2} |a_{k,\,r}| < +\infty,\ \sum_{k\geq 0,r\geq 1} (\sum_{t\geq k+1} |a_{\varepsilon_1 t,\,\varepsilon_2 r}|)^2 < +\infty,$$
$$\sum_{k\geq 1,r\geq 0} (\sum_{s\geq r+1} |a_{\varepsilon_1 k,\,\varepsilon_2 s}|)^2 < +\infty,\ \sum_{k,r\geq 0} (\sum_{t\geq k+1,s\geq r+1} |a_{\varepsilon_1 t,\,\varepsilon_2 s}|)^2 < +\infty.$$

To obtain the convergence of the above sums, we use the following inequality: if $a, b, c, d \geq 0$, then $(1+\sqrt{ab})(1+\sqrt{cd}) \leq (1+a+c)(1+b+d)$, which implies, for $t, s, t', s' \geq 0$:

$$(1+\inf(t,t'))\,(1+\inf(s,s')) \leq (1+\sqrt{tt'})\,(1+\sqrt{ss'}) \leq (1+t+s)\,(1+t'+s').$$

Hence we have:

$$\sum_{k\geq 0,r\geq 0} (\sum_{t\geq k,s\geq r} |a_{t,s}|)^2 = \sum_{k\geq 0,r\geq 0} (\sum_{t,t'\geq k,\,s,s'\geq r} |a_{t,s}||a_{t',s'}|)$$
$$\leq \sum_{t,t'\geq 0,s,s'\geq 0} |a_{t,s}||a_{t',s'}| \sum_k 1_{0\leq k\leq \inf(t,t')} \sum_r 1_{0\leq r\leq \inf(s,s')}$$
$$= \sum_{t,t'\geq 0,s,s'\geq 0} |a_{t,s}||a_{t',s'}|\,(1+\inf(t,t'))\,(1+\inf(s,s'))$$
$$\leq \sum_{t,t',s,s'\geq 0} |a_{t,s}||a_{t',s'}|(1+t+s)\,(1+t'+s') = (\sum_{t\geq 0,s\geq 0} (1+t+s)\,|a_{t,s}|)^2.$$

An analogous bound is valid for the indices with $\pm$ signs. Therefore, convergence holds if (13) is satisfied and we get $\sum_{\underline{t}\in\mathbb{Z}^2} (1+\|\underline{t}\|)\,|a_{\underline{t}}|$ as a bound for the norm of the vectors $u^0_{\varepsilon_1,0}, u^{\varepsilon_1,0}_{\varepsilon_1,0}, u^0_{0,\varepsilon_2}, u^{0,\varepsilon_2}_{0,\varepsilon_2}, u^0_{\varepsilon_1,\varepsilon_2}, u^{\varepsilon_1,0}_{\varepsilon_1,\varepsilon_2}, u^{0,\varepsilon_2}_{\varepsilon_1,\varepsilon_2}, u^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1,\varepsilon_2}$. □

Countable Lebesgue spectrum

We suppose now that the action generated by T_1 and T_2 on the Hilbert space $\mathcal{H}$ has a countable Lebesgue spectrum. With the notation 1.4, there exists a countable set $(\psi_j, j \in J)$ in $\mathcal{H}$ such that the family of vectors $\{T_1^k T_2^r \psi_j, j \in J, (k,r) \in \mathbb{Z}^2\}$ is an orthonormal basis of $\mathcal{H}$.

The representation of f in the orthonormal basis $(T_1^k T_2^r \psi_j, j \in J, (k,r) \in \mathbb{Z}^2)$ is given by $f = \sum_j f_j = \sum_{j\in J} (\sum_{(k,r)\in\mathbb{Z}^2} a_{j,(k,r)}\, T_1^k T_2^r \psi_j)$, with $a_{j,(k,r)} = \langle f,\, T_1^k T_2^r \psi_j \rangle$.

Recall that

$$M_\theta(f) = \sum_{j\in J} \gamma_j(\theta)\psi_j, \text{ with } \gamma_j(\theta) = \sum_{\underline{k}\in\mathbb{Z}^d} a_{j,\underline{k}}\, e^{2\pi i\langle \underline{k},\theta\rangle}.$$

Using the previous results, we have under the convergence conditions:

$$\begin{aligned} f_j &= \gamma_j(\theta)\psi_j + (I - e^{2\pi i\theta_1}T_1)u_{j,\theta} + (I - e^{2\pi i\theta_2}T_2)v_{j,\theta}, \forall j \in J, \\ f &= M_\theta(f) + (I - e^{2\pi i\theta_1}T_1)\sum_{j\in J} u_{j,\theta} + (I - e^{2\pi i\theta_2}T_2)\sum_{j\in J} v_{j,\theta}. \end{aligned}$$

Lemma 1.9. *Suppose that the following condition is satisfied:*

$$\sum_j \sum_{\underline{k}\in\mathbb{Z}^2} (1 + \|\underline{k}\|)\, |a_{j,\underline{k}}| < \infty. \tag{15}$$

Then there are $v, u_1, u_2 \in \mathcal{H}$ such that the family $\{T^{\underline{n}}v, \underline{n} \in \mathbb{Z}^d\}$ is orthogonal

$$f = v + (I - T_1)u_1 + (I - T_2)u_2, \tag{16}$$

and $\sigma^2(f) = \|v\|^2 = \sum_j (\sum_{\underline{k}\in\mathbb{Z}^2} a_{j,\underline{k}})^2$. Moreover, we have $\sigma^2(f) = 0$ if and only if f is a mixed coboundary: there exist $u_1, u_2 \in \mathcal{H}$ such that

$$f = (I - T_1)u_1 + (I - T_2)u_2. \tag{17}$$

More generally, for every θ, the rotated variance $\sigma^2_\theta(f)$ is null if and only if there are $u_{1,\theta}, u_{2,\theta} \in \mathcal{H}$ such that $f = (I - e^{2\pi i\theta_1}T_1)u_{1,\theta} + (I - e^{2\pi i\theta_2}T_2)u_{2,\theta}$.

2. Multidimensional actions by endomorphisms on tori

We consider now $\mathbb{Z}^d$-actions or $\mathbb{N}^d$-actions given by automorphisms or endomorphisms of the ρ-dimensional torus. The first three subsections are preparatory for the proof of the CLT for such actions.

2.1. Preliminaries.

Notation 2.1. Let $\mathcal{M}^*(\rho, \mathbb{Z})$ denote the semigroup of $\rho \times \rho$ non singular matrices with coefficients in $\mathbb{Z}$ and $GL(\rho, \mathbb{Z})$ the group of matrices with coefficients in $\mathbb{Z}$ such that $\det A = \pm 1$.

Every A in $\mathcal{M}^*(\rho, \mathbb{Z})$ defines a surjective endomorphism of $\mathbb{T}^\rho$, hence a measure preserving transformation on $(\mathbb{T}^\rho, \mu)$ and a dual endomorphism on the group of characters of $\mathbb{T}^\rho$ identified with $\mathbb{Z}^\rho$ (action by the transposed of A). If A is in $GL(\rho, \mathbb{Z})$, it defines an automorphism of $\mathbb{T}^\rho$.

For simplicity, we denote with the same notation the matrix A, its action on the torus and the dual endomorphism. Since we are composing commuting matrices, there is no problem with the transposition. If f is a function on the torus, Af stands for the function $x \to f(Ax)$.

After choosing a system $A_1, ..., A_d$ of generators, every element in $\mathcal{S}$ can be represented as $A^{\underline{n}} := A_1^{n_1}...A_d^{n_d}$, $\underline{n} = (n_1, ..., n_d) \in \mathbb{N}^d$, and we obtain an action of $\mathbb{N}^d$ by endomorphisms on $\mathbb{T}^\rho$. We use also the notation $T^{\underline{n}}f$.

It is well known that ergodicity for the action of a single $A \in \mathcal{M}^*(\rho, \mathbb{Z})$ on $(\mathbb{T}^\rho, \mu)$ is equivalent to the absence of eigenvalue root of 1 for A. Recall also Kronecker's result: an integer matrix with all eigenvalues on the unit circle has all eigenvalues roots of unity.

In what follows the data will be a finite set of commuting matrices A_i in $\mathcal{M}^*(\rho, \mathbb{Z})$ for $\rho \geq 1$ and $\mathcal{S}$ the semigroup generated. We suppose that the group $\check{S}$ generated by the matrices A_i in $GL(\rho, \mathbb{Q})$ is torsion-free.

Since $\check{S}$ is finitely generated and torsion-free, it has a system of d independent generators (not necessarily in $\mathcal{S}$) and it is isomorphic to $\mathbb{Z}^d$. The rank of the action of $\mathcal{S}$ is d.

Embedding of a semigroup of endomorphisms in a group

Lemma 2.2. *Let $\mathcal{S}$ be a commutative semigroup of endomorphisms on a compact abelian group G with dual group H. There is a compact abelian group $\tilde{G}$ such that $\mathcal{S}$ is embedded in a group $\tilde{\mathcal{S}}$ of automorphisms of $\tilde{G}$. If G is connected, then $\tilde{G}$ is also connected.*

If $\mathcal{S}$ is a finitely generated commutative semigroup of endomorphisms of $\mathbb{T}^\rho$, it can be embedded in a group $\tilde{\mathcal{S}}$ of automorphisms, isomorphic to $\mathbb{Z}^d$, acting on a compact abelian connected group $\tilde{G}$ which contains $\mathbb{T}^\rho$ as a factor.

Proof. The construction is done in a discrete group $\tilde{H}$ such that H is isomorphic to a subgroup of $\tilde{H}$. The group $\tilde{H}$ is the quotient of the group $\{(h, A)\}, h \in H, A \in \mathcal{S}$ (endowed with the additive law on the components) by the equivalence relation:

$$[(h, A) \sim (h', A')] \Leftrightarrow [A'h = Ah'].$$

The map $h \in H \to (h, Id)/\sim$ is injective. The elements $A \in \mathcal{S}$ act on $\tilde{H}$ by $(h, B)/\sim \to (Ah, B)/\sim$. One checks immediately the stability of the equivalence classes. We can identify $\mathcal{S}$ and its image.

For $A \in \mathcal{S}$, the automorphism $(h, B)/\sim \to (h, AB)/\sim$ is the inverse of $(h, B)/\sim \to (Ah, B)/\sim$.

If H is torsion free, then $\tilde{H}$ is also torsion free, and therefore its dual, the compact abelian group $\tilde{G}$, is connected.

In the case of endomorphisms A_i of $\mathbb{T}^\rho$, the construction can be describe in the following way. Let $\tilde{G}$ be the compact dual group of the discrete group $\tilde{\mathbb{Z}}^\rho := \{\underline{k}\,\Pi p_i^{\ell_i}, \underline{k} \in \mathbb{Z}^\rho, \ell_i \in \mathbb{Z}\}$, where p_i is the determinant of A_i, for each i. $\mathbb{Z}^\rho$ is a subgroup of $\tilde{\mathbb{Z}}^\rho$ and $\tilde{G}$ has $\mathbb{T}^\rho$ as a factor.

The dual action of $\mathcal{S}$ on $\mathbb{Z}^\rho$ is embedded in a group of automorphisms $\tilde{\mathcal{S}}$ acting on $\tilde{\mathbb{Z}}^\rho$ and this defines by duality a group (still denoted $\tilde{\mathcal{S}}$) acting by automorphisms on G. □

$\tilde{\mathcal{S}}$ is isomorphic to the subgroup $\check{S}$ generated by the matrices A_i in $GL(\rho, \mathbb{Q})$. Since a matrix $A \in \mathcal{M}^*(\rho, \mathbb{Z})$ is uniquely determined by the corresponding measure preserving transformation on $(\mathbb{T}^\rho, \mu)$, we can identify $\mathcal{S}$ and the associated commutative semigroup of measure preserving transformations acting on $(\mathbb{T}^\rho, \mu)$.

The spectral analysis for $\mathcal{S}$, as for $\tilde{\mathcal{S}}$, takes place in $\mathbb{T}^d$, where d is the rank of $\mathcal{S}$.

The Lebesgue spectrum property as in Section 1 can be applied to the action of $\tilde{\mathcal{S}}$. It is equivalent to the fact that the $\mathbb{Z}^d$-action $\tilde{\mathcal{S}}$ on $\tilde{\mathbb{Z}}^\rho \setminus \{\underline{0}\}$ is free.

