Local limits of conditioned Galton-Watson trees I: the infinite spine case.

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ABSTRACT. We give a necessary and sufficient condition for the convergence in distribution of a conditioned Galton-Watson tree to Kesten's tree. This yields elementary proofs of Kesten's result as well as other known results on local limits of conditioned Galton-Watson trees. We then apply this condition to get new results in the critical case, with general offspring distribution, and in the sub-critical cases, with generic offspring distribution, on the limit in distribution of a Galton-Watson tree conditioned on having a large number of individuals with out-degree in a given set.

1. INTRODUCTION

Galton-Watson (GW) processes constitute a very simple model of population growth where all individuals give birth independently of each others to a random number of children with the same offspring distribution p. This population growth can be described by a genealogical tree τ that we call the GW tree. It is well-known that in the sub-critical case (the mean number of children of a single individual is strictly less than 1) and in the (non-degenerate) critical case (the mean number of children of an individual is 1) the population becomes a.s. instinct. However, one can define in these two cases a tree τ^* with an infinite spine, that we call Kesten's tree in this paper, which can be seen as the tree conditioned on non-extinction, defined as the local limit in distribution of the tree τ conditioned to reach height n, when n tends to infinity, see KESTEN [15]. This result is recalled here in Section 2.4. The tree τ^* happens to be the size-biased tree already studied earlier, see e.g. HAWKES [9], JOFFE and WAUGH [12] as well as LYONS, PEMANTLE and PERES [18]. It also appears (for GW processes only) as a Q-process and can be viewed as a GW tree with immigration, see ATHREYA and NEY [3].

We want to stress that we only consider here local limits i.e. we look at the trees up to a fixed height h. Other limits can be considered such as scaling limits of conditioned GW trees (see [6, 17, 21]) but this is not the purpose here.

It is also known that, at least in the critical case, other conditionings such as conditioning by the total progeny, see KENNEDY [14] and GEIGER and KAUFMANN [8], or by the number of leaves, see CURIEN and KORTCHEMSKI [5], lead to the same local limit in distribution. See also the survey from JANSON [11].

For all those cases, the conditioning event can be written as $\{\tau \in \mathbb{A}_n\}$ with $\mathbb{A}_n$ of the form:

$$\mathbb{A}_n = \{\mathbf{t}, \ A(\mathbf{t}) \geq n\} \quad \text{or} \quad \mathbb{A}_n = \{\mathbf{t}, \ A(\mathbf{t}) = n\},$$

where $A : \mathbf{t} \mapsto A(\mathbf{t})$ is a functional defined of the set of trees and satisfying an additive property, see Equation (9). We prove our following main result, see Theorem 3.1 for a precise statement, which unifies all the previous conditionings and gives a necessary and sufficient condition to obtain Kesten's tree as a limit. In the non-degenerate critical case, if A satisfies

the additive property (9) (with some aperiodic condition for the converse), then the following two statements are equivalent:

- $\lim_{n\to+\infty} \mathbb{P}(\tau \in \mathbb{A}_{n+1})/\mathbb{P}(\tau \in \mathbb{A}_n) = 1$,
- The distribution of τ conditionally on $\{\tau \in \mathbb{A}_n\}$ converges to the distribution of Kesten's tree τ^*.

Using this result, we give elementary proofs for the convergence in distribution to Kesten's tree τ^* of the GW tree conditioned on:

(i) Extinction after or at a large time (sub-critical and critical case), with $A(\mathbf{t}) = H(\mathbf{t})$ the height of the tree $\mathbf{t}$ and conditioning event $\{H(\tau) = n\}$ or $\{H(\tau) \geq n\}$. See Sections 4.1 and 4.2.

(ii) Large total population size (critical case), with $A(\mathbf{t}) = \mathrm{Card}\,(\mathbf{t})$ the total size of the tree and conditioning event $\{\mathrm{Card}\,(\tau) = n\}$ or $\{\mathrm{Card}\,(\tau) \geq n\}$. See Section 4.3.

(iii) Large number of leaves (critical case), with $A(\mathbf{t}) = L_0(\mathbf{t})$ the total number of leaves of $\mathbf{t}$ and conditioning event $\{L_0(\tau) = n\}$ or $\{L_0(\tau) \geq n\}$. See Section 4.4.

Let us mention that assertion (i) with the conditioning event $\{H(\tau) \geq n\}$ was first proved by KESTEN [15] in the critical case under a finite variance condition, and in [11], Theorem 7.1, in full generality. Property (ii) is also proved in full generality in [11], Theorem 7.1 (the sub-critical case is also studied in [11], see the discussion below). Finally, assertion (iii) with the conditioning event $\{L_0(\tau) = n\}$ has been proved by CURIEN and KORTCHEMSKI [5], Theorem 4.1, in the critical and finite variance case only.

