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INHOMOGENEOUS NAVIER-STOKES EQUATIONS IN THE HALF-SPACE, WITH ONLY BOUNDED DENSITY

RAPHAËL DANCHIN AND PING ZHANG

Abstract. In this paper, we establish the global existence of small solutions to the inhomogeneous Navier-Stokes system in the half-space. The initial density only has to be bounded and close enough to a positive constant, and the initial velocity belongs to some critical Besov space. With a little bit more regularity for the initial velocity, those solutions are proved to be unique. In the last section of the paper, our results are partially extended to the bounded domain case.

Keywords: Inhomogeneous Navier-Stokes equations, Stokes system, critical Besov spaces, half-space, Lagrangian coordinates.

AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

We are concerned with the global well-posedness issue for the initial boundary value problem pertaining to the following incompressible inhomogeneous Navier-Stokes equations:

$$\left\{\begin{array}{lll}\partial_t\rho+\mathrm{div}(\rho u)=0 & \text{in} & \mathbb{R}_+\times\Omega,\\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mu\Delta u+\nabla\Pi=0 & \text{in} & \mathbb{R}_+\times\Omega,\\ \mathrm{div}\, u=0 & \text{in} & \mathbb{R}_+\times\Omega,\\ u=0 & \text{on} & \mathbb{R}_+\times\partial\Omega,\\ \rho|_{t=0}=\rho_0,\quad \rho u|_{t=0}=\rho_0u_0 & \text{in} & \Omega,\end{array}\right. \tag{1.1}$$

where $\rho=\rho(t,x)\in\mathbb{R}_+,\ u=u(t,x)\in\mathbb{R}^d$ and $\Pi=\Pi(t,x)\in\mathbb{R}$ stand for the density, velocity field and pressure of the fluid, respectively, depending on the time variable $t\in\mathbb{R}_+$ and on the space variables $x\in\Omega$. The positive real number μ stands for the viscosity coefficient. We mainly consider the case where Ω is the half-space $\mathbb{R}^d_+$, except in the last section of the paper where it stands for a smooth bounded domain of $\mathbb{R}^d$ $(d\geq 2)$.

The above system describes a fluid that is incompressible but has nonconstant density. Basic examples are mixture of incompressible and non reactant flows, flows with complex structure (e.g. blood flow or model of rivers), fluids containing a melted substance, etc.

A number of recent works have been dedicated to the mathematical study of the above system. Global weak solutions with finite energy have been constructed by J. Simon in [19] (see also the book by P.-L. Lions [17] for the variable viscosity case). In the case of smooth data with no vacuum, the existence of strong unique solutions goes back to the work of O. Ladyzhenskaya and V. Solonnikov in [15]. More recently, the first author [8] established the well-posedness of the above system in the whole space $\mathbb{R}^d$ in the so-called *critical functional framework* for small perturbations of some positive constant density. The basic idea is to use functional spaces that have the same *scaling invariance* as (1.1), namely

$$(\rho,u,\Pi)(t,x)\longmapsto(\rho,\lambda u,\lambda^2\Pi)(\lambda^2t,\lambda x),\qquad (\rho_0,u_0)(x)\longmapsto(\rho_0,\lambda u_0)(\lambda x). \tag{1.2}$$

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More precisely, in [8], global well-posedness was established assuming that

$$\|\rho_0-1\|_{\dot B^{\frac d2}_{2,\infty}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}+\mu^{-1}\|u_0\|_{\dot B^{\frac d2}_{2,1}(\mathbb{R}^d)}\ll 1.$$

Above $\dot B^\sigma_{p,r}(\mathbb{R}^d)$ stands for a homogeneous Besov space on $\mathbb{R}^d$ (see Definition 2.1 below). This result was extended to more general Besov spaces by H. Abidi in [1], and H. Abidi and M. Paicu in [2], and to the half-space setting in [10]. The smallness assumption on the initial density was removed recently in [3, 4].

Given that in all those works the density has to be at least in the Besov space $\dot B^{\frac dp}_{p,\infty}$, one cannot capture discontinuities across an hypersurface. In effect, the Besov regularity of the characteristic function of a smooth domain is only $\dot B^{\frac1p}_{p,\infty}$. Therefore, those results do not apply to a mixture of two fluids with different densities.

Very recently, the first author and P. Mucha [11] noticed that it was possible to establish existence *and* uniqueness of a solution in the case of a small discontinuity, in a critical functional framework. More precisely, the global existence and uniqueness was established for any data (ρ_0,u_0) such that for some $p\in[1,2d)$ and small enough constant c, we have

$$\|\rho_0-1\|_{\mathcal{M}(\dot B^{-1+\frac dp}_{p,1}(\mathbb{R}^d))}+\mu^{-1}\|u_0\|_{\dot B^{-1+\frac dp}_{p,1}(\mathbb{R}^d)}\leq c. \tag{1.3}$$

Above, $\|\cdot\|_{\mathcal{M}(\dot B^{-1+\frac dp}_{p,1}(\mathbb{R}^d))}$ is the *multiplier norm* associated to the Besov space $\dot B^{-1+\frac dp}_{p,1}(\mathbb{R}^d)$, which turns out to be finite for characteristic functions of C^1 domains whenever $p>d-1$. Therefore, initial densities with a discontinuity across an interface may be considered (although the jump has to be small owing to (1.3)). As observed later on in [12], large discontinuities may be considered if the initial velocity is smoother. In fact, therein, *any* initial density bounded and bounded away from 0 is admissible. Let us emphasize that in both works ([11] and [12]), using Lagrangian coordinates was the key to the proof of uniqueness.

A natural question is whether it is still possible to get existence *and* uniqueness in a critical functional framework where ρ_0 is only bounded and bounded away from zero. As regards existence, a positive answer has been given recently by J. Huang, M. Paicu and the second author in [14], in the whole space setting, and uniqueness was obtained if assuming slightly more regularity for the velocity field. Let us emphasize that once again using Lagrangian coordinates is the key to uniqueness. Therefore, assumptions on the initial velocity have to ensure the velocity u to have gradient in $L^1_{loc}(\mathbb{R}_+;L^\infty(\mathbb{R}^d))$ in order that Eulerian and Lagrangian formulations of the system are equivalent. While this property of the velocity field holds true if u_0 is in $\dot B^{-1+\frac dp}_{p,1}(\mathbb{R}^d)$, it fails if u_0 is only in $\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d)$ for some $r>1$. As a matter of fact, the question of uniqueness in a critical Besov framework for the velocity is open unless $r=1$ (this latter case requires stronger assumptions on the density, as pointed out in [11]).

In the present work, we aim at extending the results of [14] to the half-space setting. Because we shall consider only perturbations of the reference density 1, it is natural to set

$a = 1/\rho - 1$ so that System (1.1) translates into

$$
\left\{
\begin{array}{lll}
\partial_t a + u \cdot \nabla a = 0 & \text{in} & \mathbb{R}_+ \times \Omega, \\
\partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \mu \Delta u) = 0 & \text{in} & \mathbb{R}_+ \times \Omega, \\
\operatorname{div} u = 0 & \text{in} & \mathbb{R}_+ \times \Omega, \\
u = 0 & \text{on} & \mathbb{R}_+ \times \partial\Omega, \\
(a,u)|_{t=0} = (a_0, u_0) & \text{in} & \Omega.
\end{array}
\right. \tag{1.4}
$$

As in the whole space case considered in [14], the functional framework for solving (1.4) is motivated by classical maximal regularity estimates for the evolutionary Stokes system. In effect, the velocity field may be seen as the solution to the following system:

$$
\partial_t u - \mu \Delta u + \nabla \Pi = -u \cdot \nabla u + a(\mu \Delta u - \nabla \Pi), \qquad \operatorname{div} u = 0. \tag{1.5}
$$

In the whole space case, we have for any $1 < p, r < \infty$ and $t > 0$,

$$
\|(\partial_t u, \mu \nabla^2 u, \nabla \Pi)\|_{L^r_t(L^p)} \leq C\Big(\mu^{1-\frac{1}{r}} \|u_0\|_{\dot{B}^{2-\frac{2}{r}}_{p,r}} + \|u \cdot \nabla u\|_{L^r_t(L^p)} + \|a(\mu \Delta u - \nabla \Pi)\|_{L^r_t(L^p)}\Big)
$$

where we have used the notation $\|z\|_{L^r_t(L^p)} \stackrel{\text{def}}{=} \|z\|_{L^r((0,t);L^p(\mathbb{R}^d))}\cdot$

Given that $\|a(t)\|_{L^\infty} = \|a_0\|_{L^\infty}$ for all time, it is clear that the last term may be absorbed by the l.h.s. if $\|a_0\|_{L^\infty}$ is small enough. Now, it is standard (see Corollary A.1 below) that in the simpler case where u just solves the heat equation with initial data u_0, having Δu in $L^r(0,T;L^p)$ is equivalent to $u_0 \in \dot{B}^{-1+\frac{d}{p}}_{p,r}$ *provided* $-1 + \frac{d}{p} = 2 - \frac{2}{r}\cdot$ This implies that p and r have to be interrelated through $p = \frac{dr}{3r-2}$, and thus $\frac{d}{3} < p < d$.

Here we aim at extending this simple idea to the half-space setting, or to C^2 bounded domains.

In the half-space case, according to the above heuristics and because homogeneous Dirichlet boundary conditions are prescribed for the velocity, the natural solution space for $(u, \nabla\Pi)$ is

$$
X^{p,r}_T \stackrel{\text{def}}{=} \Big\{(u, \nabla \Pi) \ \text{ with } \ u \in \mathcal{C}([0,T]; \dot{\mathcal{B}}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla \Pi \in L^r(0,T; L^p_+)\Big\},
$$

where $L^p_+ \stackrel{\text{def}}{=} L^p(\mathbb{R}^d_+)$ denotes the Lebesgue space over $\mathbb{R}^d_+$, and $\dot{\mathcal{B}}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)$ stands for the set of divergence free vector fields on $\mathbb{R}^d_+$ with Besov regularity $\dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)$ and null trace at the boundary (the exact meaning will be given in Definition 2.3 below).

We also introduce the following norm for all $T > 0$:

$$
\|(u, \nabla \Pi)\|_{X^{p,r}_T} \stackrel{\text{def}}{=} \mu^{1-\frac{1}{r}} \|u\|_{L^\infty(0,T; \dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+))} + \|(\partial_t u, \mu \nabla^2 u, \nabla \Pi)\|_{L^r(0,T;L^p_+)}, \tag{1.6}
$$

and agree that $X^{p,r}$ and $\|\cdot\|_{X^{p,r}}$ correspond to the above definition with $T = +\infty$.

Before stating our main results, let us clarify what we mean by *a weak solution to* (1.4):

Definition 1.1. *A global weak solution of* (1.4) *is any couple* (a, u) *satisfying:*

- *for any test function* $\phi \in C^\infty_c([0,\infty) \times \overline{\Omega})$, *there holds*

$$
\begin{gathered}
\int_0^\infty \!\! \int_\Omega a(\partial_t \phi + u \cdot \nabla \phi)\, dx\, dt + \int_\Omega \phi(0,\cdot) a_0 \, dx = 0, \\
\int_0^\infty \!\! \int_\Omega \phi \operatorname{div} u \, dx\, dt = 0,
\end{gathered} \tag{1.7}
$$

- *for any vector valued function* $\Phi = (\Phi^1, \cdots, \Phi^d) \in C_c^\infty([0,\infty)\times\overline{\Omega})$, *one has*

$$\int_0^\infty\!\!\int_\Omega \Big\{u\cdot\partial_t\Phi + u\otimes u : \nabla\Phi + (1+a)\big(\mu\Delta u - \nabla\Pi\big)\Phi\Big\}\,dx\,dt + \int_\Omega u_0\cdot\Phi(0,\cdot)\,dx = 0. \tag{1.8}$$

Our main statement reads:

Theorem 1.1. *Let* $a_0 \in L^\infty_+$ *and* $u_0 \in \dot{\mathcal{B}}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+) \cap \dot{\mathcal{B}}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)$ *with* $p \stackrel{\text{def}}{=} \frac{dr}{3r-2} \le \widetilde{p} \le \frac{dr}{r-1}$ *and* $r\in(1,\infty)$. *There exist two positive constants* $c_0 = c_0(r,d)$ *and* $c_1 = c_1(r,d)$ *so that if*

$$\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)} \le c_0\mu \quad\text{and}\quad \|a_0\|_{L^\infty_+} \le c_1 \tag{1.9}$$

then (1.4) *has a global solution* $(a,u,\nabla\Pi)$ *in the meaning of Definition 1.1 with* $\Omega = \mathbb{R}^d_+$, *satisfying* $\|a(t)\|_{L^\infty_+} = \|a_0\|_{L^\infty_+}$ *for all* $t>0$ *and* $u \in X^{p,r}\cap X^{\widetilde{p},r}$. *Moreover, there exist* $C_1 = C_1(r,d)$ *and* $C_2 = C_2(r,d)$ *so that*

$$\|(u,\nabla\Pi)\|_{X^{p,r}} + \mu^{1-\frac{1}{2r}}\|\nabla u\|_{L^{2r}(\mathbb{R}_+;L_+^{\frac{dr}{2r-1}})} \le C_1\mu^{1-\frac1r}\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)} \tag{1.10}$$

and, if $\frac{1}{\alpha} \stackrel{\text{def}}{=} \frac{1}{\widetilde{p}} - \frac{r-1}{dr}$,

$$\|(u,\nabla\Pi)\|_{X^{\widetilde{p},r}} + \mu^{1-\frac{1}{2r}}\|\nabla u\|_{L^{2r}(\mathbb{R}_+;L^\alpha_+)} \le C_2\mu^{1-\frac1r}\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}. \tag{1.11}$$

If in addition $\widetilde{p} > d$ *then* $\nabla u \in L^q(\mathbb{R}_+;L^\infty_+)$ *with* $q = \frac{2\widetilde{p}}{d+(\frac{2}{r}-1)\widetilde{p}}$ *and*

$$\mu^{\frac1q}\|\nabla u\|_{L^q(\mathbb{R}_+;L^\infty_+)} \le C_2\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}, \tag{1.12}$$

and uniqueness holds true.

Remark 1.1. *In contrast with the whole space case in* [14, 18], *it is not clear that one may improve the* isotropic *smallness condition* (1.9) *to an* anisotropic *one allowing for arbitrarily large vertical velocity. The reason why is that even for solutions to the linear Stokes system in the half-space, the horizontal component of the velocity* u^h *depends on both* u_0^h *and* u_0^d *(see the formula in Theorem 2.1 below).*

Remark 1.2. *We shall extend this statement to a more general critical (or almost critical) Besov setting, see Theorems 5.1 and 5.2 below. We could also deal with the local well-posedness of* (1.4) *with general velocity and small inhomogeneity. For a clear presentation, we skip the details here.*

Let us briefly describe the plan of the rest of the paper. The next section is devoted to the linearized velocity equation of (1.4) *in the half-space*, that is the evolutionary Stokes system. We first derive an explicit solution formula in the spirit of that of S. Ukai in [20], and then deduce maximal regularity type estimates similar to those of the whole space. We consider the general situation with prescribed (possibly nonzero) value for $\operatorname{div} u$ as it will be needed when reformulating (1.4) in Lagrangian coordinates. The next two sections are devoted to the proof of the existence part of Theorem 1.1, first under a stronger assumption on the density, and next in the rough case corresponding to the hypotheses of the theorem. The case of more general Besov spaces will be examined in Section 5. The proof of uniqueness is postponed in Section 6. In the final section, we partially generalize Theorem 1.1 to the bounded domain setting. Some technical lemmas related to maximal regularity and $L^p(L^q)$ estimates for the heat equation in the whole space (or Stokes system in bounded domains) are presented in Appendix.

2. The evolutionary Stokes system in the half-space

This section is devoted to the study of following system in the half-space:

$$
\left\{\begin{array}{lll}
\partial_t u-\mu\Delta u+\nabla\Pi=f & \text{in} & \mathbb{R}_+\times\mathbb{R}^d_+,\\
\operatorname{div} u=g & \text{in} & \mathbb{R}_+\times\mathbb{R}^d_+,\\
u=0 & \text{on} & \mathbb{R}_+\times\partial\,\mathbb{R}^d_+,\\
u|_{t=0}=u_0 & \text{on} & \mathbb{R}^d_+\,.
\end{array}\right. \tag{2.1}
$$

We shall first derive an explicit formula for the solution to this system, and next prove the key *a priori* estimates that are needed for getting the main results of our paper.

2.1. **A solution formula.** This part extends a prior work by S. Ukai [20] (see also [7]) to the case where there is a source term f in the velocity equation, and where the divergence constraint is nonhomogeneous. Let us recall that in [20], it was assumed that $f=0$ and $g=0$ (but u need not be zero at the boundary), and that in [7] nonzero f was considered (but still u is divergence free and f has trace zero). Furthermore, the gradient of the pressure was not computed therein, a computation that turns out to be essential for us as $\nabla\Pi$ appears in the right-hand side of the velocity equation (1.5).

Before writing out the formula, let us introduce a few notations. We denote $\Delta=\sum_{\ell=1}^d\partial^2_{x_\ell}$ and $\Delta_h=\sum_{\ell=1}^{d-1}\partial^2_{x_\ell}$, and define $|D|^{\pm1}$ and $|D_h|^{\pm1}$ to be the Fourier multipliers with symbol

$$
|\xi_h|^{\pm1}=\Big(\sum_{i=1}^{d-1}\xi_i^2\Big)^{\pm\frac12}\quad\text{and}\quad |\xi|^{\pm1}=\Big(\sum_{i=1}^{d}\xi_i^2\Big)^{\pm\frac12},\quad\text{respectively.}
$$

The notations R_j and S_j stand for the Riesz transforms over $\mathbb{R}^d$ and $\mathbb{R}^{d-1}_h$, namely

$$
R_j\stackrel{\text{def}}{=}\partial_j|D|^{-1}\ \text{ for }\ j=1,\cdots,d\qquad\text{and}\qquad S_j\stackrel{\text{def}}{=}\partial_j|D_h|^{-1}\ \text{ for }\ j=1,\cdots,d-1.
$$

We further set $R_h\stackrel{\text{def}}{=}(R_1,\cdots,R_{d-1})$ and $S\stackrel{\text{def}}{=}(S_1,\cdots,S_{d-1})$.

As in [20], for $u=(u^h,u^d)$ with $u^h=(u^1,\cdots,u^{d-1})$, we define the operators V_d and V_h by

$$
V_du\stackrel{\text{def}}{=}-S\cdot u^h+u^d\quad\text{and}\quad V_hu\stackrel{\text{def}}{=}u^h+Su^d. \tag{2.2}
$$

We shall see later on that both V_hu and V_du satisfy a heat equation, this is the main motivation for considering those two quantities.

We also denote $x=(x_h,x_d)$ with $x_h=(x_1,\cdots,x_{d-1})$. Let r be the restriction operator from $\mathbb{R}^d$ to $\mathbb{R}^d_+$, that is $rf\stackrel{\text{def}}{=}f|_{\mathbb{R}^d_+}$, and $e_0(f),e_a(f),e_s(f)$ be the extension operators given by

$$
\begin{aligned}
&e_0(f)=\left\{\begin{array}{ll} f & \text{for }\ x_d\geq0,\\ 0 & \text{for }\ x_d<0,\end{array}\right.\qquad e_a(f)=\left\{\begin{array}{ll} f(x) & \text{for }\ x_d\geq0,\\ -f(x_h,-x_d) & \text{for }\ x_d<0,\end{array}\right.\\
&\text{and}\qquad e_s(f)=\left\{\begin{array}{ll} f(x) & \text{for }\ x_d\geq0,\\ f(x_h,-x_d) & \text{for }\ x_d<0.\end{array}\right.
\end{aligned} \tag{2.3}
$$

When solving (2.1), we shall repeatedly consider the following two equations:

$$
(\partial_d+|D_h|)w=|D_h|f\quad\text{in}\quad\mathbb{R}^d_+,\quad\text{and}\quad\gamma w=0\quad\text{on}\quad\partial\,\mathbb{R}^d_+\,. \tag{2.4}
$$

$$
(\partial_d+|D_h|)v=f\quad\text{in}\quad\mathbb{R}^d_+,\quad\text{and}\quad\gamma v=0\quad\text{on}\quad\partial\,\mathbb{R}^d_+\,. \tag{2.5}
$$

Here and in what follows, γ stands for the trace operator on $\partial\,\mathbb{R}^d_+$.

Finally we denote by H the harmonic extension operator from the hyperplane $\partial\,\mathbb{R}^d_+$ to the half-space $\mathbb{R}^d_+$. More precisely, for any given $b:\partial\,\mathbb{R}^d_+\to\mathbb{R}$, we set Hb to be the unique solution going to zero at ∞ of

$$\Delta Hb=0 \quad\text{in}\quad \mathbb{R}^d_+,\quad\text{and}\quad \gamma Hb=b\quad\text{on}\quad \partial\,\mathbb{R}^d_+. \tag{2.6}$$

Introducing the Fourier transform $\mathcal{F}_h$ with respect to the horizontal component x_h, the function $\mathcal{F}_h(Hb)$ is explicitly given by the formula

$$\mathcal{F}_h(Hb)(\xi_h,x_d)=e^{-|\xi_h|x_d}\mathcal{F}_hb(\xi_h),\quad \xi_h\in\mathbb{R}^{d-1},\ x_d>0, \tag{2.7}$$

hence in particular

$$(\partial_d+|D_h|)Hb=0\quad\text{in}\quad\mathbb{R}^d_+. \tag{2.8}$$

Lemma 2.1. *For any smooth enough data f decaying at ∞, Equation* (2.4) *has a unique solution going to* 0 *at* ∞, *and Equation* (2.5) *has a unique solution with gradient going to* 0 *at infinity. Furthermore, denoting by U and P the solution operators for* (2.4) *and* (2.5), *one has*

$$Uf=r\,R_h\cdot S\big(R_h\cdot S\,e_a(f)+R_de_s(f)\big)\quad\textit{and}\quad Pf(x_h,x_d)=\int_0^{x_d}(I-U)f(x_h,y_d)\,dy_d \tag{2.9}$$

and the following identities are satisfied:

1. $\nabla_hU=U\nabla_h$;
2. $\partial_dU=(I-U)|D_h|$;
3. $\nabla_hP=SU$;
4. $\partial_dP=I-U$;
5. $\Delta P=\partial_d-|D_h|$;
6. $[P,\partial_d]=-H\gamma$.

Proof. If w is a solution to (2.4) then it also satisfies the following Poisson equation:

$$\begin{cases}-\Delta w=(|D_h|-\partial_d)|D_h|f\\ w|_{x_d=0}=0,\end{cases}$$

the unique solution (decaying to 0 at infinity) of which is given by

$$w=r(-\Delta)^{-1}e_a\big((|D_h|-\partial_d)|D_h|f\big).$$

As $e_a(|D_h|^2f)=|D_h|^2e_a(f)$ and $e_a(\partial_d|D_h|f)=\partial_d(e_s(|D_h|f))=\partial_d|D_h|e_s(f)$, we get the formula for Uf.

It is obvious that U commutes with ∇_h. As regards commutation with ∂_d, we notice that, by definition of Uf,

$$\partial_dUf+|D_h|Uf=|D_h|f,$$

which yields (2).

