Low Space Complexity Multiplication over Binary Fields with Dickson Polynomial Representation

Anwar Hasan, Christophe Negre

Low Space Complexity Multiplication over Binary Fields with Dickson Polynomial Representation

M. Anwar Hasan* and Christophe Negre**

Abstract

We study Dickson bases for binary field representation. Such a representation seems interesting when no optimal normal basis exists for the field. We express the product of two field elements as Toeplitz or Hankel matrix-vector products. This provides a parallel multiplier which is subquadratic in space and logarithmic in time. Using the matrix-vector formulation of the field multiplication, we also present sequential multiplier structures with linear space complexity.

Index Terms

Binary field, Dickson basis, Toeplitz matrix, multiplier, parallel, sequential.

1 Introduction

Finite field arithmetic is extensively used in cryptography. For public key cryptosystems, the size (i.e. the number of elements) of the field may be quite large, say 2^{2048}. Finite field multiplication over such a large field requires a considerable amount of resources (time or space). For binary extension fields, used in many practical public key cryptosystems, field elements can be represented with respect to a normal basis, where squaring operations are almost free of cost. In order to reduce the cost of multiplication over the extension field, instead of using an arbitrary normal basis, it is desirable to use an optimal normal basis. The latter however does not exist for all extension fields, in which case one may use Dickson bases [2], [8] and develop an efficient field multiplier.

In this paper first we consider subquadratic space complexity bit parallel multipliers using the Dickson basis. To this end, using low weight Dickson polynomials, we formulate the problem of field multiplication as a product of a Toeplitz or Hankel matrix and a vector, and apply subquadratic space complexity

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algorithm for the product [3], which gives us a subquadratic space complexity field multiplier. Using the matrix-vector product formulation, we then develop sequential multipliers. For such multipliers, we consider both bit-serial and bit-parallel output formats.

The article is organized as follows. In Section 2 we present some general results on Dickson polynomials. Then in Section 3 we give a matrix-vector product approach for field multiplication using the Dickson basis representation. We use low weight Dickson polynomials and present parallel multipliers of subquadratic space complexity. In Section 4, we develop sequential multipliers that have linear space complexity. We wind up this article with a brief conclusion in Section 5.

2 DICKSON POLYNOMIALS

Dickson polynomials over finite fields were introduced by L.E. Dickson in [2]. These polynomials have several applications and interesting properties, the main one being a permutation property over finite fields. For a complete explanation on this the reader may refer to [7]. Our interest here concerns the use of Dickson polynomial for finite field representation for efficient binary field multiplication. There are two kinds of Dickson polynomials, and there are several ways to define and construct both of them. We give here the definition of [7] of the first kind Dickson polynomials.

Definition 1: [Dickson Polynomial[7] page 9] Let R be a ring and $a \in R$. The Dickson polynomial of the first kind $D_n(X, a)$ is defined by

$$D_n(X,a) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \begin{pmatrix} n-i \\ n \end{pmatrix} (-a)^i X^{n-2i}. \tag{1}$$

For $n = 0$, we set $D_0(X, a) = 2$ and for $n = 1$ we have $D_1(X, a) = X$.

In this paper, we will consider only $\beta_i = D_i(X, 1)$ the Dickson polynomials in $\mathbb{F}_2[X]$.

Theorem 1: Let P be an irreducible polynomial of degree n in $\mathbb{F}_2[X]$. The system $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ forms a basis of $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P)$ over $\mathbb{F}_2$.

Proof: For a detailed proof we refer to [6], we just give a brief explanation here. Using (1), we can see that a $\beta_i = X^i +$ *terms of lower degree* . This implies that the conversion matrix from $\{\beta_1, \ldots, \beta_n\}$ to $\{X, \ldots, X^n\}$, is lower triangular with 1 on the diagonal. The conversion matrix is thus invertible and since $\{X, \ldots, X^n\}$ form a basis of $\mathbb{F}_{2^n}$ then $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ is a basis of $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P)$. □

The following theorem will be extensively used for the construction of subquadratic multipliers in the Dickson basis.