In fact the conditioning on the large total population size or on the large number of leaves are particular cases of conditioning trees on large number of individuals with a given number of children. This corresponds to the functional $A(t) = L_{\mathcal{A}}(\mathbf{t})$ which gives the total number of individuals of the tree $\mathbf{t}$ whose number of children belongs to a given set $\mathcal{A}$ of nonnegative integers. Such conditioning has already been studied by RIZZOLO [21], see also MIMAMI [19], but for global scaling limits and not local limits. We obtain the convergence in distribution to Kesten's tree τ^* of a critical GW tree without any additional moment condition on the offspring distribution, conditioned on:

(iv) Large number of individuals with number of children in a given set $\mathcal{A}$ (critical case), with $A(\mathbf{t}) = L_{\mathcal{A}}(\mathbf{t})$ and conditioning event $\{L_{\mathcal{A}}(\tau) = n\}$ or $\{L_{\mathcal{A}}(\tau) \geq n\}$.

When $0 \in \mathcal{A}$, $L_{\mathcal{A}}(\tau)$ is distributed according to the total progeny of another critical GW tree, which allows to use (ii), but when $0 \notin \mathcal{A}$, another method is needed. Let us remark that the total progeny ($\mathcal{A} = \mathbb{N}$), the number of leaves ($\mathcal{A} = \{0\}$) and the number of internal nodes ($\mathcal{A} = \mathbb{N} \setminus \{0\}$) are particular cases of this conditioning.

We then study the subcritical case and define a one-parameter family $(p_\theta, \theta \in I)$ of distributions on the set of integers such that the GW tree τ associated with the offspring distribution p and the GW tree τ_θ associated with the offspring distribution p_θ have the same conditional distributions given $L_{\mathcal{A}}$, see Proposition 5.7. This generalizes Kennedy's transformation [14] concerning the total progeny, and the pruning of ABRAHAM, DELMAS and HE [2] concerning the number of leaves. According to [11], we say that p is generic (with respect to $\mathcal{A}$) if there exists θ_c such that p_{θ_c} is critical. We then immediately deduce, see Corollary 5.9, that if p is generic, then the distribution of τ conditionally on $\{L_{\mathcal{A}}(\tau) = n\}$ (in the aperiodic case) or on $\{L_{\mathcal{A}}(\tau) \geq n\}$ converges to the distribution of the Kesten's tree $\tau^*_{\theta_c}$ associated with the critical offspring distribution p_{θ_c}. When there is no such θ_c, then a condensation phenomenon may appear: JONSSON and STEFANSSON [13] or [11] proved for the conditioning on the total progeny that the limiting tree in that case is not Kesten's tree but a tree with a unique node

with an infinite number of offsprings. We shall investigate this condensation phenomenon for other conditionings in a forthcoming paper [1].

Finally, we consider another conditioning which does not enter in the framework of Theorem 3.1 : conditioning on the size on the n-th generation. However, we can adapt the proof of Theorem 3.1 to get an analogous result in that case, see Proposition 6.1. We apply this result to a geometric offspring distribution where explicit computations can be performed to prove that the corresponding tree conditioned on the n-th generation being of size n converges in distribution to Kesten's tree.

The paper is organized as follows. In Section 2, we recall the framework we use for discrete trees and define the GW tree τ and Kesten's tree τ^* associated with offspring distribution p. In Section 3, we state and prove the necessary and sufficient condition for convergence in distribution of the conditioned tree to Kesten's tree. We apply this result in Section 4 to recover the classical results on critical conditioned GW trees and we study in Section 5 the case of the number of individuals with out-degree in a given set for the critical and sub-critical case. Finally, we study in Section 6 the conditioning on the size of the n-th generation of the tree.

2. Technical background on GW trees

2.1. **First notations.** We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers and by $\mathbb{N}^* = \{1, 2, \ldots\}$ the set of positive integers.

If K is a subset of $\mathbb{N}^*$, we call the span of K the greatest common divisor of K. If X is an integer-valued random variable, we call the span of X the span of $\{n > 0, \ \mathbb{P}(X = n) > 0\}$ the restriction to $\mathbb{N}^*$ of its support.

