



HAL
open science

Complexity and regularity of maximal energy domains for the wave equation with fixed initial data

Yannick Privat, Emmanuel Trélat, Enrique Zuazua

► **To cite this version:**

Yannick Privat, Emmanuel Trélat, Enrique Zuazua. Complexity and regularity of maximal energy domains for the wave equation with fixed initial data. 2013. hal-00813647v1

HAL Id: hal-00813647

<https://hal.science/hal-00813647v1>

Preprint submitted on 16 Apr 2013 (v1), last revised 21 Feb 2014 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Complexity and regularity of maximal energy domains for the wave equation with fixed initial data

Yannick Privat* Emmanuel Trélat† Enrique Zuazua‡§

Abstract

We consider the homogeneous wave equation on a bounded open connected subset Ω of \mathbb{R}^n . The initial data being specified we address an optimal observation problem. Namely, we consider the problem of determining a measurable subset ω of Ω maximizing the L^2 norm of the restriction of the corresponding solution to ω over a time interval $[0, T]$, among all possible subsets of a given measure. We show that this problem always admits at least one solution. We prove that, if the initial conditions satisfy some analyticity assumptions then the optimal set is unique and it has a finite number of connected components. In contrast, we construct smooth but not analytic initial conditions for which the optimal set is of Cantor type and in particular has an infinite number of connected components.

Keywords: wave equation, optimal domain, Cantor set, Fourier series, calculus of variations.

AMS classification: 35L05 26A30 49K20 49J30

1 Introduction

Let $n \geq 1$ be an integer. Let T be a positive real number and Ω be an open bounded connected subset of \mathbb{R}^n . We consider the homogeneous wave equation with Dirichlet boundary conditions

$$\begin{aligned} \partial_{tt}y - \Delta y &= 0 && \text{in } \mathbb{R}_+^* \times \Omega, \\ y &= 0 && \text{on } \mathbb{R}_+^* \times \partial\Omega. \end{aligned} \tag{1}$$

For all initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique solution y of (1) in the space $\mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))$, such that $y(0, x) = y^0(x)$ and $\partial_t y(0, x) = y^1(x)$ for almost every $x \in \Omega$.

Note that the energy of the solution y over the whole domain Ω , defined by

$$E(t) = \frac{1}{2} \int_{\Omega} |y(t, x)|^2 dx + \frac{1}{2} \|y_t(t, x)\|_{H^{-1}(\Omega)}^2,$$

is a constant function of t .

Of course this conservation property is no longer true when considering the integral over a proper subset of Ω . But, from a control theoretical point of view and, in particular, motivated by

*CNRS, Université Pierre et Marie Curie (Univ. Paris 6), UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (yannick.privat@upmc.fr).

†Université Pierre et Marie Curie (Univ. Paris 6) and Institut Universitaire de France, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (emmanuel.trelat@upmc.fr).

‡BCAM - Basque Center for Applied Mathematics, Mazarredo, 14 E-48009 Bilbao-Basque Country-Spain

§Ikerbasque, Basque Foundation for Science, Alameda Urquijo 36-5, Plaza Bizkaia, 48011, Bilbao-Basque Country-Spain (zuazua@bcamath.org).

the problem of optimal observation or optimal placement of observers, it is interesting to consider such energies over a certain horizon of time. This is why we define the functional

$$G_T(\chi_\omega) = \int_0^T \int_\omega y(t, x)^2 dx dt = \int_0^T \int_\Omega \chi_\omega(x) y(t, x)^2 dx dt, \quad (2)$$

where ω is any arbitrary measurable subset of Ω of positive measure, and χ_ω is the characteristic function of ω .

Of course, similar problems can be formulated in other functional settings. For instance we could consider initial data (y_0, y_1) in the energy space $H_0^1(\Omega) \times L^2(\Omega)$ and observed quantities of the form $\int_0^T \int_\Omega \chi_\omega(x) y_t(t, x)^2 dx dt$ or $\int_0^T \int_\Omega \chi_\omega(x) |\nabla y(t, x)|^2 dx dt$. But, for the sake of simplicity, we shall focus on the L^2 -setting above.

For every subset ω , the quantity $G_T(\chi_\omega)$ gives an account of the amount of energy of the solution y is concentrated on ω , over the time interval $[0, T]$.

In this paper we address the problem of determining the optimal shape and location of the subdomain ω of a given measure, maximizing $G_T(\chi_\omega)$.

Optimal design problem (P). *Let $L \in (0, 1)$ and $T > 0$ be fixed. Given $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, we investigate the problem of maximizing $G_T(\chi_\omega)$ over all possible measurable subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$, where $y \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))$ is the solution of (1) such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$.*

Note that the optimal set ω , whenever it exists, depends on the initial data under consideration. This problem is a mathematical benchmark, in view of addressing other more intricate optimal design problems where one could as well search for the optimal set ω for a certain class of initial data. The problem we address here, where the initial data are fixed and therefore we are consider a single solution is simpler but, as we shall see, it reveals some interesting properties.

In this article we provide a complete mathematical analysis of the optimal design problem (P). The article is structured as follows. In Section 2 and, in particular, in Theorem 1 we give a sufficient condition ensuring the existence and uniqueness of an optimal set. More precisely, we prove that, if the initial data under consideration belong to a suitable class of analytic functions, then there always exist a unique solution ω , which has a finite number of connected components. In Section 3, we investigate the sharpness of the assumptions made in Theorem 1. More precisely, in Theorem 2 we build initial data (y^0, y^1) of class \mathcal{C}^∞ such that the problem (P) has a unique solution ω , which is a fractal set and thus has an infinite number of connected components. In Section 4, we present some possible generalizations of the results in this article with further potential applications.

2 Existence and uniqueness results

2.1 Existence

Throughout the section, we fix initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. For almost every $x \in \Omega$, we define

$$\varphi(x) = \int_0^T y(t, x)^2 dt \quad (3)$$

where $y \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))$ is the solution of (1) such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$. Note that the function φ is integrable on Ω , and that

$$G_T(\chi_\omega) = \int_\omega \varphi(x) dx, \quad (4)$$

for every measurable subset ω of Ω .

Proposition 1. For any fixed initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ the optimal design problem (\mathcal{P}) has at least one solution. Moreover, there exists a real number λ such that χ_{ω^*} is a solution of (\mathcal{P}) if and only if

$$|\omega^*| = L|\Omega| \quad \text{and} \quad \{\varphi > \lambda\} \subset \omega^* \subset \{\varphi \geq \lambda\}. \quad (5)$$

In other words, any optimal set, solution of (\mathcal{P}) , is characterized in terms of a level sets of the function φ .