It is clear that $|D_h|Pf$ satisfies (2.4), hence $|D_h|Pf=Uf$ and $\nabla_hPf=SUf$. Similarly, the equation for Pf yields

$$\partial_dPf=f-|D_h|Pf=f-Uf,$$

hence integrating with respect to the vertical variable gives the expression for P.

The next item is a direct consequence of the definition of P (just apply $\partial_d - |D_h|$ to the equation). Finally, we have by definition of $P\partial_d f$,

$$(\partial_d + |D_h|)P\partial_d f = \partial_d f = (\partial_d + |D_h|)f - |D_h|f.$$

Hence, using (2.8),

$$(\partial_d + |D_h|)(P\partial_d f - f + H\gamma f) = -|D_h|f \quad \text{and} \quad (P\partial_d f - f + H\gamma f)|_{x_d=0} = 0.$$

Therefore, using the definition of U, one may write

$$P\partial_d f - f + H\gamma f = -Uf.$$

Because $\partial_d P = I - U$, it is easy to complete the proof of the last item. □

Remark 2.1. *For functions vanishing at $x_d = 0$, operator U coincides with the expression*

$$rR_h \cdot S(R_h \cdot S + R_d)e_0$$

that has been introduced in [20] and plays the role of the left-inverse of $(\mathrm{Id} + |D_h|^{-1}\partial_d)$ for functions of $\mathbb{R}^d_+$ vanishing at $\partial\,\mathbb{R}^d_+$. Our definition of operator U is slightly more general as it allows us to consider functions that do not vanish at $x_d = 0$.

The main result of this subsection reads:

Theorem 2.1. *Given smooth and decaying data u_0, f and g with $g = \operatorname{div} Q$, the unique solution $(u, \nabla\Pi)$ of (2.1) is given by*

$$\begin{aligned}
u^h =& re^{\mu t\Delta}e_a(V_h u_0) - \mu SPg - SUe^{\mu t\Delta}e_a(V_d u_0)\\
&- SU\int_0^t e^{\mu(t-\tau)\Delta}e_a\big(\widetilde{N}f + Gk\big)\,d\tau + r\int_0^t e^{\mu(t-\tau)\Delta}e_a\big(\widetilde{M}f + SGk\big)\,d\tau,\\
u^d =& \mu Pg + Ue^{\mu t\Delta}e_a(V_d u_0) + U\int_0^t e^{\mu(t-\tau)\Delta}e_a\big(\widetilde{N}f + Gk\big)\,d\tau,
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
\nabla_h\Pi =& r\big(R_h \cdot S(SR_d + R_h)\big)\big(S \cdot e_a(f^h) + e_s(f^d)\big) + S(U - I)Gk\\
&+ \mu(\nabla_h - S\partial_d)g + S\partial_t\big(S \cdot UQ^h + (I - U)Q^d - H\gamma Q^d\big) + SU\widetilde{N}f\\
&+ \big(r(|D_h| - \partial_d)\nabla_h + SU\Delta\big)\Big(e^{\mu t\Delta}e_a(V_d u_0) + \int_0^t e^{\mu(t-\tau)\Delta}e_a\big(\widetilde{N}f + Gk\big)\,d\tau\Big),\\
\partial_d\Pi =& f^d + \mu(\partial_d - |D_h|)g - \partial_t\big(S \cdot UQ^h + (I - U)Q^d - H\gamma Q^d\big) - U\big(\widetilde{N}f + Gk\big)\\
&- \big(r(|D_h| - \partial_d)|D_h| + U\Delta\big)\Big(e^{\mu t\Delta}e_a(V_d u_0) + \int_0^t e^{\mu(t-\tau)\Delta}e_a\big(\widetilde{N}f + Gk\big)\,d\tau\Big),
\end{aligned} \tag{2.11}$$

where Gk, $\widetilde{M}f$ and $\widetilde{N}f$ are given by

$$\begin{aligned}
&Gk = -r\big(R_d - R_h \cdot S\big)\big(R_h \cdot e_a(k^h) + R_d e_s(k^d)\big) \quad \text{with} \quad k = \partial_t Q - \mu\nabla g,\\
&\widetilde{N}f = r\big\{\big[1 + R_d^2 - R_d R_h \cdot S\big]e_s(f^d) + R_d^2 S \cdot e_a(f^h) + R_d R_h \cdot e_a(f^h)\big\},\\
&\widetilde{M}f = rS\big[R_d - R_h \cdot S\big]\big(R_h \cdot e_a(f^h) + R_d e_s(f^d)\big) + V_h f.
\end{aligned} \tag{2.12}$$

Proof. We shall essentially follow the arguments in [7, 20]. Note that setting

$$u_{new}(t,x)=\mu u_{old}(\mu^{-1}t,x),\quad \Pi_{new}(t,x)=\Pi_{old}(\mu^{-1}t,x),\quad f_{new}(t,x)=f_{old}(\mu^{-1}t,x) \tag{2.13}$$

reduces the study to the case $\mu=1$, an assumption that we are going to make in the rest of the proof.

The basic idea is to reduce the study to that of the heat equation for the auxiliary functions V_hu and V_du. As a first step, let us compute Π in terms of $\operatorname{div} f, g$ and of its trace at $\partial\mathbb{R}^d_+$. Taking space divergence to (2.1) yields

$$\begin{cases} -\Delta\Pi=\partial_t g-\Delta g-\operatorname{div} f \quad \text{in} \quad \mathbb{R}^d_+,\\ \gamma\Pi=b \quad \text{on} \quad \partial\mathbb{R}^d_+ \end{cases}\qquad \Pi\to 0\quad\text{as}\quad |x|\to\infty,$$

the solution of which is given by

$$\Pi=r(-\Delta)^{-1}e_a(\operatorname{div}(k-f))+Hb\quad\text{with}\ \ k=\partial_tQ-\nabla g,$$

which along with (2.8) implies that

$$\begin{aligned}(\partial_d+|D_h|)\Pi &= -r(\partial_d+|D_h|)(-\Delta)^{-1}e_a(\operatorname{div}(f-k))\\ &= Me_a(\operatorname{div}(f-k))\end{aligned} \tag{2.14}$$

with

$$Mh\stackrel{\text{def}}{=}-r(\partial_d+|D_h|)(-\Delta)^{-1}h. \tag{2.15}$$

Let $z\stackrel{\text{def}}{=}(\partial_d+|D_h|)u^d$ and $Nf\stackrel{\text{def}}{=}(\partial_d+|D_h|)f^d-\partial_dMe_a(\operatorname{div} f)$. We infer from (2.1) and (2.14) that

$$\begin{cases}\partial_tz-\Delta z=Nf+\partial_dMe_a(\operatorname{div} k)\quad\text{in}\quad \mathbb{R}_+\times\mathbb{R}^d_+,\\ (z-g)|_{x_d=0}=0,\\ z|_{t=0}=(\partial_d+|D_h|)u^d_0.\end{cases}$$

Note from the definition of M in (2.15) that

$$\partial_dMe_a(h)=e_a(h)-r|D_h|(\partial_d+|D_h|)(-\Delta)^{-1}\big[e_a(h)\big]. \tag{2.16}$$

Therefore, because

$$e_a(\operatorname{div} k)=\operatorname{div}_h e_a(k^h)+\partial_de_s(k^d) \tag{2.17}$$

and

$$Me_a(\operatorname{div} k)=Gk, \tag{2.18}$$

we get

$$\begin{aligned}Nf&=|D_h|f^d-r\operatorname{div}_h e_a(f^h)+r|D_h|(\partial_d+|D_h|)(-\Delta)^{-1}\big[\operatorname{div}_h e_a(f^h)+\partial_de_s(f^d)\big]\\ &=|D_h|\widetilde{N}f.\end{aligned}$$

Now, using (2.16) with $h=\partial_tg-\Delta g=\operatorname{div} k$ and (2.17), we thus obtain

$$\begin{cases}\partial_t(z-g)-\Delta(z-g)=|D_h|\big(\widetilde{N}f+Gk\big) & \text{in}\quad \mathbb{R}_+\times\mathbb{R}^d_+,\\ (z-g)|_{x_d=0}=0 & \text{in}\quad \mathbb{R}_+\times\partial\mathbb{R}^d_+,\\ (z-g)|_{t=0}=|D_h|V_du_0 & \text{in}\quad \mathbb{R}^d_+.\end{cases}$$

Taking advantage of the solution formula for the heat equation in $\mathbb{R}^d_+$ with homogeneous Dirichlet boundary conditions, we deduce that

$$(z-g)(t) = r|D_h|\Big(e^{t\Delta}e_a(V_d u_0) + \int_0^t e^{(t-\tau)\Delta}e_a\big(\widetilde{N}f + Gk\big)\,d\tau\Big).$$

As $z-g = (|D_h|+\partial_d)(u^d - Pg)$ and $u^d - Pg$ vanishes at $x_d = 0$, keeping in mind the definition of U, we get the second equality of (2.10).

To derive the solution formula for u^h, we look at the equation satisfied by $V_h u$. Thanks to (2.1), (2.12) and (2.14), we get, observing

$$V_h f - S(|D_h|+\partial_d)\Pi = V_h f - SMe_a(\operatorname{div}(f-k)) = \widetilde{M}f + SGk,$$

that

$$\begin{cases} \partial_t V_h u - \Delta V_h u = \widetilde{M}f + SGk & \text{in } \ \mathbb{R}_+\times\mathbb{R}^d_+,\\ V_h u|_{x_d=0} = 0 & \text{in } \ \mathbb{R}_+\times\mathbb{R}^d_+,\\ V_h u|_{t=0} = V_h u_0 & \text{in } \ \mathbb{R}^d_+, \end{cases}$$

so that

$$V_h u = re^{t\Delta}e_a(V_h u_0) + r\int_0^t e^{(t-\tau)\Delta}e_a\big(\widetilde{M}f + SGk\big)\,d\tau. \tag{2.19}$$

Because $u^h = V_h u - Su^d$, combining the above identity and the second formula of (2.10), we obtain the solution formula for u^h.

Let us finally derive (2.11). By virtue of (2.1) and (2.10), we may write

$$\begin{aligned}\partial_d\Pi &= f^d - (\partial_t - \Delta)u^d\\ &= f^d - (\partial_t-\Delta)Pg - (\partial_t-\Delta)U\Big(e^{t\Delta}e_a(V_d u_0) + \int_0^t e^{(t-\tau)\Delta}e_a\big(\widetilde{N}f + Gk\big)\,d\tau\Big).\end{aligned}$$

However, because

$$\Delta Uh = r(\partial_d - |D_h|)|D_h|h,$$

one may write

$$(\partial_t - \Delta)Uh = U(\partial_t-\Delta)h + r(|D_h| - \partial_d)|D_h|h + U\Delta h.$$

Note also that, by virtue of Lemma 2.1,

$$\partial_t Pg = \partial_t(P\operatorname{div}_h Q^h + P\partial_d Q^d) = \partial_t(S\cdot UQ^h + (I-U)Q^d - H\gamma Q^d)$$

and that

$$\Delta Pg = (\partial_d - |D_h|)g,$$

which ensures that

$$\begin{aligned}\partial_d\Pi = f^d &- \partial_t(S\cdot UQ^h + (I-U)Q^d - H\gamma Q^d) + (\partial_d - |D_h|)g - U\big(\widetilde{N}f + Gk\big)\\ &- \big(r(|D_h|-\partial_d)|D_h| + U\Delta\big)\Big(e^{t\Delta}e_a(V_d u_0) + \int_0^t e^{(t-\tau)\Delta}e_a(\widetilde{N}f + Gk))\,d\tau\Big)\cdot\end{aligned} \tag{2.20}$$

On the other hand, by virtue of (2.14), one has

$$|D_h|\Pi = -\partial_d\Pi + Me_a(\operatorname{div}(f-k)),$$

and

$$\begin{aligned}Me_a(\operatorname{div} f) &= -(\partial_d + |D_h|)(-\Delta)^{-1}\big(\operatorname{div}_h e_a(f^h) + \partial_d e_s(f^d)\big)\\ &= e_s(f^d) - (\partial_d + |D_h|)(-\Delta)^{-1}\big(\operatorname{div}_h e_a(f^h) + |D_h| e_s(f^d)\big),\end{aligned}$$

which together with (2.20) gives rise to (2.11). This completes the proof of Theorem 2.1. □

2.2. **A priori estimates.** Let us first briefly recall the definition of homogeneous Besov spaces in $\mathbb{R}^d$. Let $\chi : \mathbb{R}^d \to [0,1]$ be a smooth nonincreasing radial function supported in $B(0,1)$ and such that $\chi \equiv 1$ on $B(0,1/2)$, and let

$$\varphi(\xi) \stackrel{\text{def}}{=} \chi(\xi/2) - \chi(\xi).$$

The homogeneous Littlewood-Paley decomposition of any tempered distribution u on $\mathbb{R}^d$ is defined by

$$\dot\Delta_k u \stackrel{\text{def}}{=} \varphi(2^{-k}D)u = \mathcal{F}^{-1}\big(\varphi(2^{-k}\cdot)\mathcal{F}u\big), \qquad k \in \mathbb{Z}$$

where $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^d$.

Definition 2.1. *For any $s \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$, the homogeneous Besov space $\dot B^s_{p,r}(\mathbb{R}^d)$ stands for the set of tempered distributions f so that*

$$\|f\|_{\dot B^s_{p,r}(\mathbb{R}^d)} \stackrel{\text{def}}{=} \big\|2^{sk}\|\dot\Delta_k f\|_{L^p(\mathbb{R}^d)}\big\|_{\ell^r(\mathbb{Z})} < \infty$$

and for all smooth compactly supported function θ over $\mathbb{R}^d$, we have

$$\lim_{\lambda\to+\infty} \theta(\lambda D) f = 0 \quad \text{in} \quad L^\infty(\mathbb{R}^d). \tag{2.21}$$

Remark 2.2. *Condition* (2.21) *means that functions in homogeneous Besov spaces are required to have some decay at infinity (see* [5] *for more details). In particular, we have*

$$f = \sum_{k\in\mathbb{Z}} \dot\Delta_k f \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \tag{2.22}$$

whenever f satisfies (2.21)*. In this paper, we will only consider exponents $s < d/p$ so that for f with finite $\dot B^s_{p,r}(\mathbb{R}^d)$ semi-norm,* (2.21) *and* (2.22) *are equivalent.*

The homogeneous Besov spaces on the half-space are defined by restriction:

Definition 2.2. *For any $s \in \mathbb{R}$, and $(p,r) \in [1,+\infty]^2$ we denote by $\dot B^s_{p,r}(\mathbb{R}^d_+)$ the set of distributions u on $\mathbb{R}^d_+$ admitting some extension $\widetilde u \in \dot B^s_{p,r}(\mathbb{R}^d)$ on $\mathbb{R}^d$. Then we set*

$$\|u\|_{\dot B^s_{p,r}(\mathbb{R}^d_+)} \stackrel{\text{def}}{=} \inf \|\widetilde u\|_{\dot B^s_{p,r}(\mathbb{R}^d)}$$

where the infimum is taken on all the extensions of u in $\dot B^s_{p,r}(\mathbb{R}^d)$.

We also need to introduce some spaces of divergence free vector fields vanishing at the boundary $\partial\mathbb{R}^d_+$. We proceed as follows:

Definition 2.3. *For $1 < p < \infty$ and $0 < s < 2$, we denote by $\dot{\mathcal{B}}^s_{p,r}(\mathbb{R}^d_+)$ the completion of the set of divergence free vector fields with coefficients in $W^{2,p}(\mathbb{R}^d_+) \cap W^{1,p}_0(\mathbb{R}^d_+)$ (where $W^{1,p}_0(\mathbb{R}^d_+)$ stands for the subspace of $W^{1,p}(\mathbb{R}^d_+)$ functions with null trace at $\partial\mathbb{R}^d_+$) for the norm $\|\cdot\|_{\dot B^s_{p,r}(\mathbb{R}^d_+)}$.*

It is classical (see e.g. [10]) that spaces $(\dot B^s_{p,r}(\mathbb{R}^d_+))^d$ (with the divergence free condition) and $\dot{\mathcal{B}}^s_{p,r}(\mathbb{R}^d_+)$ coincide whenever $1 < p, r < \infty$ and $0 < s < 1/p$.

The following result extends Lemma 3.2 of [20] to the context of Besov spaces.

Lemma 2.2. *Operators R_h, R_d and S map $L^p(\mathbb{R}^d)$ in itself for any $1<p<\infty$, and, with no restriction on s,p,r, we have*

$$\|\mathcal{R}z\|_{\dot B^s_{p,r}(\mathbb{R}^d)}\leq C\|z\|_{\dot B^s_{p,r}(\mathbb{R}^d)}\quad\text{for}\ \ \mathcal{R}\in\{R_h,R_d,S\}.$$

Operators V_h, V_d, U, G, $\widetilde M$ and $\widetilde N$ map L^p_+ in itself if $1<p<\infty$, and $\dot B^s_{p,r}(\mathbb{R}^d_+)$ in itself if $1<p,r<\infty$ and $0<s<2$.

Proof. The result in the Lebesgue spaces just follows from the fact that all those operators are combinations of Riesz transforms so that Calderon-Zygmund theorem applies. The result in homogeneous Besov spaces stems from the fact that the Riesz operators are Fourier multipliers of degree 0, hence map any homogeneous Besov space in itself. □

We are now ready to establish a first family of a priori estimates for System (2.1).

Proposition 2.1. *Let $1<p,r<\infty$ and the data u_0, f, g fulfill the following hypotheses:*

- $u_0\in\dot{\mathcal{B}}^{2-\frac2r}_{p,r}(\mathbb{R}^d_+)$;
- $f\in L^r(\mathbb{R}_+;L^p_+)$;
- $\nabla g\in L^r(\mathbb{R}_+;L^p_+)$, $g=\operatorname{div}Q$ *with* $\partial_tQ\in L^r(\mathbb{R}_+;L^p_+)$ *and the following compatibility conditions are fulfilled*[1]

$$\gamma u_0^d=0,\quad g|_{t=0}=0,\quad\text{and}\quad\partial_t(\gamma Q^d)=0.\tag{2.23}$$

Then System (2.1) *has a unique solution* $(u,\nabla\Pi)$ *with*

$$u\in\mathcal{C}_b(\mathbb{R}_+;\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d_+))\quad\text{and}\quad\partial_tu,\nabla^2u,\nabla\Pi\in L^r(\mathbb{R}_+;L^p_+),$$

with also $\nabla u\in L^q(\mathbb{R}_+;L^m_+)$ *whenever* $q\in[r,\infty)$ *and* $m\in[p,\infty]$ *satisfy*

$$0\leq1-\frac2r+\frac2q\leq\frac dp\quad\text{and}\quad\frac dm=\frac dp-1+\frac2r-\frac2q\cdot$$

Furthermore, the following inequality is fulfilled for all $t>0$:

$$\mu^{1-\frac1r}\|u\|_{L^\infty_t(\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d_+))}+\mu^{1-\frac1r+\frac1q}\|\nabla u\|_{L^q_t(L^m_+)}+\|(\partial_tu,\mu\nabla^2u,\nabla\Pi)\|_{L^r_t(L^p_+)}\\ \leq C\Big(\mu^{1-\frac1r}\|u_0\|_{\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d_+)}+\|(f,\mu\nabla g,\partial_tQ)\|_{L^r_t(L^p_+)}\Big)\cdot\tag{2.24}$$

Finally, if $g\equiv0$ then we have $u\in\mathcal{C}(\mathbb{R}_+;\dot{\mathcal{B}}^{2-\frac2r}_{p,r}(\mathbb{R}^d_+))$.

Remark 2.3. *As regards the bounds for ∇u, we shall often use the following two cases:*

- $p=\frac{dr}{3r-2}$, $q=2r$ *and* $m=\frac{dr}{2r-1}$,
- $p>d$, $q=\frac{2p}{d+(\frac2r-1)p}$ *and* $m=+\infty$.

Proof. We concentrate on the proof of the estimates in $L^r(0,T;L^p_+)$ for ∇^2u and $\nabla\Pi$. Indeed, once the pressure has been determined, u may be seen as a solution of the heat equation with source term in $L^r(\mathbb{R}_+;L^p_+)$, the solution of which is given by

$$u(t)=r\bigg(e^{\mu t\Delta}e_a(u_0)+\int_0^te^{\mu(t-\tau)\Delta}e_a(f-\nabla\Pi)\,d\tau\bigg)\cdot\tag{2.25}$$

---

[1]The condition on u_0^d is ensured by the fact that $u_0\in\dot{\mathcal{B}}^{2-\frac2r}_{p,r}(\mathbb{R}^d_+)$. As for Q, it stems from the fact that $\operatorname{div}Q$ is quite smooth.

Therefore combining Corollary A.1, Lemma A.1 and Lemma 2.2 allows to bound $u(t)$ in $\dot B^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)$ in terms of the data and of the norm of $\nabla\Pi$ in $L^r(0,t;L^p_+)$. In addition, because any function in $L^r(\mathbb{R}_+;L^p_+)$ may be approximated by smooth functions compactly supported in $\mathbb{R}_+\times\mathbb{R}^d_+$, and because u_0 may be approximated by functions in $W^{2,p}(\mathbb{R}^d_+)\cap W^{1,p}_0(\mathbb{R}^d_+)$, the above formula guarantees that u is continuous in time, with values in $\dot B^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)$ (or in $\dot{\mathcal{B}}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)$ if $\operatorname{div} u\equiv 0$).

In what follows, we assume that $\mu=1$, which is not restrictive owing to the change of variables (2.13). Of course, when proving estimates, one may consider separately the three cases where only one element of the triplet (u_0,f,g) is nonzero, a consequence of the fact that (2.1) is linear.