2.2. **The set of discrete trees.** We recall Neveu's formalism [20] for ordered rooted trees. We let

$$\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$$

be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. For $u \in \mathcal{U}$ let $|u|$ be the length or generation of u defined as the integer n such that $u \in (\mathbb{N}^*)^n$. If u and v are two sequences of $\mathcal{U}$, we denote by uv the concatenation of the two sequences, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$. The set of ancestors of u is the set:

(1) $$A_u = \{v \in \mathcal{U}; \text{there exists } w \in \mathcal{U}, \ w \neq \emptyset, \text{ such that } u = vw\}.$$

The most recent common ancestor of a subset $\mathbf{s}$ of $\mathcal{U}$, denoted by $\mathrm{M}(\mathbf{s})$, is the unique element u of $\bigcap_{u \in \mathbf{s}} A_u$ with maximal length $|u|$.

A tree $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies:

- $\emptyset \in \mathbf{t}$,
- If $u \in \mathbf{t}$, then $A_u \subset \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists a non-negative integer $k_u(\mathbf{t})$ such that, for every positive integer i, $ui \in \mathbf{t}$ iff $1 \leq i \leq k_u(\mathbf{t})$.

The integer $k_u(\mathbf{t})$ represents the number of offspring of the vertex $u \in \mathbf{t}$. The vertex $u \in \mathbf{t}$ is called a leaf if $k_u(\mathbf{t}) = 0$. The vertex $\emptyset$ is called the root of $\mathbf{t}$. Let us remark that, for a tree $\mathbf{t}$, we have

(2) $$\sum_{u \in \mathbf{t}} k_u = \mathrm{Card}\,(\mathbf{t}) - 1.$$

Let $\mathbf{t}$ be a tree. The set of its leaves is $\mathcal{L}_0(\mathbf{t}) = \{u \in \mathbf{t}; k_u(\mathbf{t}) = 0\}$, its height is defined by

$$H(\mathbf{t}) = \sup\{|u|, \ u \in \mathbf{t}\}$$

and can be infinite. For $u \in \mathbf{t}$, we define the sub-tree $\mathcal{S}_u(\mathbf{t})$ of $\mathbf{t}$ "above" u as:

$$\mathcal{S}_u(\mathbf{t}) = \{v \in \mathcal{U}, \ uv \in \mathbf{t}\}.$$

We denote by $\mathbb{T}$ the set of trees, by

$$\mathbb{T}_0 = \{\mathbf{t} \in \mathbb{T}; \mathrm{Card}\,(\mathbf{t}) < +\infty\}$$

the subset of finite trees, by

$$\mathbb{T}^{(h)} = \{\mathbf{t} \in \mathbb{T}; H(\mathbf{t}) \leq h\}$$

the subset of trees with height at most $h \in \mathbb{N}$, and by

$$\mathbb{T}_1 = \{\mathbf{t} \in \mathbb{T}; \lim_{n \to +\infty} |\mathrm{M}(\{u \in \mathbf{t}; |u| = n\})| = +\infty\}$$

the subset of trees with a unique infinite spine. Notice that $\mathbb{T}_0$ and $\mathbb{T}^{(h)}$ are countable and $\mathbb{T}_1$ is uncountable. For $h \in \mathbb{N}$ the restriction function r_h from $\mathbb{T}$ to $\mathbb{T}$ is defined by:

$$r_h(\mathbf{t}) = \{u \in \mathbf{t}, \ |u| \leq h\}.$$

We endow the set $\mathbb{T}$ with the ultrametric distance

$$d(\mathbf{t}, \mathbf{t}') = 2^{-\max\{h \in \mathbb{N}, \ r_h(\mathbf{t}) = r_h(\mathbf{t}')\}}.$$

A sequence $(\mathbf{t}_n, n \in \mathbb{N})$ of trees converges to a tree $\mathbf{t}$ with respect to the distance d if and only if, for every $h \in \mathbb{N}$,

$$r_h(\mathbf{t}_n) = r_h(\mathbf{t}) \qquad \text{for } n \text{ large enough.}$$

The Borel σ-field associated with the distance d is the smallest σ-field containing the singletons for which the restrictions functions $(r_h, h \in \mathbb{N})$ are measurable. With this distance, the restriction functions are contractant. Since $\mathbb{T}_0$ is dense in $\mathbb{T}$ and $(\mathbb{T}, d)$ is complete, we get that $(\mathbb{T}, d)$ is a Polish metric space.