Proof. The proof is based on a simple symmetrization argument. For every subset U of Ω , we denote by U_S the ball centered at 0 having the same Lebesgue measure than U . We recall that, for every nonnegative Lebesgue measurable function u defined on Ω and vanishing on its boundary, denoting by $\Omega(c) = \{x \in \Omega \mid u(x) \geq c\}$ its level sets, the Schwarz rearrangement of u is the function u_S defined on Ω_S by

$$u_S(x) = \sup\{c \mid x \in (\Omega(c))_S\}.$$

The function u_S is built from u by rearranging the level sets of u into balls having the same Lebesgue measure (see, e.g., [5, Chapter 2]).

Now, let ω be a measurable subset of Ω of Lebesgue measure $|\omega| = L|\Omega|$. Using the Hardy-Littlewood inequality (see [5, Theorem 2.1.4]), one has

$$\int_{\omega} \varphi(x) dx = \int_{\Omega} \chi_{\omega}(x) \varphi(x) dx \leq \int_{\Omega_S} \chi_{\omega_S}(x) \varphi_S(x) dx = \int_{\omega_S} \varphi_S(x) dx.$$

Since φ_S is the spherical decreasing rearrangement of φ , there exists $\lambda \in \mathbb{R}$ such that

$$\{\varphi_S > \lambda\} \subset \omega_S \subset \{\varphi_S \geq \lambda\}.$$

By a property of the Schwarz rearrangement, the functions φ and φ_S are equimeasurable, which means that their level sets have the same Lebesgue measure. Therefore, for every subset ω^* of Ω such that (5) holds, one has $\int_{\omega_S} \varphi_S(x) dx = \int_{\omega^*} \varphi(x) dx$. We have thus proved that for every subset ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$, one has

$$\int_{\omega} \varphi(x) dx \leq \int_{\omega^*} \varphi(x) dx,$$

whence the result. \square

Remark 1. Relaxation. In Calculus of variations, it is usual to introduce a relaxed formulation of the problem (\mathcal{P}) . Defining the set of unknowns

$$\mathcal{U}_L = \{\chi_{\omega} \in L^{\infty}(\Omega, \{0, 1\}) \mid |\omega| = L|\Omega|\},$$

the relaxation procedure consists in considering the convex closure of \mathcal{U}_L for the L^{∞} weak star topology, that is

$$\overline{\mathcal{U}}_L = \{a \in L^{\infty}(\Omega, [0, 1]) \mid \int_{\Omega} a(x) dx = L|\Omega|\},$$

and then in extending the functional G_T to $\overline{\mathcal{U}}_L$ by setting

$$G_T(a) = \int_{\Omega} a(x) \varphi(x) dx,$$

for every $a \in \overline{\mathcal{U}}_L$. The relaxed version of (\mathcal{P}) is then defined as the problem of maximizing G_T over $\overline{\mathcal{U}}_L$. Since $a \mapsto G_T(a)$ is clearly continuous for the L^{∞} weak star topology, we claim that

$$\min_{\chi_{\omega} \in \mathcal{U}_L} G_T(\chi_{\omega}) = \min_{a \in \overline{\mathcal{U}}_L} G_T(a).$$

It is easy to characterize all solutions of the relaxed problem. Indeed, adapting the proof of Proposition 1, one gets that a is solution of the relaxed problem if and only if $a = 0$ on the set $\{\varphi < \lambda\}$, $a = 1$ on the set $\{\varphi > \lambda\}$, and $a(x) \in [0, 1]$ for almost every $x \in \{\varphi = \lambda\}$ and $\int_{\Omega} a(x) dx = L|\Omega|$. This shows that the optimal solution of the relaxed problem is a classical set and therefore also a solution of the original unrelaxed problem.

2.2 Uniqueness

Let us now discuss the issue of the uniqueness of the optimal set.

Note that the characterization of the solutions of (\mathcal{P}) in Proposition 1 enables situations where (at least) two different subsets ω_1 and ω_2 maximize the functional G_T over the class of Lebesgue measurable subsets of Ω of measure $L|\Omega|$. It occurs if and only if the set $\{\varphi = \lambda\}$, where λ is the real positive number introduced in Proposition 1, has a positive Lebesgue measure. In the next theorem, we provide sufficient regularity conditions on the initial data (y^0, y^1) of the wave equation (1) to guarantee the uniqueness of the solution of (\mathcal{P}) .

We denote by $A = -\Delta$ the Dirichlet-Laplacian with domain $D(A) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\omega)\}$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ when the domain Ω is smooth of class C^2 .

Theorem 1. *If Ω has a nontrivial boundary of class C^∞ and if there exists $R > 0$ such that*

$$\sum_{j=0}^{+\infty} \frac{R^j}{j!} \left(\|A^{j/2}y^0\|_{L^2}^2 + \|A^{(j-1)/2}y^1\|_{L^2}^2 \right)^{1/2} < +\infty, \quad (6)$$

then the problem (\mathcal{P}) has a unique¹ solution χ_ω , where ω is a measurable subset of Ω of Lebesgue measure $L|\Omega|$, satisfying moreover the following properties:

- ω is semi-analytic², and has a finite number of connected components;
- If Ω is symmetric with respect to an hyperplane, σ being the symmetry operator, and if $y^0 \circ \sigma = y^0$ and $y^1 \circ \sigma = y^1$, then ω enjoys the same symmetry property;
- There exists $\eta > 0$ such that $d(\omega, \partial\Omega) > \eta$, where d denotes the Euclidean distance on \mathbb{R}^n .

Proof of Theorem 1. First of all we claim that, under the additional assumption (6), the corresponding solution y of the wave equation is analytic over $\mathbb{R}^+ \times \Omega$. Indeed, we first note that the quantity

$$\|A^{j/2}y(t, \cdot)\|_{L^2}^2 + \|A^{(j-1)/2}\partial_t y(t, \cdot)\|_{L^2}^2$$

is constant with respect to t (it is a higher-order energy over the whole domain Ω). Then, since Ω has a smooth boundary, it follows from (6) and from the Sobolev imbedding theorem (see for example [1]) that there exists $C > 0$ such that

$$\|y^{(k)}(t, \cdot)\|_\infty \leq C \frac{(2n+k)!}{R^{2n+k}},$$

¹Similarly to the definition of elements of L^p -spaces, the subset ω is unique within the class of all measurable subsets of Ω quotiented by the set of all measurable subsets of Ω of zero measure.