Step 1. Case $u_0\equiv 0$ *and* $f\equiv 0$. Then the formula for u^d given by Theorem 2.1 reduces to

$$u^d=Pg+U\int_0^t e^{(t-\tau)\Delta}e_a(Gk)\,d\tau,$$

and using the algebraic relations provided by Lemma 2.1 thus yields

$$\begin{aligned}
&\nabla^2_h u^d = SU\nabla_h g+U\int_0^t e^{(t-\tau)\Delta}\nabla^2_h e_a(Gk)\,d\tau,\\
&\nabla_h\partial_d u^d=(I-U)\nabla_h g+(I-U)\int_0^t e^{(t-\tau)\Delta}\nabla_h|D_h|e_a(Gk)\,d\tau,\\
&\partial_d^2 u^d=(\partial_d+(U-I)|D_h|)g+\int_0^t e^{(t-\tau)\Delta}\partial_d|D_h|e_a(Gk)\,d\tau+(I-U)\int_0^t e^{(t-\tau)\Delta}\Delta_h e_a(Gk)\,d\tau.
\end{aligned}$$

The important fact is that all the terms corresponding initially to Pg may be written $A\nabla g$ where A stands for some 0-th order operator for which Lemma 2.2 applies. A similar observation holds for the terms with the time integral so that applying Lemma A.1 eventually yields

$$\|\nabla^2 u^d\|_{L^r_T(L^p_+)}\le C\big(\|\nabla g\|_{L^r_T(L^p_+)}+\|Gk\|_{L^r_T(L^p_+)}\big). \tag{2.26}$$

At this point, we use Lemma 2.2 to bound the right-hand side by $\|(\nabla g,\partial_t Q)\|_{L^r_T(L^p_+)}$, and we thus get

$$\|\nabla^2 u^d\|_{L^r_T(L^p_+)}\le C\|(\nabla g,\partial_t Q)\|_{L^r_T(L^p_+)}. \tag{2.27}$$

It is clear that $\nabla^2 u^h$ also satisfies (2.27): indeed (2.10) gives

$$\nabla^2 u^h=r\int_0^t e^{(t-\tau)\Delta}\nabla^2 Se_a(Gk)\,d\tau-S\nabla^2 u^d. \tag{2.28}$$

Let us now concentrate on the pressure. Keeping in mind (2.20) and (2.23), we may write

$$\begin{aligned}
\partial_d\Pi=(\partial_d-|D_h|)g-S\cdot U\partial_t Q^h-(I-U)\partial_t Q^d-UGk\\
-\big(r(|D_h|-\partial_d)|D_h|+U\Delta\big)\int_0^t e^{(t-\tau)\Delta}e_a(Gk)\,d\tau.
\end{aligned}$$

Therefore, combining Lemmas 2.2 and A.1 gives

$$\|\partial_d\Pi\|_{L^r_T(L^p_+)}\le C\|(\partial_t Q,\nabla g)\|_{L^r_T(L^p_+)}. \tag{2.29}$$

Finally, because

$$\nabla_h\Pi=-SGk-S\partial_d\Pi, \tag{2.30}$$

it is clear that $\nabla_h\Pi$ also satisfies (2.29).

Step 2. Case $f\equiv 0$ and $g\equiv 0$. With no loss of generality, one may assume that $u_0\in W^{1,p}_0(\mathbb{R}^d_+)\cap W^{2,p}(\mathbb{R}^d_+)$. From Theorem 2.1 and Lemma 2.1, we readily get

$$\begin{aligned}\nabla_h^2 u^d &= Ue^{t\Delta}\nabla_h^2 e_a(V_d(u_0)),\\ \nabla_h\partial_d u^d &= (I-U)e^{t\Delta}\nabla_h|D_h|e_a(V_d(u_0)),\\ \partial_d^2 u^d &= re^{t\Delta}\partial_d|D_h|e_a(V_d(u_0))+r(I-U)e^{t\Delta}e_a(\Delta_h V_d(u_0)).\end{aligned}$$

Therefore, combining Corollary A.1 and Lemma 2.2,

$$\begin{aligned}\|\nabla^2 u^d\|_{L^r_T(L^p_+)} \le\ & C\Big(\|e_a(\nabla_h^2 V_d(u_0))\|_{\dot B^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)}\\ &+\|e_a(\nabla_h|D_h|V_d(u_0))\|_{\dot B^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)}+\|e_s(\partial_d|D_h|V_d(u_0))\|_{\dot B^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)}\Big)\\ \le\ & C\|u_0\|_{\dot B^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)}.\end{aligned}$$

Note that in order to bound the last term, we used the fact that because $V_d u_0$ is null at the boundary, we have

$$|D_h|\partial_d e_a(V_d(u_0)) = e_s(\partial_d|D_h|V_d(u_0))\in \dot B^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d).$$

Owing to (2.28), $\nabla^2 u^h$ satisfies the same inequality. Finally,

$$\partial_d\Pi = -\big(r(|D_h|-\partial_d)|D_h|+U\Delta\big)e^{t\Delta}e_a(V_d u_0)$$

hence, according to Lemma 2.2 and Corollary A.1,

$$\begin{aligned}\|\partial_d\Pi\|_{L^r_T(L^p_+)} &\le C\|\nabla^2 e^{t\Delta}e_a(V_d u_0)\|_{L^r_T(L^p)}\\ &\le C\|u_0\|_{\dot B^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d_+)}.\end{aligned}$$

Of course, (2.30) implies that $\nabla_h\Pi$ has the same bound.

Step 3. Case $u_0\equiv 0$ and $g\equiv 0$. As in the previous steps, owing to

$$\begin{aligned}\nabla^2 u^h &= r\int_0^t e^{(t-\tau)\Delta}\nabla^2 e_a(\widetilde{M}f)\,d\tau - S\nabla^2 u^d,\\ \nabla_h\Pi &= SGf - S\partial_d\Pi,\end{aligned}$$

it suffices to bound $\nabla^2 u^d$ and $\partial_d\Pi$. The formulae for the second spatial derivatives of u^d now read

$$\begin{aligned}\nabla_h^2 u^d &= U\int_0^t e^{(t-\tau)\Delta}\nabla_h^2 e_a(\widetilde{N}f)\,d\tau,\\ \nabla_h\partial_d u^d &= (I-U)\int_0^t e^{(t-\tau)\Delta}\nabla_h|D_h|e_a(\widetilde{N}f)\,d\tau,\\ \partial_d^2 u^d &= r\int_0^t e^{(t-\tau)\Delta}\partial_d|D_h|e_a(\widetilde{N}f)\,d\tau+(I-U)\int_0^t e^{(t-\tau)\Delta}\Delta_h e_a(\widetilde{N}f)\,d\tau.\end{aligned}$$

Therefore applying Lemmas 2.2 and A.1,

$$\begin{aligned}\|\nabla^2 u^d\|_{L^r_T(L^p_+)} &\le C\|e_a(\widetilde{N}f)\|_{L^r_T(L^p_+)}\\ &\le C\|f\|_{L^r_T(L^p_+)}.\end{aligned}$$

For the pressure, we have

$$\partial_d\Pi - f^d = -U\widetilde{N}f - \big(r(|D_h|-\partial_d)|D_h| + U\Delta\big)\int_0^t e^{(t-\tau)\Delta}e_a(\widetilde{N}f)\,d\tau,$$

therefore, using once again Lemmas 2.2 and A.1, we obtain

$$\begin{aligned}\|\partial_d\Pi - f^d\|_{L^r_T(L^p_+)} &\le C\big(\|U\widetilde{N}f\|_{L^r_T(L^p_+)} + \|e_a(\widetilde{N}f)\|_{L^r_T(L^p)}\big)\\ &\le C\|f\|_{L^r_T(L^p_+)}.\end{aligned}$$

Step 4. Estimates for ∇u. The starting point is the following classical Gagliardo-Nirenberg inequality on $\mathbb{R}^d$:

$$\|\nabla z\|_{L^m(\mathbb{R}^d)} \le C\|z\|^{1-\theta}_{\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d)}\|\nabla^2 z\|^\theta_{L^p(\mathbb{R}^d)} \tag{2.31}$$

with $\theta\in(0,1],\ m\ge p,\ 0\le 1-\frac2r+\frac{2\theta}{r}\le\frac dp$ and $\frac dm = \frac dp - 1 + \frac2r - \frac{2\theta}{r}\cdot$

If $\theta\in(0,1)$ then this inequality may be easily proved by decomposing u into low and high frequencies by means of an homogeneous Littlewood-Paley decomposition (see e.g. [5] Chap. 2 for the proof of similar inequalities). The case $\theta=1$ corresponds to the classical Sobolev inequality. We omit the proof as it is standard.

We claim that this inequality extends to the half-space setting if considering functions $u\in W^{1,p}_0(\mathbb{R}^d_+)\cap W^{2,p}(\mathbb{R}^d_+)$. In effect, we observe that for such functions we have the following identities:

$$\nabla_h(e_au) = e_a(\nabla_h u)\quad\text{and}\quad \partial_d(e_au) = e_s(\partial_d u).$$

As $\nabla_h(e_au)$ also has null trace at $\partial\,\mathbb{R}^d_+$, one can thus write

$$\nabla_h^2(e_au) = e_a(\nabla_h^2 u)\quad\text{and}\quad \partial_d\nabla_h(e_au) = e_s(\partial_d\nabla_h u).$$

Even though $\partial_d(e_au)$ need not be zero at $\partial\,\mathbb{R}^d_+$, it is symmetric with respect to the vertical variable, whence

$$\partial_d^2(e_au) = e_a(\partial_d^2 u).$$

Applying (2.31) to $z=e_au$, and the above relations for the second order derivatives, we thus gather

$$\begin{aligned}\|\nabla u\|_{L^m_+} &\le \|\nabla(e_au)\|_{L^m(\mathbb{R}^d)},\\ &\le C\|e_au\|^{1-\theta}_{\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d)}\|\nabla^2(e_au)\|^\theta_{L^p(\mathbb{R}^d)},\\ &\le C\|u\|^{1-\theta}_{\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d_+)}\|\nabla^2 u\|^\theta_{L^p_+}.\end{aligned}$$

Hence, taking the L^q norm with respect to time of both sides (with $q=r/\theta$), we discover that

$$\|\nabla u\|_{L^q_T(L^m_+)} \le C\|u\|^{1-\theta}_{L^\infty_T(\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d_+))}\|\nabla^2 u\|^\theta_{L^r_T(L^p_+)}.$$

Bounding the right-hand side according to the previous steps leads to the desired estimate of $\|\nabla u\|_{L^q_T(L^m_+)}$ in (2.24). □

In order to solve System (1.4) for more general data, it will be suitable to extend the above estimates to the case where the index of regularity of u_0 *is not* related to r. This motivates the following statement:

Proposition 2.2. *Let $1<p,r<\infty$ and $0<s<2$. Let (u_0,f,g) satisfy the compatibility conditions of Proposition 2.1, and be such that*

$$u_0\in\dot{\mathcal{B}}^s_{p,r}(\mathbb{R}^d_+),\quad t^\alpha(f,\nabla g,\partial_t Q)\in L^r(\mathbb{R}_+;L^p_+)\ \text{ with }\ \alpha\overset{\text{def}}{=}1-\frac{s}{2}-\frac{1}{r}.$$

Then System (2.1) has a unique solution $(u,\nabla\Pi)$ with $t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\in L^r(\mathbb{R}_+;L^p_+)$ and for all $T>0$,

$$\|t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\|_{L^r_T(L^p_+)}\leq C\Big(\|u_0\|_{\dot{B}^s_{p,r}(\mathbb{R}^d_+)}+\|t^\alpha(f,\nabla g,\partial_tQ)\|_{L^r_T(L^p_+)}\Big).\tag{2.32}$$

Furthermore, the following properties hold true:

1. *For any couple (p_2,r_2) so that*
$$s<1+\frac{d}{p}-\frac{d}{p_2}<2+\frac{2}{r_2}-\frac{2}{r},\quad p_2\geq p,\quad r_2\geq r$$
we have $t^\beta\nabla u\in L^{r_2}(\mathbb{R}_+;L^{p_2}_+)$ with $\beta=\frac12-\frac{d}{2p_2}+\frac{d}{2p}-\frac{s}{2}-\frac{1}{r_2}$, and
$$\|t^\beta\nabla u\|_{L^{r_2}_T(L^{p_2}_+)}\leq C\Big(\|u_0\|_{\dot{B}^{s-\frac{d}{p}+\frac{d}{p_2}}_{p_2,r_2}(\mathbb{R}^d_+)}+\|t^\alpha(f,\nabla g,\partial_tQ)\|_{L^r_T(L^p_+)}\Big).$$
2. *For any couple (p_3,r_3) so that*
$$s<\frac{d}{p}-\frac{d}{p_3}<2+\frac{2}{r_3}-\frac{2}{r},\quad p_3\geq p,\quad r_3\geq r,$$
we have $t^\gamma u\in L^{r_3}(\mathbb{R}_+;L^{p_3}_+)$ with $\gamma=-\frac{d}{2p_3}+\frac{d}{2p}-\frac{s}{2}-\frac{1}{r_3}$, and
$$\|t^\gamma u\|_{L^{r_3}_T(L^{p_3}_+)}\leq C\Big(\|u_0\|_{\dot{B}^{s-\frac{d}{p}+\frac{d}{p_3}}_{p_3,r_3}(\mathbb{R}^d_+)}+\|t^\alpha(f,\nabla g,\partial_tQ)\|_{L^r_T(L^p_+)}\Big).$$

Proof. Let us first assume that only g is nonzero. Then we start with the formula

$$t^\alpha\nabla^2_hu^d=SUt^\alpha\nabla_hg+Ut^\alpha\int_0^te^{(t-\tau)\Delta}\nabla^2_he_a(Gk)\,d\tau.$$

From Lemmas 2.2 and A.2, we immediately infer that, if $\alpha r'<1$,

$$\|t^\alpha\nabla^2_hu^d\|_{L^r_t(L^p_+)}\leq C\big(\|t^\alpha\nabla_hg\|_{L^r_t(L^p_+)}+\|t^\alpha e_a(Gk)\|_{L^r_t(L^p(\mathbb{R}^d))}\big),$$

whence

$$\|t^\alpha\nabla^2_hu^d\|_{L^r_t(L^p_+)}\leq C\|t^\alpha(\nabla g,\partial_tQ)\|_{L^r_t(L^p_+)}.\tag{2.33}$$

Similarly, as

$$t^\alpha\nabla_h\partial_du^d=(I-U)t^\alpha\nabla_hg+(I-U)t^\alpha\int_0^te^{(t-\tau)\Delta}\nabla_h|D_h|e_a(Gk)\,d\tau,$$

and

$$t^\alpha\partial_d^2u^d=(\partial_d+(U-I)|D_h|)t^\alpha g+t^\alpha\int_0^te^{(t-\tau)\Delta}\partial_d|D_h|e_a(Gk)\,d\tau$$
$$+(I-U)t^\alpha\int_0^te^{(t-\tau)\Delta}\Delta_he_a(Gk)\,d\tau,$$

it is clear that $\|t^\alpha\nabla^2u^d\|_{L^r_t(L^p_+)}$ is bounded by the right-hand side of (2.33). Because

$$t^\alpha\nabla^2u^h=r\,t^\alpha\int_0^te^{(t-\tau)\Delta}\nabla^2Se_a(Gk)\,d\tau-t^\alpha S\nabla^2u^d,$$

the same inequality holds true for $t^\alpha\nabla^2 u^h$.

In order to bound the pressure, we use the fact that

$$t^\alpha\partial_d\Pi = t^\alpha(\partial_d-|D_h|)g - t^\alpha S\cdot U\partial_t Q^h-(I-U)t^\alpha\partial_t Q^d - t^\alpha UGk$$
$$-\big(r(|D_h|-\partial_d)|D_h|+U\Delta\big)t^\alpha\int_0^t e^{(t-\tau)\Delta}e_a(Gk)\,d\tau.$$

Note that the terms in the right-hand side may be handled by means of Lemmas 2.2 and A.2, exactly as we did for $t^\alpha\nabla_h^2 u^d$. Therefore, we have

$$\|t^\alpha\partial_d\Pi\|_{L^r_t(L^p_+)}\leq C\|t^\alpha(\nabla g,\partial_t Q)\|_{L^r_t(L^p_+)}.$$

Owing to (2.30), it is clear that $t^\alpha\nabla_h\Pi$ satisfies the same inequality.

Let us now consider the case $f\equiv 0,$ $g\equiv 0$ and $u_0\in W^{1,p}_0(\mathbb{R}^d_+)\cap W^{2,p}(\mathbb{R}^d_+)$ (with no loss of generality). As usual, because one may go from u^d to u^h through

$$u^h = re^{t\Delta}e_a(V_h u_0)-Su^d,$$

we concentrate on $t^\beta\nabla^2 u^d$. We start with the formula

$$t^\alpha\nabla_h^2 u^d(t) = Ut^\alpha e^{t\Delta}e_a(\nabla_h^2 V_d u_0) = Ut^\alpha e^{t\Delta}\nabla_h^2 V_d e_a(u_0),$$

which, in view of Lemmas 2.2 and A.5 ensures that

$$\|t^\alpha\nabla_h^2 u^d\|_{L^r_t(L^p_+)}\leq C\|V_d e_a(u_0)\|_{\dot B^s_{p,r}(\mathbb{R}^d)}\leq C\|u_0\|_{\dot B^s_{p,r}(\mathbb{R}^d_+)}\quad\text{with}\quad \alpha = 1-\frac{s}{2}-\frac{1}{r}\cdot$$

Similarly, we have

$$t^\alpha\nabla_h\partial_d u^d = (I-U)t^\alpha e^{t\Delta}\nabla_h|D_h|e_a(V_d u_0),$$

hence

$$\|t^\alpha\nabla_h\partial_d u^d\|_{L^r_t(L^p_+)}\leq C\|u_0\|_{\dot B^s_{p,r}(\mathbb{R}^d_+)}.\tag{2.34}$$

Finally, $t^\alpha\partial_d^2 u^d$ satisfies

$$\begin{aligned} t^\alpha\partial_d^2 u^d &= rt^\alpha e^{t\Delta}e_s(\partial_d|D_h|V_d u_0)+(I-U)t^\alpha e^{t\Delta}e_a(\Delta_h V_d u_0)\\ &= rt^\alpha e^{t\Delta}\partial_d|D_h|e_a(V_d u_0)+(I-U)t^\alpha e^{t\Delta}e_a(\Delta_h V_d u_0)\end{aligned}$$

because $e_s(\partial_d V_d u_0)=\partial_d e_a(V_d u_0)$ owing to the fact that $V_d u_0$ vanishes on $\partial\,\mathbb{R}^d_+$. Hence $t^\alpha\partial_d^2 u^d$ satisfies (2.34), too, and we conclude that

$$\|t^\alpha\nabla^2 u^d\|_{L^r_t(L^p_+)}\leq C\|u_0\|_{\dot B^s_{p,r}(\mathbb{R}^d_+)}.\tag{2.35}$$

Bounding $\nabla\Pi$ is strictly analogous.

In order to prove the estimate for $t^\alpha\nabla^2 u$ in the case $g\equiv 0$ and $u_0\equiv 0,$ we use the fact that

$$t^\alpha\nabla_h^2 u^d = Ut^\alpha\int_0^t e^{(t-\tau)\Delta}\nabla_h^2 e_a(\widetilde N f)\,d\tau,\quad t^\alpha\nabla_h\partial_d u^d = (I-U)\,t^\alpha\int_0^t e^{(t-\tau)\Delta}\nabla_h|D_h|e_a(\widetilde N f)\,d\tau$$
$$\text{and}\quad t^\alpha\partial_d^2 u^d = r\int_0^t e^{(t-\tau)\Delta}\partial_d|D_h|e_a(\widetilde N f)\,d\tau+(I-U)\,t^\alpha\int_0^t e^{(t-\tau)\Delta}\Delta_h e_a(\widetilde N f)\,d\tau.$$

Then combining Lemmas 2.2 and A.2 readily gives

$$\|t^\alpha\nabla^2 u^d\|_{L^r_t(L^p_+)}\leq C\|t^\alpha f\|_{L^r_t(L^p_+)}.$$

Bounding $t^\alpha\nabla^2 u_h$ and $t^\alpha\nabla\Pi$ works the same.

Let us finally go to the proof of estimates for $t^\beta\nabla u$ and $t^\gamma u$. By virtue of (2.25) and of the definition of $\mathcal{B}$ (see the appendix), we have

$$t^\beta\nabla u(t)=r\Big(t^\beta\nabla e^{t\Delta}e_a(u_0)+t^\beta\mathcal{B}e_a(f-\nabla\Pi)\Big)\cdot$$

Therefore, applying Lemmas A.3 and A.5 yields

$$\|t^\beta\nabla u\|_{L^{r_2}_t(L^{p_2}_+)}\leq C\Big(\|e_a(u_0)\|_{\dot B^{s_2}_{p_2,r_2}(\mathbb{R}^d)}+\|t^\alpha e_a(f-\nabla\Pi)\|_{L^r_t(L^p(\mathbb{R}^d))}\Big),$$

whenever $p_2\geq p,\ r_2\geq r,\ s_2=\frac{d}{p_2}-\frac{d}{p}+s,$

$$\beta=\alpha+\frac d2\Big(\frac1p-\frac1{p_2}\Big)-\frac12+\frac1r-\frac1{r_2}=\frac12-\frac{s_2}{2}-\frac1{r_2}$$

and

$$s<1+\frac dp-\frac{d}{p_2}<2+\frac{2}{r_2}-\frac2r\cdot$$

Combining with the fact that e_a is continuous on functions of $\dot B^{s_2}_{p_2,r_2}(\mathbb{R}^d_+)$ with null trace at the boundary, and with (2.32), we get

$$\|t^\beta\nabla u\|_{L^{r_2}_t(L^{p_2}_+)}\leq C\Big(\|u_0\|_{\dot B^{s_2}_{p_2,r_2}(\mathbb{R}^d_+)}+\|t^\alpha(f,\nabla g,\partial_tQ)\|_{L^r_t(L^p_+)}\Big)\cdot$$

Finally, in order to bound $t^\gamma u$, we use the formula

$$t^\gamma u(t)=r\Big(t^\gamma e^{t\Delta}e_a(u_0)+t^\gamma\mathcal{C}e_a(f-\nabla\Pi)\Big)\cdot$$

Applying Lemmas A.4 and A.5 yields

$$\|t^\gamma u\|_{L^{r_3}_t(L^{p_3}_+)}\leq C\Big(\|e_a(u_0)\|_{\dot B^{s_3}_{p_3,r_3}(\mathbb{R}^d)}+\|t^\alpha e_a(f-\nabla\Pi)\|_{L^p_t(L^r(\mathbb{R}^d))},$$

with $p_3,\geq p,\ r_3\geq r,\ s_3=\frac{d}{p_3}-\frac dp+s,$

$$\gamma=\alpha+\frac d2\Big(\frac1p-\frac1{p_3}\Big)-1+\frac1r-\frac1{r_3}=-\frac{s_3}{2}-\frac1{r_3}$$

and

$$s<\frac dp-\frac{d}{p_3}<2+\frac{2}{r_3}-\frac2r\cdot$$

Combining with the fact that e_a is continuous on functions of $\dot B^{s_3}_{p_3,r_3}(\mathbb{R}^d_+)$ with null trace, and with (2.32), we get

$$\|t^\gamma u\|_{L^{r_3}_t(L^{p_3}_+)}\leq C\Big(\|u_0\|_{\dot B^{s_3}_{p_3,r_3}(\mathbb{R}^d_+)}+\|t^\gamma(f,\nabla g,\partial_tQ)\|_{L^r_t(L^p_+)}\Big).$$

This completes the proof of the proposition. □

3. Existence of smooth solutions

As a first step for proving Theorem 1.1, we here establish the global existence of strong solutions for (1.4) *in the case of a globally Lipschitz bounded density.* As for the velocity, we assume that it has slightly sub-critical regularity. Here is our statement (recall that the space $X^{p,r}$ has been defined just above (1.6)):

Theorem 3.1. *Let* $a_0\in W^{1,\infty}(\mathbb{R}^d_+)$ *and* $u_0\in\dot{\mathcal{B}}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)\cap\dot{\mathcal{B}}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)$ *with* $p=\frac{dr}{3r-2}$, $r\in(1,\infty)$ *and* $d<\widetilde{p}\leq\frac{dr}{r-1}\cdot$ *There exist two positive constants* $c_0=c_0(r,d)$ *and* $c_1=c_1(r,d)$ *so that if*

$$\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)}\leq c_0\mu\quad\text{and}\quad\|a_0\|_{L^\infty_+}\leq c_1 \tag{3.1}$$

then (1.4) *has a unique global solution* $(a,u,\nabla\Pi)$ *with* $a\in L^\infty_{loc}(\mathbb{R}_+;W^{1,\infty}(\mathbb{R}^d_+))$,

$$(u,\nabla\Pi)\in X^{p,r}\cap X^{\widetilde{p},r}\quad\text{and}\quad\nabla u\in L^{2r}(\mathbb{R}_+;L_+^{\frac{dr}{2r-1}})\cap L^q(\mathbb{R}_+;L^\infty_+)\quad\text{with}\quad q=\frac{2\widetilde{p}}{d+(\frac{2}{r}-1)\widetilde{p}}\cdot$$

In addition, there exist $C_1=C_1(r,d)$ *and* $C_2=C_2(r,d)$ *so that*

$$\|(u,\nabla\Pi)\|_{X^{p,r}}+\mu^{1-\frac{1}{2r}}\|\nabla u\|_{L^{2r}(\mathbb{R}_+;L_+^{\frac{dr}{2r-1}})}\leq C_1\mu^{1-\frac{1}{r}}\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)}, \tag{3.2}$$

$$\|(u,\nabla\Pi)\|_{X^{\widetilde{p},r}}+\mu^{1-\frac{1}{2r}}\big(\|\nabla u\|_{L^{2r}_t(L^\alpha_+)}+\mu^{\frac{1}{q}-\frac{1}{2r}}\|\nabla u\|_{L^q_t(L^\infty_+)}\big)\leq C_2\mu^{1-\frac{1}{r}}\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}, \tag{3.3}$$

where α *satisfies* $\frac{1}{\widetilde{p}}=\frac{1}{\alpha}+\frac{r-1}{dr}\cdot$

Proof. The general strategy is the same as in [10]: we set $(a^0,u^0,\nabla\Pi^0)=(0,0,0)$ and solve inductively the following *linear* system:

$$\left\{\begin{array}{l}\partial_t a^{n+1}+u^n\cdot\nabla a^{n+1}=0,\\ \partial_t u^{n+1}-\mu\Delta u^{n+1}+\nabla\Pi^{n+1}=F^n\\ \operatorname{div}\,u^{n+1}=0,\\ u^{n+1}|_{\partial\mathbb{R}^d_+}=0,\\ (a^{n+1},u^{n+1})|_{t=0}=(a_0,u_0),\end{array}\right. \tag{3.4}$$

with $F^n\stackrel{\text{def}}{=}a^{n+1}(\mu\Delta u^n-\nabla\Pi^n)-u^n\cdot\nabla u^n$.