Consider the closed ball $B(\mathbf{t}, 2^{-h}) = \{\mathbf{t}' \in \mathbb{T}; d(\mathbf{t}, \mathbf{t}') \leq 2^{-h}\}$ for some $\mathbf{t} \in \mathbb{T}$ and $h \in \mathbb{N}$ and notice that:

$$B(\mathbf{t}, 2^{-h}) = r_h^{-1}(\{r_h(\mathbf{t})\}).$$

Since the distance is ultrametric, the closed balls are open and the open balls are closed, and the intersection of two balls is either empty or one of them. We deduce that the family $((r_h^{-1}(\{\mathbf{t}\}), \mathbf{t} \in \mathbb{T}^{(h)}), h \in \mathbb{N})$ is a π-system, and Theorem 2.3 in [4] implies that this family is convergence determining for the convergence in distribution. Let $(T_n, n \in \mathbb{N}^*)$ and T be $\mathbb{T}$-valued random variables. We denote by dist (T) the distribution of the random variable T (which is uniquely determined by the sequence of distributions of $r_h(T)$ for every $h \geq 0$), and we denote

$$\mathrm{dist}\,(T_n) \xrightarrow[n \to +\infty]{} \mathrm{dist}\,(T)$$

for the convergence in distribution on the sequence $(T_n, n \in \mathbb{N}^*)$ to T.

We deduce from the Portmanteau theorem that the sequence $(T_n, n \in \mathbb{N}^*)$ converge in distribution to T if and only if for all $h \in \mathbb{N}$, $\mathbf{t} \in \mathbb{T}^{(h)}$:

$$\lim_{n \to +\infty} \mathbb{P}(r_h(T_n) = \mathbf{t}) = \mathbb{P}(r_h(T) = \mathbf{t}).$$

As we shall only consider $\mathbb{T}_0$-valued random variables that converge in distribution to a $\mathbb{T}_1$-valued random variable, we shall give an other characterization of convergence in distribution

that holds for this restriction. To present this result, we introduce some notations. If $\mathbf{t}, \mathbf{s} \in \mathbb{T}$ and $x \in \mathcal{L}_0(\mathbf{t})$ we denote by:

$$\mathbf{t} \circledast (\mathbf{s}, x) = \{u \in \mathbf{t}\} \cup \{xv, v \in \mathbf{s}\}$$

the tree obtained by grafting the tree $\mathbf{s}$ on the leaf x of the tree $\mathbf{t}$. For every $\mathbf{t} \in \mathbb{T}$ and every $x \in \mathcal{L}_0(\mathbf{t})$, we shall consider the set of trees obtained by grafting a tree on the leaf x of $\mathbf{t}$:

$$\mathbb{T}(\mathbf{t}, x) = \{\mathbf{t} \circledast (\mathbf{s}, x), \ \mathbf{s} \in \mathbb{T}\}.$$

It is easy to see that $\mathbb{T}(\mathbf{t}, x)$ is closed. It is also open, as for all $\mathbf{s} \in \mathbb{T}(\mathbf{t}, x)$ we have that $B(\mathbf{s}, 2^{-H(\mathbf{t})-1}) \subset \mathbb{T}(\mathbf{t}, x)$.

Moreover, notice that the set $\mathbb{T}_1$ is a Borel subset of the set $\mathbb{T}$.

Lemma 2.1. *Let $(T_n, n \in \mathbb{N}^*)$ and T be $\mathbb{T}$-valued random variables which belong a.s. to $\mathbb{T}_0 \bigcup \mathbb{T}_1$. The sequence $(T_n, n \in \mathbb{N}^*)$ converges in distribution to T if and only if for every $\mathbf{t} \in \mathbb{T}_0$ and every $x \in \mathcal{L}_0(\mathbf{t})$, we have:*

(3) $$\lim_{n \to +\infty} \mathbb{P}(T_n \in \mathbb{T}(\mathbf{t}, x)) = \mathbb{P}(T \in \mathbb{T}(\mathbf{t}, x)) \quad and \quad \lim_{n \to +\infty} \mathbb{P}(T_n = \mathbf{t}) = \mathbb{P}(T = \mathbf{t}).$$

Proof. The subclass $\mathcal{F} = \{\mathbb{T}(\mathbf{t}, x), \ \mathbf{t} \in \mathbb{T}_0, \ x \in \mathcal{L}_0(\mathbf{t})\} \cup \{\{\mathbf{t}\}, \ \mathbf{t} \in \mathbb{T}_0\}$ of the Borel sets on $T_0 \bigcup \mathbb{T}_1$ forms a π-system since we have

$$\mathbb{T}(\mathbf{t}_1, x_1) \cap \mathbb{T}(\mathbf{t}_2, x_2) = \begin{cases} \mathbb{T}(\mathbf{t}_1, x_1) & \text{if } \mathbf{t}_1 \in \mathbb{T}(\mathbf{t}_2, x_2), \\ \mathbb{T}(\mathbf{t}_2, x_2) & \text{if } \mathbf{t}_2 \in \mathbb{T}(\mathbf{t}_1, x_1), \\ \{\mathbf{t}_1\} & \text{if } \mathbf{t}_1 = \mathbf{t}_2 \text{ and } x_1 \neq x_2, \\ \emptyset & \text{in the other cases.} \end{cases}$$