²A subset ω of \mathbb{R}^n is said to be semi-analytic if it can be written in terms of equalities and inequalities of analytic functions, that is, for every $x \in \omega$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} (with $1 \leq i \leq p$ and $1 \leq j \leq q$) such that

$$\omega \cap U = \bigcup_{i=1}^p \{y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0, j = 1, \dots, q\}.$$

We recall that such semi-analytic (and more generally, subanalytic) subsets enjoy nice properties, for instance they are stratifiable in the sense of Whitney (see [4, 7]).

for every $t \geq 0$ and every integer k . The analyticity property of y follows. As a consequence, the function φ defined by (3) is analytic on Ω . Hence φ cannot be constant on a subset of positive measure (otherwise by analyticity it would be constant on Ω and hence equal to 0 due to the boundary conditions). This ensures the uniqueness of the optimal set ω .

The first additional property follows from the analyticity properties. The symmetry property follows from the fact that $\varphi \circ \sigma(x) = \varphi(x)$ for every $x \in \Omega$. If ω were not symmetric with respect to this hyperplane, the uniqueness of the solution of the first problem would fail, which is a contradiction. Moreover, since $\varphi(x) = \int_0^T |y(t, x)|^2 dt = 0$ for every $x \in \partial\Omega$, it follows that φ reaches its global minimum on the boundary of Ω . \square

Remark 2. Condition (6) guarantees both the analyticity of the initial data and the boundary compatibility conditions that are required to ensure the analyticity of the solution. Note that the analyticity of (y^0, y^1) by itself is not sufficient to ensure the analyticity of the corresponding solution y of the wave equation since boundary singularities associated to the lack of boundary compatibility conditions propagate inside the domain according to the d'Alembert formula.

Remark 3. The solution of the problem (\mathcal{P}) depends on the initial data under consideration. But there are infinitely many initial data $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ leading to the same function φ and thus to the same solution(s) of (\mathcal{P}) . Indeed, consider for the sake of simplicity, the one dimensional case $\Omega = (0, \pi)$ with $T = 2\pi$. By expanding the initial data y^0 and y^1 as

$$y^0(x) = \sum_{j=1}^{+\infty} a_j \sin(jx) \quad \text{and} \quad y^1(x) = \sum_{j=1}^{+\infty} j b_j \sin(jx),$$

it follows that the functional $G_T(a)$ has the very simple expression

$$G_T(a) = \pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \int_0^\pi a(x) \sin^2(jx) dx, \quad (7)$$

and the function φ writes

$$\varphi(x) = \pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \sin^2(jx). \quad (8)$$

This justifies the claim above. Similar considerations have been discussed in [14, Section 1.2].

2.3 Non-uniqueness

We have seen that the optimal solution may not be unique whenever the function φ is constant on some subset of Ω of positive measure. More precisely, assume that φ is constant, equal to c , on some subset I of Ω of positive measure $|I|$. If $|\{\varphi \geq c\}| < L|\Omega| < |\{\varphi > c\}|$ then there exists an infinite number of measurable subsets ω of Ω maximizing (4), all of them containing the subset $\{\varphi > c\}$. The part of ω lying in $\{\varphi = c\}$ can indeed be chosen arbitrarily.

Note that there is no simple characterization of all initial data for which this non-uniqueness phenomenon occurs, however to get convinced that this may indeed happen it is convenient to consider the one-dimensional case where T is moreover an integer multiple of 2π . Indeed in that case the functional G_T reduces to (7) and the corresponding function φ reduces to (8). Using the notations of Remark 3, and noting that

$$\frac{\pi}{2} \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) = \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2,$$

one gets

$$\varphi(x) = \frac{\pi}{2} \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) - \frac{\pi}{2} \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \cos(2jx),$$

for almost every $x \in (0, \pi)$. Then φ can be written as a Fourier series whose sine Fourier coefficients vanish and cosine coefficients are nonpositive and summable. Hence, to provide an explicit example where the non-uniqueness phenomenon occurs, consider any nontrivial even function ψ of class C^∞ on \mathbb{R} whose support is contained in $[-\alpha, \alpha]$ for some $\alpha \in (0, \pi/4)$. The C^1 regularity ensures that its Fourier coefficients are summable. To ensure the nonpositivity of its Fourier coefficients, it suffices to consider the π -periodic function φ defined on $(0, \pi)$ by the convolution

$$\varphi(x) = \int_{\mathbb{R}} \psi(y)\psi(\pi - y) dy - \int_{\mathbb{R}} \psi(y)\psi(x - y) dy.$$

Indeed, the function φ defined in such a way vanishes at $x = 0$ and $x = \pi$, is of class C^∞ on $(0, \pi)$, with support contained in $[0, 2\alpha] \cup [\pi - 2\alpha, \pi]$, and all its Fourier coefficients are nonpositive. More precisely φ has a Fourier series expansion of the form

$$\varphi(x) = \left(\int_{\mathbb{R}} \psi(y) dy \right)^2 - \sum_{j=1}^{+\infty} \beta_j \cos(2jx),$$

with $\beta_j \geq 0$ for every $j \geq 1$. To construct an example where the solution of (\mathcal{P}) is not unique, it suffices to define the initial data y^0 and y^1 by their Fourier expansion, and with the notations of Remark 3 in such a way that $\frac{\pi}{2}(a_j^2 + b_j^2) = \beta_j$, for every $j \geq 1$. Since the function φ vanishes (at least) on $[2\alpha, \pi - 2\alpha]$, it suffices to choose $L > 4\alpha/\pi$ and it follows that a is a solution of the relaxed problem $\min_{a \in \bar{u}_L} G_T(a)$ introduced in Remark 1 if and only if the three following conditions hold:

- (i) $a(x) = 1$ on $\text{supp } \varphi$,
- (ii) $a(x) \in [0, 1]$ for almost every $x \in (0, \pi) \setminus \text{supp } \varphi$,
- (iii) $\int_0^\pi a(x) dx = L\pi$.

The non uniqueness of solutions is thus obvious.

2.4 Numerical simulations

We provide hereafter a numerical illustration of the results presented in this section. According to Theorem 1, the optimal domain is characterized as a level set of the function φ . Some numerical simulations are provided on Figure 1, with $\Omega = (0, \pi)^2$, $L = 0.6$, $T = 3$, $y^1 = 0$ and

$$y^0(x) = \sum_{n,k=1}^{N_0} a_{n,k} \sin(nx_1) \sin(kx_2),$$

where $N_0 \in \mathbb{N}^*$ and $(a_{n,k})_{n,k \in \mathbb{N}^*}$ are real numbers. The level set is numerically computed using a simple dichotomy procedure.