Definition 2.3. We say that $\underline{n} \to A^{\underline{n}}$ is *totally ergodic* if $A_1^{n_1} ... A_d^{n_d}$ is ergodic for every $\underline{n} = (n_1, ..., n_d) \neq \underline{0}$.

Total ergodicity is equivalent to the property that $A^{\underline{n}}$ has no eigenvalue root of 1, for $\underline{n} \neq \underline{0}$. Replacing $\underline{n}$ by a multiple, there is no $\underline{n} \neq \underline{0}$ such that $A^{\underline{n}}$ has a fixed vector $v \neq \{0\}$. In other words, total ergodicity is equivalent to say that the $\mathbb{Z}^d$-action $\underline{n} = (n_1, ..., n_d) : (v \to A^{\underline{n}} v)$ on $\mathbb{Z}^\rho \setminus \{\underline{0}\}$ is free.

Using a common triangular representation over $\mathbb{C}$ for the commuting matrices A_j, one sees that if $\lambda_{1j}, ..., \lambda_{\rho j}$ are the eigenvalues of A_j (with multiplicity), for $j = 1, ..., d$, this is equivalent to $(\prod_{j=1}^d \lambda_{ij}^{n_j} = 1 \Rightarrow (n_1, ..., n_d) = \underline{0})$, $\forall i \in \{1, ..., \rho\}$.

Lemma 2.4. *Let $B \in \mathcal{M}^*(\rho, \mathbb{Z})$ be a matrix with irreducible (over $\mathbb{Q}$) characteristic polynomial P. Let $\{A_1, ..., A_d\}$ be d matrices in $\mathcal{M}^*(\rho, \mathbb{Z})$ commuting with B. They generates a commutative semigroup of endomorphisms on $\mathbb{T}^\rho$ which is totally ergodic, if and only if for any $\underline{n} \in \mathbb{Z}^d \setminus \{\underline{0}\}$, $A^{\underline{n}} \neq Id$.*

Proof. Since P is irreducible, the eigenvalues of B are distinct. It follows that (on $\mathbb{C}$) the matrices A_i are simultaneously diagonalizable, hence are pairwise commuting. Now suppose that there are $\underline{n} \in \mathbb{Z}^d \setminus \{\underline{0}\}$ and $v \in \mathbb{Z}^\rho \setminus \{\underline{0}\}$ such that $A^{\underline{n}} v = v$. Let W be the subspace of $\mathbb{R}^\rho$ generated by v and its images by B. The restriction of $A^{\underline{n}}$ to W is the identity. W is B-invariant, the characteristic polynomial of the restriction of B to W has rational coefficients and factorizes P. By the assumption of irreducibility over $\mathbb{Q}$, this implies $W = \mathbb{R}^\rho$. Therefore $A^{\underline{n}}$ is the identity. □

Lemma 2.5. *The following conditions are equivalent for a $\mathbb{Z}^d$-action T by automorphisms on a compact abelian group:*
i) T is totally ergodic;
ii) T is 2-mixing[2];
iii) T has the Lebesgue spectrum property.

Proof. The free action property expressed in terms of orbits gives immediately the Lebesgue spectrum property. Mixing of order 2 is a priori stronger than total ergodicity. At last the implication $(iii) \Rightarrow (ii)$ is a general fact, as remarked in Section 1. □

[2] Mixing of order 2 is expressed by (6) (here $\mathcal{H} = L_0^2(\mathbb{T}^\rho, \mu)$) or equivalently by $\lim_{\underline{n} \to \infty} \mu(B_1 \cap T^{-\underline{n}} B_2) = \mu(B_1)\, \mu(B_2)$, $\forall\, B_1, B_2 \in \mathcal{A}$.

Remark 2.6. Finding the dimension of $\mathcal{S}$ and computing a set of independent generators can be very difficult in practice. For $\rho = 3$, we will give explicit examples in the Appendix. Given a finite set of commuting matrices in dimension ρ with determinant 1 for $\rho > 3$, it can be difficult and even computationally impossible to find independent generators.

In some cases the problem can be easier with endomorphisms. For instance, let $p_i, i = 1, ..., d$ be coprime positive integers and $A_i : x \to q_i x \bmod 1$ the corresponding endomorphisms acting on $\mathbb{T}^1$. Then the A_i's give a system of independent generators of the group $\tilde{\mathcal{S}}$ generated on the compact abelian group dual of $\tilde{\mathbb{Z}}^\rho := \{\underline{k}\,\Pi q_i^{\ell_i}, \underline{k} \in \mathbb{Z}^\rho, \ell_i \in \mathbb{Z}\}$. More generally we have:

Corollary 2.7. *Let $\mathcal{S}$ be the semigroup generated by d matrices $\{A_1, ..., A_d\}$ in $\mathcal{M}^*(\rho, \mathbb{Z})$ with the irreducibility property like in Lemma 2.4, with determinant q_i. If the numbers $\log|q_i|$ are linearly independent over $\mathbb{Q}$, then $\mathcal{S}$ is totally ergodic.*

Proof. If $\mathcal{S}$ is not totally ergodic, from the hypothesis, by Lemma 2.4 there exists $\underline{n} \in \mathbb{Z}^d \backslash \{\underline{0}\}$ such that $A^{\underline{n}} = Id$; therefore $\sum n_i \log|q_i| = 0$, contrary to the assumptions. □

Notation 2.8. When $\mathcal{S}$ is a totally ergodic group of automorphisms, J will denote a *section* of its action on $\mathbb{Z}^\rho \backslash \{\underline{0}\}$, i.e., a subset $J \subset \mathbb{Z}^\rho \backslash \{\underline{0}\}$ such that every $\underline{k} \in \mathbb{Z}^\rho \backslash \{\underline{0}\}$ can be written in a unique way as $\underline{k} = A_1^{n_1} ... A_d^{n_d}\, \underline{j}$, with $\underline{j} \in J$ and $(n_1, ..., n_d) \in \mathbb{Z}^d$.

Remark 2.9. It is useful to choose the section in the following way. For a fixed $\underline{\ell}$, the set $\{A^{\underline{k}}\underline{\ell}, \underline{k} \in \mathbb{Z}^d\}$ is discrete and $\lim_{\|\underline{k}\| \to \infty} \|A^{\underline{k}}\underline{\ell}\| = +\infty$. Therefore the minimum of the norm is achieved for some value of $\underline{k}$. We can choose an element $\underline{j}$ in each class modulo the action of $\mathcal{S}$ on $\mathbb{Z}^\rho$, which achieves the minimum of the norm. By this choice, we have

$$\|\underline{j}\| \leq \|A^{\underline{k}}\underline{j}\|, \forall \underline{j} \in J, \underline{k} \in \mathbb{Z}^d, \tag{18}$$

If $\mathcal{K}_0$ denotes the closed subspace of $L^2_0(\mathbb{T}^\rho)$ generated by $\psi_{\underline{j}}(x) := \mathrm{e}^{2\pi i \langle \underline{j}, x\rangle}$ for $\underline{j} \in J$, the subspaces $A_1^{n_1} ... A_d^{n_d} \mathcal{K}_0$ are pairwise orthogonal, since $A_1^{n_1} ... A_d^{n_d} J \cap A_1^{n'_1} ... A_d^{n'_d} J = \emptyset$, for $(n_1, ..., n_d) \neq (n'_1, ..., n'_d)$.

Let f be in $L^2(\mathbb{T}^\rho)$. Recall that the Fourier coefficient of f are denoted $c_f(\underline{k}) = \int_{\mathbb{T}^\rho} e^{-2\pi i \langle \underline{k}, x\rangle} f(x)\, dx$. Then, with the convention of Notation 2.1, the decorrelation is

$$\langle f, A^{\underline{n}} f\rangle = \sum_{\underline{\ell} \in \mathbb{Z}^\rho} c_f(A^{\underline{n}}\underline{\ell})\, \overline{c_f(\underline{\ell})}. \tag{19}$$

2.2. **Rate of decorrelation.**

If a $\mathbb{N}^d$-action by endomorphisms is mixing, a question is the rate of decorrelation for regular functions on $\mathbb{T}^\rho$. A key lemma for the decorrelation property is the following:

Lemma 2.10. *(Leonov* [18]*, Katznelson* [14, Lemma 3]*) If B is a $\rho \times \rho$ matrix with integral coefficients and V a m-dimensional eigenspace of B such that $V \cap \mathbb{Z}^\rho = \{0\}$, then there exists a constant C such that, for every $\underline{j} \in \mathbb{Z}^\rho \backslash \{\underline{0}\}$, the distance (for the euclidian norm) $d(\underline{j}, V)$ of $\underline{j}$ to V satisfies $d(\underline{j}, V) \geq C\|\underline{j}\|^{-m}$.*

The rate of decorrelation for sufficiently regular functions is related to following lemma (cf. D. Damjanović and A. Katok [6]).

Lemma 2.11. *If $(A^{\underline{n}}, \underline{n} \in \mathbb{Z}^d)$ is a totally ergodic $\mathbb{Z}^d$-action on $\mathbb{T}^\rho$ by automorphisms, there are $\tau > 0$ and $C > 0$, such that for all $\underline{k} \in \mathbb{Z}^\rho \setminus \{\underline{0}\}$ and for all $\underline{n} \in \mathbb{Z}^d$*

$$\|A^{\underline{n}}\underline{k}\| \geq Ce^{\tau\|\underline{n}\|}\|\underline{k}\|^{-\rho}. \tag{20}$$

Proof. We give a proof which follows closely the proof of Lemma 4.3 in [6]. We suppose the action is irreducible.

The Lyapunov exponents of $A^{\underline{n}}$ are $\chi_i(\underline{n}) = \sum_{j=1}^d n_j \ln|\lambda_{ij}|$, where $\underline{n} = (n_1, ..., n_d) \in \mathbb{Z}^d, i = 1, ..., \rho$, and $\lambda_{1j}, ..., \lambda_{\rho j}$ are the eigenvalues of A_j, for $j = 1, ..., d$. For $\underline{t} = (t_1, ..., t_d) \in \mathbb{R}^d$, let $\chi_i(\underline{t}) = \sum_{j=1}^d t_j \ln|\lambda_{ij}|$.

We have $\sum_i \chi_i(\underline{n}) = \ln|\det A^{\underline{n}}| \geq 0$, for $\underline{n} \in \mathbb{Z}^d$. This inequality extends to $\mathbb{R}^d$ since, if $(\underline{n}_\ell/q_\ell)$ is a sequence of rational vectors approximating $\underline{t}$ with components $n_{j,\ell}/q_\ell$, $n_{j,\ell} \in \mathbb{Z}, j = 1, ..., d$, $q_\ell \in \mathbb{Z}^+$, then

$$\sum_i \chi_i(\underline{t}) = \lim_\ell \sum_i \chi_i(\underline{n}_\ell/q_\ell) = \lim_\ell \frac{1}{q_\ell} \sum_i \chi_i(\underline{n}_\ell) \geq 0.$$

Let $\tau := \min_t \max_i \chi_i(\underline{t})$, for $\underline{t}$ on the unit sphere U of $\mathbb{R}^d$, and let t_0 be a point on U where this minimum is achieved. Let us show that $\tau > 0$.

If $\tau \leq 0$, then $\chi_i(\underline{t}_0) \leq 0$, for $i = 1, ..., \rho$. Since $\sum_i \chi_i(\underline{t}) \geq 0$ for all $\underline{t} \in \mathbb{R}^d$, it follows that $\chi_i(\underline{t}_0) = 0$ for $i = 1, ..., \rho$ and consequently $\tau = 0$. This implies that, for the points of the half line $L := \{s\underline{t}_0, s \in \mathbb{R}\}$, $\chi_i(s\underline{t}_0) = 0$ for all i.