Theorem 2: We denote $\beta_i = D_i(X, 1)$ the n-th Dickson polynomial in $\mathbb{F}_2[X]$. Then for all $i, j \geq 0$ the following equation holds

$$\beta_i \beta_j = \beta_{i+j} + \beta_{|i-j|}. \tag{2}$$

Proof: We will show it by induction on i and j. We can easily check that equation (2) holds for $i, j \leq 1$. We suppose that the equation is true for all $i, j \leq n$ and we prove that the equation is true for $i, j \leq n+1$. We first prove it for $i = n+1$ and $j \leq n$. We have

$$\begin{aligned} \beta_{n+1}\beta_j &= (X\beta_n + \beta_{n-1})\beta_j \\ &= X(\beta_{n+j} + \beta_{|n-j|}) + (\beta_{n-1+j} + \beta_{|n-1-j|}), \end{aligned}$$

by induction hypothesis. Now we have

$$\begin{aligned} \beta_{n+1}\beta_j &= (X\beta_{n+j} + \beta_{n+j-1}) + (X\beta_{|n-j|} + \beta_{|n-1-j|}) \\ &= \beta_{n+1+j} + \beta_{|n+1-j|}. \end{aligned}$$

For the other case $i = n+1$ and $j = n+1$, the product $\beta_{n+1}\beta_{n+1}$ is obtained using similar tricks. □

3 Field Multiplication Using Low Weight Dickson Polynomials

In this section we consider multiplication of two elements of the binary field $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P)$ where the polynomial P is a low weight Dickson polynomial. In particular we consider two and three-term Dickson polynomials P, i.e., Dickson binomials and trinomials. Like low weight conventional polynomials the use of low weight Dickson polynomials is expected to yield lower space complexity multipliers.

In Table 1 we give the degree $n \in [160, 285]$ of field $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P)$ where P is a low weight Dickson polynomial. Specifically, since no irreducible Dickson binomials were available, we have looked for *Almost Dickson binomials (ADB)* with irreducibible P satisfying $P \times (X+1) = \beta_{n+1} + 1$. We also give Dickson trinomials (DT) of the form $P = \beta_n + \beta_k + 1$ with $k \leq n/2$. For the purpose of comparison, we mention also whether for each degree an ONB of type I or II exists (marked as NI and NII).

Our main goal here is to express the product of two elements, represented in the Dickson basis, as a Toeplitz or Hankel matrix-vector products. Recall that an $n \times n$ Toeplitz matrix $T = [t_{i,j}]$ satisfies $t_{i,j} = t_{i+1,j+1}$ and a Hankel matrix $H = [h_{i,j}]$ satisfies $h_{i,j} = h_{i+1,j-1}$. We will then use the subquadratic Toeplitz matrix-vector product of [3] to design a subsquadratic multiplier.

3.1 Irreducible Dickson binomials

In this subsection we focus on finite fields $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P)$ where P is a two-term polynomial of the form $P = \beta_n + 1$ where a β_n is the n-th Dickson polynomial.

The elements of $\mathbb{F}_{2^n}$ are expressed in the Dickson basis $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$. Now, our main goal is to show that the product of two elements A and B in $\mathbb{F}_{2^n}$ can be computed as a matrix-vector product $M_A \cdot B$ where M_A is a sum of a Toeplitz matrix and an essentially Hankel matrix.

Table 1

Irreducible Dickson binomials and trinomials

n		n		n		n	
163	DT	195		226	DT, NI	257	
164		196	NI	227	DT	258	
165	DT	197	DT	228		259	DT
166		198	ADB	229	DT	260	DT
167	ADB,DT	199	DT	230	NII	261	DT,NII
168		200	DT	231	DT, NII	262	ADB
169		201	DT	232		263	DT
170	DT	202	DT	233	NII	264	
171	DT	203	DT	234		265	DT
172	DT, NI	204		235	DT	266	
173	ADB,DT,NII	205	DT	236		267	
174	NII	206		237	DT	268	ADB,DT,NI
175	DT	207	DT	238	ADB	269	DT
176	DT	208	DT	239	DT, NII	270	ADB,NII
177		209	NII	240		271	DT
178	DT,NI	210	NI,NII	241	DT	272	DT
179	DT,NII	211	DT	242	DT	273	DT, NII
180	NI	212	DT	243	NII	274	DT
181	DT	213		244	DT	275	DT
183	DT,NII	214		245	DT,NII	276	
184	DT	215	DT	246		277	DT
185		216		247	DT	278	NII
186	NII	217		248	DT	279	DT
187	DT	218	DT	249		280	DT
188	DT	219	DT	250	DT	281	NII
189	DT, NII	220	DT	251	DT,NII	282	
190		221	DT,NII	252	ADB	283	DT
191	DT,NII	222		253	DT	284	
192		223	DT	254	NII	285	DT
193	DT	224	DT	255	DT		
194	DT,NII	225	DT	256	DT		