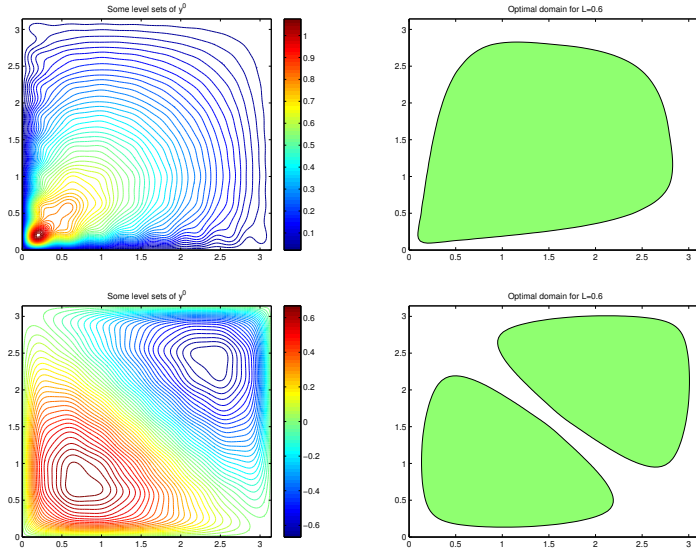


Figure 1: $\Omega = (0, \pi)^2$ with Dirichlet boundary conditions, $L = 0.6$, $T = 3$ and $y^1 = 0$. At the top: $N_0 = 15$ and $a_{n,k} = 1/[n^2 + k^2]$. At the bottom: $N_0 = 15$ and $a_{n,k} = [1 - (-1)^{n+k}]/[n^2 k^2]$. On the left: some level sets of y^0 . On the right: optimal domain (in green) for the corresponding choice of y^0 .

3 On the complexity of the optimal set

3.1 Main result

It is interesting to raise the question of the complexity of the optimal sets solutions of the problem (\mathcal{P}) . In Theorem 1 we proved that, if the initial data belong to some analyticity spaces, then the (unique) optimal set ω is the union of a finite number of connected components. Hence, analyticity implies finiteness and it is interesting to wonder whether this property still holds true for less regular initial data.

In what follows we show that, in the one dimensional case and for particular values of T , there exist C^∞ initial data for which the optimal set ω has a fractal structure and, more precisely, is of Cantor type.

The proof of the following theorem is quite technical and relies on a careful harmonic analysis construction. In order to facilitate the use of Fourier series, it is more convenient to assume hereafter that $\Omega = (0, 2\pi)$.

Theorem 2. *Let $\Omega = (0, 2\pi)$ and let $T > 0$ be an integer multiple of 4π . There exist C^∞ initial data (y^0, y^1) defined on Ω for which the problem (\mathcal{P}) has a unique solution ω ; moreover ω has a fractal structure and in particular it has an infinite number of connected components.*

Remark 4. It is interesting to note that the generalization of this fractal optimal set to the hypercube $\Omega = (0, 2\pi)^n$ is not immediate since the solutions of the multi-dimensional wave equation fail to be time-periodic.

Proof. Without loss of generality, we set $T = 4\pi$. Denote by $(e_n)_{n \in \mathbb{N}^*}$ the eigenfunctions of the

Dirichlet Laplacian on Ω defined by

$$e_n(x) = \sin(nx/2).$$

Note that for a given $n \in \mathbb{N}^*$, e_n is associated to the eigenvalue $n^2/4$.

Using the same kind of computations as those of Remark 3, one gets that

$$\begin{aligned} \varphi(x) &= 2\pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \sin^2(jx/2) \\ &= \pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) - \pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \cos(jx). \end{aligned}$$

Note that the coefficients $(a_j^2 + b_j^2)$ are nonnegative and of converging sum. The construction of the set ω having an infinite number of connected components is based on the following result.

Proposition 2. *There exist a measurable open subset C of $[-\pi, \pi]$, of Lebesgue measure $|C| \in (0, 2\pi)$, and a smooth function f on $[-\pi, \pi]$, satisfying the following properties:*

- C is of fractal type, and in particular has an infinite number of connected components;
- $f(x) > 0$ for every $x \in C$, and $f(x) = 0$ for every $x \in [-\pi, \pi] \setminus C$;
- f is even;
- for every integer n ,

$$\alpha_n = \int_{-\pi}^{\pi} f(x) \cos(nx) dx > 0;$$

- The series $\sum a_n$ is convergent.

Indeed, consider the function f introduced in this proposition and choose the initial data y^0 and y^1 such that their Fourier coefficient a_j and b_j verify $\pi(a_j^2 + b_j^2) = \alpha_j$ for every $j \in \mathbb{N}^*$. Let us extend the characteristic function χ_C of C as a 2π -periodic function on \mathbb{R} , and denote by \tilde{C} the set $[0, 2\pi] \cap \{\chi_C = 1\}$.

Thus, one has

$$\varphi(x) = \sum_{j=1}^{+\infty} \alpha_j - \sum_{j=1}^{+\infty} \alpha_j \cos(jx).$$

Notice that the function φ constructed in such a way is nonnegative, verifies $\varphi(0) = \varphi(\pi) = 0$ and there exists $\lambda > 0$ such that the set $\omega = \{\varphi \geq \lambda\}$ has an infinite number of connected components. In other words the optimal set ω can be chosen as the complement of the fractal set \tilde{C} in $[0, 2\pi]$.

Theorem 2 follows from that result. Lemma 2 is proved in the next subsection. □

3.2 Proof of Proposition 2

There are many possible variants of such a construction. We provide hereafter one possible way of proving this result.

Let $\alpha \in (0, 1/3)$. We assume that α is a rational number, that is, $\alpha = \frac{p}{q}$ where p and q are relatively prime integers, and moreover we assume that $p + q$ is even. Let us first construct

the fractal set $C \subset [-\pi, \pi]$. Since C will be symmetric with respect to 0, we describe below the construction of $C \cap (0, \pi)$. Set $s_0 = 0$ and

$$s_k = \pi - \frac{\pi}{2^k}(\alpha + 1)^k,$$

for every $k \in \mathbb{N}^*$. Around every such point s_k , $k \in \mathbb{N}^*$, we define the interval

$$I_k = \left[s_k - \frac{\pi}{2^k} \alpha (1 - \alpha)^k, s_k + \frac{\pi}{2^k} \alpha (1 - \alpha)^k \right]$$

of length $|I_k| = \frac{\pi}{2^{k-1}} \alpha (1 - \alpha)^k$.