There are non zero integer vectors either in L or arbitrary close to L. More precisely, let $t_{0,j}, j = 1, ..., d$, be the coordinates of $\underline{t}_0$. By Dirichlet's theorem, there are sequences of integers $(p_{k,j})$ and (q_k), with $q_k > 0$ and $\lim_k q_k = +\infty$, such that $|p_{k,j} - q_k t_{0,j}| \leq 1/q_k^{1/d}$, for $j = 1, ..., d$. Since $\sum_j \ln|\lambda_{i,j}|\, t_{0,j} = \chi_i(\underline{t}_0) = 0$, for $i = 1, ..., \rho$, we have

$$\begin{aligned} |\sum_{j=1}^d \ln|\lambda_{i,j}|\, p_{k,j}| \quad &\leq q_k |\sum_j \ln|\lambda_{i,j}|\, (p_{k,j}/q_k - t_{0,j})| + q_k\, |\sum_j \ln|\lambda_{i,j}|\ t_{0,j}| \\ &\leq \sum_j |\ln|\lambda_{i,j}||\, |p_{k,j} - q_k t_{0,j}| + 0 \leq (\sum_j |\ln|\lambda_{i,j}||)\, q_k^{-1/d} \to 0. \end{aligned}$$

It follows that, for the sequence $(\underline{p}_k)$ in $\mathbb{Z}^d$, $(A^{\underline{p}_k})$ is a sequence of integer matrices whose eigenvalues tend to 1 in absolute value as $k \to \infty$. Now, by an improvement of Kronecker's result (P. E. Blanksby and H. L. Montgomery [3]), for every $\rho \geq 1$, there exists a number $b(\rho) > 1$ such that any integer matrix in $GL(\rho, \mathbb{Z})$ with all eigenvalues in absolute value less than $b(\rho)$ has its eigenvalues roots of unity. Thus we conclude that, for k big enough, the eigenvalues of $A^{\underline{n}_k}$ are roots of unity, contrary to the assumption of total ergodicity. Therefore, $\tau > 0$.

For $\underline{k} \in \mathbb{Z}^\rho \setminus \{\underline{0}\}$, let $\underline{k}_i$ be the projections of $\underline{k}$ to the corresponding proper direction. Individual proper directions are irrational and, due to the irreducibility assumption, each of the projections $\underline{k}_i$ is nontrivial. By applying Lemma 2.10 we have $\|\underline{k}_i\| \geq C\|\underline{k}\|^{-\rho}$ for each i and we obtain:

$$\|A^{\underline{n}}\underline{k}\| \geq C\sum_{i=1}^{\rho}\|A^{\underline{n}}\underline{k}_i\| = C\sum_{i=1}^{\rho}\|\underline{k}_i\|\exp\chi_i(\underline{n}) \geq Ce^{\tau\|\underline{n}\|}\min_{1\leq i\leq\rho}\|\underline{k}_i\| \geq Ce^{\tau\|\underline{n}\|}\|\underline{k}\|^{-\rho}.$$

□

Notice that for endomorphisms Inequality (20) is valid for $\underline{n} \in \mathbb{N}^d$. For the proof, we replace the unit sphere U by U_+ the set of vectors with non negative coordinates in the unit sphere of $\mathbb{R}^d$.

Regularity and Fourier series

We need some results from the theory of approximation of functions by trigonometric polynomials. For the sake of completeness, we give a short proof.

For $f \in L^2(\mathbb{T}^d)$, the rectangular Fourier partial sums of f are denoted by $S_{N_1,\dots,N_d}(f)$. Its *integral modulus of continuity* is defined as

$$\omega_2(\delta_1,\cdots,\delta_d,f) = \sup_{|\tau_1|\leq\delta_1,\dots,|\tau_d|\leq\delta_d}\|f(x_1+\tau_1,\cdots,x_d+\tau_d)-f(x_1,\cdots,x_d)\|_2.$$

Let $J_{N_1,\dots,N_d}(t_1,\cdots,t_d) = K^2_{N_1,\dots,N_d}(t_1,\cdots,t_d)/\|K_{N_1,\dots,N_d}\|^2_{L^2(\mathbb{T}^d)}$ be the d-dimensional Jackson's kernel, where $K_{N_1,\dots,N_d}$ is the d-dimensional Fejér Kernel.

Clearly, $J_{N_1,\dots,N_d}(t_1,\cdots,t_d) = J_{N_1}(t_1)\cdots J_{N_d}(t_d)$. It is known that the 1-dimensional Jackson's kernel satisfies the following moment relations:

$$\int_0^{\frac{1}{2}} t^k\, J_N(t)\,dt = O(N^{-k}),\ \forall N\geq 1,\ k=0,1,2. \tag{21}$$

Lemma 2.12. *There exists a positive constant C_d such that, for every $f \in L^2(\mathbb{T}^d)$, for every $N_1,\dots,N_d \geq 1$, $\|J_{N_1,\dots,N_d} * f - f\|_2 \leq C_d\,\omega_2(\frac{1}{N_1},\cdots,\frac{1}{N_d},f)$.*

Proof. Since $\omega_2(\delta_1,\cdots,\delta_d,f)$ is increasing and subadditive with respect to δ_i, we have for any positive numbers λ_i: $\omega_2(\lambda_1\delta_1,\cdots,\lambda_d\delta_d,f) \leq (\lambda_1+1)\cdots(\lambda_d+1)\,\omega_2(\delta_1,\cdots,\delta_d,f)$.

Using this inequality and (21), we obtain:

$$\begin{aligned}
\|J_{N_1,\dots,N_d} * f - f\|_2 &\leq \textstyle\int_{[-\frac{1}{2},\frac{1}{2}[^d} J_{N_1,\dots,N_d}(\tau_1,\cdots,\tau_d)\,\|f(.-\tau_1,\cdots,.-\tau_d)-f\|_{L^p}\,d\tau_1\cdots d\tau_d\\
&\leq 2^d\textstyle\int_{[0,\frac{1}{2}[^d} J_{N_1,\dots,N_d}(\tau_1,\cdots,\tau_d)\,\omega_2(\tau_1,\cdots,\tau_d,f)\,d\tau_1\cdots d\tau_d\\
&= 2^d\textstyle\int_{[0,\frac{1}{2}[^d} J_{N_1,\dots,N_d}(\tau_1,\cdots,\tau_d)\ \omega_2(\frac{N_1\tau_1}{N_1},\cdots,\frac{N_d\tau_d}{N_d},f)\,d\tau_1\cdots d\tau_d\\
&\leq 2^d\omega_2(\tfrac{1}{N_1},\cdots,\tfrac{1}{N_d},f)\textstyle\int_{[0,\frac{1}{2}[^d}(N_1\tau_1+1)\cdots(N_d\tau_d+1)\,J_{N_1,\dots,N_d}(\tau_1,\cdots,\tau_d)\,d\tau_1\cdots d\tau_d\\
&= 2^d\omega_2(\tfrac{1}{N_1},\cdots,\tfrac{1}{N_d},f)\ \textstyle\prod_{i=1}^d\int_0^{\frac{1}{2}}(N_i\tau_i+1)J_{N_i}(\tau_i)\,d\tau_i \leq C_d\,\omega_2(\tfrac{1}{N_1},\cdots,\tfrac{1}{N_d},f).
\end{aligned}$$

□

Proposition 2.13. *There exists a positive constant C_d, such that, for every $f \in L^2(\mathbb{T}^d)$ and $N_1, \dots, N_d \geq 1$, we have $\|f - S_{N_1,\dots,N_d}(f)\|_2 \leq C_d\, \omega_2(\frac{1}{N_1}, \cdots, \frac{1}{N_d}, f)$.*

Proof. For every d-dimensional trigonometric polynomial P of degree at most $N_1 \times \cdots \times N_d$, we have: $\|f - S_{N_1,\dots,N_d}(f)\|_2 \leq \|f - P\|_2$. The result follows then from Lemma 2.12. □

Notation 2.14. *(Regular functions)* We introduce below several regularity conditions for real functions f in $L^2(\mathbb{T}^\rho)$:

$$\|f - s_{N,\dots,N}(f)\|_2 \leq R(f)\,(\ln N)^{-\alpha}, \text{ with } \alpha > 1, \tag{22}$$

$$|c_f(\underline{k})| = O(\|k\|)^{-\beta}, \text{ with } \beta > \rho, \tag{23}$$

(24) There are $\alpha > 1$ and $C(f) < +\infty$ such that $\omega_2(\delta, ..., \delta, f) \leq C(f)\,(\ln \frac{1}{\delta})^{-\alpha}, \forall \delta > 0$.

One easily checks that (23) implies (22). By Proposition 2.13, Condition (24) on the modulus of continuity implies (22).

In what follows in this subsection, $\underline{n} \to A^{\underline{n}}$ is a totally ergodic $\mathbb{Z}^d$-action by *automorphisms* on $\mathbb{T}^\rho$. We denote simply by $|.|$ the norm of an integral vector. Recall that we do not write the transposition for the dual action of $A^{\underline{n}}$.

Proposition 2.15. *If $f \in L^2_0(\mathbb{T}^\rho)$ satisfies (23), we have, for constants $C, \tau > 0$.*

$$|\langle A^{\underline{n}} f, f\rangle| \leq C\, e^{\frac{\rho-\beta}{\rho+1}\tau\|\underline{n}\|}, \ \forall \underline{n} \in \mathbb{Z}^d. \tag{25}$$

Proof. For $L > 0$, by (19) we have :

$$|\langle A^{\underline{n}} f, f\rangle| \leq \|f\|_2\, [\sum_{|\underline{k}|>L} |c_f(\underline{k})| + \sum_{|\underline{k}|\leq L} |c_f(A^{\underline{n}}\underline{k})|] = (1) + (2).$$

By estimating separately (1) and (2) and choosing L, we get a bound for the decorrelation. Indeed, since, by Lemma 2.11

$$\sum_{|\underline{k}|>L} |\underline{k}|^{-\beta} \sim C L^{\rho-\beta}, \ \sum_{|\underline{k}|\leq L} |A^{\underline{n}}\underline{k}|^{-\beta} \sim C e^{-\beta\tau|n|} L^{\rho(1+\beta)},$$

With $L^{\rho-\beta} = e^{-\beta\tau|n|} L^{\rho(1+\beta)}$, i.e., $L = e^{\frac{\tau}{\rho+1}|\underline{n}|}$, we obtain: $(1) + (2) = O(e^{\frac{\rho-\beta}{\rho+1}\tau||n||})$. □

The proof of the following proposition is like that of the analogous result in [18].

Proposition 2.16. *Let $\mathcal{N}_1$ be a subset of $\mathbb{Z}^\rho$. Let $f \in L^2_0(\mathbb{T}^\rho)$ satisfying (22) and $f_1(x) := \sum_{\underline{n}\in\mathcal{N}_1} c_{\underline{n}}(f) e^{2\pi i\langle \underline{n}, x\rangle}$. Then there is a finite constant $B(f)$ depending only on $R(f)$ such that*

$$|\langle A^{\underline{n}} f_1, f_1\rangle| \leq B(f)\|f_1\|_2 \|\underline{n}\|^{-\alpha}, \ \forall \underline{n} \neq \underline{0}. \tag{26}$$

Proof. It suffices to prove the result for f, since, by setting $c_f(n) = 0$ outside N_1, we obtain (26) with the same constant $B(f)$ as shown by the proof. Let λ, b, d such that $1 < \lambda < e^\tau,\ 1 < b < \lambda^{\frac{1}{\rho}},\ \lambda b^{-\rho} = d > 1$. We have for $\underline{n} \in \mathbb{Z}^d$:

$$\langle A^{\underline{n}} f, f\rangle = \sum_{\underline{k}\in\mathbb{Z}^\rho} c_{\underline{k}}(f)\,\overline{c}_{A^{\underline{n}}\underline{k}}(f) = \sum_{|\underline{k}|<b^{|\underline{n}|}} + \sum_{|\underline{k}|\geq b^{|\underline{n}|}} . \tag{27}$$

From Inequality (20) of Lemma 2.11, we deduce that, if $|\underline{k}| < b^{|\underline{n}|}$, then $|A^{\underline{n}}\underline{k}| \geq D\lambda^{\underline{n}}\,|\underline{k}|^{-\rho} \geq D\lambda^{|\underline{n}|}\,b^{-\rho|\underline{n}|} = Dd^{|\underline{n}|},\ \underline{n} \neq \underline{0}$. It follows, for the first sum:

$$|\sum_{|\underline{k}|<b^{|\underline{n}|}} c_{\underline{k}}(f)\ \overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq (\sum_{|\underline{k}|<b^{|\underline{n}|}} |c_{\underline{k}}(f)|^2)^{\frac{1}{2}}\ (\sum_{|\underline{k}|<b^{|\underline{n}|}} |c_{A^{\underline{n}}\underline{k}}(f)|^2)^{\frac{1}{2}} \leq \|f\|_2 \sum_{|\underline{m}|>Dd^{|\underline{n}|}} |c_{\underline{m}}(f)|^2.$$