ADB=Dickson Binomial,DT=Dickson Trinomial,NI=ONBI and NII=ONBII

If we multiply two elements A and B expressed in $\mathcal{B}$ and if we use Theorem 2 we get the following

$$\begin{aligned} AB &= (\textstyle\sum_{i=1}^{n} a_i\beta_i) \times \left(\sum_{i=j}^{n} b_j\beta_j\right) \\ &= \underbrace{\left(\sum_{i,j=1}^{n} a_i b_j \beta_{i+j}\right)}_{S_1} + \underbrace{\left(\sum_{i,j=1}^{n} a_i b_j \beta_{|i-j|}\right)}_{S_2} \end{aligned} \tag{3}$$

Now we express each sum S_1 and S_2 as matrix-vector products. Let us begin with S_1. We remark that S_1 has a similar expression as product of two polynomials of the same degree. In other words, S_1 can

be computed as $Z_A \cdot B$ where

$$Z_A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & a_1 & 0 \\ a_n & \cdots & \cdots & a_2 & a_1 \\ 0 & a_n & \cdots & a_3 & a_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \begin{matrix} \leftarrow \beta_1 \\ \leftarrow \beta_2 \\ \vdots \\ \leftarrow \beta_n \\ \leftarrow \beta_{n+1} \\ \leftarrow \beta_{n+2} \\ \vdots \\ \leftarrow \beta_{2n} \end{matrix} \tag{4}$$

We reduce the matrix Z_A modulo $P = \beta_n + 1$ to get non-zero coefficients only on rows corresponding to $\beta_1, \ldots, \beta_n$. We use the fact that β_{n+i} for $i \geq 0$ satisfies

$$\beta_{n+i} = \beta_i \beta_n + \beta_{n-i} = \beta_i + \beta_{n-i}.$$

This equation is a simple consequence of (2) and that $\beta_n = 1 \mod P$. This implies that the rows corresponding to β_{n+i} are reduced into two rows one corresponding to β_i and the other to β_{n-i}. After performing this reduction and removing zero rows we get

$$S_1 = Z_A \cdot B = \underbrace{\begin{bmatrix} a_n & a_{n-1} & \cdots & a_2 & a_1 \\ a_1 & a_n & \cdots & a_3 & a_2 \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & a_1 & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{S_{1,1}}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & \cdots & a_n & a_{n-1} \\ \vdots & & & & \vdots \\ 0 & a_n & \cdots & a_3 & a_2 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{S_{1,2}}$$

Finally, we get an expression of S_1 as matrix-vector product where the matrix is a sum of a Toeplitz and an essentially Hankel matrix.

Now we do the same for S_2. We split S_2 into two sums

$$S_2 = \underbrace{\left(\sum_{k=1}^{n} \sum_{j=1}^{n-k} a_{j+k} b_j \beta_k \right)}_{S_{2,1}} + \underbrace{\left(\sum_{k=1}^{n} \sum_{j=k}^{n} a_{j-k} b_j \beta_k \right)}_{S_{2,2}}. \tag{5}$$

We express $S_{2,1}$ and $S_{2,2}$ as matrix-vector products

$$S_{2,1} = \begin{bmatrix} a_2 & a_3 & \cdots & a_{n-1} & a_n & 0 \\ a_3 & a_4 & \cdots & a_n & 0 & 0 \\ \vdots & & & & \vdots & \\ a_n & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \tag{6}$$

$$S_{2,2} = \begin{bmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & a_1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}. \tag{7}$$

So now we have each of S_1 and S_2 in the required form. We finally write $S_{1,1} + S_{2,2} = T_A \cdot B$ where T_A is a Toeplitz matrix and $S_{1,2} + S_{2,1} = H_A \cdot B$ where H_A is an Hankel matrix. We obtain

$$A \times B = T_A \cdot B + H_A \cdot B \tag{8}$$

as stated at the beginning of the current subsection.

3.2 Dickson trinomials

Now we assume that the field $\mathbb{F}_{2^n}$ is defined by a three-term irreducible Dickson trinomial P

$$P = 1 + \beta_k + \beta_n, \text{ with } k \leq n/2.$$

The elements in $\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(P)$ are expressed in the Dickson basis $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$. Our aim is to express the product of two elements A and B of $\mathbb{F}_{2^n}$ as Toeplitz or Hankel matrix-vector product. We use again the expression of the product $C = A \times B = S_1 + S_2$ given in equation (3). Similar to the previous subsection, here we express S_1 and S_2 as matrix-vector product separately. Specifically

1) The sum S_1 is expressed as $Z_A \cdot B$ where Z_A is given in (4)
2) For S_2 we use the expression of (5) and we put this expression in a matrix-vector product form.

$$S_2 = \left(\begin{bmatrix} a_2 & a_3 & \cdots & a_n & 0 \\ a_3 & a_4 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_1 & \cdots & a_{n-1} \\ 0 & 0 & \cdots & a_{n-2} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_1 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \right) \cdot B. \tag{9}$$