Lemma 1. *We have the following properties:*

- $\inf I_1 > \alpha\pi$;
- $\sup I_k < \inf I_{k+1} < \pi$ for every $k \in \mathbb{N}^*$.

Proof. Since $\alpha < 1/3$ it follows that $\inf I_1 = \pi - \frac{\pi}{2}(\alpha + 1) > \alpha\pi$. For the second property, note that the inequality $\sup I_k < \inf I_{k+1}$ is equivalent to

$$\alpha(1 - \alpha)^{k-1}(3 - \alpha) < (\alpha + 1)^k,$$

which holds true for every $k \in \mathbb{N}^*$ since $\alpha(3 - \alpha) < \alpha + 1$. □

It follows in particular from that lemma that the intervals I_k are two by two disjoint. Now, we define the set C by

$$C \cap (0, \pi) = [0, \alpha\pi] \cup \bigcup_{k=1}^{+\infty} I_k.$$

The resulting set C (symmetric with respect to 0) is then of fractal type and has an infinite number of connected components (see Figure 2).

We now define the function f such that f is continuous, piecewise affine, equal to 0 outside C , and such that $f(s_k) = b_k$ for every $k \in \mathbb{N}$, where the b_k are positive real numbers to be chosen (see Figure 2).

Let us compute the Fourier series of f . Since f is even, its sine coefficients are all equal to 0. In order to compute its cosine coefficients, we will use the following result.

Lemma 2. *Let $a \in \mathbb{R}$, $\ell > 0$ and $b > 0$. Let g be the function defined on \mathbb{R} by*

$$g(x) = \begin{cases} \frac{2b}{\ell}(x - a + \frac{\ell}{2}) & \text{if } a - \frac{\ell}{2} \leq x \leq a, \\ \frac{2b}{\ell}(a + \frac{\ell}{2} - x) & \text{if } a \leq x \leq a + \frac{\ell}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, g is a positive triangle of height b above the interval $[a - \frac{\ell}{2}, a + \frac{\ell}{2}]$. Then

$$\int_{\mathbb{R}} g(x) \cos(nx) dx = \frac{4b}{\ell n^2} \cos(na) \left(1 - \cos \frac{n\ell}{2} \right),$$

for every $n \in \mathbb{N}^$.*

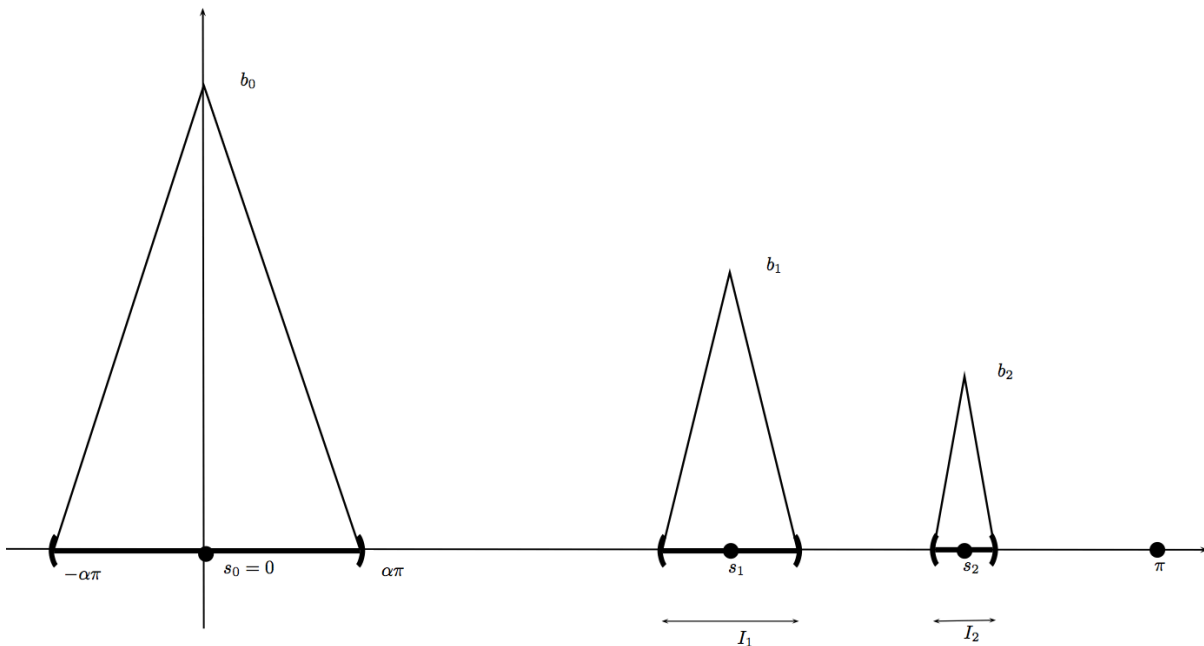


Figure 2: Drawing of the function f and of the set C

It follows from this lemma that

$$\int_0^{\alpha\pi} f(x) \cos(nx) dx = \frac{b_0}{\alpha\pi n^2} (1 - \cos(n\alpha\pi)), \quad (9)$$

and

$$\int_{I_k} f(x) \cos(nx) dx = \frac{2^{k+1}b_k}{\alpha(1-\alpha)^k \pi n^2} \cos\left(n\pi - \frac{n\pi}{2^k}(\alpha+1)^k\right) \left(1 - \cos\left(\frac{n\pi}{2^k}\alpha(1-\alpha)^k\right)\right), \quad (10)$$

for every $k \in \mathbb{N}^*$. Note that

$$\left| \int_{I_k} f(x) \cos(nx) dx \right| \leq \frac{4b_k}{\alpha\pi n^2} \left(\frac{2}{1-\alpha}\right)^k, \quad (11)$$

for every $k \in \mathbb{N}^*$. Formally, the n^{th} cosine Fourier coefficient of f is given by

$$a_n = \int_{-\pi}^{\pi} f(x) \cos(nx) dx = 2 \int_0^{\alpha\pi} f(x) \cos(nx) dx + 2 \sum_{k=1}^{+\infty} \int_{I_k} f(x) \cos(nx) dx.$$

Our next task consists of choosing adequately the positive real numbers b_k , $k \in \mathbb{N}$, so that the series appearing in the above formal expression of a_n is convergent, a_n is nonnegative, and the series of general term a_n is convergent.