Both the global existence and uniqueness of a solution to (3.4) in the spaces given in Theorem 3.1 follows from basic results for the transport equation (given that ∇u^n is indeed in $L^1_{loc}(\mathbb{R}^+;L^\infty_+)$) and from the solution formula for the evolutionary Stokes system in the half-space. So, as a first step, we focus on the proof of uniform estimates in $L^\infty_{loc}(\mathbb{R}_+;W^{1,\infty}(\mathbb{R}^d_+))$ for the density, and in $X^{p,r}\cap X^{\widetilde{p},r}$ for $(u,\nabla\Pi)$.

Step 1. Uniform estimates. It is obvious that

$$\|a^n(t)\|_{L^\infty_+}=\|a_0\|_{L^\infty_+}\quad\text{for all}\quad n\in\mathbb{N}\quad\text{and}\quad t\in\mathbb{R}^+, \tag{3.5}$$

and that

$$\|\nabla a^{n+1}(t)\|_{L^\infty_+}\leq e^{\int_0^t\|\nabla u^n\|_{L^\infty_+}\,d\tau}\|\nabla a_0\|_{L^\infty_+}. \tag{3.6}$$

We shall prove inductively that for all $n\in\mathbb{N}$,

$$\|(u^n,\nabla\Pi^n)\|_{X^{p,r}_t}+\mu^{1-\frac{1}{2r}}\|\nabla u^n\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}\leq C\mu^{1-\frac{1}{r}}\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)} \tag{3.7}$$

$$\begin{aligned}\|(u^n,\nabla\Pi^n)\|_{X^{\widetilde{p},r}_t}+\mu^{1-\frac{1}{2r}}\big(\|\nabla u^n\|_{L^{2r}_t(L^\alpha_+)}+\mu^{\frac{1}{q}-\frac{1}{2r}}\|\nabla u^n\|_{L^q_t(L^\infty_+)}\big)&\\ \leq C\mu^{1-\frac{1}{r}}\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}&\end{aligned} \tag{3.8}$$

for some large enough constant C.

Applying Proposition 2.1 we see that the left-hand sides of (3.7) and (3.8) are bounded (up to some harmless factor) by $\mu^{1-\frac1r}\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)}+\|F^n\|_{L^r_t(L^p_+)}$ and $\mu^{1-\frac1r}\|u_0\|_{\dot B^{-1+\frac dp}_{\widetilde p,r}(\mathbb{R}^d_+)}+\|F^n\|_{L^r_t(L^{\widetilde p}_+)}$, respectively. Now, by virtue of Hölder inequality (here we use that $p=\frac{dr}{3r-2}$ and the assumption $\widetilde p\leq\frac{dr}{r-1}$ comes into play), we have

$$\|F^n\|_{L^r_t(L^p_+)}\leq\|a^{n+1}\|_{L^\infty_t(L^\infty_+)}\big(\mu\|\Delta u^n\|_{L^r_t(L^p_+)}+\|\nabla\Pi^n\|_{L^r_t(L^p_+)}\big)+\|u^n\|_{L^{2r}_t(L_+^{\frac{dr}{r-1}})}\|\nabla u^n\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}.$$

and

$$\|F^n\|_{L^r_t(L^{\widetilde p}_+)}\leq\|a^{n+1}\|_{L^\infty_t(L^\infty_+)}\big(\mu\|\Delta u^n\|_{L^r_t(L^{\widetilde p}_+)}+\|\nabla\Pi^n\|_{L^r_t(L^{\widetilde p}_+)}\big)+\|u^n\|_{L^{2r}_t(L_+^{\frac{dr}{r-1}})}\|\nabla u^n\|_{L^{2r}_t(L^\alpha_+)}.$$

Therefore, using the Sobolev embedding

$$W_0^{1,\frac{dr}{2r-1}}(\mathbb{R}^d_+)\hookrightarrow L^{\frac{dr}{r-1}}(\mathbb{R}^d_+),\tag{3.9}$$

we arrive at

$$\begin{aligned}\|(u^{n+1},\nabla\Pi^{n+1})\|_{X^{p,r}_t}+\mu^{1-\frac1{2r}}\|\nabla u^{n+1}\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}&\leq C\Big(\mu^{1-\frac1r}\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)}\\&+\|a^{n+1}\|_{L^\infty_t(L^\infty_+)}\big(\mu\|\Delta u^n\|_{L^r_t(L^p_+)}+\|\nabla\Pi^n\|_{L^r_t(L^p_+)}\big)+\|\nabla u^n\|^2_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}\Big),\end{aligned}$$

$$\begin{aligned}&\|(u^{n+1},\nabla\Pi^{n+1})\|_{X^{\widetilde p,r}_t}+\mu^{1-\frac1{2r}}\big(\|\nabla u^{n+1}\|_{L^{2r}_t(L^\alpha_+)}+\mu^{\frac1q-\frac1{2r}}\|\nabla u^{n+1}\|_{L^q_t(L^\infty_+)}\big)\\&\quad\leq C\Big(\mu^{1-\frac1r}\|u_0\|_{\dot B^{-1+\frac dp}_{\widetilde p,r}(\mathbb{R}^d_+)}+\|a^{n+1}\|_{L^\infty_t(L^\infty_+)}\big(\mu\|\Delta u^n\|_{L^r_t(L^{\widetilde p}_+)}+\|\nabla\Pi^n\|_{L^r_t(L^{\widetilde p}_+)}\big)\\&\qquad+\|\nabla u^n\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}\|\nabla u^n\|_{L^{2r}_t(L^\alpha_+)}\Big).\end{aligned}$$

Now, the induction hypotheses (3.5), (3.7) and (3.8) thus imply

$$\begin{aligned}&\|(u^{n+1},\nabla\Pi^{n+1})\|_{X^{p,r}_t}+\mu^{1-\frac1{2r}}\|\nabla u^{n+1}\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}\\&\qquad\leq C(1+\|a_0\|_{L^\infty_+}+\mu^{-1}\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)})\mu^{1-\frac1r}\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)}\\&\|(u^{n+1},\nabla\Pi^{n+1})\|_{X^{\widetilde p,r}_t}+\mu^{1-\frac1{2r}}\big(\|\nabla u^{n+1}\|_{L^{2r}_t(L^\alpha_+)}+\mu^{\frac1q-\frac1{2r}}\|\nabla u^{n+1}\|_{L^q_t(L^\infty_+)}\big)\\&\qquad\leq C\big(1+\|a_0\|_{L^\infty_+}+\mu^{-1}\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)}\big)\mu^{1-\frac1r}\|u_0\|_{\dot B^{-1+\frac dp}_{\widetilde p,r}(\mathbb{R}^d_+)}.\end{aligned}$$

Therefore, if c_0 and c_1 in (3.1) are small enough then we get (3.7) and (3.8) at rank $n+1$.

Step 2. Convergence of the sequence. Let $\check p$ be some real number in $(\frac d2,\frac{dr}{2r-1})$ such that in addition[2] $p\leq\check p\leq\widetilde p$. We are going to show that $(a^n)_{n\in\mathbb{N}}$ and $(u^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{C}_b([0,T]\times\mathbb{R}^d_+)$ and $X^{\check p,r}_T$, respectively, for all $T>0$. Of course, by interpolation, we easily find out that the bounds for $(a^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ in $X^{\check p,r}_T$ are the same as in $X^{p,r}_T\cap X^{\widetilde p,r}_T$.

As regards $(a^n)_{n\in\mathbb{N}}$, we use the fact that

$$\partial_t\delta a^n+u^n\cdot\nabla\delta a^n=-\delta u^{n-1}\cdot\nabla a^n\quad\text{with}\quad\delta a^n\stackrel{\text{def}}{=}a^{n+1}-a^n\quad\text{and}\quad\delta u^n\stackrel{\text{def}}{=}u^{n+1}-u^n.$$

[2]It would be natural to take $\check p=p$ but we do not know how to handle the case $r\geq2$ with this value of $\check p$.

Hence, using standard estimates for the transport equation, we get for all positive T,

$$\|\delta a^n(T)\|_{L^\infty_+} \leq \int_0^T \|\delta u^{n-1}\|_{L^\infty_+}\|\nabla a^n\|_{L^\infty_+}\,dt.$$

Now, arguing exactly as in the proof of (2.31), we get the following Gagliardo-Nirenberg inequality[3] for functions z vanishing at $\partial\,\mathbb{R}^d_+$:

$$\|z\|_{L^\infty_+} \leq C\|z\|^{\theta}_{\dot B^{2-\frac{2}{r}}_{\check p,r}(\mathbb{R}^d_+)}\|\nabla^2 z\|^{1-\theta}_{L^{\check p}_+} \quad \text{with } \theta \stackrel{\text{def}}{=} \Bigl(\check p-\frac{d}{2}\Bigr)\frac{r}{\check p}\cdot \tag{3.10}$$

Hence

$$\|\delta a^n(T)\|_{L^\infty_+} \leq \int_0^T \|\nabla a^n\|_{L^\infty_+}\|\delta u^{n-1}\|^{\theta}_{\dot B^{2-\frac{2}{r}}_{\check p,r}(\mathbb{R}^d_+)}\|\nabla^2\delta u^{n-1}\|^{1-\theta}_{L^{\check p}_+}\,dt$$

Taking advantage of Young's inequality we thus get for all positive ε,

$$\|\delta a^n\|_{L^\infty(0,T\times\mathbb{R}^d_+)} \leq \varepsilon\|\nabla^2\delta u^{n-1}\|_{L^r_T(L^{\check p}_+)} + C_\varepsilon\int_0^T \|\nabla a^{n-1}\|^{\frac{2\check p}{(2\check p-d)r}}_{L^\infty_+}\|\delta u^{n-1}\|_{\dot B^{2-\frac{2}{r}}_{\check p,r}(\mathbb{R}^d_+)}\,dt. \tag{3.11}$$

Next, we use the fact that

$$\begin{cases} \partial_t\delta u^n - \mu\Delta\delta u^n + \nabla\delta\Pi^n = \delta F^n\\ \operatorname{div}\delta u^n = 0\\ u^n|_{t=0} = 0 \quad \text{and} \quad u^n|_{\partial\,\mathbb{R}^d_+} = 0\end{cases}$$

with

$$\delta F^n \stackrel{\text{def}}{=} (\mu\Delta u^n - \nabla\Pi^n)\delta a^n + a^n(\mu\Delta\delta u^{n-1} - \nabla\delta\Pi^{n-1}) - u^n\cdot\nabla\delta u^{n-1} - \delta u^{n-1}\cdot\nabla u^{n-1}.$$

Applying Proposition 2.1, we see that for some constant $C_0 = C_0(p,d)$,

$$\|(\delta u^n,\nabla\delta\Pi^n)\|_{X^{\check p,r}_T} + \mu^{1-\frac{1}{2r}}\|\nabla\delta u^n\|_{L^{2r}_T(L^m_+)} \leq C_0\|\delta F^n\|_{L^r_T(L^{\check p}_+)} \quad \text{with } \frac{d}{m} = \frac{d}{\check p} - 1 + \frac{1}{r}\cdot \tag{3.12}$$

Therefore, in order to prove the convergence of the sequence, it is only a matter of getting suitable estimates in $L^r(0,T;L^{\check p}_+)$ for the terms in δF^n. We easily get

$$\begin{aligned}\|\delta F^n\|_{L^r_T(L^{\check p}_+)} \leq \Bigl(&\|\delta a^n\|_{L^\infty(0,T\times\mathbb{R}^d_+)}\|\mu\Delta u^n - \nabla\Pi^n\|_{L^r_T(L^{\check p}_+)}\\ &+\|a^n\|_{L^\infty(0,T\times\mathbb{R}^d_+)}\|\mu\Delta\delta u^{n-1} - \nabla\delta\Pi^{n-1}\|_{L^r_T(L^{\check p}_+)}\\ &+\|u^n\|_{L^{2r}_T(L^{\frac{dr}{r-1}}_+)}\|\nabla\delta u^{n-1}\|_{L^{2r}_T(L^m_+)} + \|\delta u^{n-1}\|_{L^{2r}_T(L^{m^*})}\|\nabla u^{n-1}\|_{L^{2r}_T(L^{\frac{dr}{2r-1}}_+)}\Bigr),\end{aligned}$$

with[4] m^* such that $\frac{d}{m^*} = \frac{d}{\check p} - 2 + \frac{1}{r}\cdot$

At this point, one may combine (3.9), the following Sobolev embedding

$$W^{1,m}_0(\mathbb{R}^d_+) \hookrightarrow L^{m^*}_+(\mathbb{R}^d_+)$$

and the conservation of the L^∞ norm of a^n. We end up with

$$\begin{aligned}\|\delta F^n\|_{L^r_T(L^{\check p}_+)} \leq \Bigl(&\|\delta a^n\|_{L^\infty(0,T\times\mathbb{R}^d_+)}\|(u^n,\nabla\Pi^n)\|_{X^{\check p,r}_T} + \|a_0\|_{L^\infty_+}\|(\delta u^{n-1},\nabla\delta\Pi^{n-1})\|_{X^{\check p,r}_T}\\ &+(\|\nabla u^{n-1}\|_{L^{2r}_T(L^{\frac{dr}{2r-1}}_+)} + \|\nabla u^n\|_{L^{2r}_T(L^{\frac{dr}{2r-1}}_+)})\|\nabla\delta u^{n-1}\|_{L^{2r}_T(L^m_+)}\Bigr)\cdot\end{aligned}$$

[3] It suffices to apply the corresponding inequality in $\mathbb{R}^d$ to function $e_a(z)$.

[4] Here $\check p < \frac{dr}{2r-1}$ comes into play.

Keeping in mind the estimates that have been established in the previous step and assuming that the smallness condition over a_0 and u_0 in (3.1) is satisfied with small enough constants c_0 and c_1, we eventually get

$$C_0\|\delta F^n\|_{L^r_T(L^{\check p}_+)}\leq \frac12\big(\|(\delta u^{n-1},\nabla\delta\Pi^{n-1})\|_{X^{\check p,r}_T}+\mu^{1-\frac1{2r}}\|\nabla\delta u^{n-1}\|_{L^{2r}_T(L^m_+)}\big)$$
$$+C\mu^{1-\frac1r}\|u_0\|_{\dot B^{2-\frac2r}_{p,r}(\mathbb{R}^d_+)\cap\dot B^{2-\frac2r}_{\tilde p,r}(\mathbb{R}^d_+)}\|\delta a^n\|_{L^\infty(0,T\times\mathbb{R}^d_+)}.$$

Now, plugging (3.11) with suitably small ε in the above inequality, and resuming to (3.12) yields

$$\|(\delta u^n,\nabla\delta\Pi^n)\|_{X^{\check p,r}_T}+\mu^{1-\frac1{2r}}\|\nabla\delta u^n\|_{L^{2r}_T(L^m_+)}\leq\frac34\big(\|(\delta u^{n-1},\nabla\delta\Pi^{n-1})\|_{X^{\check p,r}_T}$$
$$+\mu^{1-\frac1{2r}}\|\nabla\delta u^{n-1}\|_{L^{2r}_T(L^m_+)}\big)+C\int_0^T A(t)\|(\delta u^{n-1},\nabla\delta\Pi^{n-1})\|_{X^{\check p,r}_t}\,dt$$

where A is a continuous function of t that depends only on the norm of the data. Summing up over $n\geq1$ and remembering that $(u^0,\nabla\Pi^0)\equiv0$, we thus get

$$\sum_{n\geq1}\Big(\|(\delta u^n,\nabla\delta\Pi^n)\|_{X^{\check p,r}_T}+\mu^{1-\frac1{2r}}\|\nabla\delta u^n\|_{L^{2r}_T(L^m_+)}\Big)\leq\|(u^1,\nabla\Pi^1)\|_{X^{\check p,r}_T}+\mu^{1-\frac1{2r}}\|\nabla u^1\|_{L^{2r}_T(L^m_+)}$$
$$+4C\int_0^T A(t)\|(u^1,\nabla\Pi^1)\|_{X^{\check p,r}_t}\,dt+4C\int_0^T A(t)\sum_{n\geq1}\|(\delta u^n,\nabla\delta\Pi^n)\|_{X^{\check p,r}_t}\,dt.$$

Hence, using Gronwall lemma implies that

$$\sum_{n\geq1}\Big(\|(\delta u^n,\nabla\delta\Pi^n)\|_{X^{\check p,r}_T}+\mu^{1-\frac1{2r}}\|\nabla\delta u^n\|_{L^{2r}_T(L^m_+)}\Big)\leq\Bigg(\|(u^1,\nabla\Pi^1)\|_{X^{\check p,r}_T}$$
$$+\mu^{1-\frac1{2r}}\|\nabla u^1\|_{L^{2r}_T(L^m_+)}+4C\int_0^T A(t)\|(u^1,\nabla\Pi^1)\|_{X^{\check p,r}_t}\,dt\Bigg)e^{4C\int_0^T A(t)\,dt}.$$

This obviously entails that $(u^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X^{\check p,r}_T$ for all $T>0$. Then resuming to (3.11) implies that $(a^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}_b([0,T]\times\mathbb{R}^d_+)$ for all $T>0$.

Step 3. End of the proof of the theorem. Granted with the convergence result of the previous step, and the uniform bounds of the first step, it is not difficult to pass to the limit in (3.4): we conclude that the triplet $(a,u,\nabla\Pi)$ with $a=\lim_{n\to+\infty}a^n$, $u=\lim_{n\to+\infty}u^n$, and $\nabla\Pi=\lim_{n\to+\infty}\nabla\Pi^n$, satisfies (1.4) in the sense of Definition 1.1. In addition, as $(u^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in the space $X^{p,r}\cap X^{\widetilde p,r}$ which possesses the Fatou property, one may conclude that $(u,\nabla\Pi)\in X^{p,r}\cap X^{\widetilde p,r}$ and[5] that (3.2), (3.3) are fulfilled. Similarly, the uniform bounds for a^n allow to conclude that $a\in L^\infty_{loc}(\mathbb{R}_+;W^{1,\infty}(\mathbb{R}^d_+))$. This completes the proof of Theorem 3.1 □

4. Proving the existence part of the main theorem

This section is dedicated to the proof of the existence part of Theorem 1.1. It is mostly based on *a priori* estimates for smooth solutions –the same as in the previous section, and on compactness arguments.

[5]Rigorously speaking we do not get the time continuity for u, but it may be recovered afterward from Proposition 2.1 by observing that u satisfies an evolutionary Stokes equation with source term in $L^r(\mathbb{R}_+;L^p_+\cap L^{\widetilde p}_+)$ and initial data in $\dot{\mathcal{B}}^{2-\frac2r}_{p,r}(\mathbb{R}^d_+)\cap\dot{\mathcal{B}}^{2-\frac2r}_{\widetilde p,r}(\mathbb{R}^d_+)$.

Step 1. Constructing a sequence of smooth solutions. This is only a matter of smoothing out the data (a_0, u_0) so as to apply Theorem 3.1. We proceed as follows:

- Let $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi(x) = 1$ for $|x| \leq 1$. We extend a_0 to $\widetilde{a}_0$ on $\mathbb{R}^d$ by symmetry then use convolution of $\chi(x/n)\widetilde{a}_0$ with a sequence of nonnegative radially symmetric mollifiers, then restrict to the half-space. We get a sequence $(a_0^n)_{n\in\mathbb{N}}$ in $W^{1,\infty}(\mathbb{R}^d_+)$ with the same lower and upper bounds as a_0, and satisfying $a_0^n \to a_0$ a. e. on $\mathbb{R}^d_+$.
- As $u_0 \in \dot{\mathcal{B}}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)$, one may take the zero extension $\widetilde{u}_0 = e_0(u_0)$ of u_0 over the whole space and then set
$$u_0^n = r\biggl(\sum_{|j|\leq n} \dot{\Delta}_j \widetilde{u}_0\biggr)\cdot$$
It is obvious that each term u_0^n is smooth and $\gamma u_0 \equiv 0$. Furthermore, $(u_0^n)_{n\in\mathbb{N}}$ converges to u_0 in $\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+) \cap \dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)$. Of course, one may find $\check{p}$ in $(d, \frac{dr}{r-1})$ so that each u_0^n belongs to $\dot{\mathcal{B}}^{-1+\frac{d}{p}}_{\check{p},r}(\mathbb{R}^d_+)$.