Now we replace S_1 and S_2 by their corresponding expressions given above in $A \times B = S_1 + S_2$. We get

$$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ a_{n-1} & \cdots & \cdots & 0 \\ a_n & \cdots & \cdots & a_1 \\ 0 & a_n & \cdots & a_2 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} + \begin{bmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_1 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \right) \cdot B \quad . \tag{10}$$

$$+ \begin{bmatrix} a_2 & a_3 & \cdots & a_{n-1} & a_n & 0 \\ a_3 & a_4 & \cdots & a_n & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ a_n & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = A \times B$$

In (10) the addition of two $2n \times n$ Toeplitz matrices results in one single $2n \times n$ Toeplitz matrix. The latter can be horizontally split in the middle to obtain two $n \times n$ Toeplitz matrices, say T_{up} and T_{down}, which can be then multiplied separately with vector $(b_1, \ldots, b_n)$ with a total cost of two $n \times n$ Toeplitz matrix-vector products.

The other $2n \times n$ Hankel matrix in (10) has all zero in the lower n rows, contributing nothing to the cost of the matrix-vector multiplication. Thus, the total computational cost of (10) is no more than three $n \times n$ Toeplitz or Hankel matrix-vector products.

The reduction:

The resulting expression of C in (10) is an unreduced form of $A \times B$, since it has non zero coefficients c_i on rows $i = n+1, \ldots, 2n$. These coefficients are obtained by multiplying T_{down} with vector $(b_1, b_2, \ldots, b_n)$, and must be reduced modulo $P = \beta_n + \beta_k + 1$, to get an expression of C in $\mathcal{B}$. We have

$$\begin{aligned} \beta_i &= \beta_n \beta_{i-n} + \beta_{2n-i} = (\beta_k + 1)\beta_{i-n} + \beta_{2n-i} \\ &= \beta_{i-n+k} + \beta_{|i-n-k|} + \beta_{i-n} + \beta_{2n-i}. \end{aligned}$$

We reduce the expressing of $C = \sum_{i=1}^{2n} c_i \beta_i$ by replacing each β_i for $i > n$ by the expression given above. Since we assume $k < \frac{n}{2}$ this process must be done two times to get a reduced expression of C. A circuit can be designed to perform this process which requires $6n - k$ XOR gates and is performed in time $3T_X$ (see [6] for details).

3.3 Parallel multiplier

We can design multiplier using the expression of the multiplication in $\mathbb{F}_{2^n}$ as a Toeplitz or Hankel matrix-vector product (TMVP). Specifically we use the Toeplitz or Hankel matrix-vector multiplier presented

in [3] to perform these products. In Table 2, we recall the complexity of the TMVP multiplier established by Fan and Hasan [3].

Table 2

Asymptotic complexity of TMVP multiplier

	2-way split method	3-way split method
# AND	$n^{\log_2(3)}$	$n^{\log_3(6)}$
# XOR	$5.5n^{\log_2(3)} - 6n + 0.5$	$\frac{24}{5}n^{\log_3(6)} - 5n + \frac{1}{5}$
Delay	$T_A + 2\log_2(n)T_X$	$T_A + 3\log_3(n)T_X$

In the case of Dickson binomials, to compute the matrix-vector products of (8) we need two TMVP multipliers in parallel. Each of them can use 2-way or 3-way split approach of [3]. We also need additionnal $2n$ XOR gates to compute the coefficient of T_A and add the result of the two matrix-vector products.

In the case Dickson trinomials, as specified in Subsection 3.2, three TMVPs are done in parallel using 2-way or 3-way split approach of [3]. We also need to perform a reduction using the circuit depicted in [6]. We obtain the complexities of Table 3 below where the second left most column indicates b-way splits with the value of b being either 2 or 3.