By Parseval inequality and (22), there is a finite constant $B_1(f)$ such that, for $|\underline{n}| \neq 0$:

$$(\sum_{|\underline{m}|>Dd^{|\underline{n}|}} |c_{\underline{m}}(f)|^2)^{\frac{1}{2}} \qquad \leq \|f - s_{[Dd^{|\underline{n}|}],\ldots,[Dd^{|\underline{n}|}]}\|_2 \leq \frac{R(f)}{(\ln[Dd^{|\underline{n}|}])^\alpha} \leq B_1(f)|\underline{n}|^{-\alpha}. \tag{28}$$

From the previous inequalities, it follows:

$$|\sum_{|\underline{k}|<b^{|\underline{n}|}} c_n(f)\,\overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq B_1(f)\|f\|_2|\underline{n}|^{-\alpha}, \forall |\underline{n}| \neq 0. \tag{29}$$

Now we bound the second sum in (27):

$$|\sum_{|\underline{k}|\geq b^{|\underline{n}|}} c_{\underline{k}}(f)\,\overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq (\sum_{|\underline{k}|\geq b^{|\underline{n}|}} |c_{\underline{k}}(f)|^2)^{\frac{1}{2}}\ (\sum_{\underline{k}\in\mathbb{Z}^\rho} |c_{A^{\underline{n}}\underline{k}}(f)|^2)^{\frac{1}{2}} \leq (\sum_{|\underline{k}|\geq b^{|\underline{n}|}} |c_{\underline{k}}(f)|^2)^{\frac{1}{2}} (\sum_{m\in\mathbb{Z}^\rho} |c_m(f)|^2)^{\frac{1}{2}}.$$

Analogous to (28), we have: $(\sum_{|\underline{k}|\geq b^{|\underline{n}|}} |c_{\underline{k}}(f)|^2)^{\frac{1}{2}} \leq R(f)\,(\ln b^{|\underline{n}|})^{-\alpha} \leq B_2(f)\,|\underline{n}|^{-\alpha}$; hence:

$$|\sum_{|\underline{k}|\geq b^{|\underline{n}|}} c_{\underline{k}}(f)\,\overline{c}_{A^{\underline{n}}\underline{k}}(f)| \leq B_2(f)\|f\|_2|\underline{n}|^{-\alpha},\ |\underline{n}| \neq 0. \tag{30}$$

Taking $B(f) = B_1(f) + B_2(f)$, (26) follows from (27), (29), (30). □

The following theorem allows polynomial approximations in the proof of the CLT.

Theorem 2.17. *If f satisfies (22), (in particular if f satisfies the regularity condition (24)), then $\sum_{\underline{n}\in\mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| < \infty$, the variance $\sigma^2(f)$ exists, $\sigma^2(f) = \sum_{\underline{n}\in\mathbb{Z}^d} \langle A^{\underline{n}} f, f\rangle$, the density φ_f of the spectral measure of f is continuous.*

Moreover, there is a constant C such that, if $\mathcal{N}$ is any subset of $\mathbb{Z}^\rho$ and $f_1(x) = \sum_{\underline{k}\in\mathcal{N}} c_{\underline{k}}(f) e^{2\pi i\langle \underline{k}, x\rangle}$, then $\sigma(f - f_1) \leq C\|f - f_1\|_2$.

Proof. The Fourier coefficients of φ_f are $\langle A^{\underline{n}} f, f\rangle$. The previous proposition implies [(22) $\Rightarrow$ $\sum_{\underline{n}\in\mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| < +\infty$] and the second statement. □

The same conclusion holds if f satisfies $\|f\|_c := \sum_{\underline{\ell}\in\mathbb{Z}^\rho} |c_f(\underline{\ell})| < +\infty$, with the inequality $\sigma(f-f_1) \le C\|f-f_1\|_c$.

Indeed, by total ergodicity, for every $\underline{\ell} \in \mathbb{Z}^\rho$ the map $\underline{n} \in \mathbb{Z}^d \to A^{\underline{n}}\underline{\ell} \in \mathbb{Z}^\rho$ is injective, and therefore $\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(A^{\underline{n}}\ell)| \le \sum_{\underline{k}\in\mathbb{Z}^\rho} |c_f(\underline{k})|$. Using (19), we have:

$$\sum_{\underline{n}\in\mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| \le \sum_{\underline{n}\in\mathbb{Z}^d} \Big(\sum_{\underline{\ell}\in\mathbb{Z}^\rho} |c_f(A^{\underline{n}}\underline{\ell})|\, |c_f(\underline{\ell})|\Big)$$
$$= \sum_{\underline{\ell}\in\mathbb{Z}^\rho} \Big(\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(A^{\underline{n}}\underline{\ell})|\Big) |c_f(\underline{\ell})| \le \Big(\sum_{\underline{\ell}\in\mathbb{Z}^\rho} |c_f(\underline{\ell})|\Big)^2 = \|f\|_c^2.$$

Coboundary characterization

Using the previous result on the decay of correlation, let us give a sufficient condition on the Fourier series of f for the coboundary characterization. Recall that J denotes a section of the $\mathbb{Z}^d$-action by automorphisms on $\mathbb{Z}^\rho$ (cf. Remark 2.9).

The sufficient condition (15) given in Lemma 1.9 for the coboundary representation in the framework of automorphisms reads

$$\sum_{j\in J} \sum_{\underline{k}\in\mathbb{Z}^d} (1+\|\underline{k}\|)\, |c_f(A^{\underline{k}} j)| < \infty. \tag{31}$$

Theorem 2.18. *If $|c_f(\underline{k})| = O(\|k\|)^{-\beta}$, with $\beta > \rho$, we have $\sigma^2(f) = 0$ if and only if f is a mixed coboundary: there are continuous functions u_i, $i = 1, ..., d$ such that*

$$f = \sum_{i=1}^{d} (I - A_i) u_i. \tag{32}$$

Proof. Let $\varepsilon \in]0, \beta-\rho[$ and $\delta := (\beta-\rho-\varepsilon)/(\beta(1+\rho))$, we have $\delta\beta\rho - \beta(1-\delta) = -(\rho+\varepsilon)$.

There is a constant C_1 such that $\|\underline{k}\|\, e^{-\delta\beta\tau\|\underline{k}\|} \le C_1$, $\forall \underline{k} \in \mathbb{Z}^d$.

According to (20), we have $|c_f(A^{\underline{k}}\underline{j})| \le C\|A^{\underline{k}}\underline{j}\|^\beta \le Ce^{-\beta\tau\|\underline{k}\|}\, \|\underline{j}\|^{\beta\rho}$; hence

$$e^{\delta\beta\tau\|\underline{k}\|}\, |c_f(A^{\underline{k}} j)|^\delta \le C\|\underline{j}\|^{\delta\beta\rho}. \tag{33}$$

Recall for every $\underline{\ell} \in Z^\rho\backslash\{\underline{0}\}$ there is a unique pair $(k, j) \in Z^d \times (Z^\rho\backslash\{\underline{0}\})$ such that $A^{\underline{k}}\underline{j} = \underline{\ell}$. Therefore we have, using Inequality (18) (see Remark 2.9):

$$\sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|\underline{k}\|\, |c_f(A^{\underline{k}}\underline{j})| = \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|\underline{k}\|\, |c_f(A^{\underline{k}}\underline{j})|^\delta |c_f(A^{\underline{k}}\underline{j})|^{1-\delta}$$
$$\le C_1 \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} e^{\delta\beta\tau\|\underline{k}\|}\, |c_f(A^{\underline{k}}\underline{j})|^\delta\, |c_f(A^{\underline{k}}\underline{j})|^{1-\delta} \le C_2 \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|\underline{j}\|^{\delta\beta\rho}\, |c_f(A^{\underline{k}}\underline{j})|^{1-\delta}$$
$$\le C_2 \sum_{\underline{j}\in J} \sum_{\underline{k}\in\mathbb{Z}^d} \|A^{\underline{k}}\underline{j}\|^{\delta\beta\rho}\, |c_f(A^{\underline{k}}\underline{j})|^{1-\delta} \text{ by (33) and (18)}$$
$$= C_2 \sum_{\underline{\ell}\in\mathbb{Z}^\rho\backslash\{\underline{0}\}} \|\underline{\ell}\|^{\delta\beta\rho}\, |c_f(\underline{\ell})|^{1-\delta} \le C_3 \sum_{\underline{\ell}\in\mathbb{Z}^\rho\backslash\{\underline{0}\}} \|\underline{\ell}\|^{\delta\beta\rho-\beta(1-\delta)} \le C_3 \sum_{\underline{\ell}\in\mathbb{Z}^\rho\backslash\{\underline{0}\}} \|\underline{\ell}\|^{-(\rho+\varepsilon)} < +\infty.$$

This implies that (31) is satisfied. Moreover, since here the functions involved in the proof of Lemmas 1.8 and 1.9 are characters, hence continuous and bounded, we get also continuity of the functions u_i in the representation (17). □

Approximation by M_θ

In the algebraic setting of endomorphisms of tori, the family (ψ_j) of the general theory (Subsection 1.2) is $(e_{\underline{j}})$ where, for $\underline{\ell} \in \mathbb{Z}^\rho$, $e_{\underline{\ell}}$ denotes the exponential $e_{\underline{\ell}}(x) = e^{2\pi i\langle \underline{\ell}, x\rangle}$. For automorphisms (up to a not written transposition) we have

$$a_{\underline{j},n} = \langle f, T^{\underline{n}} e_{\underline{j}}\rangle = c_f(T^{\underline{n}}\,\underline{j}),$$
$$\gamma_{\underline{j}}(\theta) = \sum_{\underline{n}\in\mathbb{Z}^d} c_f(T^{\underline{n}}\,\underline{j})\, e^{2\pi i\langle \underline{n},\theta\rangle},\ M_\theta f = \sum_{\underline{j}} \gamma_{\underline{j}}(\theta)\, e_{\underline{j}}.$$

Hence, $M_\theta f$ is defined for every θ, if $\sum_{\underline{j}\in J}\big(\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(T^{\underline{n}}\,\underline{j})|^2\big) < \infty$.

Let us assume that f has an absolutely convergent Fourier series. Since every $\underline{k} \in \mathbb{Z}^\rho$ can be written in an unique way as $\underline{k} = T^{\underline{n}}\,\underline{j}$, with $\underline{j} \in J$ and $\underline{n} \in \mathbb{Z}^d$, we have the inequality:

$$\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(T^{\underline{n}}\,\underline{j})| \le \sum_{\underline{j}\in J}\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(T^{\underline{n}}\,\underline{j})| = \sum_{\underline{k}\in\mathbb{Z}^\rho} |c_f(\underline{k})| = \|f\|_c.$$

Then, for every $\underline{j} \in J$, the series defining $\gamma_{\underline{j}}$ is uniformly converging, $\gamma_{\underline{j}}$ is continuous and $\sum_{\underline{j}\in J} \|\gamma_{\underline{j}}\|_\infty \le \sum_{\underline{j}\in J}\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(T^{\underline{n}}\,\underline{j})| = \|f\|_c$. We have also:

$$\begin{aligned}\sum_{\underline{j}} |\gamma_{\underline{j}}(\theta)|^2 \quad &\le \|f\|_c \sum_{\underline{j}} |\gamma_{\underline{j}}(\theta)| \le \|f\|_c \sum_{\underline{j}\in J}\sum_{\underline{n}\in\mathbb{Z}^d} |c_f(T^{\underline{n}}\,\underline{j})| \\ &= \|f\|_c \sum_{\underline{k}\in\mathbb{Z}^\rho} |c_f(\underline{k})| = \|f\|_c^2.\end{aligned}$$

The function $\sum_{\underline{j}\in J} |\gamma_{\underline{j}}|^2$ is continuous, as a sum of a uniformly converging series of continuous functions.

The functions φ_f and $\sum_{\underline{j}\in J} |\gamma_{\underline{j}}|^2$ are equal a.e. Moreover $\sum_{\underline{j}\in J} |\gamma_{\underline{j}}|^2$ and φ_f by Theorem 2.17 are both continuous, hence they are equal and one easily proves:

Proposition 2.19. *The following approximation holds for every θ, if f has an absolutely convergent Fourier series:*

$$\frac{1}{n^d}\Big\| \sum_{0\le \ell_1,\dots,\ell_d\le n-1} \mathrm{e}^{2\pi i\langle \underline{\ell},\theta\rangle} T^{\underline{\ell}}(f - M_\theta f)\Big\|_2^2 \to 0. \tag{34}$$

2.3. **Mixing, moments and cumulants, application to the CLT.**

Before we continue studying actions by automorphisms, we recall in this subsection some needed general results on mixing of all orders, moments and cumulants.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $T : \underline{n} \to T^{\underline{n}}$ be a measure preserving $\mathbb{N}^d$- or $\mathbb{Z}^d$-action on $(\Omega, \mathcal{A}, \mu)$.