For every $h \in \mathbb{N}$ and every $\mathbf{t} \in \mathbb{T}^{(h)}$, we have that $\mathbf{t}'$ belongs to $r_h^{-1}(\{\mathbf{t}\}) \bigcap \mathbb{T}_1$ if and only if $\mathbf{t}'$ belongs to some $\mathbb{T}(\mathbf{s}, x)$ where x is a leaf of $\mathbf{t}$ such that $|x| = h$ and $\mathbf{s}$ belongs to $r_h^{-1}(\{\mathbf{t}\}) \bigcap \mathbb{T}_0$ such that x is also a leaf of $\mathbf{s}$. Since $\mathbb{T}_0$ is countable, we deduce that $\mathcal{F}$ generates the Borel σ-field on $\mathbb{T}_0 \cup \mathbb{T}_1$. In particular $\mathcal{F}$ is a separating class on $\mathbb{T}_0 \bigcup \mathbb{T}_1$.

Since $A \in \mathcal{F}$ is closed and open as well, according to Theorem 2.3 of [4], to prove that the family $\mathcal{F}$ is a convergence determining class, it is enough to check that for all $\mathbf{t} \in \mathbb{T}_0 \bigcup \mathbb{T}_1$ and $h \in \mathbb{N}$, there exists $A \in \mathcal{F}$ such that:

(4) $$\mathbf{t} \in A \subset B(\mathbf{t}, 2^{-h}).$$

If $\mathbf{t} \in \mathbb{T}_0$, this is clear as $\{\mathbf{t}\} = B(\mathbf{t}, 2^{-h})$ for all $h > H(\mathbf{t})$. If $\mathbf{t} \in \mathbb{T}_1$, for all $\mathbf{s} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{s})$ such that $\mathbf{t} \in \mathbb{T}(\mathbf{s}, x)$, we have $\mathbf{t} \in \mathbb{T}(\mathbf{s}, x) \subset B(\mathbf{t}, 2^{-|x|})$. Since we can find such a $\mathbf{s}$ and x such that $|x|$ is arbitrary large, we deduce that (4) is satisfied. This proves that the family $\mathcal{F}$ is a convergence determining class on $\mathbb{T}_0 \bigcup \mathbb{T}_1$.

Since, for $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$ the sets $\mathbb{T}(\mathbf{t}, x)$ and $\{\mathbf{t}\}$ are open and closed, we deduce from the Portmanteau Theorem that if $(T_n, n \in \mathbb{N}^*)$ converges in distribution to T, then (3) holds for every $\mathbf{t} \in \mathbb{T}_0$ and every $x \in \mathcal{L}_0(\mathbf{t})$. $\square$

2.3. **GW trees.** Let $p = (p(n), n \in \mathbb{N})$ be a probability distribution on the set of the non-negative integers. We assume that

(5) $$p(0) > 0, \ p(0) + p(1) < 1, \ \text{and} \ \mu := \sum_{n=0}^{+\infty} np(n) < +\infty.$$

A $\mathbb{T}$-valued random variable τ is a Galton-Watson (GW) tree with offspring distribution p if the distribution of $k_\emptyset(\tau)$ is p and for $n \in \mathbb{N}^*$, conditionally on $\{k_\emptyset(\tau) = n\}$, the sub-trees

$(\mathcal{S}_1(\tau), \mathcal{S}_2(\tau), \ldots, \mathcal{S}_n(\tau))$ are independent and distributed as the original tree τ. Equivalently, for every $h \in \mathbb{N}^*$ and every $\mathbf{t} \in \mathbb{T}^{(h)}$, we have

$$\mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{u \in r_{h-1}(\mathbf{t})} p(k_u(\mathbf{t})).$$

In particular, the restriction of the distribution of τ on the set $\mathbb{T}_0$ is given by:

$$(6) \qquad \forall \mathbf{t} \in \mathbb{T}_0, \quad \mathbb{P}(\tau = \mathbf{t}) = \prod_{u \in \mathbf{t}} p(k_u(\mathbf{t})).$$

The GW tree is called critical (resp. sub-critical, super-critical) if $\mu = 1$ (resp. $\mu < 1$, $\mu > 1$).

2.4. **Conditioning on non-extinction.** Let p be an offspring distribution satisfying Assumption (5) with $\mu \leq 1$ (i.e. the associated GW process is critical or sub-critical). We denote by $p^* = (p^*(n) = np(n)/\mu, n \in \mathbb{N})$ the corresponding size-biased distribution.