Step 2. Uniform estimates. Let us solve system (1.4) with regularized initial data (a_0^n, u_0^n) according to Theorem 3.1. We get a global solution $(a^n, u^n, \nabla\Pi^n)$ in $L^\infty_{loc}(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}^d_+)) \times (X^{p,r} \cap X^{\check{p},r})$ satisfying
$$\|a^n\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d_+)} = \|a_0^n\|_{L^\infty(\mathbb{R}^d_+)} \leq \|a_0\|_{L^\infty(\mathbb{R}^d_+)},$$
and also
$$\|(u^n, \nabla\Pi^n)\|_{X^{p,r}} + \mu^{1-\frac{1}{2r}}\|\nabla u^n\|_{L^{2r}(L_+^{\frac{dr}{2r-1}})} \leq C\mu^{1-\frac{1}{r}}\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,r}(\mathbb{R}^d_+)}. \tag{4.1}$$
In addition, by following the computations leading to (3.8), it is not difficult to see that the assumption that $\widetilde{p} > d$ is not needed if it is only a matter of getting a control on the norm of the solution in $X^{\widetilde{p},r}$. Therefore we also have
$$\|(u^n, \nabla\Pi^n)\|_{X^{\widetilde{p},r}} \leq C\mu^{1-\frac{1}{r}}\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{\widetilde{p},r}(\mathbb{R}^d_+)}. \tag{4.2}$$
Of course, if $\widetilde{p} > d$, then we also have a bound for ∇u^n in $L^q(\mathbb{R}_+; L^\infty_+)$.

Step 3. The proof of convergence. Owing to the low regularity of a_0, it is not clear that one may still use stability estimates in order to prove the convergence of the sequence defined in the previous step. In effect, as pointed out in the previous section, there is a loss of one derivative in the stability estimates for the density. Therefore, we shall use compactness arguments instead, borrowed from [14]. For completeness, we outline the proof here.

According to the previous step, $(\partial_t u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^r(\mathbb{R}_+; L^p_+)$. Combining with (4.1), (4.2), Ascoli-Arzela Theorem and compact embeddings in Besov spaces, we conclude that there exists a subsequence, of $(a^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ (still denoted by $(a^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$) and some $(a, u, \nabla\Pi)$ with $a \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d_+)$,
$$\nabla u \in L^{2r}(\mathbb{R}_+; L_+^{\frac{dr}{2r-1}}) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla\Pi \in L^r(\mathbb{R}_+; L^p_+ \cap L^{\widetilde{p}}_+)$$
(and also $\nabla u \in L^q(\mathbb{R}_+; L^\infty_+)$ if $\widetilde{p} > d$), such that
$$\begin{aligned} &a^n \rightharpoonup a \quad \text{weak } * \text{ in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^d_+), \\ &\nabla^2 u^n \rightharpoonup \nabla^2 u \quad \text{and} \quad \nabla\Pi^n \rightharpoonup \nabla\Pi \quad \text{weakly in } L^r(\mathbb{R}_+; L^p_+), \end{aligned} \tag{4.3}$$

with in addition for all small enough $\eta > 0$,

$$
\begin{aligned}
& u^n \to u \quad \text{strongly} \quad \text{in } L^{2r}_{loc}(\mathbb{R}_+; L_{loc}^{\frac{dr}{r-1}-\eta}(\mathbb{R}^d_+)),\\
& \nabla u^n \to \nabla u \quad \text{strongly} \quad \text{in } L^{2r}_{loc}(\mathbb{R}_+; L_{loc}^{\frac{dr}{2r-1}-\eta}(\mathbb{R}^d_+)).
\end{aligned}
\tag{4.4}
$$

By construction, $(a^n, u^n, \nabla\Pi^n)$ satisfies

$$
\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}^d_+} a^n(\partial_t\phi + u^n\cdot\nabla\phi)\,dx\,dt + \int_{\mathbb{R}^d_+} \phi(0,x)a_0^n(x)\,dx = 0,\\
& \int_0^\infty \int_{\mathbb{R}^d_+} \phi \operatorname{div} u^n\,dx\,dt = 0 \qquad \text{and}\\
& \int_0^\infty \int_{\mathbb{R}^d_+} \Big\{u^n\cdot\partial_t\Phi + ((u^n\otimes u^n):\nabla\Phi\big) + (1+a^n)(\mu\nabla u^n - \nabla\Pi^n)\cdot\Phi\Big\}\,dx\,dt\\
& \qquad\qquad + \int_{\mathbb{R}^d_+} u_0^n\cdot\Phi(0,x)\,dx = 0,
\end{aligned}
\tag{4.5}
$$

for all test functions ϕ, Φ given by Definition 1.1.

Putting (4.3) and (4.4) together, it is easy to pass to the limit in all the terms of (4.5), except in $a^n(\mu\Delta u^n - \nabla\Pi^n)$. To handle that term, it suffices to show that $a^n \to a$ in $L^m_{loc}(\mathbb{R}_+\times\mathbb{R}^d_+)$ for some $m \geq r'$.

Now, it is easy to observe from the transport equation that

$$
\partial_t(a^n)^2 + \operatorname{div}(u^n(a^n)^2) = 0,
$$

from which, (4.3) and (4.4), we deduce that

$$
\partial_t\overline{a^2} + \operatorname{div}(u\overline{a^2}) = 0,
\tag{4.6}
$$

where we denote by $\overline{a^2}$ the weak $*$ limit of $((a^n)^2)_{n\in\mathbb{N}}$.

Thanks to (4.3), (4.4) and (4.5), there holds

$$
\partial_t a + \operatorname{div}(ua) = 0
$$

in the sense of distributions. Moreover, as $\nabla u \in L^{2r}(\mathbb{R}_+; L^{\frac{dr}{2r-1}}(\mathbb{R}^d_+))$ and $\operatorname{div} u = 0$, we infer by a mollifying argument as that in [17] that

$$
\partial_t a^2 + \operatorname{div}(ua^2) = 0.
\tag{4.7}
$$

Subtracting (4.7) from (4.6), we obtain

$$
\partial_t(\overline{a^2} - a^2) + \operatorname{div}(u(\overline{a^2} - a^2)) = 0,
\tag{4.8}
$$

from which and Theorem II.2 of [17] concerning the uniqueness of solutions to transport equations, we infer

$$
(\overline{a^2} - a^2)(t,x) = 0 \quad \text{a. e.} \quad x\in\mathbb{R}^d_+ \quad \text{and} \quad t\in\mathbb{R}^+.
$$

Together with the fact that $(a^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\mathbb{R}_+\times\mathbb{R}^d_+)$, this implies that

$$
a^n \;\to\; a \quad \text{strongly in} \quad L^m_{loc}(\mathbb{R}_+\times\mathbb{R}^d) \quad \text{for all} \quad m < \infty.
\tag{4.9}
$$

Granted with this new information, it is now easy to pass to the limit in (4.5). Therefore $(a, u, \nabla\Pi)$ satisfies (1.7) and (1.8). Moreover, thanks to (4.1) and (4.2), there hold (1.10), (1.11) and (1.12). Besides, as $(u, \nabla\Pi)$ satisfies (1.5) and the r.h.s. is in $L^r(\mathbb{R}_+; L^p_+\cap L^{\widetilde{p}}_+)$, the time continuity for u stems from Proposition 2.1. This completes the proof of the existence part of Theorem 1.1.

5. More general data

Until now, we assumed that p and r where interrelated through

$$-1+\frac{d}{p}=2-\frac{2}{r}\cdot \tag{5.1}$$

It is natural to investigate whether the inhomogeneous Navier-Stokes equations may still be solved with initial data (a_0,u_0) in $L^\infty_+\times\dot{\mathcal{B}}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)$ if (5.1) is *not* satisfied.

The case where $1<r<2p/(3p-d)$ or, equivalently $p<\frac{dr}{3r-2}$ is not so interesting because, by embedding one may find some $p^*\in(p,d)$ so that $u_0\in\dot{\mathcal{B}}^{-1+\frac{d}{p^*}}_{p^*,r}(\mathbb{R}^d)$ and (5.1) is fulfilled by (p^*,r).

The case where $r>2p/(3p-d)$ is more involved and cannot be solved by taking advantage of embeddings. In order to explain how this may be overcome anyway, let us first focus on the toy case where u satisfies the basic heat equation

$$\partial_t u-\Delta u=0\quad\text{in}\quad\mathbb{R}_+\times\mathbb{R}^d$$

with initial data $u^0\in\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d)$. Then by using embedding in $\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},\widetilde{r}}(\mathbb{R}^d)$ for any $\widetilde{p}\geq p$ and $\widetilde{r}\geq r$, we easily get (see Lemma A.5 in the appendix) that

(1) $t^\alpha\nabla^2 u\in L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))$ with $\alpha=\frac32-\frac{d}{2p}-\frac1r$ if $p>d/3$,

(2) $t^\beta\nabla u\in L^{r_2}(\mathbb{R}_+;L^{p_2}(\mathbb{R}^d))$ with $\beta=1-\frac{d}{2p_2}-\frac{1}{r_2}$ if $p_2\geq p,\ r_2\geq r$ and $p_2>d/2$,

(3) $t^\gamma u\in L^{r_3}(\mathbb{R}_+;L^{p_3}(\mathbb{R}^d))$ with $\gamma=\frac12-\frac{d}{2p_3}-\frac{1}{r_3}$ if $p_3\geq p,\ r_3\geq r$ and $p_3>d$.

As pointed out in Proposition 2.2, those properties are still true for the free solution to the Stokes system in the half-space. Keeping in mind that we want to apply those types of estimates to System (1.5), we see that we need to be able to handle also the Stokes system with some source term f satisfying $t^\alpha f\in L^r(\mathbb{R}_+;L^p_+)$. Still in the simpler case of the heat equation:

$$\partial_t v-\Delta v=f\quad\text{in}\quad\mathbb{R}_+\times\mathbb{R}^d\quad\text{with}\quad t^\alpha f\in L^r(\mathbb{R}_+;L^p_+),$$

it has been observed by the second author and collaborators in [14] (see also the Appendix) that if $\alpha r'<1$ then

(1) $t^\alpha\nabla^2 v\in L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))$,

(2) $t^\beta\nabla v\in L^{r_2}(\mathbb{R}_+;L^{p_2}(\mathbb{R}^d))$ with $\beta=\alpha+\frac d2\bigl(\frac1p-\frac1{p_2}\bigr)-\frac12+\frac1r-\frac1{r_2}$ if $p_2\geq p,\ r_2\geq r$ and $\frac dp-\frac{d}{p_2}<1+\frac2{r_2}-\frac2r$,

(3) $t^\gamma v\in L^{r_3}(\mathbb{R}_+;L^{p_3}(\mathbb{R}^d))$ with $\gamma=\alpha+\frac d2\bigl(\frac1p-\frac1{p_3}\bigr)-1+\frac1r-\frac1{r_3}$ if $p_3\geq p,\ r_3\geq r$ and $p_3>d$ and $\frac dp-\frac d{p_3}<2+\frac2{r_3}-\frac2r\cdot$

In the case we are interested in, owing to the presence of $u\cdot\nabla u$ in (1.5) and to Hölder inequality, the following supplementary relations have to be fulfilled:

$$\frac1p=\frac1{p_2}+\frac1{p_3},\quad\frac1r=\frac1{r_2}+\frac1{r_3}\quad\text{and}\quad\alpha=\beta+\gamma. \tag{5.2}$$

Under the first two conditions, if se set

$$\alpha\stackrel{\text{def}}{=}\frac32-\frac{d}{2p}-\frac1r,\quad\beta\stackrel{\text{def}}{=}1-\frac{d}{2p_2}-\frac1{r_2},\quad\gamma\stackrel{\text{def}}{=}\frac12-\frac{d}{2p_3}-\frac1{r_3}, \tag{5.3}$$

then the relationships above between α, β and γ are satisfied. Let us emphasize that if (p,r) with $d/3<p<d$ has been chosen so that $r>2p/(3p-d)$ (which is equivalent to $0<\alpha<1-1/r$), then one may take any (p_2,r_2) such that $p_2\geq p,\ p_2>d/2,\ r_2\geq r$ and

$$\frac{d}{p}-1+\frac{2}{r}-\frac{2}{r_2}<\frac{d}{p_2}<2-\frac{2}{r_2}\cdot \tag{5.4}$$

The assumption on (p,r) ensures that such a couple exists. This motivates the following statement:

Theorem 5.1. *Assume that $r\in(1,\infty)$ and $p\in(d/3,d)$ satisfy $r>2p/(3p-d)$. Then for any data $u_0\in\dot{\mathcal{B}}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)$ and $a_0\in L^\infty_+$ fulfilling the smallness condition* (1.9)*, System* (1.4) *has a global solution (a,u) with $\|a(t)\|_{L^\infty_+}=\|a_0\|_{L^\infty_+}$ for all $t\in\mathbb{R}_+$, and*

$$t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\in L^r(\mathbb{R}_+;L^p_+),\quad t^\beta\nabla u\in L^{r_2}(\mathbb{R}_+;L^{p_2}_+),\quad t^\gamma u\in L^{r_3}(\mathbb{R}_+;L^{p_3}_+) \tag{5.5}$$

with α,β,γ defined in (5.3)*, (p_2,r_2) satisfying* (5.4)*, and (p_3,r_3) defined in* (5.2)*.*

Furthermore, for any $1<\sigma<\frac{r}{1+\alpha r}$, the fluctuation $u-u_L$ belongs to $\mathcal{C}(\mathbb{R}_+;\dot{\mathcal{B}}^{2-\frac{2}{\sigma}}_{p,\sigma}(\mathbb{R}^d_+))$ where u_L stands for the "free solution" to the evolutionary Stokes system, namely

$$\partial_t u_L-\mu\Delta u_L+\nabla\Pi_L=0,\qquad \operatorname{div}u_L=0,\qquad u_L|_{t=0}=u_0. \tag{5.6}$$

Proof. For simplicity, we just treat the case $\mu=1$. The usual rescaling gives the result in the general case.

We smooth out the initial velocity u_0 into a sequence $(u_0^n)_{n\in\mathbb{N}}$ satisfying the assumptions of theorem 1.1: we take p_0 so that $-1+\frac{d}{p_0}=2-\frac{2}{r}$ and require $u_{0,n}$ to be in $\dot{\mathcal{B}}^{-1+\frac{d}{p_0}}_{p_0,r}(\mathbb{R}^d_+)\cap\dot{\mathcal{B}}^{-1+\frac{d}{p_0}}_{p,r}(\mathbb{R}^d_+)$, and to converge to u_0 in $\dot{B}^{-1+\frac{d}{p_0}}_{p,r}(\mathbb{R}^d_+)$. To construct $u_{0,n}$, one may first consider the zero extension $\tilde u_0=e_0(u_0)$ of u_0 to $\mathbb{R}^d$, then approximate it by compactly supported divergence free vector-fields by means of the stream function.

Step 1. Uniform estimates. The corresponding solution $(a^n,u^n,\nabla\Pi^n)$ satisfies in particular

$$\partial_t u^n,\nabla^2u^n,\nabla\Pi^n\in L^r(\mathbb{R}_+;L^{p_0}_+\cap L^p_+).$$

Hence Hölder inequality ensures that $t^\alpha(\partial_t u^n,\nabla^2u^n,\nabla\Pi^n)\in L^r(0,T;L^p_+)$ for all $T>0$. A similar argument ensures that $t^\beta\nabla u^n$ and $t^\gamma u^n$ are in $L^{r_2}(0,T;L^{p_2}_+)$ and $L^{r_3}(0,T;L^{p_3}_+)$, respectively, for all $T>0$. Now, because $(a^n,u^n,\nabla\Pi^n)$ satisfies

$$\partial_t u^n-\Delta u^n+\nabla\Pi^n=F^n\stackrel{\text{def}}{=}a^n(\Delta u^n-\nabla\Pi^n)-u^n\cdot\nabla u^n,$$

Proposition 2.2 implies that

$$\begin{aligned}Z_n(t)\stackrel{\text{def}}{=}&\|t^\gamma u^n\|_{L^{r_3}_t(L^{p_3}_+)}+\|t^\beta\nabla u^n\|_{L^{r_2}_t(L^{p_2}_+)}+\|t^\alpha(\partial_t u^n,\nabla^2u^n,\nabla\Pi^n)\|_{L^r_t(L^p_+)}\\ &\leq C\big(\|u_0^n\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)}+\|t^\alpha F^n\|_{L^r_t(L^p_+)}\big).\end{aligned}$$

Taking advantage of Hölder inequality, of the relationship between (r_2,p_2) and (r_3,p_3), and of the conservation of $\|a^n(t)\|_{L^\infty_+}$, we may write

$$\|t^\alpha F^n\|_{L^r_t(L^p_+)}\leq\|a_0\|_{L^\infty_+}\big(\|t^\alpha\Delta u^n\|_{L^r_t(L^p_+)}+\|t^\alpha\nabla\Pi^n\|_{L^r_t(L^p_+)}\big)+\|t^\gamma u^n\|_{L^{r_3}_t(L^{p_3}_+)}\|t^\beta\nabla u^n\|_{L^{r_2}_t(L^{p_2}_+)}.$$

Therefore taking c_0 small enough in (1.9), we get that

$$Z_n(t)\leq C\big(\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)}+Z_n^2(t)\big),$$

so that as long as $\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)} \leq \frac{1}{4C^2}$, we have

$$Z_n(t) \leq 2C\|u_0\|_{\dot B^{-1+\frac dp}_{p,r}(\mathbb{R}^d_+)}.$$

Step 2. Convergence. Hence sequence $(a^n,u^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in the space (5.5). In order to complete the proof of existence, we have to establish convergence, up to extraction, to a solution $(a,u,\nabla\Pi)$ of (1.4) in the desired functional space. For that, we first notice that

$$\Delta u^n = t^{-\alpha}\,(t^\alpha \Delta u^n),$$

Hölder inequality guarantees that $(\Delta u^n)_{n\in\mathbb{N}}$ is bounded in $L^\sigma_{loc}(\mathbb{R}_+;L^p_+)$ for any $\sigma<r/(1+\alpha r)$. Note that because $\alpha+1/r<1$, one may take $\sigma>1$. Similarly, we have $(\partial_t u^n)_{n\in\mathbb{N}}$ and $(\nabla\Pi^n)_{n\in\mathbb{N}}$ bounded in $L^\sigma_{loc}(\mathbb{R}_+;L^p_+)$. Therefore, setting $\widetilde u^n \stackrel{\rm def}{=} u^n-u^n_L$ where u^n_L stands for the solution to (5.6) with u^n_0 instead of u_0, we conclude that $(\widetilde u^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}_+;\dot B^{2-\frac2\sigma}_{p,\sigma}(\mathbb{R}^d_+))$.

Similarly, writing that $\nabla u^n = t^{-\beta}\,(t^\beta\nabla\widetilde u^n)$ and that $u^n=t^{-\gamma}\,(t^\gamma u^n)$, we get $(\nabla u^n)_{n\in\mathbb{N}}$ and $(u^n)_{n\in\mathbb{N}}$ bounded in $L^{\sigma_2}(\mathbb{R}_+;L^{p_2}_+)$ and $L^{\sigma_3}(\mathbb{R}_+;L^{p_3}_+)$, respectively. As we may choose σ_2 and σ_3 as close to (but smaller than) $r_2/(1+\beta r_2)$ and $r_3/(1+\gamma r_3)$ as we want, one may ensure that

$$\frac{1}{\sigma_2}+\frac{1}{\sigma_3}<1. \tag{5.7}$$

Now, combining with the boundedness of $(\Delta\widetilde u^n)_{n\in\mathbb{N}}$ in $L^\sigma_{loc}(\mathbb{R}_+;L^p_+)$ and using Arzela-Ascoli theorem, we conclude that, up to extraction, sequence $(a^n,u^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ converges weakly to some triple $(a,u,\nabla\Pi)$ with $a\in L^\infty(\mathbb{R}_+\times\mathbb{R}^d_+)$,

$$t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\in L^r(\mathbb{R}_+;L^p_+),\quad t^\beta\nabla u\in L^{r_2}(\mathbb{R}_+;L^{p_2}_+),\quad t^\gamma u\in L^{r_3}(\mathbb{R}_+;L^{p_3}_+).$$

More precisely, we have

$$\begin{aligned}&a^n \rightharpoonup a \quad \text{weak } * \text{ in } L^\infty(\mathbb{R}_+\times\mathbb{R}^d_+),\\ &\nabla^2u^n\rightharpoonup\nabla^2u \quad\text{and}\quad \nabla\Pi^n\rightharpoonup\nabla\Pi \quad \text{weakly in } L^\sigma(\mathbb{R}_+;L^p_+),\end{aligned}$$

with in addition for all small enough $\eta>0$,

$$\begin{aligned}&u^n\to u \quad\text{strongly in } L^{\sigma_3}_{loc}(\mathbb{R}_+;L^{p_3-\eta}_{loc}(\mathbb{R}^d_+)),\\ &\nabla u^n\to\nabla u \quad\text{strongly in } L^{\sigma_2}_{loc}(\mathbb{R}_+;L^{p_2-\eta}_{loc}(\mathbb{R}^d_+)).\end{aligned}$$

Because (5.7) is satisfied, passing to the limit in System (1.4) follows from the same arguments as in the proof of Theorem 1.1. That the constructed solution has all the properties listed in Theorem 5.1 is left to the reader. This completes the proof of existence. □

As in Theorem 1.1, assuming just critical regularity for the velocity does not seem to be enough to ensure uniqueness. At least, the constructed velocity u does not satisfy $\nabla u\in L^1_{loc}(\mathbb{R}_+;L^\infty_+)$ so that we cannot resort to the Lagrangian approach. This motivates the following statement.

Theorem 5.2. *In addition to the hypotheses of Theorem 5.1, assume that u_0 belongs to $\dot{\mathcal{B}}^{-1+\frac dp}_{\widetilde p,r}(\mathbb{R}^d_+)$ for some finite $\widetilde p>p$. Then (1.4) has a global solution $(a,u,\nabla\Pi)$ fulfilling the properties of Theorem 5.1 and, in addition,*

$$t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\in L^r(\mathbb{R}_+;L^{\widetilde p}_+),\quad t^\beta\nabla u\in L^{r_2}(\mathbb{R}_+;L^{\widetilde p_2}_+),\quad t^\gamma u\in L^{r_3}(\mathbb{R}_+;L^{\widetilde p_3}_+)$$

with α, β and γ defined as previously in (5.3),

$$\frac{1}{\widetilde{p}_2}-\frac{1}{p_2}=\frac{1}{\widetilde{p}}-\frac{1}{p}\quad\text{and}\quad\frac{1}{\widetilde{p}_3}-\frac{1}{p_3}=\frac{1}{\widetilde{p}}-\frac{1}{p},\tag{5.8}$$

whenever p_2 and r_2 may be chosen so that (5.4) *is fulfilled and*

$$\frac{2d}{p}-\frac{d}{\widetilde{p}}-\frac{d}{p_2}\leq 1+\frac{2}{r_2}-\frac{2}{r}\cdot\tag{5.9}$$

If in addition $\widetilde{p}>d$ then there exists some positive $\delta<\alpha$ and $s>r$ so that

$$t^\delta\nabla u\in L^s(\mathbb{R}_+;L^\infty_+),\tag{5.10}$$

and the constructed solution is unique in its class of regularity.