Table 3

Comparison of Subquadratic Space Complexity Parallel Multipliers

	b	Space		Time
		# AND	# XOR	
DB	2	$2n^{\log_2(3)}$	$11n^{\log_2(3)} - 11n$	$(2\log_2(n)+1)T_X$ $+T_A$
	3	$2n^{\log_3(6)}$	$48/5n^{\log_3(6)}$ $-11n + 3/5$	$(3\log_3(n)+1)T_X$ $+T_A$
DT	2	$\frac{7}{3}n^{\log_2(3)}$	$\frac{38,5}{3}n^{\log_2(3)}$ $-6,5n - k + 2,5$	$(2\log_2(n)+6)T_X$ $+T_A$
	3	$2n^{\log_3(6)}$	$48/5n^{\log_3(6)}$ $-3n - k + 7/5$	$(3\log_3(n)+5)T_X$ $+T_A$
ONBI [4]	2	$n^{\log_2(3)} + n$	$5.5n^{\log_2(3)}$ $-4n - 0.5$	$(2\log_2(n)+1)T_X$ $+T_A$
	3	$n^{\log_3(6)} + n$	$24/5n^{\log_3(6)}$ $-3n - 4/5$	$(3\log_3(n)+1)T_X$ $+T_A$
ONBII [5], [9]†	2	$n^{\log_2(3)}$	$6n^{\log_2(3)} - 3n +$ $(8n+2)\log_2(2n+1)$	$(3\log_2(n)+1)T_X$ $+T_A$
ONBII [5], [11]‡	3	$n^{\log_3(6)}$	$\frac{72}{15}n^{\log_3(6)} - \frac{7}{3}n - 1 +$ $(8n+2)\log_2(2n+1)$	$(4\log_3(n)+1)T_X$ $+T_A$

The row of Table 3 labelled by † (resp. ‡) refers to the method of [5] combined to the polynomial multiplication of [9] (resp. [11]).

In a recent paper Mullin *et al.* [8] pointed out that there were some links between the Dickson basis and the normal basis. In practice, a Dickson basis is interesting when no optimal normal basis exists for the considered field.

This is the case for NIST recommended binary fields $\mathbb{F}_{2^{163}}$ and $\mathbb{F}_{2^{283}}$. Using Table 1, we can remark that NIST fields can be constructed with Dickson trinomials, and thus we obtain a subquadratic multiplier in each of these cases.

4 Sequential Multipliers

In this section, we present sequential multipliers. Each of these multipliers takes $O(n)$ clock cycles but has a space complexity of $O(n)$.

4.1 Using irreducible Dickson binomials

4.1.1 Multiplier with bit serial output

In the sequel, we denote the entry at (i, j) of the Toeplitz and the Hankel matrices of (8) as $T_{i,j}$ and $H_{i,j}$, respectively. We also denote the rows of the Toeplitz matrix as $T_{1,\star}, T_{2,\star}, \ldots, T_{n,\star}$ and those of the essentially Hankel matrix as $H_{1,\star}, H_{2,\star}, \ldots, H_{n,\star}$. Thus we can write

$$A \times B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} T_{1,\star} + H_{1,\star} \\ T_{2,\star} + H_{2,\star} \\ \vdots \\ T_{n,\star} + H_{n,\star} \end{bmatrix} \cdot B. \tag{11}$$

We remark that

a) $T_{n,\star}$ consists of the coordinates of input A that are rotated left one position, i.e., $T_{n,\star} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_n \end{bmatrix}$. On the other hand, $H_{n,\star}$ is the all zero row vector and $H_{n-1,\star} = \begin{bmatrix} 0 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}$.

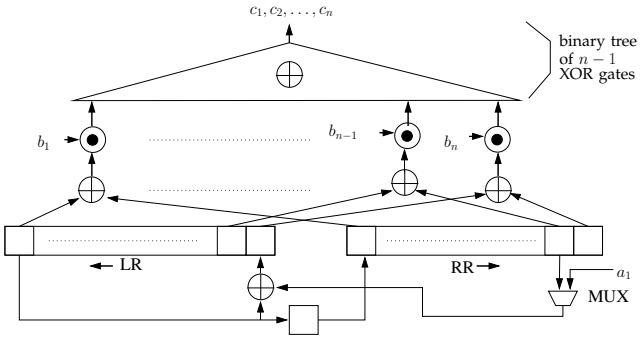
b) Given $T_{i,\star}$ and $H_{i,n-1}$, we can express $T_{i-1,\star}$ for $1 < i \leq n$ as $T_{i-1,\star} = \begin{bmatrix} T_{i,2} & T_{i,3} & \cdots & T_{i,n} & T_{i,1} + H_{i,n-1} \end{bmatrix}$. Furthermore, given the row $H_{i,\star}$ and the entry $T_{i+1,1}$ we can express $H_{i-1,\star}$ for $1 < i \leq n-1$ as follows

$$H_{i-1,\star} = \begin{bmatrix} T_{i+1,1} & H_{i,1} & \cdots & H_{i,n-2} & H_{i,n-1} \end{bmatrix}.$$

The following diagram (Figure 1) corresponds to a sequential structure to realize the multiplication $C = A \times B$ in accordance with (11). In the initial clock cycle, the left side register (LR) in the diagram is loaded with $T_{n,\star}$ and the right side register (RR) with $H_{n,\star}$. In this cycle, rows $T_{n,\star}$ and $H_{n,\star}$ are added and an inner product is performed to yield $c_n = (T_{n,\star} + H_{n,\star}) \cdot B$. Also, in this cycle the output of MUX is a_1 (and in other cycles the MUX output is the second right most bit of RR). In the next cycle, RR is loaded with $H_{n-1,\star}$ and LR is shifted left to generate $T_{n-1,\star}$ eventually yielding c_{n-1}.