Definition 2.20. *The action by T is r-mixing, $r > 1$, if for all sets $B_1, ..., B_r \in \mathcal{A}$*

$$\lim_{\substack{\min \|\underline{n}_\ell - \underline{n}_{\ell'}\| \to \infty \\ 1 \le \ell < \ell' \le r}} \mu(\bigcap_{\ell=1}^{r} T^{-\underline{n}_\ell} B_\ell) = \prod_{\ell=1}^{r} \mu(B_\ell).$$

Mixing of order $r \geq 2$ implies mixing of order r' for $2 \leq r' \leq r$.

Reminders on moments and cumulants

For the sake of completeness we gather some facts about moments and cumulants. They are essentially classical. A large part can be found in [18] and in the references given therein. Implicitly we assume existence of moments of all orders when they are used.

For a real random variable Y (or for a probability distribution on $\mathbb{R}$), the cumulants (or semi-invariants) can be formally defined as the coefficients $c^{(r)}(Y)$ of the cumulant generating function $t \to \ln \mathbb{E}(e^{tY}) = \sum_{r=0}^{\infty} c^{(r)}(Y) \frac{t^r}{r!}$, i.e.,

$$c^{(r)}(Y) = \frac{\partial^r}{\partial^r t} \ln \mathbb{E}(e^{tY})|_{t=0}.$$

Similarly the joint cumulant of a random vector $(X_1, ..., X_r)$ is defined by

$$c(X_1, ..., X_r) = \frac{\partial^r}{\partial t_1 ... \partial t_r} \ln \mathbb{E}(e^{\sum_{j=1}^{r} t_j X_j})|_{t_1 = ... = t_r = 0}.$$

This definition can be given as well for a finite measure on $\mathbb{R}^r$.

One easily checks that the joint cumulant of $(Y, ..., Y)$ (r copies of Y) is $c^{(r)}(Y)$.

For any subset $I = \{i_1, ..., i_p\} \subset J_r := \{1, ..., r\}$, we put

$$m(I) = m(i_1, ..., i_p) := \mathbb{E}(X_{i_1} ... X_{i_p}), \ s(I) = s(i_1, ..., i_p) := c(X_{i_1}, ..., X_{i_p}).$$

The cumulants of a process $(X_j)_{j \in \mathcal{J}}$, where $\mathcal{J}$ is a set of indexes, is the family

$$\{c(X_{i_1}, ..., X_{i_r}), (i_1, ..., i_r) \in \mathcal{J}^r, r \geq 1\}.$$

The following formulas link moments and cumulants and vice-versa:

$$c(X_1, ..., X_r) = s(J_r) = \sum_{\mathcal{P}} (-1)^{p-1} (p-1)! \, m(I_1) ... m(I_p), \tag{35}$$

$$\mathbb{E}(X_1 ... X_r) = m(J_r) = \sum_{\mathcal{P}} s(I_1) ... s(I_p). \tag{36}$$

where in both formulas, $\mathcal{P} = \{I_1, I_2, ..., I_p\}$ runs through the set of partitions of $J_r = \{1, ..., r\}$ into $p \leq r$ non empty intervals.[3]

[3]About cumulants and for (35) and (36), see references quoted in [18] or [12].

Now, let be given a random process $(X_{\underline{k}})_{\underline{k}\in\mathbb{Z}^d}$, where for $\underline{k}\in\mathbb{Z}^d$, $X_{\underline{k}}$ is a real random variable, and a summation kernel R with finite support in $\mathbb{Z}^d$ and values in $\mathbb{R}^+$. (For examples of summation kernels, see Section 1, in particular Proposition 1.7). Let us consider the process defined for $\underline{k}\in\mathbb{Z}^d$ by

$$Y_k=\sum_{\underline{\ell}\in\mathbb{Z}^d} R(\underline{\ell}+\underline{k})\,X_{\underline{\ell}},\ \underline{k}\in\mathbb{Z}^d.$$

By permuting summation and integral, we easily obtain:

$$c(Y_{\underline{k}_1},...,Y_{\underline{k}_r})=\sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1},...,X_{\underline{\ell}_r})\,R(\underline{\ell}_1+\underline{k}_1)...R(\underline{\ell}_r+\underline{k}_r).$$

In particular, we have for $Y=\sum_{\underline{\ell}\in\mathbb{N}^d} R(\underline{\ell})\,X_{\underline{\ell}}$:

$$c^{(r)}(Y)=c(Y,...,Y)=\sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1},...,X_{\underline{\ell}_r})\,R(\underline{\ell}_1)...R(\underline{\ell}_r). \tag{37}$$

Limiting distribution and cumulants

For our purpose, we state in terms of cumulants a particular case of a theorem of M. Fréchet and J. Shohat, generalizing classical results of A. Markov. Using the formulas linking moments and cumulants, the content of "a generalized statement of the second limit-theorem" given in [9] can be expressed in the particular case of convergence toward a normal distribution in the following way:

Theorem 2.21. *Let $(Z^n, n\geq 1)$ be a sequence of centered r.r.v. such that*

$$\lim_n c^{(2)}(Z^n)=\sigma^2,\ \lim_n c^{(r)}(Z^n)=0, \forall r\geq 3, \tag{38}$$

then (Z^n) tends in distribution to $\mathcal{N}(0,\sigma^2)$. (If $\sigma=0$, then the limit is δ_0).

It implies the following result, a slight extension of Theorem 7 in [17]:

Theorem 2.22. *Let $(X_{\underline{k}})_{\underline{k}\in\mathbb{Z}^d}$ be a random process and $(R_n)_{n\geq 1}$ a summation sequence on $\mathbb{Z}^d$. Let $(Y^n)_{n\geq 1}$ be the process defined by $Y^n=\sum_{\underline{\ell}} R_n(\underline{\ell})\,X_{\underline{\ell}}, n\geq 1$. Under the assumptions $\lim_n \|Y^n\|_2=+\infty$ and*

$$\sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1},...,X_{\underline{\ell}_r})\,R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)=o(\|Y^n\|_2^r), \forall r\geq 3, \tag{39}$$

$\frac{Y^n}{\|Y^n\|_2}$ tends in distribution to $\mathcal{N}(0,1)$ when n tends to ∞.

Proof. Let $\beta_n:=\|Y^n\|_2=\|\sum_{\underline{\ell}} R_n(\underline{\ell})\,X_{\underline{\ell}}\|_2$ and $Z_n=\beta_n^{-1}Y^n$.

We have using (37), $c^{(r)}(Z^n)=\beta_n^{-r}\sum_{(\underline{\ell}_1,...,\underline{\ell}_r)\in(\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1},...,X_{\underline{\ell}_r})\,R(\underline{\ell}_1)...R(\underline{\ell}_r)$. The theorem follows then from the assumption (39) by Theorem 2.21 applied to (Z_n). □

For $f \in L_0^\infty$, the space of measurable essentially bounded functions on (Ω, μ) with $\int f\, d\mu = 0$, we apply the definition of moment and cumulant to $(T^{\underline{n}_1} f, ..., T^{\underline{n}_r} f)$ and put

$$m_f(\underline{n}_1, ..., \underline{n}_r) = \int T^{\underline{n}_1} f ... T^{\underline{n}_r} f\, d\mu, \quad s_f(\underline{n}_1, ..., \underline{n}_r) := c(T^{\underline{n}_1} f, ..., T^{\underline{n}_r} f). \tag{40}$$

In order to show that the cumulants of a mixing system of all orders is asymptotically null, we need the following lemma.

Lemma 2.23. *For every sequence $(\underline{n}_1^k, ..., \underline{n}_r^k)$ in $(\mathbb{Z}^d)^r$, there are a subsequence with possibly a permutation of indices (still written $(\underline{n}_1^k, ..., \underline{n}_r^k)$), an integer $\kappa(r) \in [1, r]$ and a subdivision $1 = r_1 < r_2 < ... < r_{\kappa(r)-1} < r_{\kappa(r)} \leq r$ of $\{1, ..., r\}$, such that*

$$\lim_k \min_{1 \leq s \neq s' \leq \kappa(r)} \|\underline{n}_{r_s}^k - \underline{n}_{r_{s'}}^k\| = \infty, \tag{41}$$

$$\underline{n}_j^k = \underline{n}_{r_s}^k + \underline{a}_j, \text{ for } r_s < j < r_{s+1},\ s = 1, ..., \kappa(r) - 1, \text{ and for } r_{\kappa(r)} < j \leq r, \tag{42}$$

where $\underline{a}_j$ is a constant integral vector.

If the sequence $(\underline{n}_1^k, ..., \underline{n}_r^k)$ satisfy $\lim_k \max_{i \neq j} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$, then the construction can be done in such a way that $\kappa(r) > 1$.

Remark that if $\sup_k \max_{i \neq j} \|\underline{n}_i^k - \underline{n}_j^k\| < \infty$, then $\kappa(r) = 1$ so that (41) is void and that (42) is void for the indexes such that $r_{s+1} = r_s + 1$.

Proof. The proof is by induction. The result is clear for $r = 2$. Suppose we have construct the subsequence for the sequence of $r-1$-tuples $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$.

Let $1 \leq r_1 < r_2 < ... < r_{\kappa(r-1)} \leq r - 1$ be the corresponding subdivision of $\{1, ..., r - 1\}$, as stated above for the sequence $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$. If the sequence $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$ satisfy $\lim_k \max_{1 < i < j \leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$, then $\kappa(r-1) > 1$ by construction in the induction process.

Now we consider $(\underline{n}_1^k, ..., \underline{n}_r^k)$. If $\lim_k \|\underline{n}_r^k - \underline{n}_i^k\| = +\infty$, for all $i = 1, ..., r - 1$, then we have just to take $1 \leq r_1 < r_2 < ... < r_{\kappa(r-1)} < r_{\kappa(r)} = r$ as new subdivision of $\{1, ..., r\}$.

If $\liminf_k \|\underline{n}_r^k - \underline{n}_{i_s}^k\| < +\infty$, for some $s \leq \kappa(r-1)$, then along a new subsequence (still denoted with the same notation) we have $\underline{n}_r^k = \underline{n}_{i_s}^k + \underline{a}_r$, where $\underline{a}_r$ is a constant integral vector. After changing the labels, we insert n_r in the subdivision for $\{1, ..., r-1\}$ and obtain the new subdivision for $\{1, ..., r\}$.

For the last condition on κ, suppose that $\lim_k \max_{1 < i < j \leq r} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$.

Then if $\liminf_k \max_{1 < i < j \leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| < +\infty$, necessarily, $\kappa(r) > 1$. If, on the contrary, the sequence $(\underline{n}_1^k, ..., \underline{n}_{r-1}^k)$ satisfy $\lim_k \max_{1 < i < j \leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$, then $\kappa(r-1) > 1$ so that $\kappa(r) \geq \kappa(r-1) > 1$. □

Lemma 2.24. *If a $\mathbb{Z}^d$-dynamical system is mixing of order $r \geq 2$, then, for any $f \in L_0^\infty$,*

$$\lim_{\max_{i \neq j} \|\underline{n}_i - \underline{n}_j\| \to \infty} s_f(\underline{n}_1, ..., \underline{n}_r) = 0. \tag{43}$$

Proof. We give a sketch of the proof. The notation s_f was introduced in (40). Suppose that (43) does not hold. Then there is $\varepsilon > 0$ and a sequence of r-tuples $(\underline{n}_1^k = \underline{0}, ..., \underline{n}_r^k)$ such that $|s_f(\underline{n}_1^k, ..., \underline{n}_r^k)| \geq \varepsilon$ and $\max_{i \neq j} \|\underline{n}_i^k - \underline{n}_j^k\| \to \infty$ (we use stationarity).

By taking a subsequence (but keeping the same notation), we can assume that, for two fixed indexes i, j, $\lim_k \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$.