Let us first consider the integral (9) (first peak). It is clearly nonnegative for every $n \in \mathbb{N}^*$, and is positive except whenever n is a multiple of $2q$. Taking advantage of the rationality of α , we can moreover derive an estimate from below, as follows. Set

$$\sigma_0 = \min \left\{ 1 - \cos\left(n\frac{p}{q}\pi\right) \mid n = 1, \dots, 2q-1 \right\}.$$

One has $\sigma_0 > 0$, and there holds

$$\int_0^{\alpha\pi} f(x) \cos(nx) dx \geq \frac{b_0\sigma_0}{\alpha\pi n^2}, \quad (12)$$

for every $n \in \mathbb{N}^* \setminus (2q\mathbb{N}^*)$. At this step, assume that

$$b_k \leq \left(\frac{1-\alpha}{2}\right)^k \frac{1}{2^k} \frac{\sigma_0 b_0}{8}, \quad (13)$$

for every $k \in \mathbb{N}^*$ ($b_0 > 0$ is arbitrary). Under this assumption, using (11) it follows that the formal expression of a_n above is well defined, and that

$$\left| \sum_{k=1}^{+\infty} \int_{I_k} f(x) \cos(nx) dx \right| \leq \frac{1}{2} \frac{b_0\sigma_0}{\alpha\pi n^2} \leq \frac{1}{2} \int_0^{\alpha\pi} f(x) \cos(nx) dx,$$

for every $n \in \mathbb{N}^* \setminus (2q\mathbb{N}^*)$, ensuring therefore $a_n > 0$ for such integers n .

If $n = 2rq$, with $r \in \mathbb{N}^*$, then the integral (9) vanishes. We then focus on the second peak, that is, on the integral (10) with $k = 1$. Since $n = 2rq$, its value is

$$\int_{I_1} f(x) \cos(nx) dx = \frac{4b_1}{\alpha(1-\alpha)\pi n^2} \cos\left(2rq\pi - rq\pi\left(\frac{p}{q} + 1\right)\right) \left(1 - \cos\left(rq\pi\frac{p}{q}\left(1 - \frac{p}{q}\right)\right)\right).$$

Since $p + q$ is even, it follows that $\cos\left(2rq\pi - rq\pi\left(\frac{p}{q} + 1\right)\right) = 1$. Hence, we have

$$\int_{I_1} f(x) \cos(nx) dx = \frac{4b_1}{\alpha(1-\alpha)\pi n^2} \left(1 - \cos\left(r\pi\frac{p}{q}(q-p)\right)\right) \geq 0.$$

Moreover, since the integers p and q are relatively prime integers and $q - p$ is even, in this last expression one has $\cos\left(r\pi\frac{p}{q}(q-p)\right) = 1$ if and only if r is multiple of q , that is, if and only if n is multiple of $2q^2$. As before we derive an estimate from below, setting

$$\sigma_1 = \min \left\{ 1 - \cos\left(r\pi\frac{p}{q}(q-p)\right) \mid r = 1, \dots, 2q-1 \right\}.$$

One has $\sigma_1 > 0$, and there holds

$$\int_{I_1} f(x) \cos(nx) dx \geq \frac{4b_1\sigma_1}{\alpha(1-\alpha)\pi n^2}, \quad (14)$$

for every $n \in (2q\mathbb{N}^*) \setminus (2q^2\mathbb{N}^*)$. At this step, additionally to (13) assume that

$$b_k \leq \left(\frac{1-\alpha}{2}\right)^{k-1} \frac{1}{2^{k+1}} b_1\sigma_1, \quad (15)$$

for every $k \geq 2$. Under this assumption, using (11) it follows that

$$\left| \sum_{k=2}^{+\infty} \int_{I_k} f(x) \cos(nx) dx \right| \leq \frac{1}{2} \frac{4b_1\sigma_1}{\alpha(1-\alpha)\pi n^2} \leq \frac{1}{2} \int_{I_1} f(x) \cos(nx) dx,$$

for every $n \in (2q\mathbb{N}^*) \setminus (2q^2\mathbb{N}^*)$, ensuring therefore $a_n > 0$ for such integers n .

The construction can be easily iterated. At iteration m , assume that $n = 2rq^m$, with $r \in \mathbf{N}^*$. Then the integrals over the m first peaks vanish, that is,

$$\int_0^{\alpha\pi} f(x) \cos(nx) dx = \int_{I_k} f(x) \cos(nx) dx = 0$$

for every $k = 1, \dots, m-1$. We then focus on the $(m+1)^{\text{th}}$ peak, that is, on the integral (10) with $k = m$. Since $n = 2rq^m$, its value is

$$\begin{aligned} \int_{I_m} f(x) \cos(nx) dx &= \frac{2^{m+1}b_m}{\alpha(1-\alpha)^m\pi n^2} \cos\left(2rq^m\pi - \frac{rq^m\pi}{2^{m-1}}\left(\frac{p}{q} + 1\right)^m\right) \\ &\quad \times \left(1 - \cos\left(\frac{rq^m\pi}{2^{m-1}}\frac{p}{q}\left(1 - \frac{p}{q}\right)^m\right)\right). \end{aligned}$$

Since $p + q$ is even, it follows that

$$\cos\left(2rq^m\pi - \frac{rq^m\pi}{2^{m-1}}\left(\frac{p}{q} + 1\right)^m\right) = 1,$$

and hence,

$$\int_{I_m} f(x) \cos(nx) dx = \frac{2^{m+1}b_m}{\alpha(1-\alpha)^m\pi n^2} \left(1 - \cos\left(\frac{r\pi}{2^{m-1}}\frac{p}{q}(q-p)^m\right)\right) \geq 0.$$

Moreover, since the integers p and q are relatively prime integers and $q - p$ is even, it follows easily that q and $\left(\frac{q-p}{2}\right)^m$ are relatively prime integers, and therefore this last expression vanishes if and only if r is multiple of q , that is, if and only if n is multiple of $2q^{m+1}$. Setting

$$\sigma_m = \min\left\{1 - \cos\left(\frac{r\pi}{2^{m-1}}\frac{p}{q}(q-p)^m\right) \mid r = 1, \dots, 2q-1\right\},$$

one has $\sigma_m > 0$ and

$$\int_{I_m} f(x) \cos(nx) dx \geq \frac{2^{m+1}b_m\sigma_m}{\alpha(1-\alpha)^m\pi n^2},$$

for every $n \in (2q^m\mathbf{N}^*) \setminus (2q^{m+1}\mathbf{N}^*)$. Additionally to (13), (15) and the following iterative assumptions, we assume that

$$b_k \leq \left(\frac{1-\alpha}{2}\right)^{k-m} \frac{1}{2^{k-m+2}} b_m \sigma_m, \quad (16)$$

for every $k \geq m+1$. Under this assumption, using (11) it follows that

$$\left| \sum_{k=m+1}^{+\infty} \int_{I_k} f(x) \cos(nx) dx \right| \leq \frac{1}{2} \frac{2^{m+1}b_m\sigma_m}{\alpha(1-\alpha)^m\pi n^2} \leq \frac{1}{2} \int_{I_m} f(x) \cos(nx) dx,$$

for every $n \in (2q^m\mathbf{N}^*) \setminus (2q^{m+1}\mathbf{N}^*)$, ensuring therefore $a_n > 0$ for such integers n .