For each of the following $n-2$ clock cycles, LR is shifted left, RR is shifted right, their contents are bit-wise added and an inner product is performed to produce one coordinate of C. The space and time complexity of the architecture of Fig. 1 is given in Table 4.

Figure 1. Sequential multiplier with bit serial out using Dickson binomials



4.1.2 Sequential multiplier with bit parallel output

Referring to (8) we denote the columns of a Toeplitz matrix as $T_{\star,i}$ for $1 \leq i \leq n$ and those of an essentially Hankel matrix as $H_{\star,i}$ for $1 \leq i \leq n$. Thus we can write $A \times B = \left[\begin{array}{cccc} T_{\star,1} + H_{\star,1} & T_{\star,2} + H_{\star,2} & \cdots & T_{\star,n} + H_{\star,n} \end{array} \right] \cdot B$, i.e.,

$$C = \sum_{i=1}^{n} b_i (T_{\star,i} + H_{\star,i}). \tag{12}$$

We remark that

a) $T_{\star,1} = \left[\begin{array}{cccc} a_n & a_1 & \cdots & a_{n-1} \end{array} \right]^t$ and $H_{\star,1} = \left[\begin{array}{cccccc} a_2 & a_3 & \cdots & a_{n-1} & 0 & 0 \end{array} \right]^t$

b) Given the column $T_{\star,i}$ and the entry $H_{1,i-1}$, we can express $T_{\star,i+1}$ as $T_{\star,i+1} = \left[\begin{array}{c} T_{n,i} + H_{1,i-1}, T_{1,i}, \cdots, T_{n-1,i} \end{array} \right]^t, 1 \leq i \leq n$, where $H_{1,0}$ is assumed to be a_1.

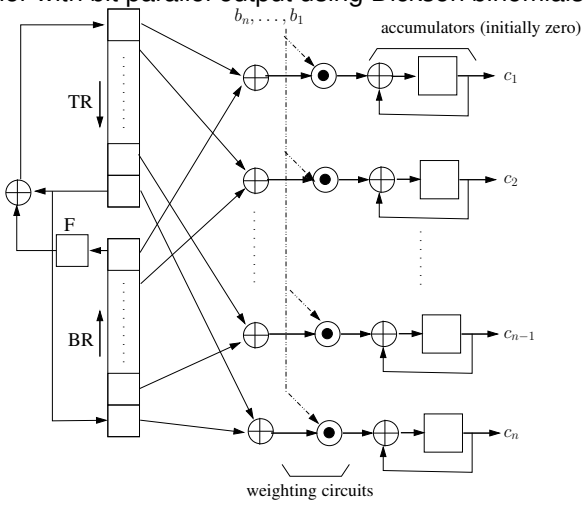
Additionally, given column $H_{\star,i}$ and entry $T_{n,i}$, we can express $H_{\star,i+1}$ as $H_{\star,i+1} = \left[\begin{array}{c} H_{2,i}, H_{3,i}, \cdots, H_{n-1,i}, T_{n,i}, 0 \end{array} \right]^t, 1 \leq i \leq n$.

In the following diagram (Fig. 2), the column vectors $T_{\star,1}$ and $H_{\star,1}$ are initially loaded into the top register (TR) and the bottom register (BR) respectively. The one-bit feedback cell F is initialized with $H_{1,0} = a_1$. If TR is shifted downward and BR upward with the feedback connections as shown in the diagram, the new contents of TR and BR will be $T_{\star,2}$ and $H_{\star,2}$ respectively. Note that BR is an $n-1$ bits long shift register, since $H_{n,i} = 0$ for $1 \leq i \leq n$. With additional shifts on TR and BR, the remaining columns of the Toeplitz and the essentially Hankel matrices are generated.

Each corresponding pair of columns (i.e., $T_{\star,i}$ and $H_{\star,i}$) are added and the resulting columns are multiplied with b_i (in the diagram these are shown using an array of XOR and AND gates).

The weighted columns are accumulated in accordance with (12) to produce the desired output C in a total of n clock cycles. The delay and the space complexity of this architecture are given in Table 4.