From Lemma 2.23, it follows that there is a subdivision $1 = r_1 < r_2 < ... < r_{\kappa(r)-1} < r_{\kappa(r)} \leq r$ and constant integer vectors $\underline{a}_j$ such that

$$\lim_k \min_{1 \leq s \neq s' \leq \kappa(r)} \|\underline{n}_{r_s}^k - \underline{n}_{r_{s'}}^k\| = \infty, \tag{44}$$

$$\underline{n}_j^k = \underline{n}_{r_s}^k + \underline{a}_j, \text{ for } r_s < j < r_{s+1}, \ s = 1, ..., \kappa(r) - 1, \text{ and for } r_{\kappa(r)} < j \leq r. \tag{45}$$

Let $d\mu_k(x_1, ..., x_r)$ denote the probability measure on $\mathbb{R}^r$ defined by the distribution of the random vector $(T^{\underline{n}_1^k} f(.), ..., T^{\underline{n}_r^k} f(.))$. We can extract a converging subsequence from the sequence (μ_k), as well as for the moments of order $\leq r$.

Let us denote $\nu(x_1, ..., x_r)$ (resp. $\nu(x_{i_1}, ..., x_{i_p})$) the limit of $\mu_k(x_1, ..., x_r)$ (resp. of its marginal measures $\mu_k(x_{i_1}, ..., x_{i_p})$ for $\{i_1, ..., i_p\} \subset \{1, ..., r\}$).

Let $\varphi_i, i = 1, ..., r$, be continuous functions with compact support on $\mathbb{R}$. Mixing of order r and condition (44) imply

$$\nu(\varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_r) = \lim_k \int_{\mathbb{R}^d} \varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_r \, d\mu_k = \lim_k \int \prod_{i=1}^r \varphi_i(f(T^{\underline{n}_i^k} x)) \, d\mu(x)$$

$$= \lim_k \int \Big[\prod_{s=1}^{\kappa(r)-1} \prod_{r_s \leq j < r_{s+1}} \varphi_j(f(T^{\underline{n}_s^k + \underline{a}_j} x)) \Big] \prod_{\kappa(r) \leq j \leq r} \varphi_j(f(T^{\underline{n}_{\kappa(r)}^k + \underline{a}_j} x)) \, d\mu(x)$$

$$= \Big[\prod_{s=1}^{\kappa(r)-1} \Big(\int \prod_{r_s \leq j < r_{s+1}} \varphi_j(f(T^{\underline{a}_j} x)) \, d\mu(x) \Big) \Big] \Big[\int \prod_{\kappa(r) \leq j \leq r} \varphi_j(f(T^{\underline{a}_j} x)) \, d\mu(x) \Big].$$

Therefore ν is the product of marginal measures corresponding to disjoint subsets: at least there are $I_1 = \{i_1, ..., i_p\}, I_2 = \{i'_1, ..., i'_{p'}\} \subset J_r = \{1, ..., r\}$, two non empty subsets such that (I_1, I_2) is a partition of J_r and $d\nu(x_1, ..., x_r) = d\nu(x_{i_1}, ..., x_{i_p}) \times d\nu(x_{i'_1}, ..., x_{i'_{p'}})$.

Putting $\Phi(t_1, ..., t_r) = \ln \int e^{\sum t_j x_j} \, d\mu(x_1, ..., x_r)$ and the analogous formulas for $\nu(x_{i_1}, ..., x_{i_p})$ and $\nu(x_{i'_1}, ..., x_{i'_p})$, we obtain: $\Phi(t_1, ..., t_r) = \Phi(t_{i_1}, ..., t_{i_p}) + \Phi(t_{i'_1}, ..., t_{i'_p})$. It implies that the derivative $\frac{\partial^r}{\partial t_1 ... \partial t_r} \Phi(t_1, ..., t_r)|_{t_1 = ... = t_r = 0}$ is 0. Hence $c(\nu(x_1, ..., x_r)) = 0$.

But this contradicts $\liminf_k |s_f(\underline{n}_1^k, ..., \underline{n}_r^k)| > 0$. □

Application to d-dimensional actions by endomorphisms

For an action of $\mathbb{N}^d$ by commuting endomorphisms, on (G, μ), a compact abelian group with its Haar measure, the method of moments as in [17] can be used for the CLT

when mixing of all orders is satisfied. It gives immediately the CLT for trigonometric polynomials.

Theorem 2.25. *Let $\underline{n} : (n_1, ..., n_d) \to T^{\underline{n}} = T_1^{n_1}...T_d^{n_d}$ be a $\mathbb{N}^d$-action by commuting endomorphisms on a compact abelian group G which is mixing of all orders. Let $(R_n)_{n\geq 1}$ be a summation sequence on $\mathbb{N}^d$ and let f be a trigonometric polynomial. If $\lim_n \| \sum_\ell R_n(\underline{\ell})\, T^{\underline{\ell}} f\|_2 = \infty$, then the CLT is satisfied by the sequence*

$$\Big(\frac{\sum_{\underline{\ell}} R_n(\underline{\ell})\, T^{\underline{\ell}} f}{\| \sum_{\underline{\ell}} R_n(\underline{\ell})\, T^{\underline{\ell}} f\|_2}\Big)_{n\geq 1}.$$

Proof. For an action by endomorphisms of compact abelian groups, the moments of the process $(f(T^{\underline{n}}.))_{\underline{n}\in\mathbb{Z}^d}$ for a trigonometric polynomial $f(x) = \sum_{\underline{k}\in\Lambda} c_{\underline{k}}(f)\, \chi_{\underline{k}}(x)$ are:

$$m_f(\underline{n}_1, ..., \underline{n}_r) = \int f(T^{\underline{n}_1}x)...f(T^{\underline{n}_r}x)\, dx = \sum_{\underline{k}_1,...,\underline{k}_r\in\Lambda} c_{k_1}...c_{k_r} 1_{T^{n_1}\chi_{\underline{k}_1}...T^{\underline{n}_r}\chi_{\underline{k}_r}=\mathbf{1}}.$$

For r fixed, the function $(\underline{k}_1, ..., \underline{k}_r) \to m_f(\underline{k}_1, ..., \underline{k}_r)$ takes a finite number of values, since by the above formula m_f is a sum with coefficients 0 or 1 of the products $c_{k_1}...c_{k_r}$ which belong to a finite set. The cumulants of a given order according to (35) take also a finite number of values.

Therefore, since mixing of all orders implies by Lemma 2.24

$$\lim_{\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\|\to\infty} s_f(\underline{\ell}_1, ..., \underline{\ell}_r) = 0,$$

there is M_r such that $s_f(\underline{\ell}_1, ..., \underline{\ell}_r) = 0$ for $\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\| > M_r$.

We apply Theorem 2.26. Let us check (39). Using (37), we obtain that

$$\begin{aligned} |\sum s_f(\underline{\ell}_1, ..., \underline{\ell}_r)\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)| \quad &= |\sum s_f(\underline{0}, \underline{\ell}_2 - \underline{\ell}_1, ..., \underline{\ell}_r - \underline{\ell}_1)\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)| \\ &\leq \sum_{\|\underline{\ell}'_2\|,...,\|\underline{\ell}'_r\|\leq M_r} |s_f(\underline{0}, \underline{\ell}'_2, ..., \underline{\ell}'_r)|\, \sup_n \|R_n\|_\infty^r. \end{aligned}$$

Since the summation sequences are supposed to be bounded, $\sum s_f(\underline{\ell}_1, ..., \underline{\ell}_r)\, R_n(\underline{\ell}_1)...R_n(\underline{\ell}_r)$ is bounded and (39) is satisfied. □

2.4. **CLT for abelian groups of toral automorphisms.**

A method for a proof of the CLT is to use a martingale-type property when it holds. Such a property is satisfied by the subclass of actions by automorphisms satisfying the K-property and this is a way to prove the CLT in that case as shown in [4]. Let us mention that, for $\mathbb{Z}^d$-action by automorphisms on zero-dimensional compact abelian groups, the K-property (or property of completely positive entropy) is equivalent to mixing of all orders (cf. [23]). We will rather focus here on an extension of the method of r-mixing used by Leonov for a single ergodic automorphism and use it for abelian groups of toral automorphisms.

The method of Leonov

The proof of the CLT given by Leonov in [17] for a single ergodic automorphism T of a compact abelian group G is based on the computation of the moments, when f is trigonometric polynomial. It uses the fact that T is mixing of all orders, a property shown by Rohlin [20], consequence of the K-property for ergodic automorphisms.

For $\mathbb{Z}^d$-actions by automorphisms on connected compact abelian groups, in particular on tori, the method of moments can also be used, since the mixing property of all orders holds (Theorem 2.26 below). First we prove a CLT for trigonometric polynomials using the mixing property, then the result is extended to regular functions by approximation.

Mixing of $\mathbb{Z}^d$-actions by automorphisms on tori

For $\mathbb{Z}^d$-actions by automorphisms by automorphisms on compact abelian groups, $d > 1$, mixing of all orders is not always satisfied (cf. [15], [24]). We restrict ourselves to toral automorphisms.

In 1992, W. Philip [19] and K. Schmidt and T. Ward [23] used results about the number of solutions of S-units equations in the study of semigroups of endomorphisms or automorphisms on compact abelian groups. The method relies on results which appeared in a series of papers since the years 80's.

For $\mathbb{Z}^d$-actions by automorphisms, the mixing of all orders property is a consequence of algebraic results on S-units. The following result based on a theorem on S-units (cf. [21]) is shown in [23]:

Theorem 2.26. *([23]) Let $\underline{n} \to T^{\underline{n}}$ be a mixing $\mathbb{Z}^d$-action on a compact, connected, abelian group G. Then it is r-mixing for every $r \geq 2$.*

In particular, if $A_1, ..., A_d$ are commuting matrices in $GL(\rho, \mathbb{Z})$ acting as automorphisms of the torus $\mathbb{T}^\rho$, $\rho \geq 1$. If the $\mathbb{Z}^d$-action on the torus $\underline{n} \in \mathbb{Z}^d \to (x \to A^{\underline{n}}x \bmod 1)$ is mixing, then it is r-mixing for every $r \geq 2$.

With the notations of Lemma 2.2, we have:

Corollary 2.27. *Let $\mathcal{S}$ be a semigroup of endomorphisms on $\mathbb{T}^\rho$ generating a $\mathbb{Z}^d$-action by automorphisms on $\tilde{\mathbb{Z}}^\rho$. If this action is totally ergodic, it is mixing of all orders.*

Proof. The group G in Lemma 2.2 is connected and Theorem 2.26 applies to the group $\tilde{\mathcal{S}}$ of automorphisms of G in which $\mathcal{S}$ is embedded. The action of $\tilde{\mathcal{S}}$ is mixing of all orders, hence also the action of $\mathcal{S}$. □

CLT for $\mathbb{Z}^d$-action by automorphisms

In the sequel of this section, $A_1, ..., A_d$ will be commuting matrices in $SL(\rho, \mathbb{Z})$, $A^{\underline{n}}$ stands for $A_1^{n_1}...A_d^{n_d}$, $\underline{n} = (n_1, ..., n_d)$, and we suppose that the corresponding $\mathbb{Z}^d$-action on the torus $\mathbb{T}^\rho$ is totally ergodic,

By Theorem 2.17, if f satisfies (22), then $\sum_{\underline{n}\in\mathbb{Z}^d} |\langle A^{\underline{n}} f, f\rangle| < +\infty$, the Fourier series $\sum_{\underline{n}\in\mathbb{Z}^d} \langle A^{\underline{n}} f, f\rangle\, e^{2\pi i\langle \underline{n}, t\rangle}$ of the spectral density φ_f is uniformly convergent, the variance $\sigma^2(f)$ is well defined and is equal to $\sigma^2(f) = \sum_{\underline{n}\in\mathbb{Z}^d} \langle A^{\underline{n}} f, f\rangle = \varphi_f(0)$.

Theorem 2.28. *Let $\underline{n} \to A^{\underline{n}}$ be a totally ergodic $\mathbb{Z}^d$-action by commuting matrices on $\mathbb{T}^\rho$. Let $(D_n)_{n\geq 1}$ be a Følner sequence of $\mathbb{Z}^d$.*

1) If f satisfies (22), in particular if f satisfies the regularity condition (24), we have[4] *$\sigma^2(f) = \varphi_f(0)$ and*

$$|D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}\in D_n} f(A^{\underline{\ell}}.) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \sigma^2(f)).$$

2) If f satisfies (23) (i.e., $|c_f(\underline{k})| = O(\|k\|)^{-\beta}$, with $\beta > \rho$), then $\sigma^2(f) = 0$ if and only if there are continuous functions u_t on $\mathbb{T}^\rho$, for $t = 1, ..., d$, such that $f = \sum_{t=1}^d (I - A_t)u_t$.