Remark 5.1. *Despite the appearances, it is always possible to take $\tilde{p}>d$ in the above statement. At first sight it seems not obvious because a necessary condition for having $\tilde{p}>d$ in* (5.9) *is that $p>\frac{dr}{2r-1}$. However, by embedding, one may always find some $p_1\in(\frac{dr}{2r-1},d)$ so that $u_0\in\dot{B}^{-1+\frac{d}{p_1}}_{p_1,r}(\mathbb{R}^d_+)\cap\dot{B}^{-1+\frac{d}{p_1}}_{\widetilde{p},r}(\mathbb{R}^d_+)$ and thus replace p by p_1.*

Proof. The general scheme for proving existence is exactly the same as in the previous statement. Therefore we remain at the level of a priori estimates. Let Z and $\widetilde{Z}$ be defined on $\mathbb{R}_+$ by

$$\begin{aligned}Z(t)&\stackrel{\text{def}}{=}\|t^\gamma u\|_{L^{r_3}_t(L^{p_3}_+)}+\|t^\beta\nabla u\|_{L^{r_2}_t(L^{p_2}_+)}+\|t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\|_{L^r_t(L^p_+)}\\ \widetilde{Z}(t)&\stackrel{\text{def}}{=}\|t^\gamma u\|_{L^{r_3}_t(L^{\widetilde{p}_3}_+)}+\|t^\beta\nabla u\|_{L^{r_2}_t(L^{\widetilde{p}_2}_+)}+\|t^\alpha(\partial_t u,\nabla^2u,\nabla\Pi)\|_{L^r_t(L^{\widetilde{p}}_+)}\cdot\end{aligned}$$

According to the proof of Theorem 5.1 and under Condition (1.9), we have for some constant $C=C(p,r,d)$,

$$Z(t)\leq C\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d_+)}.\tag{5.11}$$

Next, keeping in mind our assumptions on the Lebesgue exponents p, p_2, r_2, $\widetilde{p}$ and $\widetilde{p}_2$, Proposition 2.2 ensures that for some constant $C=C(p,\widetilde{p},r,d)$,

$$\widetilde{Z}(t)\leq C\Big(\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}+\|t^\alpha\,a(\Delta u-\nabla\Pi)\|_{L^r_t(L^{\widetilde{p}}_+)}+\|t^\alpha\,u\cdot\nabla u\|_{L^r_t(L^{\widetilde{p}}_+)}\Big).$$

Because $\|a(t)\|_{L^\infty_+}$ is constant during the evolution, under Condition (1.9), the first term of the right-hand side may be absorbed by the left-hand side. As for the last term, we use Hölder inequality and the fact that

$$\alpha=\beta+\gamma,\quad\frac{1}{r}=\frac{1}{r_2}+\frac{1}{r_3}\quad\text{and}\quad\frac{1}{\widetilde{p}}=\frac{1}{\widetilde{p}_2}+\frac{1}{p_3}\cdot$$

We thus end up with

$$\widetilde{Z}(t)\leq C\Big(\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}+\|t^\beta\nabla u\|_{L^{r_2}_t(L^{\widetilde{p}_2}_+)}\|t^\gamma u\|_{L^{r_3}_t(L^{p_3}_+)}\Big),$$

whence, if c is small enough in (1.9),

$$\widetilde{Z}(t)\leq C\|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}(\mathbb{R}^d_+)}.\tag{5.12}$$

In order to prove (5.10), we first have to check whether one may take $\widetilde{p}>d$, knowing that Conditions (5.4), (5.8) and (5.9) have to be fulfilled. This is in fact equivalent to $p>\frac{dr}{2r-1}$.

Assuming from now on that this condition is fulfilled, and taking $\widetilde{p} > d$, we may use the following Gagliardo-Nirenberg inequality for all functions[6] of $L_+^{\widetilde{p}_3} \cap W_0^{1,\widetilde{p}}(\mathbb{R}_+^d) \cap W^{2,\widetilde{p}}(\mathbb{R}_+^d)$:

$$\|\nabla u\|_{L_+^\infty} \leq C\|\nabla^2 u\|_{L_+^{\widetilde{p}}}^\theta \|u\|_{L_+^{\widetilde{p}_3}}^{1-\theta} \quad \text{with} \quad \theta \stackrel{\text{def}}{=} \frac{1+\frac{d}{\widetilde{p}}-\frac{d}{p_2}}{2-\frac{d}{p_2}}\cdot \tag{5.13}$$

Then, using Hölder inequality and (5.12), we readily get

$$\|t^\delta \nabla u\|_{L_t^s(L_+^\infty)} \leq C\|t^\alpha \nabla^2 u\|_{L_t^r(L_+^{\widetilde{p}})}^\theta \|t^\gamma u\|_{L_t^{r_3}(L_+^{\widetilde{p}_3})}^{1-\theta} \leq C'\|u_0\|_{\dot{B}_{\widetilde{p},r}^{-1+\frac{d}{\widetilde{p}}}(\mathbb{R}_+^d)} < \infty$$

with

$$\delta \stackrel{\text{def}}{=} \theta\alpha + (1-\theta)\gamma \quad \text{and} \quad \frac{1}{s} = \frac{\theta}{r} + \frac{1-\theta}{r_3}\cdot \tag{5.14}$$

Because $r_3 > r$ and $\gamma < \alpha$, it is obvious that $s > r$ and $\delta < \alpha$. This completes the proof of the second part of the statement. Proving uniqueness is postponed to the next section. □

6. Uniqueness

This section is devoted to proving the uniqueness parts of Theorems 1.1 and 5.2. As in [11, 14], it strongly relies on the fact that for smooth enough solutions, one may use the *Lagrangian formulation* of (1.4), which turns out to be equivalent to (1.4).

6.1. **Lagrangian coordinates.** Before going into the detailed proof of uniqueness, we here recall some basic facts concerning Lagrangian coordinates. Throughout, we are given some smooth enough solution $(a, u, \nabla\Pi)$ to (1.4) (typically we assume that $(u, \nabla\Pi) \in X_T^{p,r} \cap X_T^{\widetilde{p},r}$ with (p, r) and $\widetilde{p} > d$ as in the statement of Theorem 1.1). Then we set

$$b(t,y) \stackrel{\text{def}}{=} a(t, X(t,y)), \quad v(t,y) \stackrel{\text{def}}{=} u(t, X(t,y)) \quad \text{and} \quad P(t,y) \stackrel{\text{def}}{=} \Pi(t, X(t,y))$$

where, for any $y \in \mathbb{R}^d$, $X(\cdot, y)$ stands for the solution to the following ordinary differential equation on $[0, T]$:

$$\frac{dX(t,y)}{dt} = u(t, X(t,y)), \qquad X(t,y)|_{t=0} = y. \tag{6.1}$$

Therefore we have the following relation between the *Eulerian coordinates* x and the *Lagrangian coordinates* y:

$$x = X(t,y) = y + \int_0^t v(\tau, y)\, d\tau. \tag{6.2}$$

Let $Y(t,\cdot)$ be the inverse mapping of $X(t,\cdot)$, then $D_x Y(t,x) = (D_y X(t,y))^{-1}$ with $x = X(t,y)$. Furthermore, if

$$\int_0^T \|\nabla v(t)\|_{L^\infty}\, dt < 1 \tag{6.3}$$

then one may write

$$D_x Y = (\mathrm{Id} + (D_y X - \mathrm{Id}))^{-1} = \sum_{k=0}^\infty (-1)^k \Big(\int_0^t D_y v(\tau, y)\, d\tau\Big)^k. \tag{6.4}$$

[6] As know usual, this inequality may be deduced from the classical one on $\mathbb{R}^d$, taking advantage of the antisymmetric extension operator.

Setting $A(t,y) \stackrel{\text{def}}{=} (D_y X(t,y))^{-1} = D_x Y(t,x)$ for $x = X(t,y)$, one may prove (see the Appendix of [11]) that

$$\nabla_x u(t,x) = {}^T\!A(t,y)\nabla_y v(t,y) \quad \text{and} \quad \operatorname{div}_x u(t,x) = \operatorname{div}_y(A(t,y)v(t,y)). \tag{6.5}$$

By the chain rule, we also have[7]

$$\operatorname{div}_y\big(Av\big) = {}^T\!A : \nabla_y v = D_y v : A = \operatorname{Tr}(D_y v \cdot A). \tag{6.6}$$

As in [12], we denote

$$\nabla_u \stackrel{\text{def}}{=} {}^T\!A\cdot\nabla_y, \quad \operatorname{div}_u \stackrel{\text{def}}{=} \operatorname{div}(A\cdot) \quad \text{and} \quad \Delta_u \stackrel{\text{def}}{=} \operatorname{div}_u \nabla_u.$$

Note that for any $t>0$, the solution of (1.4) obtained in Theorem 1.1 satisfies the smoothness assumption of Proposition 2 in [12], so that $(b, v, \nabla P)$ satisfies

$$\begin{cases} b_t = 0 & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^d_+,\\ \partial_t v - (1+b)(\Delta_u v - \nabla_u P) = 0 & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^d_+,\\ \operatorname{div}_u v = 0 & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^d_+,\\ v = 0 & \text{on} \quad \mathbb{R}_+\times\partial\,\mathbb{R}^d_+,\\ (b,v)|_{t=0} = (a_0, u_0) & \text{in} \quad \mathbb{R}^d_+, \end{cases} \tag{6.7}$$

which is the Lagrangian formulation of (1.4).

6.2. **Proving uniqueness : the "smooth case".** Here we prove the uniqueness part of Theorem 1.1. Thanks to the usual rescaling (2.13), one may assume with no loss of generality that $\mu = 1$. Let $(a_i, u_i, \Pi_i), i = 1,2,$ be two solutions of (1.4) satisfying (1.10), (1.11) and (1.12).

For $i = 1,2,$ let X_i be the flow of u_i (defined in (6.2)) and denote by (v_i, P_i) the corresponding velocity and pressure in Lagrangian coordinates. Let

$$\delta v \stackrel{\text{def}}{=} v_2 - v_1, \quad \delta P = P_2 - P_1.$$

Observing that $b \equiv a_0$, we see that $(\delta v, \nabla\delta P)$ satisfies

$$\begin{cases} \partial_t\delta v - \Delta\delta v + \nabla\delta P = a_0(\Delta\delta v - \nabla\delta P) + \delta f_1 + \delta f_2 \stackrel{\text{def}}{=} \delta F & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^d_+,\\ \operatorname{div}\,\delta v = \delta g = \operatorname{div}\delta R & \text{in} \quad \mathbb{R}_+\times\mathbb{R}^d_+, \end{cases} \tag{6.8}$$

where

$$\begin{aligned} \delta f_1 &\stackrel{\text{def}}{=} (1+a_0)[(\operatorname{Id} - {}^T\!A_2)\nabla\delta P - \delta A\nabla P_1] \quad \text{with} \quad \delta A \stackrel{\text{def}}{=} A_2 - A_1,\\ \delta f_2 &\stackrel{\text{def}}{=} (1+a_0)\operatorname{div}[(A_2{}^T\!A_2 - \operatorname{Id})\nabla\delta v + (A_2{}^T\!A_2 - A_1{}^T\!A_1)\nabla v_1],\\ \delta g &\stackrel{\text{def}}{=} (\operatorname{Id} - {}^T\!A_2) : \nabla\delta v - {}^T\delta A : \nabla v_1,\\ \delta R &\stackrel{\text{def}}{=} (\operatorname{Id} - A_2)\delta v - \delta A\, v_1. \end{aligned} \tag{6.9}$$

[7]Here and in what follows, we always denote by ${}^T\!A$ the transpose matrix of A, and $\big(\nabla u\big)_{i,j} = \big(\partial_i u^j\big)_{1\le i,j\le d}$, and $Du = {}^T\nabla u = \big(\partial_j u^i\big)_{1\le i,j\le d}$.

As $\gamma\delta R^d\equiv 0$ and $\delta v|_{t=0}=0$, applying Proposition 2.1 with $p=\frac{dr}{3r-2}$, and $(q,m)=(2r,\frac{dr}{2r-1})$ or $(q,m)=(r,\frac{dr}{2r-2})$ implies that

$$\|\nabla\delta v\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}+\|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})}+\|(\partial_t\delta v,\nabla^2\delta v,\nabla\delta P)\|_{L^r_t(L^p_+)}$$
$$\leq C\Big(\|a_0(\Delta\delta v-\nabla\delta P)\|_{L^r_t(L^p_+)}+\|(\delta f_1,\delta f_2,\nabla\delta g,\partial_t\delta R)\|_{L^r_t(L^p_+)}\Big)\cdot$$

Of course, the first term in the right-hand side may be absorbed by the left-hand side if the constant c_1 is small enough in (1.9). So let us now bound the other terms. In what follows, we will use repeatedly that (see e.g [11])

$$\delta A(t)=\Big(\int_0^t D\delta v\,d\tau\Big)\Big(\sum_{k\geq1}\sum_{0\leq j<k}C_1^jC_2^{k-1-j}\Big)\quad\text{with}\quad C_i(t)\stackrel{\text{def}}{=}\int_0^t Dv_i\,d\tau. \tag{6.10}$$

Bounds for δg. The definition of δg implies that

$$\|\nabla\delta g\|_{L^r_t(L^p_+)}\leq\|\nabla A_2\otimes\nabla\delta v\|_{L^r_t(L^p_+)}+\|(\mathrm{Id}-A_2)\otimes\nabla^2\delta v\|_{L^r_t(L^p_+)}$$
$$+\|\nabla\delta A\otimes\nabla v_1\|_{L^r_t(L^p_+)}+\|\delta A\otimes\nabla^2 v_1\|_{L^r_t(L^p_+)}.$$

Therefore using Hölder inequality, (6.3) and (6.10), we get

$$\|\nabla\delta g\|_{L^r_t(L^p_+)}\leq C\Big(\|\nabla A_2\|_{L^\infty_t(L^d_+)}\|\nabla\delta v\|_{L^r(L_+^{\frac{dr}{2r-2}})}+\|(\mathrm{Id}-A_2)\|_{L^\infty_t(L^\infty_+)}\|\nabla^2\delta v\|_{L^r_t(L^p_+)}$$
$$+\|\nabla\delta A\|_{L^\infty_t(L^p_+)}\|\nabla v_1\|_{L^r_t(L^\infty_+)}+\|\delta A\|_{L^\infty_t(L_+^{\frac{dr}{2r-2}})}\|\nabla^2 v_1\|_{L^r_t(L^d_+)}\Big)\cdot$$

Remark that for θ being determined by $\frac1d=\frac\theta p+\frac{1-\theta}{\tilde p}$,

$$\begin{aligned}
&\|\nabla A_2\|_{L^\infty_t(L^d_+)}\leq C\int_0^t\|\nabla^2 v_2\|_{L^d_+}\,dt'\leq Ct^{1-\frac1r}\|\nabla^2v_2\|^\theta_{L^r_t(L^p_+)}\|\nabla^2v_2\|^{1-\theta}_{L^r_t(L^{\tilde p}_+)},\\
&\|(\mathrm{Id}-A_2)\|_{L^\infty_t(L^\infty_+)}\leq Ct^{1-\frac1q}\|\nabla v_2\|_{L^q_t(L^\infty_+)},\\
&\|\nabla\delta A\|_{L^\infty_t(L^p_+)}\leq Ct^{1-\frac1r}\|\nabla^2\delta v\|_{L^r_t(L^p_+)},\\
&\|\delta A\|_{L^\infty_t(L_+^{\frac{dr}{2r-2}})}\leq Ct^{1-\frac1r}\|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})},\\
&\|\nabla^2 v_1\|_{L^r_t(L^d_+)}\leq C\|\nabla^2v_1\|^\theta_{L^r_t(L^p_+)}\|\nabla^2v_1\|^{1-\theta}_{L^r_t(L^{\tilde p}_+)}.
\end{aligned}$$

Hence we obtain

$$\|\nabla\delta g\|_{L^r_t(L^p_+)}\leq\eta_1(t)\big(\|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})}+\|\nabla^2\delta v\|_{L^r_t(L^p_+)}\big),$$

with $\eta_1(t)\to0$ as t goes to 0.

Bounds for $\partial_t\delta R$. First, we see that

$$\|\partial_t[(\mathrm{Id}-A_2)\delta v]\|_{L^r_t(L^p_+)}\leq C\big(\|\nabla v_2\otimes\delta v\|_{L^r_t(L^p_+)}+\|(\mathrm{Id}-A_2)\partial_t\delta v\|_{L^r_t(L^p_+)}\big).$$

It is easy to check that

$$\begin{aligned}
\|\nabla v_2\otimes\delta v\|_{L^r_t(L^p_+)}&\leq C\|\nabla v_2\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}\|\delta v\|_{L^{2r}_t(L_+^{\frac{dr}{r-1}})}\\
&\leq C\|\nabla v_2\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})}\|\nabla\delta v\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})},
\end{aligned}$$

and

$$\begin{aligned}\|(\mathrm{Id}-A_2)\partial_t\delta v\|_{L^r_t(L^p_+)} &\leq C\|\mathrm{Id}-A_2\|_{L^\infty_t(L^\infty_+)}\|\partial_t\delta v\|_{L^r_t(L^p_+)}\\ &\leq Ct^{1-\frac1q}\|\nabla v_2\|_{L^q_t(L^\infty_+)}\|\partial_t\delta v\|_{L^r_t(L^p_+)},\end{aligned}$$

which gives rise to

$$\|\partial_t[(\mathrm{Id}-A_2)\delta v]\|_{L^r_t(L^p_+)} \leq \eta_2(t)\big(\|\nabla\delta v\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}_+)} + \|\partial_t\delta v\|_{L^r_t(L^p_+)}\big) \quad\text{with}\quad \lim_{t\to0}\eta_2(t)=0.$$

On the other hand, thanks to (6.10), we have

$$\|\partial_t[\delta A\, v_1]\|_{L^r_t(L^p_+)} \leq C\bigg(\|v_1\otimes\nabla\delta v\|_{L^r_t(L^p_+)}+\|\delta A\,\partial_t v_1\|_{L^r_t(L^p_+)}+\bigg\|\int_0^t|\nabla\delta v|\,dt'\big|\nabla v_{1,2}\big||v_1|\bigg\|_{L^r_t(L^p_+)}\bigg)$$

where $v_{1,2}$ designates components of v_1 or v_2.

Applying Hölder and Sobolev inequalities gives

$$\begin{aligned}\|v_1\otimes\nabla\delta v\|_{L^r_t(L^p_+)} &\leq C\|v_1\|_{L^{2r}_t(L^{\frac{dr}{r-1}}_+)}\|\nabla\delta v\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}_+)}\\ &\leq C\|\nabla v_1\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}_+)}\|\nabla\delta v\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}_+)}.\end{aligned}$$

Following the computations for bounding δg, we also have

$$\begin{aligned}\|\delta A\,\partial_t v_1\|_{L^{2r}_t(L^p_+)} &\leq C\|\delta A\|_{L^\infty_t(L^{\frac{dr}{2r-2}}_+)}\|\partial_t v_1\|_{L^r_t(L^d_+)}\\ &\leq Ct^{1-\frac1r}\|\partial_t v_1\|^\theta_{L^r_t(L^p_+)}\|\partial_t v_1\|^{1-\theta}_{L^r_t(L^{\tilde p}_+)}\|\nabla\delta v\|_{L^r_t(L^{\frac{dr}{2r-2}}_+)},\end{aligned}$$

and, for α so that $\frac1\alpha+\frac{r-1}{dr}=\frac1{\tilde p}$, one has

$$\begin{aligned}\bigg\|\int_0^t|\nabla\delta v|\,dt'\big|\nabla v_{1,2}\big||v_1|\bigg\|_{L^r_t(L^p_+)} &\leq C\|\nabla\delta v\|_{L^1_t(L^{\frac{dr}{2r-2}}_+)}\|\nabla v_{1,2}\otimes v_1\|_{L^r_t(L^d_+)}\\ &\leq Ct^{\frac1{r'}}\|\nabla\delta v\|_{L^r_t(L^{\frac{dr}{2r-2}}_+)}\big(\|\nabla v_{1,2}\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}_+)}\|v_1\|_{L^{2r}_t(L^{\frac{dr}{r-1}}_+)}\big)^\theta\\ &\qquad\times\big(\|\nabla v_{1,2}\|_{L^{2r}_t(L^\alpha_+)}\|v_1\|_{L^{2r}_t(L^{\frac{dr}{d-1}}_+)}\big)^{1-\theta}.\end{aligned}$$

As a consequence, we obtain

$$\|\partial_t[\delta A\, v_1]\|_{L^r_t(L^p_+)} \leq \eta_3(t)\big(\|\nabla\delta v\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}_+)}+\|\nabla\delta v\|_{L^r_t(L^{\frac{dr}{2r-2}}_+)}\big) \quad\text{with}\quad \lim_{t\to0}\eta_3(t)=0.$$

Bounds for δf_1. We notice that

$$\delta f_1=(1+a_0)\big((\mathrm{Id}-{}^T\!A_2)\nabla\delta P-{}^T\delta A\nabla P_1\big).$$

Hence, thanks to (1.10), (1.11) and (1.12), we have

$$\begin{aligned}\|\delta f_1\|_{L^r_t(L^p_+)} &\leq C\big(\|{}^T(\mathrm{Id}-A_2)\nabla\delta P\|_{L^r_t(L^p_+)}+\|{}^T\delta A\nabla P_1\|_{L^r_t(L^p_+)}\big)\\ &\leq C\big(\|\nabla v_2\|_{L^1_t(L^\infty_+)}\|\nabla\delta P\|_{L^r_t(L^p_+)}+\|\nabla\delta v\|_{L^1_t(L^{\frac{dr}{2r-2}}_+)}\|\nabla P_1\|_{L^r_t(L^d_+)}\big)\\ &\leq C\Big(t^{\frac1{q'}}\|\nabla v_2\|_{L^q_t(L^\infty_+)}\|\nabla\delta P\|_{L^r_t(L^p_+)}+t^{\frac1{r'}}\|\nabla P_1\|_{L^r_t(L^p_+\cap L^{\tilde p}_+)}\|\nabla\delta v\|_{L^r_t(L^{\frac{dr}{2r-2}}_+)}\Big)\end{aligned}$$

so that

$$\|\delta f_1\|_{L^r_t(L^p_+)} \leq \eta_4(t)\big(\|\nabla\delta P\|_{L^r_t(L^p_+)}+\|\nabla\delta v\|_{L^r_t(L^{\frac{dr}{2r-2}}_+)}\big) \quad\text{with}\quad \lim_{t\to0}\eta_4(t)=0.$$

Bounds for δf_2. We may write

$$\|\delta f_2\|_{L^r_t(L^p_+)} \leq C\Big(\|\,\mathrm{div}((A_2{}^TA_2-\mathrm{Id})\nabla\delta v)\|_{L^r_t(L^p_+)} + \|\,\mathrm{div}\big((A_2{}^TA_2-A_1{}^TA_1)\nabla v_1\big)\|_{L^r_t(L^p_+)}\Big)\cdot$$

Therefore Hölder inequality and (6.10) imply that

$$\begin{aligned}\|\delta f_2\|_{L^r_t(L^p_+)} \leq C\Big(&\|\nabla(A_2{}^TA_2)\|_{L^\infty_t(L^d_+)}\|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})} + \|(A_2{}^TA_2-\mathrm{Id})\|_{L^\infty_t(L^\infty_+)}\|\nabla^2\delta v\|_{L^r_t(L^p_+)}\\ &+\|\nabla v_1\|_{L^r_t(L^\infty_+)}\|\nabla(A_2{}^TA_2-A_1{}^TA_1)\|_{L^\infty_t(L^p_+)} + \|\nabla^2 v_1\|_{L^r_t(L^d_+)}\|A_2{}^TA_2-A_1{}^TA_1\|_{L^\infty_t(L_+^{\frac{dr}{2r-2}})}\Big),\end{aligned}$$

which leads to

$$\begin{aligned}\|\delta f_2\|_{L^r_t(L^p_+)} \leq\,& C\Big(t^{\frac1{r'}}\big(\|\nabla^2 v_1\|_{L^r_t(L^d_+)} + \|\nabla^2 v_2\|_{L^r_t(L^d_+)}\big) + t^{\frac1{q'}}\big(\|\nabla v_1\|_{L^q_t(L^\infty_+)}\\ &+\|\nabla v_2\|_{L^q_t(L^\infty_+)}\big)\Big)\big(\|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})} + \|\nabla^2\delta v\|_{L^r_t(L^p_+)}\big)\\ \leq\,& \eta_5(t)\big(\|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})} + \|\nabla^2\delta v\|_{L^r_t(L^p_+)}\big)\quad\text{with}\quad \lim_{t\to0}\eta_5(t)=0.\end{aligned}$$

Therefore, one may conclude that

$$\begin{aligned}&\|\nabla\delta v\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})} + \|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})} + \|(\partial_t\delta v,\nabla^2\delta v,\nabla\delta P)\|_{L^r_t(L^p_+)}\\ &\qquad\leq \eta(t)\big(\|\nabla\delta v\|_{L^{2r}_t(L_+^{\frac{dr}{2r-1}})} + \|\nabla\delta v\|_{L^r_t(L_+^{\frac{dr}{2r-2}})} + \|(\partial_t\delta v,\nabla^2\delta v,\nabla\delta P)\|_{L^r_t(L^p_+)}\big)\end{aligned}$$

where $\eta(t)=\sum_{i=1}^5\eta_i(t)$, which goes to zero as $t\to0$. This yields uniqueness on a small time interval. Then a standard continuation argument yields uniqueness on the whole interval $[0,T]$. The proof of Theorem 1.1 is now complete.