Figure 2. Sequential multiplier with bit parallel output using Dickson binomials



4.2 Using irreducible Dickson trinomials

From (10) of Subsection 3.2 the coefficients $c_1, c_2, \cdots, c_n$ are given by

$$\left(\begin{bmatrix} 0 & a_1 & \cdots & a_{n-1} \\ a_1 & 0 & \cdots & a_{n-2} \\ \vdots & & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_1 \\ a_{n-1} & a_{n-2} & \cdots & 0 \end{bmatrix} + \begin{bmatrix} a_2 & a_3 & \cdots & a_n & 0 \\ a_3 & a_4 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \right) \cdot B, \tag{13}$$

and

$$\begin{bmatrix} c_{n+1} \\ c_{n+2} \\ \vdots \\ c_{2n} \end{bmatrix} = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 \\ 0 & a_n & \cdots & a_2 \\ \vdots & & & \\ 0 & 0 & \cdots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}. \tag{14}$$

Note that $c_{n+1}, c_{n+2}, \cdots, c_{2n}$ can be reduced as explained at the end of Subsection 3.2.

Below we will first present a hardware structure to generate $c_1, c_2, \ldots, c_n$ in accordance with (13). Then we will discuss how to use part of the above hardware to generate $c_{n+1}, c_{n+2}, \ldots, c_{2n}$. In practice, one can first generate $c_{n+1}, c_{n+2}, \ldots, c_{2n}$. While these n bits are reduced, one can generate $c_1, c_2, \ldots, c_n$. This overlap of operations will effectively eliminate/hide the extra time for reduction of $c_{n+1}, \ldots, c_{2n}$.

4.2.1 Sequential multiplier with bit serial output

We denote the rows of the Toeplitz and the Hankel matrices of (13) as $T_{i,\star}$ and $H_{i,\star}$ respectively. For $1 \leq i \leq n$, we can then write the following

$$\begin{aligned} T_{i+1,\star} &= \begin{bmatrix} H_{i-1,1} & T_{i,1} & \cdots & T_{i,n-2} & T_{i,n-1} \end{bmatrix}, \\ H_{i+1,\star} &= \begin{bmatrix} H_{i,2} & H_{i,3} & \cdots & H_{i,n} & 0 \end{bmatrix}, \end{aligned} \tag{15}$$

where $H_{0,1}$ is assumed to be a_1 and

$$\begin{aligned} T_{1,\star} &= \left[\begin{array}{ccccccc} 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{array}\right], \\ H_{1,\star} &= \left[\begin{array}{ccccccc} a_2 & a_3 & a_4 & \cdots & a_n & 0 \end{array}\right]. \end{aligned} \tag{16}$$

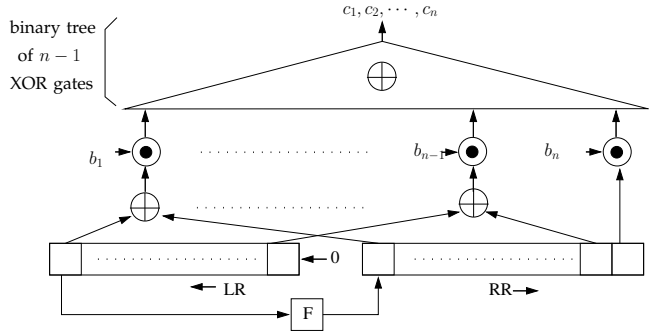
In Fig. 3 below, registers RR and LR are initialized with $T_{1,\star}$ and $H_{1,\star}$. The feedback cell F is initialized with $a_1 = H_{0,1}$. Then with the application of a shift to these registers together, the second rows of the Toeplitz and the Hankel matrices of (12) are formed in RR and LR respectively. This happens due to the fact that the shift and the feedback connection as shown in Fig. 3 essentially realize (15). The remaining rows of the two matrices are formed pair by pair with successive shifts.

Note that LR is $n-1$ bits long, since the right most bit of each row of the Hankel matrix is zero. The upper part of Fig. 3 is similar to that of Fig. 1 and is to add the corresponding rows of the Toeplitz and the Hankel matrices, followed by inner product operations to yield $c_1, c_2, \cdots, c_n$.

To generate $c_{n+1}, c_{n+2}, \ldots, c_{2n}$ using the structure in Fig. 3, we initialize RR with $[a_n, a_{n-1}, \ldots, a_1]$, which is the first row of the upper triangular Toeplitz matrix of (14). Register LR and cell F are initialized with all zeros. Then with successive shifts, RR will contain the remaining rows of the Toeplitz matrix and LR will have all zeros. This will result in $c_{n+1}, c_{n+2}, \ldots, c_2$ at the output of Fig. 3.

The time and the space complexities of the structure of Fig. 3 are given in Table 4. These exclude the cost associated with the reduction of $c_{n+1}, c_{n+2}, \cdots, c_{2n}$.