Proof. 1) Let $(\mathcal{N}_s)$ be an increasing sequence of finite sets in $\mathbb{Z}^d$ with union $\mathbb{Z}^d \setminus \{0\}$ and let $f_s(x) := \sum_{\underline{k}\in\mathcal{N}_s} c_f(\underline{k}) e^{2\pi i\langle \underline{k}, x\rangle}$ be the trigonometric polynomial obtained by restriction of the Fourier series of f to $\mathcal{N}_s$. Let Z_n^s, Z_n denote respectively

$$Z_n^s := |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}} R_n(\underline{\ell}) f_s(A^{\underline{\ell}}.),\ Z_n := |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}} R_n(\underline{\ell}) f(A^{\underline{\ell}}.).$$

By Theorem 2.17 we have $\sigma(f - f_s)^2 \leq C\|f - f_s\|_2$, where the constant C does not depend on s. It follows $\sigma^2(f) := \lim_s \sigma^2(f_s)$ and $\sigma^2(f_s) \neq 0$ for s big enough, since $\sigma^2(f) > 0$ by hypothesis.

For the kernel K_n associated to D_n, by the Følner property and (8), we have, if g satisfies (22):

$$|D_n|^{-1} \| \sum_{\underline{\ell}} R_n(\underline{\ell})\, A^{\underline{\ell}} g\|_2^2 = \int_{\mathbb{T}^d} K_n\, \varphi_g\, dt \underset{n\to\infty}{\to} \varphi_g(0) = \sigma^2(g).$$

From Theorem 2.26 (mixing of all orders) and Theorem 2.25 applied to the trigonometric polynomial f_s, it follows: $Z_n^s \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \sigma^2(f_s))$ for every s. Moreover, since

$$\begin{aligned}\limsup_n \int |Z_n^s - Z_n|_2^2\, d\mu \quad &= \limsup_n \int_{\mathbb{T}^d} K_n\, \varphi_{f-f_s}\, dt \\ &= \lim_n \int_{\mathbb{T}^d} K_n\, \varphi_{f-f_s}\, dt = \sigma^2(f - f_s) \leq C\|f - f_s\|_2^2,\end{aligned}$$

we have, for every $\varepsilon > 0$,

$$\limsup_n \mu[|Z_n^s - Z_n| > \varepsilon] \leq \varepsilon^{-2} \limsup_n \int |Z_n^s - Z_n|_2^2\, d\mathbb{P} \underset{s\to 0}{\to} 0$$

and the condition $\lim_s \limsup_n \mathbb{P}[|Z_n^s - Z_n| > \varepsilon] = 0$ is satisfied.

[4] with the convention that the limiting distribution is δ_0 if $\sigma^2(f) = 0$. The second statement gives a criterion for the non degeneracy of the limiting law

By using Theorem 3.2 in [2], we conclude $Z_n \xrightarrow[n\to\infty]{distr} \mathcal{N}(0,\sigma^2(f))$.

2) The second assertion follows from Theorem 2.18. □

The previous result is valid for the rotated sums: if f satisfies (24), then, for every θ,

$$\sigma^2_\theta(f) = \varphi_f(\theta),\ |D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}\in D_n} e^{2\pi i\langle \underline{\ell},\theta\rangle} f(A^{\underline{\ell}}.) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0,\sigma^2_\theta(f)). \tag{46}$$

If f satisfies the regularity condition (23), then $\sigma^2_\theta(f) = 0$ if and only if there are continuous functions $u_{t,\theta}$ on $\mathbb{T}^\rho$, for $t = 1, ..., d$, such that $f = \sum_{t=1}^d (I - e^{2\pi i\theta} A_t) u_{t,\theta}$.

This applies in particular when (D_n) is a sequence of d-dimensional cubes in $\mathbb{Z}^d$.

A CLT for the rotated sums for a.e. θ without regularity assumptions

For the summation sequence given by squares ($d = 2$) or cubes, a CLT for the rotated sums can be shown for a.e. θ without regularity assumptions on f. The proof relies on an extension of (34) which can be shown, for any given $f \in L^2(\mathbb{T}^\rho)$, for θ in a set of full measure. This extends results of [4].

Theorem 2.29. *Let $\underline{n} \to A^{\underline{n}}$ be a totally ergodic $\mathbb{Z}^d$-action by commuting matrices on $\mathbb{T}^\rho$. Let $(D_n)_{n\geq 1}$ be a sequence of cubes in $\mathbb{Z}^d$. Let $f \in L^2(\mathbb{T}^\rho)$. For a.e. $\theta \in \mathbb{T}^d$, we have $\sigma^2_\theta(f) = \varphi_f(\theta)$. If the variance $\sigma^2_\theta(f)$ is > 0, then*

$$|D_n|^{-\frac{1}{2}} \sum_{\underline{\ell}\in D_n} e^{2\pi i\langle \underline{\ell},\theta\rangle} f(A^{\underline{\ell}}.) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0,\sigma^2_\theta(f)).$$

Let us mention that, if we take for D_n triangles instead of squares, a CLT for the rotated sums is also valid for a.e. θ, provided f satisfies $\sum_{\underline{k}} |c_f(\underline{k})|^2 \log k_1^2 ... \log k_\rho^2 < +\infty$. More details will be given in a sequel to the present article.

Other examples of kernels

Let $(R_n)_{n\geq 1}$ be a summation sequence on $\mathbb{Z}^d$ (cf. notations of Definition 1.1). Summation on Følner sets corresponds by Fourier transform to kernels converging to the Dirac distribution δ_0. The CLT extends easily to a more general class of summation sequences.

Theorem 2.30. *Let $(R_n)_{n\geq 1}$ be a summation sequence on $\mathbb{Z}^d$ which is regular and such that $(\tilde{R}_n(t))$ weakly converges to a measure ζ on the circle. Let f be a function on $\mathbb{T}^\rho$ satisfying (24) with spectral density φ_f. If $\zeta(\varphi_f) \neq 0$, then we have*

$$\sum_{\underline{\ell}} R_n(\underline{\ell}) f(A^{\underline{\ell}}.) / (\sum_{\underline{\ell}\in\mathbb{Z}^d} |R_n(\underline{\ell})|^2)^{\frac{1}{2}} \xrightarrow[n\to\infty]{distr} \mathcal{N}(0,\zeta(\varphi_f))$$

Proof. The proof is the same as that of Theorem 2.28 and uses the convergence:

$$\lim_n \| \sum_{\underline{\ell}\in\mathbb{Z}^d} R_n(\underline{\ell})\, A^{\underline{\ell}} f\|_2^2 / \sum_{\underline{\ell}\in\mathbb{Z}^d} |R_n(\underline{\ell})|^2 = \lim_n \int_{\mathbb{T}^d} K_n(t)\, \varphi_f(t)\, dt = \zeta(\varphi_f),$$

□

Barycenter operators

The barycenter operators satisfy the condition of the previous theorem.

let $A_1, ..., A_d$ be to commuting matrices in $GL(\rho, \mathbb{Z})$ generating a totally ergodic action on $\mathbb{T}^\rho$, $\rho \geq 3$. Let P be the barycenter operator defined as in Formula (3) by:

$$Pf(x) := \sum_j p_j \, f(A_j x). \tag{47}$$

By Proposition 1.7 and Theorem 2.30 we obtain:

Theorem 2.31. *Let f be a function on $\mathbb{T}^\rho$ satisfying (24) with spectral density φ_f. Assume that $\sigma_P^2(f) := \int_{\mathbb{T}} \varphi_f(u, u, ..., u)\, du \neq 0$, then we have*

$$(4\pi)^{\frac{d-1}{2}} \, (p_1 ... p_d)^{\frac{1}{2}} \, n^{\frac{d-1}{2}} \, P^n f(.) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \sigma_P^2(f))$$

Example: let A_1, A_2 be to commuting matrices in $GL(\rho, \mathbb{Z})$ generating a totally ergodic action on $\mathbb{T}^\rho$, $\rho \geq 3$. Let P be the barycenter operator: $Pf(x) := \frac{1}{2}(f(A_1 x) + f(A_2 x))$.

If φ_f is continuous, then we have seen that $\lim_{n\to\infty} \sqrt{\pi n}\, \|P^n f\|_2^2 = \int_{\mathbb{T}} \varphi_f(u, u)\, du$. Assume that $\sigma_P^2(f) := \int_{\mathbb{T}} \varphi_f(u, u)\, du \neq 0$. It follows from Theorem 2.30, for f satisfying (24) on $\mathbb{T}^\rho$:

$$(\pi n)^{\frac{1}{4}} \, P^n f \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \sigma_P^2(f)).$$

We have $\sigma_P(f) = 0$ if and only if $\varphi_f(u, u) = 0$, for every $u \in \mathbb{T}^1$. In particular, if f is not a coboundary in the sense of the first section (cf. (17)), $\sigma_P(f) \neq 0$. Nevertheless, the condition for $\sigma_P(f) = 0$ is stronger than the coboundary condition and it can be shown that $\sigma_P(f) = 0$ if and only if f can be written $f = A_1 g - A_2 g$.

Remark. The case of commutative or amenable actions strongly differs from the case of non amenable actions for which a "spectral gap property" is often available ([11]). For action by algebraic automorphisms $A_j, j = 1, ..., d$, on the torus, the existence of a spectral gap for an operator of the form (47) when A_j's are no more supposed to commute is related to a property of the generated group G, namely that there is no factor torus on which G is virtually abelian ([1]).

2.5. **Semigroups of endomorphims.**

Part of the previous results extend to actions by endomorphisms, since the key point, mixing of all orders, follows from Corollary 2.27. For instance, we have

Proposition 2.32. *Let $\underline{n} \to A^{\underline{n}}$ be a totally ergodic $\mathbb{N}^d$-action by commuting matrices on $\mathbb{T}^\rho$. Let $(D_n)_{n\geq 1}$ be a Følner sequence of $\mathbb{N}^d$ and let f be a trigonometric polynomial.*

We have $\sigma^2(f) = \varphi_f(0)$. *If* $\sigma^2(f) \neq 0$, *then*

$$|D_n|^{-\frac{1}{2}} \sum_{\underline{\ell} \in D_n} f(A^{\underline{\ell}} .) \xrightarrow[n\to\infty]{distr} \mathcal{N}(0, \sigma^2(f)).$$

Nevertheless, a difficulty appears when the spectral density and the rate of decorrelation are involved. More precisely, let us give the expression of the decorrelation for a semigroup of measure preserving endomorphisms giving a $\mathbb{N}^d$-action $T : \underline{n} \to T^{\underline{n}}$ on a probability space.

Using an embedding $\tilde{T}$ of T in a $\mathbb{Z}^d$-action of measure preserving automorphisms, as in Subsection 2.1, we can suppose that $\tilde{T}$ acts on a probability space $(\Omega, \mathcal{A}, \mu)$ and T leaves invariant a sub σ-algebra $\mathcal{B}$ of $\mathcal{A}$: for f a real function in $L^2(\mathcal{B}, \mu)$, $f \circ T^{\underline{n}}$ is $\mathcal{B}$-measurable for every $\underline{n} \in \mathbb{N}^d$. The Fourier coefficients of the spectral measure ν_f of f for the $\mathbb{Z}^d$-action given by $\tilde{T}$ are given, for $\underline{n} = (n_1, ..., n_d) \in \mathbb{Z}^d$, with positive and negative parts n_i^+ and n_i^- respectively, by

$$\nu_f(\underline{n}) = \langle \prod_{i=1}^{d} T^{n_i^+} f, \prod_{i=1}^{d} T^{n_i^-} f \rangle.$$

Consider the case of a semigroup $\underline{n} \in \mathbb{N}^d \to A^{\underline{n}}$ of endomorphisms of $\mathbb{T}^\rho$. Then the previous formula reads:

$$\langle \prod_{i=1}^{d} A^{n_i^+} f, \prod_{i=1}^{d} A^{n_i^-} f \rangle = \sum_{\underline{k}, \underline{k}' \in \mathbb{Z}^\rho} c_f(\underline{k}) \overline{c_f}(\underline{k}') \, 1_{A^{n_i^+} \underline{k} = A^{n_i^-} \underline{k}'}. \tag{48}$$

There is a difficulty in the needed extension of Propositions 2.15 or 2.16. Considering, for instance, the case of an $\mathbb{N}^2$-action on $\mathbb{T}^\rho$ by endomorphisms, $(n_1, n_2) \to A_1^{n_1} A_2^{n_2}$, we see that Formula (19) for the decorrelation for $\underline{n} = (n_1, -n_2)$ such that $n_1, n_2 > 0$ should be modified. In this case we have $\nu_f(\underline{n}) = \langle T^{n_1} f, T^{n_2} f \rangle$ and we have to estimate the decay when $n_1 + n_2 \to \infty$ of $\sum_{\underline{k}, \underline{k}' \in \mathbb{Z}^\rho} c_f(\underline{k}) \, \overline{c_f}(\underline{k}') \, 1_{A_1^{n_1} \underline{k} = A_2^{n_2} \underline{k}'}$.