We define an infinite random tree τ^* (the size-biased tree that we call Kesten's tree in that paper), whose distribution is as follows. There exists a unique infinite sequence $(V_k, k \in \mathbb{N}^*)$ of positive integers such that, for every $h \in \mathbb{N}$, $V_1 \cdots V_h \in \tau^*$, with the convention that $V_1 \cdots V_h = \emptyset$ if $h = 0$. The joint distribution of $(V_k, k \in \mathbb{N}^*)$ and τ^* is determined recursively as follows: for each $h \in \mathbb{N}$, conditionally given $(V_1, \ldots, V_h)$ and $r_h(\tau^*)$, we have:

- The number of children $(k_v(\tau^*), \ v \in \tau^*, \ |v| = h)$ are independent and distributed according to p if $v \neq V_1 \cdots V_h$ and according to p^* if $v = V_1 \cdots V_h$.
- Given also the numbers of children $(k_v(\tau^*), \ v \in \tau^*, \ |v| = h)$, the integer V_{h+1} is uniformly distributed on the set of integers $\{1, \ldots, k_{V_1 \cdots V_h}(\tau^*)\}$.

Notice that by construction, $\tau^* \in \mathbb{T}_1$ a.s.

Following Kesten [15], the random tree τ^* can be viewed as the tree τ conditioned on non-extinction as:

$$\forall h \in \mathbb{N}^*, \ \forall \mathbf{t} \in \mathbb{T}^{(h)}, \ \mathbb{P}(r_h(\tau^*) = \mathbf{t}) = \lim_{n \to +\infty} \mathbb{P}(r_h(\tau) = \mathbf{t} \mid H(\tau) \geq n).$$

The distribution of τ^* is as follows. Almost surely, τ^* belongs to $\mathbb{T}_1$, that is

As a direct consequence we get that for all $h \in \mathbb{N}$, $\mathbf{t} \in \mathbb{T}^{(h)}$, $u \in \mathbf{t}$ such that $|u| = h$:

$$\mathbb{P}(r_h(\tau^*) = \mathbf{t}, V_1 \cdots V_h = u) = \mu^{-h} \mathbb{P}(r_h(\tau) = \mathbf{t}),$$

and for all $\mathbf{t} \in \mathbb{T}_0$, $x \in \mathcal{L}_0(\mathbf{t})$:

$$(7) \qquad \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) = \mu^{-|x|} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)).$$

Since, for $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$, $\mathbb{P}(\tau = \mathbf{t}) = \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), k_x(\tau) = 0) = \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x))p(0)$, we deduce that:

$$(8) \qquad \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) = \frac{1}{\mu^{|x|} p(0)} \mathbb{P}(\tau = \mathbf{t}).$$

Since τ^* is in $\mathbb{T}_1$ a.s., this implies that (8) with $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$ characterizes the distribution of τ^*.

3. MAIN RESULT

Let A be an integer-valued function defined on $\mathbb{T}$ which is finite on $\mathbb{T}_0$ and satisfies the following additivity property: there exists an integer-valued function D defined on $\mathbb{T}$ such that, for every $\mathbf{t} \in \mathbb{T}_0$, every $x \in \mathcal{L}_0(\mathbf{t})$ and for every $\tilde{\mathbf{t}}$ such that $A(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x))$ is large enough,

$$(9) \qquad A(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = A(\tilde{\mathbf{t}}) + D(\mathbf{t}, x).$$

Let $n_0 \in \mathbb{N} \cup \{+\infty\}$ be given. We define for all $n \in \mathbb{N}^*$, the subset of trees

$$\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}; A(\mathbf{t}) \in [n, n + n_0)\}.$$

Typical values of n_0 that will be considered are 1 and $+\infty$.

The following theorem states that the distribution of the GW tree τ conditioned to be in $\mathbb{A}_\infty$, the limit of $\mathbb{A}_n$, is distributed as τ^* as soon as the probability of $\mathbb{A}_n$ satisfies some regularity. We denote by

$$\text{dist} \, (\tau | \tau \in \mathbb{A}_n)$$

the conditional law of τ given $\{\tau \in \mathbb{A}_n\}$.

Theorem 3.1. *Assume that Assumptions (5) and (9) hold, that $\mathbb{P}(\tau \in \mathbb{A}_n) > 0$ for n large enough and that one of the two following conditions*

- *$\mu = 1$ or*
- *$\mu < 1$ and $D(\mathbf{t}, x) = |x|$ for all $\mathbf{t} \in \mathbb{T}_0$, $x \in \mathcal{L}_0(\mathbf{t})$.*

Then, if

$$(10) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(\tau \in \mathbb{A}_{n+1})}{\mathbb{P}(\tau \in \mathbb{A}_n)} = \mu,$$

we have:

$$\text{dist} \, (\tau | \tau \in \mathbb{A}_n) \underset{n \to +\infty}{\longrightarrow} \text{dist} \, (\tau^*).$$