The construction of the function f goes in such a way by iteration. By construction, its Fourier cosine coefficients a_n are positive, and moreover, the series $\sum_{n=0}^{+\infty} a_n$ is convergent. We have thus constructed a function f satisfying all requirements of the statement except the fact that f is smooth.

Let us finally show that, using appropriate convolutions, we can modify f in order to obtain a smooth function keeping all required properties. Set $f_0 = f_{[-\alpha\pi, \alpha\pi]}$ and $f_k = f_{I_k}$ for every $k \in \mathbf{N}^*$.

For every $\varepsilon > 0$, let ρ_ε be a real nonnegative function which is even, whose support is $[-\varepsilon, \varepsilon]$, whose integral over \mathbb{R} is equal to 1, and whose Fourier (cosine) coefficients are all positive. Such a function clearly exists. Indeed, only the last property is not usual, but to ensure this Fourier property it suffices to consider the convolution of any usual bump function with itself. Then, for every $k \in \mathbb{N}$, consider the (nonnegative) function \tilde{f}_k defined by the convolution $\tilde{f}_k = \rho_{\varepsilon(k)} \star f_k$, where each $\varepsilon(k)$ is chosen small enough so that the supports of all functions \tilde{f}_k are still disjoint two by two and contained in $[-\pi, \pi]$ as in Lemma 1. Then, we define the function \tilde{f} as the sum of all functions \tilde{f}_k , and we symmetrize it with respect to 0. Clearly, every Fourier (cosine) coefficient of \tilde{f} is the sum of the Fourier (cosine) coefficients of \tilde{f}_k , and thus is positive, and their sum is still convergent. The function \tilde{f} is smooth and satisfies all requirements of the statement of the proposition. This ends the proof.

4 Conclusions and further comments

4.1 Other partial differential equations

The study developed in this article can be extended in several directions and in particular to other partial differential equations (PDE)

Indeed, Proposition 1 holds by addressing the problem

$$\inf_{|\omega|=L|\Omega|} \int_{\omega} \varphi(x) dx,$$

with $\varphi \in L^1(\Omega)$. It is clear that the same result holds for other evolution PDE, either parabolic or hyperbolic.

On the other hand, the conclusion of Theorem 1 is true as soon as one is able to ensure that the function $\varphi(x) = \int_0^T |y(t, x)|^2 dt$ is analytic in Ω . Depending on the model under consideration it may however be more or less difficult to ensure this property by prescribing suitable regularity properties on the initial conditions. Let us next provide details for two examples.

Schrödinger equation. Consider the Schrödinger equation on Ω

$$i\partial_t y = \Delta y, \tag{17}$$

with Dirichlet boundary conditions, and $y(0, \cdot) = y^0(\cdot) \in L^2(\Omega, \mathbb{C})$. Clearly, Theorem 1 holds true when replacing the sufficient condition (6) with

$$\sum_{j=0}^{+\infty} \frac{R^j}{j!} \|A^{j/2} y^0\|_{L^2(\Omega)} < +\infty.$$

As concerns the complexity of the optimal set, it is clear that the construction of C^∞ initial data for which the optimal set ω has a fractal structure, made in Theorem 2 can be applied to this case as well.

The generalization of the fractal optimal set in several space dimensions is simpler in this case. Indeed, by considering cartesian products of this one-dimensional fractal set constructed in the proof of Theorem 2, it is immediate to generalize the construction to $\Omega = (0, \pi)^n$ since any solution of (17) remains periodic in this case, which ensures that G_T does not involve any additional crossed terms. As mentioned above, the problem is more complex for the wave equation because of the lack of periodicity and, accordingly, the non diagonal terms in G_T .

Heat equation. Consider the homogeneous heat equation on Ω

$$\partial_t y = \Delta y,$$

with Dirichlet boundary conditions, and $y(0, \cdot) = y^0(\cdot) \in L^2(\Omega)$. Let $(\phi_j)_{j \in \mathbb{N}^*}$ be a Hilbertian basis of $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet-Laplacian operator on Ω , associated with the negative eigenvalues $(-\lambda_j^2)_{j \in \mathbb{N}^*}$. We decompose y^0 as

$$y^0 = \sum_{j=1}^{+\infty} a_j \phi_j(\cdot)$$

with $a_j = \langle y^0, \phi_j \rangle_{L^2}$. We claim that the function φ is analytic provided that the initial datum y^0 satisfies the condition

$$\sum_{j=1}^{\infty} \lambda_j^{\frac{n-1}{2}} |a_j| < +\infty. \quad (18)$$

Indeed, using standard Sobolev estimates, one has $\|\phi_j\|_{L^\infty} \leq C \lambda_j^{n/2}$. It follows that for almost every $x \in \Omega$,

$$\begin{aligned} \varphi(x) &\leq \int_0^T \sum_{j,k=1}^{+\infty} |a_j a_k| e^{-(\lambda_j + \lambda_k)t} |\phi_j(x) \phi_k(x)| dt \\ &\leq \sum_{j,k=1}^{+\infty} |a_j a_k| \frac{(\lambda_j \lambda_k)^{n/2}}{\lambda_j + \lambda_k} \\ &\leq \frac{1}{2} \sum_{j,k=1}^{+\infty} |a_j a_k| (\lambda_j \lambda_k)^{\frac{n-1}{2}} = \frac{1}{2} \left(\sum_{j=1}^{+\infty} |a_j| \lambda_j^{\frac{n-1}{2}} \right)^2. \end{aligned}$$

The claim follows by noting that for a given $t > 0$, the function $|y(t, \cdot)|^2$ is analytic in Ω and by standard analyticity results. As a consequence, if the condition (18) holds, then the conclusion of Theorem 1 is true.

4.2 Other boundary conditions

Throughout the article, for the clarity of the exposition we restricted our study to the wave equation with Dirichlet boundary conditions. With minor changes, our results can be extended to the case of other boundary conditions. For example, in the Neumann case, it is necessary to consider initial data (y^0, y^1) in $L^2(\Omega) \times [H^1(\Omega)]'$.