Figure 3. Bit serial output sequential multiplier for Dickson trinomials



4.2.2 *Bit parallel output*

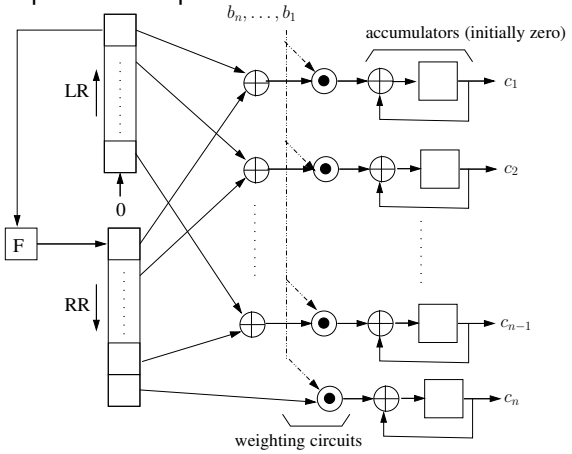
For (13), let $T_{\star,i}$ and $H_{\star,i}$ be the i-th columns of the Toeplitz and the Hankel matrices, respectively. Then (13) can be re-written as

$$\left[\begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array}\right]^t = \sum_{i=1}^{n} b_i (T_{\star,i} + H_{\star,i}).$$

In order to generate $T_{\star,i}$ and $H_{\star,i}$ we note that in (13) $T_{\star,i} = T_{i,\star}$ and $H_{\star,i} = H_{i,\star}$. In other words, the i-th column is the same as the i-th row for each of the matrices. Thus columns can be generated using the same system of feedback registers as shown in Fig. 3 earlier.

To obtain $c_1, c_2, \cdots, c_n$ in bit parallel fashion the inner product unit of Fig. 3 can be replaced with an array unit of weighting AND gates and accumulators (as used in Fig. 2). The complete diagram is shown in Fig. 4, and its space and time complexities are given in Table 4.

Figure 4. Bit parallel output sequential multiplier for Dickson trinomials



The coefficients $c_{n+1}, c_{n+2}, \ldots, c_{2n}$ from (14) will be computed with the same hardware of Fig. 2. Specifically, in Fig. 4 RR is initialized with the first column of the above matrix. The accumulators LR and F are all initialized to zero. Then in n clock cycles with weighting input as $b_n, b_{n-1}, \ldots, b_1$ the accumulators will have $c_{2n}, c_{2n-1}, \ldots, c_{n+1}$.

4.3 Complexity and comparison

In Table 4 we put the resulting complexities of the different sequential multipliers based on the Dickson basis representation. For the purpose of comparison, we also give the complexity of the method of [4] using ONB of type I and II. We remark that when no ONB is available, a Dickson binomial seems to be the best choice since Dickson trinomial require an increased number of clock cycles.

5 Conclusion

In this paper we have presented new parallel multipliers based on Dickson basis representation of binary fields. The multiplier for an irreducible Dickson binomial has a complexity similar to the subquadratic multiplier for ONB II of [4]. For an irreducible Dickson trinomial, the multiplier has a slightly more space complexity, but can still be used for fields with degree of several hundreds (for example those used in today's elliptic curve cryptographic systems).

Table 4

Complexity of sequential multipliers

Archi.	#AND	#XOR	#FF	#MUX	#CC	Delay
DB Fig 1	n	$2n$	$2n+1$	1	n	T_A+ $(1+\lceil\log_2(n)\rceil)T_X$
DB Fig 2	n	$2n$	$2n+1$	1	n	T_A+2T_X
DT Fig 3	n	$2n-2$	$2n$	0	$2n$	T_A+ $(1+\lceil\log_2(n)\rceil)T_X$
DT Fig 4	n	$2n-1$	$3n-1$	0	$2n$	T_A+T_X
DG[1]	$2n$	$4n-3$	$3n$	0	n	$2T_X+T_A$
ONBI[10]	n	$\frac{3n}{2}$	$2n$	0	n	T_A+2T_X
ONBII[12]	n	$\frac{3n-1}{2}$	$3n$	0	n	T_A+2T_X

DB=Dickson Binomial, DT=Dickson Trin., DG=General Dickson , CC=Clock Cycle.

In this paper, we have also presented sequential multipliers using the above mentioned Dickson representation. The sequential multipliers have a space complexity of $O(n)$. We have considered both bit-serial and bit-parallel output formats for the sequential multipliers. Compared to the sequential multipliers with bit-parallel output format presented in [1] and [8], the sequential multipliers presented here with the same output format reduce the number of XOR and AND gates by a factor of two or more, while keeping the number of flip-flops and clock cycles about the same.

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