Conversely, if $\text{dist} \, (\tau | \tau \in \mathbb{A}_n) \underset{n \to +\infty}{\longrightarrow} \text{dist} \, (\tau^*)$ *and if the span of $\{D(\mathbf{t}, x); \mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})\} \bigcap \mathbb{N}^*$ is one, then (10) holds.*

Recall that the local convergence in distribution towards τ^* is equivalent to

$$(11) \qquad \forall h \in \mathbb{N}^*, \ \forall \mathbf{t} \in \mathbb{T}^{(h)}, \ \lim_{n \to +\infty} \mathbb{P}(r_h(\tau) = \mathbf{t} \mid \tau \in \mathbb{A}_n) = \mathbb{P}(r_h(\tau^*) = \mathbf{t}).$$

Proof. Let us first remark that, as we supposed that $m \leq 1$, we have a.s. $\tau \in \mathbb{T}_0$ and thus we are in the setting of Lemma 2.1.

Using (6), we have for every $\mathbf{t} \in \mathbb{T}_0$, $x \in \mathcal{L}_0(\mathbf{t})$ and $\tilde{\mathbf{t}} \in \mathbb{T}_0$:

$$\mathbb{P}(\tau = \mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = \frac{1}{p(0)} \mathbb{P}(\tau = \mathbf{t}) \mathbb{P}(\tau = \tilde{\mathbf{t}}).$$

Let $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$. We deduce that for n large enough, so that we can apply Equation (9):

$$\begin{aligned}
\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \tau \in \mathbb{A}_n) &= \sum_{\tilde{\mathbf{t}} \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) \mathbf{1}_{\{n \leq A(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) < n + n_0\}} \\
&= \frac{1}{p(0)} \sum_{\tilde{\mathbf{t}} \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t}) \mathbb{P}(\tau = \tilde{\mathbf{t}}) \mathbf{1}_{\{n \leq A(\tilde{\mathbf{t}}) + D(\mathbf{t}, x) < n + n_0\}} \\
&= \frac{1}{p(0)} \mathbb{P}(\tau = \mathbf{t}) \mathbb{P}(n - D(\mathbf{t}, x) \leq A(\tau) < n + n_0 - D(\mathbf{t}, x)) \\
&= \mu^{|x|} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \mathbb{P}(\tau \in \mathbb{A}_{n - D(\mathbf{t}, x)}),
\end{aligned}$$

where we used (8) for the last equality. Therefore we have

$$(12) \qquad \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x) \mid \tau \in \mathbb{A}_n) = \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \, \mu^{|x|} \frac{\mathbb{P}(\tau \in \mathbb{A}_{n - D(\mathbf{t}, x)})}{\mathbb{P}(\tau \in \mathbb{A}_n)}.$$

Then, using (10) and that $D(\mathbf{t}, x) = |x|$ if $\mu < 1$, we obtain that:

$$(13) \qquad \lim_{n \to +\infty} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x) \mid \tau \in \mathbb{A}_n) = \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)).$$

For all $\mathbf{t} \in \mathbb{T}_0$ and all $n > A(\mathbf{t})$, we have

$$\mathbb{P}(\tau = \mathbf{t}, \tau \in \mathbb{A}_n) = \mathbb{P}(\tau = \mathbf{t}, \mathbf{t} \in \mathbb{A}_n) \leq \mathbf{1}_{\{\mathbf{t} \in \mathbb{A}_n\}} = 0$$

and thus:

$$(14) \qquad \lim_{n \to +\infty} \mathbb{P}(\tau = \mathbf{t} \mid \tau \in \mathbb{A}_n) = 0 = \mathbb{P}(\tau^* = \mathbf{t}).$$

We deduce from Lemma 2.1 that (11) holds.

Conversely, if (11) holds, then Lemma 2.1 implies that (13) and (14) hold. The fact that the span of $\{D(\mathbf{t}, x); \mathbf{t} \in \mathbb{T}_0 \text{ and } x \in \mathcal{L}_0(\mathbf{t})\} \bigcap \mathbb{N}^*$ is one and (12) imply, with Bezout theorem, that (10) holds. $\qquad \square$

4. Examples

4.1. **Conditioning on extinction after large time.** We give here a simple proof of Kesten's result for the convergence in distribution of a critical or sub-critical GW tree conditioned on non-extinction, see [15] under a finite variance condition and [11] for the general case.