6.3. **Proving uniqueness : the "rough case".** We now assume that we are given two solutions $(a_i,u_i,\Pi_i), i=1,2,$ satisfying the properties of the last part of Theorem 5.1. Of course, the difference $(\delta v,\nabla\delta P)$ between the two solutions in Lagrangian coordinates still satisfies System (6.8). In order to prove uniqueness, we shall derive suitable bounds for the following quantity:

$$\delta Z(T) \stackrel{\mathrm{def}}{=} \|t^\alpha(\partial_t\delta v,\nabla^2\delta v,\nabla\delta P)\|_{L^r_T(L^p_+)} + \|t^\beta\nabla\delta v\|_{L^{r_2}_T(L^{p_2}_+)} + \|t^\gamma\delta v\|_{L^{r_3}_T(L^{p_3}_+)}.$$

To start with , let us apply Proposition 2.2 with regularity exponent $-1+\frac dp$. We get for all positive T:

$$\delta Z(T)\leq C\big(\|a_0\,t^\alpha(\Delta\delta v-\nabla\delta P)\|_{L^r_T(L^p_+)} + \|t^\alpha(\delta f_1,\delta f_2,\nabla\delta g,\partial_t\delta R)\|_{L^r_T(L^p_+)}.\tag{6.11}$$

Again, the first term of the right-hand side may be absorbed if the constant c_1 is small enough in (1.9).

Bounds for $\nabla\delta g$. From the definition of δg, we readily have

$$\begin{aligned}\|t^\alpha\nabla\delta g\|_{L^r_T(L^p_+)} \leq\,& \|t^\alpha\nabla A_2\otimes\nabla\delta v\|_{L^r_T(L^p_+)} + \|t^\alpha(\mathrm{Id}-A_2)\otimes\nabla^2\delta v\|_{L^r_T(L^p_+)}\\ &+\|t^\alpha\nabla\delta A\otimes\nabla v_1\|_{L^r_T(L^p_+)} + \|t^\alpha\delta A\otimes\nabla^2 v_1\|_{L^r_T(L^p_+)}.\end{aligned}$$

Using Hölder inequality and (6.3), we get

$$\|t^\alpha\nabla A_2\otimes\nabla\delta v\|_{L^r_T(L^p_+)} \leq \|t^\alpha\nabla\delta v\|_{L^r_T(L^{p^*}_+)}\int_0^T\|\nabla^2 v_2\|_{L^d_+}\,dt,$$

where $p^* \stackrel{\text{def}}{=} dp/(d-p)$ stands for the Lebesgue exponent in the critical Sobolev embedding

$$W^{1,p}(\mathbb{R}^d_+) \hookrightarrow L^{p^*}_+. \tag{6.12}$$

Therefore, remembering that $\alpha + 1/r < 1$ and that $L^d_+ \hookrightarrow L^p_+ \cap L^{\widetilde{p}}_+$,

$$\|t^\alpha \nabla A_2 \otimes \nabla \delta v\|_{L^r_T(L^p_+)} \leq CT^{1-\frac{1}{r}-\alpha} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)} \|t^\alpha \nabla^2 v_2\|_{L^r_T(L^p_+ \cap L^{\widetilde{p}}_+)}.$$

Next, using the fact that

$$\|(\mathrm{Id} - A_2)(T)\|_{L^\infty_+} \leq C \int_0^T \|\nabla v_2\|_{L^\infty_+}\, dt \leq CT^{1-\frac{1}{s}-\delta} \|t^\delta \nabla v_2\|_{L^s_T(L^\infty_+)}, \tag{6.13}$$

where s and δ are defined in (5.14), we easily get

$$\|t^\alpha (\mathrm{Id} - A_2) \otimes \nabla^2 \delta v\|_{L^r_T(L^p_+)} \leq CT^{1-\frac{1}{s}-\delta} \|t^\delta \nabla v_2\|_{L^s_T(L^\infty_+)} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)}.$$

In order to bound the third term of $\nabla \delta g$, we notice from (6.10) that

$$\|\nabla \delta A\|_{L^\infty_T(L^p_+)} \leq Ct^{1-\frac{1}{r}-\alpha} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)}. \tag{6.14}$$

Therefore, setting $\frac{1}{m} = \frac{1}{r} - \frac{1}{s}$,

$$\begin{aligned} \|t^\alpha \nabla \delta A \otimes \nabla v_1\|_{L^r_T(L^p_+)} &\leq CT^{1-\frac{1}{r}-\alpha} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)} \|\tau^{\alpha-\delta}\|_{L^m(0,T)} \|t^\delta \nabla v_1\|_{L^s_T(L^\infty_+)} \\ &\leq CT^{1-\frac{1}{s}-\alpha} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)} \|t^\delta \nabla v_1\|_{L^s_T(L^\infty_+)}. \end{aligned}$$

Finally, we have from (6.10) and Sobolev embedding (6.12),

$$\|\delta A\|_{L^\infty_T(L^{p^*}_+)} \leq CT^{1-\frac{1}{r}-\alpha} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)}. \tag{6.15}$$

Hence we obtain

$$\|t^\alpha \delta A \otimes \nabla^2 v_1\|_{L^r_T(L^p_+)} \leq CT^{1-\frac{1}{r}-\alpha} \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)} \|t^\alpha \nabla^2 v_1\|_{L^r_T(L^d_+)}.$$

Putting together the four above estimates, we conclude that

$$\begin{aligned} \|t^\alpha \nabla \delta g\|_{L^r_T(L^p_+)} \leq C \|t^\alpha \nabla^2 \delta v\|_{L^r_T(L^p_+)} \Big(& T^{1-\frac{1}{r}-\alpha} \|t^\alpha (\nabla^2 v_1, \nabla^2 v_2)\|_{L^r_T(L^p_+ \cap L^{\widetilde{p}}_+)} \\ & + T^{1-\frac{1}{s}-\alpha} \|t^\delta (\nabla v_1, \nabla v_2)\|_{L^s_T(L^\infty_+)} \Big). \end{aligned} \tag{6.16}$$

Bounds for $\partial_t \delta R$. Recall that

$$\partial_t \delta R = -(\partial_t A_2)\delta v + (\mathrm{Id} - A_2)\partial_t \delta v - (\partial_t \delta A)\, v_1 - \delta A\, \partial_t v_1.$$

Now, using the expression of $\partial_t A_2$ and Hölder inequality, we get

$$\|t^\alpha\, (\partial_t A_2)\delta v\|_{L^r_T(L^p_+)} \leq C \|t^\beta \nabla v_2\|_{L^{r_2}_T(L^{p_2}_+)} \|t^\gamma \delta v\|_{L^{r_3}_T(L^{p_3}_+)}.$$

Next, we have, using (6.13)

$$\begin{aligned} \|t^\alpha (\mathrm{Id} - A_2)\partial_t \delta v\|_{L^r_T(L^p_+)} &\leq C \|\,\mathrm{Id} - A_2\|_{L^\infty_T(L^\infty_+)} \|t^\alpha \partial_t \delta v\|_{L^r_T(L^p_+)} \\ &\leq CT^{1-\frac{1}{s}-\delta} \|t^\delta \nabla v_2\|_{L^s_T(L^\infty_+)} \|t^\alpha \partial_t \delta v\|_{L^r_T(L^p_+)}. \end{aligned}$$

In order to bound the third term of $\partial_t \delta R$, we differentiate (6.10) with respect to time and easily find that

$$\|t^\alpha (\partial_t \delta A)\, v_1\|_{L^r_T(L^p_+)} \leq C \Big(\|t^\alpha v_1 \otimes \nabla \delta v\|_{L^r_T(L^p_+)} + \Big\| t^\alpha \int_0^t |\nabla \delta v|\, dt' |\nabla v_{1,2}| |v_1| \Big\|_{L^r_T(L^p_+)} \Big)$$

where $v_{1,2}$ designates components of v_1 or v_2.

On the one hand, applying Hölder inequality gives
$$\|t^\alpha\, v_1\otimes\nabla\delta v\|_{L^r_T(L^p_+)}\leq C\|t^\gamma v_1\|_{L^{r_3}_T(L^{p_3}_+)}\|t^\beta\nabla\delta v\|_{L^{r_2}_T(L^{p_2}_+)}.$$
On the other hand, we have, by virtue of (6.12)
$$\begin{aligned}\Big\|t^\alpha\int_0^t|\nabla\delta v|\,dt'\big|\nabla v_{1,2}\big||v_1|\Big\|_{L^r_T(L^p_+)}&\leq C\|t^\alpha\nabla v_{1,2}\otimes v_1\|_{L^r_T(L^d_+)}\int_0^T\|\nabla\delta v\|_{L^{p^*}_+}\,dt\\&\leq CT^{1-\frac1r-\alpha}\|t^\alpha\nabla^2\delta v\|_{L^r_T(L^p_+)}\|t^\gamma v_1\|_{L^{r_3}_T(L^{p_3}_+)}\|t^\beta\nabla v_{1,2}\|_{L^{r_2}_T(L^a_+)}\end{aligned}$$
with $\frac1a=\frac1d-\frac1{p_2}$. Given that $p<d<\widetilde p$, we have $p_2<a<\widetilde p_2$, hence $L^a_+\hookrightarrow L^{p_2}_+\cap L^{\widetilde p_2}_+$.

Finally, arguing as for bounding the last term of $\nabla\delta g$ yields
$$\|t^\alpha\delta A\,\partial_t v_1\|_{L^r_T(L^p_+)}\leq CT^{1-\frac1r-\alpha}\|t^\alpha\partial_t v_1\|_{L^r_T(L^p_+\cap L^{\widetilde p}_+)}\|t^\alpha\nabla^2\delta v\|_{L^r_T(L^p_+)}.$$
Putting all those estimates together, we conclude that
$$\|\partial_t\delta R\|_{L^r_T(L^p_+)}\leq\eta(T)\delta Z(T)\tag{6.17}$$
for some function η going to 0 at 0.

Bounds for δf_1. We notice that
$$\delta f_1=(1+a_0)\big((\mathrm{Id}-{}^T\!A_2)\nabla\delta P-{}^T\delta A\nabla P_1\big).$$
Hence it suffices to follow the computations leading to the bounds for the second and fourth terms of $\nabla\delta g$: we just have to change $\nabla^2\delta v$ and $\nabla^2 v_1$ into $\nabla\delta P$ and ∇P_1. We end up with
$$\begin{aligned}\|t^\alpha\delta f_1\|_{L^r_T(L^p_+)}\leq C\Big(&T^{1-\frac1s-\delta}\|t^\delta\nabla v_2\|_{L^s_T(L^\infty_+)}\|t^\alpha\nabla\delta P\|_{L^r_T(L^p_+)}\\&+T^{1-\frac1r-\alpha}\|t^\alpha\nabla P_1\|_{L^r_T(L^p_+\cap L^{\widetilde p}_+)}\|t^\alpha\nabla^2\delta v\|_{L^r_T(L^p_+)}\Big).\end{aligned}\tag{6.18}$$

Bounds for δf_2. As we may write
$$\begin{aligned}\|t^\alpha\delta f_2\|_{L^r_T(L^p_+)}\leq C\Big(&\|t^\alpha\nabla(A_2{}^TA_2)\otimes\nabla\delta v\|_{L^r_T(L^p_+)}+\|t^\alpha(A_2{}^TA_2-\mathrm{Id})\otimes\nabla^2\delta v\|_{L^r_T(L^p_+)}\\&+\|t^\alpha\nabla(A_2{}^TA_2-A_1{}^TA_1)\otimes\nabla v_1\|_{L^r_T(L^p_+)}+\|(A_2{}^TA_2-A_1{}^TA_1)\nabla^2 v_1\|_{L^r_T(L^p_+)}\Big),\end{aligned}$$
one may repeat the computations leading to (6.16): this only a matter of replacing everywhere A_1 and A_2 by $A_1{}^TA_1$ and $A_2{}^TA_2$, respectively. We conclude that $\|t^\alpha\delta f_2\|_{L^r_T(L^p_+)}$ is bounded by the right-hand side of (6.16).

Conclusion. Plugging (6.16), (6.17), (6.18) and the above inequality in (6.11), we conclude that whenever both v_1 and v_2 satisfy (6.3), we have
$$\delta Z(T)\leq\eta(T)\delta Z(T)$$
for some function η going to 0 at 0. This implies uniqueness on a small time interval. Then a standard continuation argument yields uniqueness on the whole interval where the two solutions are given.

7. Remarks on the bounded domain case

This section aims at extending partially our main theorem to the initial boundary value problem (1.4) in a C^2 bounded domain Ω of $\mathbb{R}^d$. Before we present the main result, let us introduce a few notation. We set

$$X^q(\Omega) \stackrel{\text{def}}{=} \Big\{ \; u \in \big(L^q(\Omega)\big)^d \; | \; \operatorname{div} u = 0 \;\text{ and }\; u \cdot \vec{n} = 0 \;\text{ on }\; \partial\Omega \; \Big\},$$

where $\vec{n}$ stands for the unit normal exterior vector at $\partial\Omega$. Denoting by P_q the projection operator from $L^q(\Omega)$ onto $X^q(\Omega)$, the Stokes operator on $L^q(\Omega)$ is the unbounded operator (see e. g. [13])

$$A_q \stackrel{\text{def}}{=} -P_q\Delta \quad \text{with domain} \quad D(A_q) \stackrel{\text{def}}{=} W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap X^q(\Omega).$$

Definition 7.1. *Let $1<q<\infty$. For $\alpha \in (0,1)$ and $s \in (1,\infty)$, we set*

$$\|u\|_{D^{\alpha,s}_{A_q}} \stackrel{def}{=} \|u\|_{L^q} + \Big(\int_0^\infty \|t^{1-\alpha} A_q e^{-tA_q} u\|^s_{L^q} \frac{dt}{t} \Big)^{\frac{1}{s}},$$

where e^{-tA_q} stands for the analytic semigroup generated by A_q. We then define the inhomogeneous fractional domain $D^{\alpha,s}_{A_q}$ as the completion of $D(A_q)$ under $\|u\|_{D^{\alpha,s}_{A_q}}$.

Remark 7.1. *Let $\alpha \in (0,1)$ and $1<q,s<\infty$. Let $\mathcal{B}^\beta_{q,s}$ be the completion of $C^\infty_c(\Omega)$ in $B^\beta_{q,s}(\mathbb{R}^d)$. Then we have*

$$\mathcal{B}^{2\alpha}_{q,s}(\Omega) \cap X^q(\Omega) \hookrightarrow D^{\alpha,s}_{A_q} \hookrightarrow B^{2\alpha}_{q,s}(\Omega) \cap X^q(\Omega).$$

If moreover $2\alpha < 1/q$ then the three sets coincide.

The main result of this section reads:

Theorem 7.1. *Let Ω be a bounded domain with C^2 boundary. Let $r \in (1,\infty)$, $a_0 \in L^\infty(\Omega)$ and $u_0 \in D^{1-\frac{1}{r},r}_{A_p}$ with $p = \frac{dr}{3r-2}\cdot$ There exists a positive constant c_0 so that if*

$$\mu\|a_0\|_{L^\infty(\Omega)} + \|u_0\|_{D^{1-\frac{1}{r},r}_{A_p}} \leq c_0\mu, \tag{7.1}$$

then (1.4) *has a global weak solution $(a,u,\nabla\Pi)$ in the sense of Definition 1.1, which satisfies*

$$\begin{aligned} \mu^{1-\frac{1}{r}}\|u\|_{L^\infty_t(\mathbb{R}_+;D^{1-\frac{1}{r},r}_{A_p})} &+ \|(\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L^r(\mathbb{R}_+;L^p(\Omega))} \\ &+ \mu^{\frac{1}{2r}}\|\nabla u\|_{L^{2r}(\mathbb{R}_+;L^{\frac{dr}{2r-1}}(\Omega))} \leq C\mu^{1-\frac{1}{r}}\|u_0\|_{D^{1-\frac{1}{r},r}_{A_p}}, \end{aligned} \tag{7.2}$$

for some sufficiently large positive constant C.

Remark 7.2. *Uniqueness would require our using Lagrangian coordinates, hence investigating the evolutionary Stokes system in a bounded domain* with non homogeneous divergence. *We leave this interesting issue to a future work.*

The proof of the theorem mainly relies on the following result (see Theorem 3.2 of [9]):

Proposition 7.1. *Let Ω be a C^2 bounded domain of $\mathbb{R}^d$ and $1<q,s<\infty$. Assume that $u_0 \in D^{1-\frac{1}{s},s}_{A_q}$ and $f \in L^s(\mathbb{R}_+;L^q(\Omega))$. Then the system*

$$\begin{cases} \partial_t u - \mu\Delta u + \nabla\Pi = f & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ \operatorname{div}\, u = 0 & \text{in} \quad \mathbb{R}_+ \times \Omega, \\ u|_{\partial\Omega} = 0 & \text{on} \quad \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

with initial data u_0 has a unique solution (u,Π) satisfying $\int_\Omega \Pi\,dx=0$ and for all $t>0$,

$$\mu^{1-\frac1s}\|u(t)\|_{D_{A_q}^{1-\frac1s,s}}+\|(\partial_t u,\mu\nabla^2u,\nabla\Pi)\|_{L^s_t(L^q(\Omega))}\leq C\big(\mu^{1-\frac1s}\|u_0\|_{D_{A_q}^{1-\frac1s,s}}+\|f\|_{L^s_t(L^q(\Omega))}\big).$$

Now we are in a position to prove Theorem 7.1.

Proof of Theorem 7.1. We first solve System (1.4) with regularized data, according to e.g. [9]. We get a sequence $(a^n,u^n,\nabla\Pi^n)_{n\in\mathbb{N}}$ of smooth solutions to (1.4). In particular, as

$$\partial_t u^n-\mu\Delta u^n+\nabla\Pi^n=F^n\stackrel{\text{def}}{=}a^n(\mu\Delta u^n-\nabla\Pi^n)-u^n\cdot\nabla u^n,$$

Proposition 7.1 implies that

$$\begin{aligned}Z_n(t)&\stackrel{\text{def}}{=}\mu^{1-\frac1r}\|u^n(t)\|_{D_{A_p}^{1-\frac1r,r}}+\|\partial_t u^n\|_{L^r_t(L^p)}+\mu\|\Delta u^n\|_{L^r_t(L^p)}+\|\nabla\Pi^n\|_{L^r_t(L^p)}\\&\leq C\big(\mu^{1-\frac1r}\|u_0\|_{D_{A_p}^{1-\frac1r,r}}+\|F^n\|_{L^r_t(L^p)}\big).\end{aligned}$$

By interpolation and embedding we have for all $z\in W^{1,p}(\Omega)$ the following Gagliardo-Nirenberg inequality:

$$\|z\|_{L^{\frac{dr}{2r-1}}(\Omega)}\leq C\|z\|^{\frac12}_{B^{1-\frac2r}_{p,r}(\Omega)}\|z\|^{\frac12}_{W^{1,p}(\Omega)}.$$

Applying this inequality to $z=\nabla u^n$ and taking advantage of Poincaré inequality (here we use that u^n vanishes at the boundary), we discover that

$$\|\nabla u^n\|_{L^{\frac{dr}{2r-1}}(\Omega)}\leq C\|u^n\|^{\frac12}_{B^{2-\frac2r}_{p,r}(\Omega)}\|\nabla^2u^n\|^{\frac12}_{L^p(\Omega)}.$$

Therefore, using Remark 7.1,

$$\|\nabla u^n\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}(\Omega))}\leq C\big(\mu^{1-\frac1r}\|u_0\|_{D_{A_p}^{1-\frac1r,r}}+\|F^n\|_{L^r_t(L^p(\Omega))}\big).\tag{7.3}$$

Now, taking advantage of Hölder inequality and of the Sobolev embedding

$$W_0^{1,\frac{dr}{3r-2}}(\Omega)\hookrightarrow L^{\frac{dr}{2r-1}}(\Omega),$$

we may write

$$\begin{aligned}\|F^n\|_{L^r_t(L^p(\Omega))}&\leq\|a^n\|_{L^\infty_t(L^\infty(\Omega))}\big(\mu\|\Delta u^n\|_{L^r_t(L^p(\Omega))}+\|\nabla\Pi^n\|_{L^r_t(L^p(\Omega))}\big)\\&\qquad+\|u^n\|_{L^{2r}_t(L^{\frac{dr}{r-1}}(\Omega))}\|\nabla u^n\|_{L^{2r}_t(L^{\frac{dr}{2r-1}}(\Omega))}\\&\leq\|a_0\|_{L^\infty(\Omega)}\big(\mu\|\Delta u^n\|_{L^r_t(L^p(\Omega))}+\|\nabla\Pi^n\|_{L^r_t(L^p(\Omega))}\big)+C\|\nabla u^n\|^2_{L^{2r}_t(L^{\frac{dr}{2r-1}}(\Omega))}.\end{aligned}$$

Therefore taking c_0 small enough in (7.1), we get by using (7.3) that

$$Z_n(t)\leq C\big(\mu^{1-\frac1r}\|u_0\|_{D_{A_p}^{1-\frac1r,r}}+\mu^{-(2-\frac1r)}Z_n^2(t)\big),$$

so that as long as $\|u_0\|_{D_{A_p}^{1-\frac1r,r}}\leq\frac{1}{4C^2}\mu$, we have

$$Z_n(t)\leq 2C\mu^{1-\frac1r}\|u_0\|_{D_{A_p}^{1-\frac1r,r}}.$$

Granted with this estimate, we can follow the lines of the proof of Theorem 1.1 to complete the proof of Theorem 7.1. ◻

Appendix A

Here we establish several $L^p - L^q$ or maximal regularity type estimates involving the heat semigroup in the whole space, or the Stokes semigroup in a bounded C^2 domain Ω. Although those estimates belong to the mathematical folklore (as a matter of fact the heat semigroup case in $\mathbb{R}^d$ has been treated in [14]), we did not find any reference where they are proved with this degree of generality.