For the following class of semigroup of endomorphisms the extension is easy:

We assume that the generators A_i of the semigroup $\mathcal{S}$ have the form $A_i = p_i B_i$, where the matrices B_i in $GL(\rho, \mathbb{Z})$ are commuting automorphisms and p_i are pairwise coprime integers.

This condition implies that the semigroup $\mathcal{S}$ is totally ergodic by Corollary 2.7.

Lemma 2.33. *If the semigroup* $\mathcal{S}$ *satisfies the previous condition, there is* $\tau > 0$ *such that* $A^{n_i^+} \underline{k} = A^{n_i^-} \underline{k}'$ *implies*

$$\sup(\|\underline{k}\|, \|\underline{k}\|') \geq e^{\tau \|n\|}. \tag{49}$$

Proof. For an integer vector $\underline{k}$, let $g(\underline{k})$ be the g.c.d. of the components k_i of $\underline{k}$. If D is an automorphism, then $g(D\underline{k}) = g(\underline{k})$. If $A^{n_i^+}\underline{k} = A^{n_i^-}\underline{k}'$, then

$$\prod p_i^{n_i^+}\underline{k} = \prod p_i^{n_i^-} B^{-n_i^+} B^{n_i^-}\underline{k}'; \tag{50}$$

hence:

$$\prod p_i^{n_i^+} g(\underline{k}) = \prod p_i^{n_i^-} g(\underline{k}').$$

Since p_i are pairwise coprime integers, this implies that $\prod p_i^{n_i^-}$ divides $g(\underline{k})$ and $\prod p_i^{n_i^+}$ divides $g(\underline{k}')$; hence (49), with $\tau = \frac{1}{2}\min_i \ln p_i$. □

Using the lemma, we can extend Propositions 2.15 or 2.16. Therefore part 1) of Theorem 2.28 and Theorem 2.30 are valid for a semigroup $\mathcal{S}$ satisfying the condition introduced above.

This result extends to a larger class of semigroups the CLT proved by T. Fukuyama and B. Petit ([10]) for semigroups generated by coprime integers on the circle. A study, using the previous techniques as well as the transfer operators, is possible for the general case of semigroups of endomorphisms and will be presented in a sequel to this paper.

3. **Appendix: examples of $\mathbb{Z}^d$-actions by automorphisms**

The aim of this section is to give explicit examples of commuting matrices generating totally ergodic $\mathbb{Z}^d$-actions on tori for $d > 1$. We mainly recall some known facts. (See in particular [13] and [6] for the construction of $\mathbb{Z}^d$-action by automorphisms on the torus.)

The construction of $\mathbb{Z}^d$-action by automorphisms on $\mathbb{T}^\rho$ is linked to the groups of units in number fields. Following [13], let us recall some facts.

Let $M \in GL(\rho, \mathbb{Z})$ be a matrix with an irreducible characteristic polynomial $P = P(M)$ and hence distinct eigenvalues. The centralizer of M in $\mathcal{M}(n, \mathbb{Q})$ can be identified with the ring of all polynomials in M with rational coefficients modulo the principal ideal generated by the polynomial $P(M)$, and hence with the field $K = \mathbb{Q}(\lambda)$, where λ is an eigenvalue of M, by the map $\gamma : p(A) \to p(\gamma)$ with $p \in \mathbb{Q}[x]$.

By Dirichlet's theorem, if P has d_1 real roots and d_2 pairs of complex conjugate roots, then there are $d_1 + d_2 - 1$ fundamental units in the group of units in the ring of integers in $K(P)$. This provides a totally ergodic $\mathbb{Z}^{d_1+d_2-1}$-action by automorphisms on $\mathbb{T}^\rho$.

Explicit computation of examples relies on an algorithm (see [5]). The first computed examples appeared with the development of the computers. Even nowadays computations are limited to low dimensional examples.

Examples for the torus $\mathbb{T}^3$

To give a concrete example for $\mathbb{T}^3$, we explicit a pair A, B of matrices in $SL(3, \mathbb{Z})$ such that $\{A, B\}$ generates a free action in $\mathbb{Z}^2$.

We start with a integer polynomial $P(X) = -X^3 + qX + n$ which is irreducible over $\mathbb{Q}$ and its companion matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ n & q & 0 \end{pmatrix}.$$

Let $K(P)$ denote the number field associated to P. Suppose that $K(P)$ belongs to any of the tables where the characteristics of the first cubic real fields $K(P)$ are listed. Let θ be a root of P. The table gives a pair of fundamental units for the group of units in the ring of integers in $K(P)$ of the form $P_1(\theta)$, $P_2(\theta)$, where P_1, P_2 are two integer polynomials. Then the matrices $A_1 = P_1(M)$ and $A_2 = P_2(M)$ give a system of generators of the group of matrices in $GL(3,\mathbb{Z})$ commuting with M, generating a totally ergodic $\mathbb{Z}^2$-action on $\mathbb{T}^3$ by automorphisms.

Explicit examples

1) (from the table in [25]) Let us consider the polynomial $P(X) = X^3 - 12X - 10$ and its companion matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 12 & 0 \end{pmatrix}.$$

Let θ be a root of P. The table gives the pair of fundamental units for the algebraic group associated to P:

$$P_1(\theta) = \theta^2 - 3\theta - 3,\ P_2(\theta) = -\theta^2 + \theta + 11.$$

The matrices $A_1 = P_1(M)$ and $A_2 = P_2(M)$ give a system of generators of the group of matrices in $GL(3,\mathbb{Z})$ commuting with M. They generate a totally ergodic action of $\mathbb{Z}^2$ by automorphisms on $\mathbb{T}^3$. They have 3 real eigenvalues and $\det(A_1) = 1, \det(A_2) = -1$.

$$A_1 = \begin{pmatrix} -3 & -3 & 1 \\ 10 & 9 & -3 \\ -30 & -26 & 9 \end{pmatrix},\quad A_2 = \begin{pmatrix} 11 & 1 & -1 \\ -10 & -1 & 1 \\ 10 & 2 & -1 \end{pmatrix}.$$

2) (from tables in [25] and in [5]) Let us consider now the polynomial $P(X) = X^3 - 9X - 2$ and its companion matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 9 & 0 \end{pmatrix}.$$

Let θ be a root of P. The table gives the pair of fundamental units for the algebraic group associated to P:

$$P_1(\theta) = 3\theta^2 - 9\theta - 1,\ P_2(\theta) = 2\theta^2 - 4\theta - 1.$$

The matrices $A_1 = P_1(M)$ and $A_2 = P_2(M)$ give a system of generators of the group of matrices in $GL(3,\mathbb{Z})$ commuting with M. They generate a totally ergodic action of $\mathbb{Z}^2$ by automorphisms on $\mathbb{T}^3$. They have 3 real eigenvalues and $\det(A_1) = 1, \det(A_2) = -1$.

$$A_1 = \begin{pmatrix} -3 & -3 & 1 \\ 10 & 9 & -3 \\ -30 & -26 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 11 & 1 & -1 \\ -10 & -1 & 1 \\ 10 & 2 & -1 \end{pmatrix}.$$

Remark that in [25] a different set of generators is given. The polynomials are

$$P_1'(\theta) = 85\theta^2 - 245\theta - 59, \ P_2'(\theta) = -18\theta^2 + 4\theta + 161.$$

The matrices $A_1' = P_1'(M)$ and $A_2' = P_2'(M)$ give another pair of generators of the group of matrices in $GL(3,\mathbb{Z})$ commuting with M. The relations between the two pairs are:

$$A_1' = A_1 A_2, \ A_2' = A_1^{-1}.$$

A simple example on $\mathbb{T}^4$

If $P(X) = X^4 + aX^3 + bX^2 + aX + 1$, the polynomial P has two real roots: $\lambda_0, \lambda_0^{-1}$, two complex conjugate roots of modulus 1: $\lambda_1, \overline{\lambda}_1$. Let $\sigma_j = \lambda_j + \overline{\lambda}_j$, $j = 0, 1$. They are roots of $Z^2 - aZ + b - 2 = 0$.

Under the conditions: $a^2 - 4b + 8 > 0$, $a > 4$, $b > 2$, $2a > b + 2$, i.e., (since $2a - 2 \leq \frac{1}{4}a^2 + 2$)

$$2 < b < \frac{1}{4}a^2 + 2, \ a > 4,$$

$\lambda_0, \lambda_0^{-1}$ are solution of $\lambda^2 - \sigma_0\lambda + 1 = 0$, and $\lambda_1, \overline{\lambda}_1$ are solutions of $\lambda^2 - \sigma_1\lambda + 1 = 0$, where

$$\sigma_0 = -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4b + 8}, \ \sigma_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b + 8}.$$

The polynomial P is not factorizable over $\mathbb{Q}$. Suppose that $P = P_1 P_2$ with P_1, P_2 with rational coefficients and degree ≥ 1. Since the roots of P are irrational, the degrees of P_1 and P_2 are 2. Necessarily their roots are, say, $\lambda_1, \overline{\lambda}_1$ for P_1, $\lambda_0, \lambda_0^{-1}$ for P_2. The sum $\lambda_1 + \overline{\lambda}_1$, root of $Z^2 - aZ + b - 2 = 0$, is not rational and the coefficients of P_1 are not rational. Let

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -a & -b & -a \end{pmatrix}, B = A + I.$$

From the irreducibility over $\mathbb{Q}$, it follows that, if there is a non zero fixed integral vector for $A^k B^\ell$, where k, ℓ are in $\mathbb{Z}$, then we have $A^k B^\ell = Id$. This implies: $\lambda_1^k (\lambda_1 - 1)^k = 1$, hence, since we have $|\lambda_1| = 1$, it follows $|\lambda_1 - 1| = 1$ which is not true.

Example: $P(X) = X^4 + 5X^3 + 7X^2 + 5X + 1$. If A is the companion matrix, then A and $A + 1$, with characteristic polynomials $P(X)$ and $X^4 + X^3 - 2X^2 + 2X - 1$ respectively, generate a $\mathbb{Z}^2$-totally ergodic action on $\mathbb{T}^4$.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -5 & -7 & -5 \end{pmatrix},\ B = A + I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -5 & -7 & -4 \end{pmatrix}.$$

This elementary example gives only a $\mathbb{Z}^2$-action on $\mathbb{T}^4$. A question is to produce an example with full dimension 3.

3) (construction by blocks)

Let M_1, M_2 be two ergodic matrices respectively of dimension d_1 and d_2. Let p_i, q_i, $i = 1, 2$ be two pairs of integers such that $p_1 q_2 - p_2 q_1 \neq 0$. On the torus $\mathbb{T}^{d_1+d_2}$ we obtain a $\mathbb{Z}^2$-totally ergodic action by taking A_1, A_2 of the following form:

$$A_1 = \begin{pmatrix} M_1^{p_1} & 0 \\ 0 & M_2^{q_1} \end{pmatrix},\ A_2 = \begin{pmatrix} M_1^{p_2} & 0 \\ 0 & M_2^{q_2} \end{pmatrix}.$$

Indeed, if there exists $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{Z}^{d_1+d_2} \setminus \{0\}$ invariant by $A_1^n A_2^\ell$, then $M_1^{np_1+\ell p_2} v_1 = v_1$, $M_2^{nq_1+\ell q_2} v_2 = v_2$, which implies $np_1 + \ell p_2 = 0$, $nq_1 + \ell q_2 = 0$; hence $n = \ell = 0$.

This is a method to obtain explicit free $\mathbb{Z}^2$-actions on $\mathbb{T}^4$. The same method gives explicit free $\mathbb{Z}^3$-actions on $\mathbb{T}^5$ (by using a $\mathbb{Z}$-action on $\mathbb{T}^2$ and a $\mathbb{Z}^2$-action on $\mathbb{T}^3$).

We do not know explicit examples of full dimension, i.e., with 3 independent generators on $\mathbb{T}^4$, or with 4 independent generators on $\mathbb{T}^5$.

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