Denoting now by A the Neumann-Laplacian, Theorem 1 still holds in this case (except the last claim, that is, the fact that there is a positive distance between the optimal set and the boundary of Ω , which is specific to the Dirichlet case).

Theorem 1 is still valid if for more general choices of boundary conditions such as Neumann, mixed Dirichlet-Neumann, or Robin boundary conditions for the wave equation, and the corresponding appropriate functional spaces are discussed in [15].

4.3 Perspectives

In this article, given fixed initial data, we have solved the problem of determining the best shape and location of a subdomain of a given measure maximizing the energy of the corresponding

solution of the wave equation with Dirichlet boundary conditions, restricted to the subdomain and over a certain horizon of time. We have discussed the question of the uniqueness of an optimal solution. We have also investigated the complexity of the optimal set, showing that it depends on the regularity of the initial data. In particular, we have constructed an example where the optimal set is fractal.

In this article we have considered the problem of optimally observing solutions. The dual problem of optimally controlling the solutions to rest can also be considered. Similar results have been established in [12, 13] and [16] in the context of optimal design of the control support.

In this paper we have considered the problem of choosing the optimal observation set for fixed initial data. But, in engineering applications this problem can be viewed as a first step towards modeling the problem of optimizing the shape and location of sensors or actuators location problems. Recall that the equation (1) is said to be *observable* on ω in time T if there exists a positive constant C such that the inequality

$$C\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_{\omega} y(t, x)^2 dx dt, \quad (19)$$

holds for every $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. This is the so-called *observability inequality* (see [10]). It is well known that within the class of C^∞ domains Ω , this observability property holds if the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [2, 3]), according to which every ray of Geometric Optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ intersects ω within time T . We denote by $C_T(\chi_\omega)$ the largest constant in the observability inequality above, that is

$$C_T(\chi_\omega) = \inf \left\{ \frac{G_T(\chi_\omega)}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2} \mid (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \setminus \{(0, 0)\} \right\}. \quad (20)$$

An a priori natural way of modeling the problem of optimal shape and placement of sensors or controllers for the wave equation consists of maximizing the functional $C_T(\chi_\omega)$ over the set of all measurable subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$. Moreover, it can be argued that the observability constant $C_T(\chi_\omega)$, appearing in the observability inequality (19), gives an account for the quality of some inverse problem consisting of reconstructing the initial data (y^0, y^1) from the observed variable $\chi_\omega y$ over $[0, T]$. This optimal design problem, settled as such, is a difficult one. We refer to [14] for a study of a simplified version in which the criterium to be optimized is reduced to a purely spectral one by some randomization procedure.

It can be also of interest in practice to address a variant of the above criterion by considering suitable class of initial data:

$$J(\chi_\omega) = \inf_{(y^0, y^1) \in \mathcal{V}} \frac{G_T(\chi_\omega)}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2},$$

where \mathcal{V} is a set of initial data. In the criterion above (observability constant), one has $\mathcal{V} = L^2 \times H^{-1}$, that is the set of *all* possible initial data, but it may be interesting to consider subsets of $L^2 \times H^{-1}$, such as:

- The subset of initial data having a certain number of nonzero Fourier components. This is the case in practice when the measurement devices can only measure, say, the first N frequencies of the solutions of the wave equation.
- A set of data in a neighborhood of a given fixed datum in $L^2 \times H^{-1}$. It may happen in practice that only such initial data be relevant for physical reasons.

- A set of initial data which is defined as a parametrized (finite or infinite dimensional) submanifold of $L^2 \times H^{-1}$, appearing in the model of some physical experiment. For instance the works in [8, 9, 11, 17] consider a finite number of possibilities for the unknown domains and run the optimization over this finite dimensional manifold.

A systematic analysis of these issues is still to be done.

Acknowledgment. The first author was partially supported by the ANR project OPTIFORM. The third author was partially supported by Grant MTM2011-29306-C02-00, MICINN, Spain, ERC Advanced Grant FP7-246775 NUMERIWAVES, ESF Research Networking Programme OPTPDE and Grant PI2010-04 of the Basque Government.

References

- [1] R.A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics **65** (1975), Academic Press, New York.
- [2] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.
- [3] N. Burq, P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes (French) [A necessary and sufficient condition for the exact controllability of the wave equation]*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 7, 749–752.
- [4] R.M. Hardt, *Stratification of real analytic mappings and images*, Invent. Math. **28** (1975).
- [5] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.
- [6] A. Henrot, M. Pierre, *Variation et optimisation de formes (French) [Shape variation and optimization] Une analyse géométrique. [A geometric analysis]*, Math. & Appl. **48**, Springer, Berlin, 2005.
- [7] H. Hironaka, *Subanalytic sets*, in: Number Theory, Algebraic Geometry and Commutative Algebra. In honor of Y. Akizuki, Tokyo (1973).
- [8] C.S. Kubrusly, H. Malebranche, *Sensors and controllers location in distributed systems - a survey*, Automatica **21** (1985), 117–128.
- [9] S. Kumar, J.H. Seinfeld, *Optimal location of measurements for distributed parameter estimation*, IEEE Trans. Autom. Contr. **23** (1978), 690–698.
- [10] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Tome 1, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], Masson (1988).
- [11] K. Morris, *Linear-quadratic optimal actuator location*, IEEE Trans. Automat. Control **56** (2011), no. 1, 113–124.
- [12] A. Münch, *Optimal location of the support of the control for the 1-D wave equation: numerical investigations*, Comput. Optim. Appl. **42** (2009), 443–470.
- [13] F. Periago, *Optimal shape and position of the support for the internal exact control of a string*, Syst. Cont. Letters **58** (2009), no. 2, 136–140.

- [14] Y. Privat, E. Trélat, E. Zuazua, *Optimal observation of the one-dimensional wave equation*, Preprint Hal (2012), to appear in *J. Fourier Anal. Appl.*
- [15] Y. Privat, E. Trélat, E. Zuazua, *Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains*, Preprint Hal (2013).
- [16] Y. Privat, E. Trélat, E. Zuazua, *Optimal location of controllers for the one-dimensional wave equation*, Preprint Hal (2012), to appear in *Ann. Inst. H. Poincaré Anal. Non Linéaire*.
- [17] D. Ucinski, M. Patan, *Sensor network design fo the estimation of spatially distributed processes*, *Int. J. Appl. Math. Comput. Sci.* **20** (2010), no. 3, 459–481.