As in [14], the key to the proof of maximal regularity estimates is the following proposition (see e. g. Th. 2.34 of [5]) enabling us to characterize Besov spaces with negative indices by means of the heat semigroup.

Proposition A.1. *Let s be a negative real number and $(p,r) \in [1,\infty]^2$. A constant C exists such that for all $\mu > 0$, we have*

$$C^{-1}\mu^{\frac{s}{2}}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \leq \Big\|\|t^{-\frac{s}{2}}e^{\mu t\Delta}f\|_{L^p(\mathbb{R}^d)}\Big\|_{L^r(\mathbb{R}_+;\frac{dt}{t})} \leq C\mu^{\frac{s}{2}}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)}.$$

We shall often use the following consequence of the above proposition:

Corollary A.1. *For any $(p,r) \in [1,\infty]^2$ with r finite, there exists a constant C so that for all $\mu > 0$,*

$$C^{-1}\|f\|_{\dot{B}^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)} \leq \mu^{\frac{1}{r}}\|e^{\mu t\Delta}f\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))} \leq C\|f\|_{\dot{B}^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)}.$$

Besides, for all $T \geq 0$, we have

$$\|e^{\mu T\Delta}f\|_{\dot{B}^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)} \leq C\|f\|_{\dot{B}^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)},$$

and, if in addition $p < \infty$ then the map $T \mapsto e^{\mu T\Delta}f$ is continuous on $\mathbb{R}_+$ with values in $\dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d)$, whenever f is in $\dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d)$.

Proof. The first item corresponds to the previous proposition with $s = -2/r$. Given that $(e^{\mu t\Delta})_{t>0}$ is a contracting semigroup over $L^p(\mathbb{R}^d)$, we readily get the second item. The continuity result is a consequence of the density of smooth functions in $\dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d)$ if both p and r are finite. $\square$

To prove Theorem 1.1, we also need the following result:

Lemma A.1. *The operator $\mathcal{A}$ defined by*

$$\mathcal{A} : f \longmapsto \Big\{t \mapsto \int_0^t \nabla^2 e^{\mu(t-\tau)\Delta}f\,d\tau\Big\} \tag{A.4}$$

is bounded from $L^r(0,T;L^p(\mathbb{R}^d))$ to $L^r(0,T;L^p(\mathbb{R}^d))$ for every $T \in (0,\infty]$ and $1 < p, r < \infty$. Moreover, there holds

$$\mu\|\mathcal{A}f\|_{L^r_T(L^p(\mathbb{R}^d))} \leq C\|f\|_{L^r_T(L^p(\mathbb{R}^d))}$$

and for all $T > 0$,

$$\mu^{1-\frac{1}{r}}\Big\|\int_0^T e^{\mu(T-t)\Delta}f\,dt\Big\|_{\dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d)} \leq C\|f\|_{L^r_T(L^p(\mathbb{R}^d))}.$$

Furthermore, the map $T \mapsto \int_0^T e^{\mu(T-t)\Delta}f\,dt$ is continuous on $\mathbb{R}_+$ with values in $\dot{B}^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d)$.

Proof. The first part of the statement is Lemma 7.3 of [16]. For establishing the second inequality, we just have to notice that

$$\Big\|\int_0^T e^{\mu(T-t)\Delta} f\,dt\Big\|_{\dot B^{2-\frac{2}{r}}_{p,r}(\mathbb{R}^d)} \leq C\|\mathcal{A}f(T)\|_{\dot B^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)}.$$

Using that $(e^{\mu t\Delta})_{t>0}$ is contracting over $L^p(\mathbb{R}^d)$, and Corollary A.1, we may write

$$\|\mathcal{A}f(T)\|_{\dot B^{-\frac{2}{r}}_{p,r}(\mathbb{R}^d)} \leq C\mu^{\frac{1}{r}}\|\mathcal{A}f\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))}.$$

By changing f to $f1_{[0,T]}$, one gets the desired inequality. The continuity result follows by density. □

From now on, to simplify the presentation, we agree that A_p denotes either the Stokes operator on $L^p(\Omega)$ with Ω a C^2 bounded domain (see just above Definition 7.1), or the heat operator on $L^p(\mathbb{R}^d)$. Lemma A.1 extends as follows:

Lemma A.2. *Let $1<p,r<\infty$ and Operator $\widetilde{\mathcal{A}}_p$ be defined by*

$$\widetilde{\mathcal{A}}_p : f \longmapsto \Big[t \to \int_0^t A_p e^{-(t-\tau)A_p} f(\tau)\,d\tau\Big].$$

Then for all real number $\alpha\in[0,1-1/r)$ there exists a constant C so that for all $T>0$,

$$\|t^\alpha\widetilde{\mathcal{A}}_p f\|_{L^r_T(L^p(\Omega))} \leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}.$$

Proof. As regards the heat semigroup in $\mathbb{R}^d$, this result has been established in [14]. We here propose another proof that also works for the Stokes semigroup in bounded domains (and, more generally, whenever maximal regularity estimates are available). Let

$$v(t) \overset{\text{def}}{=} \int_0^t e^{-(t-\tau)A_p} f(\tau)\,d\tau. \tag{A.5}$$

Because

$$\partial_t(t^\alpha v) + A_p(t^\alpha v) = t^\alpha f + \alpha t^{\alpha-1} v \quad\text{and}\quad (t^\alpha v)|_{t=0}=0,$$

we readily have

$$\|t^\alpha\widetilde{\mathcal{A}}_p v\|_{L^r_T(L^p(\Omega))} \leq C\big(\|t^\alpha f\|_{L^r_T(L^p(\Omega))} + \alpha\|t^{\alpha-1}v\|_{L^r_T(L^p(\Omega))}\big). \tag{A.6}$$

From the definition of v and the fact that $(e^{-\lambda A_p})_{\lambda>0}$ is contracting on $L^p(\Omega)$, we infer that

$$\|v(t)\|_{L^p(\Omega)} \leq \int_0^t \|f(\tau)\|_{L^p(\Omega)}\,d\tau.$$

Therefore,

$$\|t^{\alpha-1}v(t)\|_{L^p(\Omega)} \leq \int_0^t \Big(\frac{t}{\tau}\Big)^\alpha F(\tau)\,\frac{d\tau}{t} \quad\text{with}\quad F(\tau) \overset{\text{def}}{=} \|\tau^\alpha f(\tau)\|_{L^p(\Omega)},$$

that is to say,

$$\|t^{\alpha-1}v(t)\|_{L^p(\Omega)} \leq \int_0^1 (\tau')^{-\alpha} F(t\tau')\,d\tau'.$$

Hence taking the norm in $L^r(0,T)$, and using Minkowski inequality and $\alpha+1/r<1$,

$$\begin{aligned}\|t^{\alpha-1}v\|_{L^r_T(L^p(\Omega))} &\leq \int_0^1 (\tau')^{-\alpha}\biggl(\int_0^T F^r(t\tau')\,dt\biggr)^{\frac1r}d\tau'\\ &\leq \int_0^1 (\tau')^{-\alpha-1/r}\biggl(\int_0^{\tau'T} F^r(t')\,dt'\biggr)^{\frac1r}d\tau'\\ &\leq C\|F\|_{L^r(0,T)}.\end{aligned}$$

Plugging this inequality in (A.6) completes the proof. □

Remark A.1. *Applying the above result in the Stokes case implies that the function v defined in* (A.5) *and the corresponding gradient term $\nabla\Pi$ satisfy*

$$\|t^\alpha(\partial_t v,\nabla^2 v,\nabla\Pi)\|_{L^r_T(L^p(\Omega))}\leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.7}$$

The next lemma provides estimates on the gradient.

Lemma A.3. *Let $1<p,q,r<\infty$ with $p\leq q$. Let Operator $\mathcal{B}$ be defined by*

$$\mathcal{B}:f\longmapsto\biggl[t\to\int_0^t\nabla e^{-(t-\tau)A_p}f(\tau)\,d\tau\biggr]\cdot$$

(1) *If in addition $\frac dp-\frac dq\leq1$ then for all real numbers α and β satisfying*

$$\beta=\alpha+\frac d2\biggl(\frac1p-\frac1q\biggr)-\frac12\quad\text{and}\quad\alpha r'<1 \tag{A.8}$$

there exists a constant C so that for all $T>0$,

$$\|t^\beta\mathcal{B}f\|_{L^r_T(L^q(\Omega))}\leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.9}$$

(2) *If in addition $\frac dp-\frac dq<1-\frac2r$ then for all real numbers α and β satisfying*

$$\beta=\alpha+\frac d2\biggl(\frac1p-\frac1q\biggr)-\frac12+\frac1r\quad\text{and}\quad\alpha r'<1 \tag{A.10}$$

there exists a constant C so that for all $T>0$,

$$\|t^\beta\mathcal{B}f\|_{L^\infty_T(L^q(\Omega))}\leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.11}$$

(3) *More generally, for any $s\in[r,\infty]$, if $\frac dp-\frac dq<1-\frac2r+\frac2s$ and (α,β) satisfy*

$$\beta=\alpha+\frac d2\biggl(\frac1p-\frac1q\biggr)-\frac12+\frac1r-\frac1s\quad\text{and}\quad\alpha r'<1 \tag{A.12}$$

then there exists a constant C so that for all $T>0$,

$$\|t^\beta\mathcal{B}f\|_{L^s_T(L^q(\Omega))}\leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.13}$$

Proof. The limit case $\beta=\alpha$ of the first inequality is a consequence of Lemma A.2 and Sobolev embedding. To treat the case $\beta<\alpha$, the starting point is the following inequality

$$\|\nabla e^{-tA_p}f\|_{L^q(\Omega)}\leq Ct^{-\delta}\|f\|_{L^p(\Omega)}\quad\text{with}\quad\delta\stackrel{\text{def}}{=}\frac d2\biggl(\frac1p-\frac1q\biggr)+\frac12 \tag{A.14}$$

which holds true whenever $1<p\leq q<\infty$. It has been proved in [13] for the Stokes operator in bounded domains, and follows from an explicit computation for the heat operator in $\mathbb{R}^d$.

This inequality obviously implies that

$$
t^\beta\|\mathcal{B}f(t)\|_{L^q(\Omega)}\leq Ct^\beta\int_0^t(t-\tau)^{-\delta}\tau^{-\alpha}F(\tau)\,d\tau\quad\text{with}\quad F(\tau)\stackrel{\text{def}}{=}\|\tau^\alpha f(\tau)\|_{L^p(\Omega)}. \tag{A.15}
$$

Therefore, making a change of variables, and using the relationship between α and β,

$$
t^\beta\|\mathcal{B}f(t)\|_{L^q(\Omega)}\leq C\int_0^1(1-\tau')^{-\delta}(\tau')^{-\alpha}F(\tau't)\,d\tau'.
$$

Now, taking the L^r_T norm of both sides and applying Minkowski inequality implies that

$$
\|t^\beta\mathcal{B}f\|_{L^r_T(L^q(\Omega))}\leq C\int_0^1(1-\tau')^{-\delta}(\tau')^{-\alpha}\biggl(\int_0^T F^r(\tau't)\,dt\biggr)^{\frac1r}\,d\tau'.
$$

Then arguing exactly as in the proof of the previous lemma, we get Inequality (A.9) whenever (A.8) is satisfied.

In order to establish (A.11), we start from (A.15) and apply Hölder inequality. We obtain for all $t\in[0,T]$,

$$
t^\beta\|\mathcal{B}f(t)\|_{L^q(\Omega)}\leq Ct^\beta\biggl(\int_0^t(t-\tau)^{-r'\delta}\tau^{-\alpha r'}\,d\tau\biggr)^{\frac{1}{r'}}\|F\|_{L^r(0,T)}.
$$

Then making the usual change of variable and taking advantage of the relationship between α and β, and of the definition of F, we get for all $t\in[0,T]$,

$$
t^\beta\|\mathcal{B}f(t)\|_{L^q(\Omega)}\leq C\biggl(\int_0^t(1-\tau')^{-r'\delta}(\tau')^{-\alpha r'}\biggr)^{\frac{1}{r'}}\|t^\alpha f\|_{L^r_T(L^p(\Omega))}.
$$

It is now clear that we get Inequality (A.11) under the constraints of (A.10) and $\delta r'<1$.

In order to treat the general case, we have to combine the methods for proving (A.9) and (A.11). Starting from (A.15), we write

$$
I\stackrel{\text{def}}{=}\|t^\beta\mathcal{B}f(t)\|_{L^s_T(L^q(\Omega))}\leq C\biggl(\int_0^T t^{\beta s}\biggl(\int_0^t(t-\tau)^{-\delta\varphi}\,(t-\tau)^{-\delta(1-\varphi)}F(\tau)\tau^{-\alpha}\,d\tau\biggr)^s dt\biggr)^{\frac1s}
$$

where φ is a parameter in $[0,1]$, to be fixed hereafter.

Applying Hölder inequality to the inner integral, we get (with obvious notation)

$$
I^s\leq C\int_0^T t^{\beta s}\biggl(\int_0^t(t-\tau)^{-\delta\varphi\frac sr}F(\tau)\tau^{-\alpha}\,d\tau\biggr)^r\biggl(\int_0^t(t-\tau)^{-\delta(1-\varphi)(\frac sr)'}F(\tau)\tau^{-\alpha}\,d\tau\biggr)^{s-r}dt\cdot
$$

Applying again Hölder inequality, in the last integral only, we thus find that the above r.h.s. is bounded by

$$
\int_0^T t^{\beta s}\biggl(\int_0^t(t-\tau)^{-\delta\varphi\frac sr}F(\tau)\tau^{-\alpha}\,d\tau\biggr)^r\biggl(\int_0^t F^r(\tau)\,d\tau\biggr)^{\frac sr-1}\biggl(\int_0^t(t-\tau)^{-\delta(1-\varphi)(\frac sr)'r'}\tau^{-\alpha r'}\,d\tau\biggr)^{\frac{s-r}{r'}}dt\cdot
$$

Then we perform the same change of variables as above to get

$$
I^s\leq C\|F\|^{s-r}_{L^r(0,T)}\int_0^T\biggl(\int_0^1(1-\tau)^{-\delta\varphi\frac sr}F(t\tau)\tau^{-\alpha}\,d\tau\biggr)^r\biggl(\int_0^1(1-\tau)^{-\delta(1-\varphi)(\frac sr)'r'}\tau^{-\alpha r'}\,d\tau\biggr)^{\frac{s-r}{r'}}dt
$$

where we have used the fact that $\beta = \alpha + \delta - 1/r' - 1/s$. If we assume that $\alpha r' < 1$ and that $\delta(1-\varphi)(\frac{s}{r})' r' < 1$ then the last integral is bounded. Then applying Minkowski inequality to swap the integrals on $[0,T]$ and on $[0,1]$, we eventually get

$$I \leq C\|F\|_{L^r(0,T)}^{1-\frac{r}{s}} \biggl(\int_0^1 (1-\tau)^{-\delta\varphi\frac{s}{r}} \tau^{-\alpha} \biggl(\int_0^T F^r(t\tau)\,dt\biggr)^{\frac{1}{r}} d\tau\biggr)^{\frac{r}{s}},$$

whence

$$\|t^\beta \mathcal{B} f(t)\|_{L^s_T(L^q(\Omega))} \leq C\|F\|_{L^r(0,T)}^{1-\frac{r}{s}} \biggl(\int_0^1 (1-\tau)^{-\delta\varphi\frac{s}{r}} \tau^{-\alpha-\frac{1}{r}} \biggl(\int_0^{\tau T} F^r(t')\,dt'\biggr)^{\frac{1}{r}} d\tau\biggr)^{\frac{r}{s}},$$

which implies Inequality (A.13) provided $\alpha + 1/r < 1$ and $\delta\varphi s/r < 1$. In order to complete the proof, it is only a matter of taking $\varphi = \frac{r^2}{r+sr-s}$ so that the conditions $\delta(1-\varphi)(\frac{s}{r})' r' < 1$ and $\delta\varphi s/r < 1$ are equivalent. □

Finally we need a lemma involving the following operator

$$\mathcal{C} : f \longmapsto \biggl[t \to \int_0^t e^{-(t-\tau)A_p} f(\tau)\,d\tau\biggr]\cdot$$

Lemma A.4. *Let $1 < p, q, r < \infty$ with $q \geq p$.*

(1) *If in addition $\frac{d}{p} - \frac{d}{q} \leq 2$ then for all real numbers α and γ satisfying*

$$\gamma = \alpha + \frac{d}{2}\biggl(\frac{1}{p} - \frac{1}{q}\biggr) - 1 \quad \textit{and} \quad \alpha r' < 1 \tag{A.16}$$

there exists a constant C so that for all $T > 0$,

$$\|t^\gamma \mathcal{C} f\|_{L^r_T(L^q(\Omega))} \leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.17}$$

(2) *If in addition $\frac{d}{p} - \frac{d}{q} < 2 - \frac{2}{r}$ then for all real numbers α and γ satisfying*

$$\gamma = \alpha + \frac{d}{2}\biggl(\frac{1}{p} - \frac{1}{q}\biggr) - \frac{1}{r'} \quad \textit{and} \quad \alpha r' < 1 \tag{A.18}$$

there exists a constant C so that for all $T > 0$,

$$\|t^\gamma \mathcal{C} f\|_{L^\infty_T(L^q(\Omega))} \leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.19}$$

(3) *More generally, for any $s \in [r, \infty]$, if $\frac{d}{p} - \frac{d}{q} < 2 - \frac{2}{r} + \frac{2}{s}$ and (α, γ) satisfy*

$$\gamma = \alpha + \frac{d}{2}\biggl(\frac{1}{p} - \frac{1}{q}\biggr) - 1 + \frac{1}{r} - \frac{1}{s} \quad \textit{and} \quad \alpha r' < 1 \tag{A.20}$$

then there exists a constant C so that for all $T > 0$,

$$\|t^\gamma \mathcal{C} f\|_{L^s_T(L^q(\Omega))} \leq C\|t^\alpha f\|_{L^r_T(L^p(\Omega))}. \tag{A.21}$$

Proof. The limit case $\gamma = \alpha$ of the first inequality is a consequence of Lemma A.3 and Sobolev embedding. To treat the case $\gamma < \alpha$, the starting point is the following inequality

$$\|e^{-tA_p} f\|_{L^q(\Omega)} \leq Ct^{-\delta}\|f\|_{L^p(\Omega)} \quad \text{with} \quad \delta \stackrel{\text{def}}{=} \frac{d}{2}\biggl(\frac{1}{p} - \frac{1}{q}\biggr) \tag{A.22}$$

whenever $1 < p \leq q < \infty$, which has been proved in [13] for the Stokes operator in bounded domains, and follows from an explicit computation for the heat operator in $\mathbb{R}^d$.

This inequality obviously implies that

$$t^\gamma\|\mathcal{C}f(t)\|_{L^q(\Omega)}\leq Ct^\gamma\int_0^t(t-\tau)^{-\delta}\tau^{-\alpha}F(\tau)\,d\tau\quad\text{with}\quad F(\tau)\stackrel{\text{def}}{=}\|\tau^\alpha f(\tau)\|_{L^p(\Omega)}.\tag{A.23}$$

Therefore, making a change of variables, and using the relationship between α and γ,

$$t^\gamma\|\mathcal{B}f(t)\|_{L^q(\Omega)}\leq C\int_0^1(1-\tau')^{-\delta}(\tau')^{-\alpha}F(\tau't)\,d\tau'.$$

Now, taking the L^r_T norm of both sides and applying Minkowski inequality as in the above lemmas yields Inequality (A.17).

In order to establish (A.19), we start from (A.23) and apply Hölder inequality. We obtain for all $t\in[0,T]$,

$$t^\gamma\|\mathcal{B}f(t)\|_{L^q(\Omega)}\leq Ct^\gamma\biggl(\int_0^t(t-\tau)^{-r'\delta}\tau^{-\alpha r'}\,d\tau\biggr)^{\frac1{r'}}\|F\|_{L^r(0,T)}.$$

Then arguing as in the previous lemma easily leads to (A.19) under the constraints of (A.18). The general case $s\geq r$ follows from similar arguments. The details are left to the reader. □

We finally want to establish decay estimates for the free solution to heat equation in the whole space.

Lemma A.5. *Assume that $u_0\in\dot B^s_{p,r}(\mathbb{R}^d)$ with $1\leq p,r\leq\infty$. The following inequalities hold true:*

(1) *If $s<2$ then*

$$\|t^\alpha\nabla^2e^{t\Delta}u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))}\leq C\|u_0\|_{\dot B^s_{p,r}(\mathbb{R}^d)}\quad\textit{with}\quad\alpha\stackrel{def}{=}1-\frac s2-\frac1r\cdot\tag{A.24}$$

(2) *If $s<1$ then*

$$\|t^\beta\nabla e^{t\Delta}u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))}\leq C\|u_0\|_{\dot B^s_{p,r}(\mathbb{R}^d)}\quad\textit{with}\quad\beta\stackrel{def}{=}\frac12-\frac s2-\frac1r\cdot\tag{A.25}$$

(3) *If $s<0$ then*

$$\|t^\gamma e^{t\Delta}u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))}\leq C\|u_0\|_{\dot B^s_{p,r}(\mathbb{R}^d)}\quad\textit{with}\quad\gamma\stackrel{def}{=}-\frac s2-\frac1r\cdot\tag{A.26}$$

Proof. The assumption ensures that $\nabla^2u_0\in\dot B^{s-2}_{p,r}(\mathbb{R}^d)$. Because $s-2<0$, Proposition A.1 yields

$$\|t^{1-\frac s2-\frac1r}e^{t\Delta}\nabla^2u_0\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}^d))}=\|t^{1-\frac s2}\|e^{t\Delta}\nabla^2u_0\|_{L^p(\mathbb{R}^d)}\|_{L^r(\mathbb{R}_+;\frac{dt}{t})}\approx\|\nabla^2u_0\|_{\dot B^{s-2}_{p,r}(\mathbb{R}^d)}.$$

The proof of the other inequalities is totally similar. □

Remark A.2. *In this paper, we mainly consider the case where $s=-1+\frac dp\cdot$ The corresponding values of (α,β,γ) are*

$$\alpha=\frac32-\frac d{2p}-\frac1r\ \text{ if }\ p>\frac d3,\qquad\beta=1-\frac d{2p}-\frac1r\ \text{ if }\ p>\frac d2,\qquad\gamma=\frac12-\frac d{2p}-\frac1r\ \text{ if }\ p>d.$$

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