Proposition 4.1. *Let τ be a critical or sub-critical GW tree with offspring distribution p satisfying Assumption (5). Then, we have*

$$(15) \qquad \mathrm{dist}\,(\tau | H(\tau) \geq n) \underset{n \to +\infty}{\longrightarrow} \mathrm{dist}\,(\tau^*).$$

Proof. Consider $A(\mathbf{t}) = H(\mathbf{t})$ and $n_0 = +\infty$ that is $\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}; H(\mathbf{t}) \geq n\}$. Notice that in this case for a tree $\tilde{\mathbf{t}}$ such that $H(\tilde{\mathbf{t}})$ is larger than $H(\mathbf{t})$, we have for every $x \in \mathcal{L}_0(\mathbf{t})$

$$(16) \qquad A(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = A(\tilde{\mathbf{t}}) + |x|.$$

Therefore, Condition (9) is satisfied by A.

According to Theorem 3.1, it suffices to prove

$$(17) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(H(\tau) \geq n + 1)}{\mathbb{P}(H(\tau) \geq n)} = \mu$$

to get (15).

We denote by φ the generating function of p and we define recursively $\varphi_1 = \varphi$ and for $n \geq 1$, $\varphi_{n+1} = \varphi_n \circ \varphi$. As φ_n is the generating function of the distribution of $\{u \in \tau; |u| = n\}$ the number of individuals at height n, we have $\mathbb{P}(\tau \in \mathbb{A}_n) = 1 - \varphi_n(0)$. We also have $\lim_{n \to +\infty} \varphi_n(0) = 1$ and

$$\lim_{n \to +\infty} \frac{\mathbb{P}(\tau \in \mathbb{A}_{n+1})}{\mathbb{P}(\tau \in \mathbb{A}_n)} = \lim_{n \to +\infty} \frac{1 - \varphi(\varphi_n(0))}{1 - \varphi_n(0)} = \varphi'(1) = \mu$$

which is (17). $\qquad \square$

4.2. **Conditioning on extinction at large time.**

Proposition 4.2. *Let τ be a critical or sub-critical GW tree with offspring distribution p satisfying Assumption (5). Then we have*

$$\text{(18)} \qquad \text{dist} \,(\tau | H(\tau) = n) \longrightarrow \text{dist} \,(\tau^*).$$

Proof. We consider $A(t) = H(\mathbf{t})$ with $n_0 = 1$ that is $\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}; \, H(\mathbf{t}) = n\}$. Since (16) is in force, we get that Condition (9) still holds. Again it suffices to prove

$$\text{(19)} \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(H(\tau) = n + 1)}{\mathbb{P}(H(\tau) = n)} = \mu$$

to get (18). Recall notation φ_n introduced in Section 4.1 and that $\lim_{n \to +\infty} \varphi_n(0) = 1$. We have $\mathbb{P}(\tau \in \mathbb{A}_n) = \varphi_{n+1}(0) - \varphi_n(0)$ and:

$$\lim_{n \to +\infty} \frac{\mathbb{P}(\tau \in \mathbb{A}_{n+1})}{\mathbb{P}(\tau \in \mathbb{A}_n)} = \lim_{n \to +\infty} \frac{\frac{1 - \varphi(\varphi_n(0))}{1 - \varphi_n(0)} - \frac{1 - \varphi_2(\varphi_n(0))}{1 - \varphi_n(0)}}{1 - \frac{1 - \varphi(\varphi_n(0))}{1 - \varphi_n(0)}} = \frac{\mu - \mu^2}{1 - \mu} = \mu,$$

which is (19). $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

4.3. **Conditioning on the total population size, critical case.** We recover here results from Theorem 7.1 in [11] on the convergence in distribution of a critical GW tree conditioned on the size of its total progeny to Kesten's tree.

Our proof is based on Dwass formula (see [7]) that we recall now. Let $(\tau_k, k \in \mathbb{N}^*)$ be independent GW trees distributed as τ. Set $W_k = \text{Card} \,(\tau_k)$. Let $(X_k, k \in \mathbb{N}^*)$ be independent integer-valued random variables distributed according to p. For $k \in \mathbb{N}^*$ and $n \geq k$, we have:

$$\text{(20)} \qquad \mathbb{P}(W_1 + \ldots + W_k = n) = \frac{k}{n} \mathbb{P}(X_1 + \ldots + X_n = n - k).$$

We also recall some results on random walks. Let Y be an integrable random variable taking values in $\mathbb{Z}$, such that $\mathbb{E}[Y] = 0$, $\mathbb{P}(Y = 0) < 1$ and the span of $|Y|$ is 1. We consider the random walk $S = (S_n, n \in \mathbb{N})$ defined by:

$$\text{(21)} \qquad S_0 = 0 \quad \text{and} \quad S_n = \sum_{k=1}^{n} Y_k \quad \text{for } n \in \mathbb{N}^*.$$

Then the random walk S is recurrent. We define the period of S as the span of the set $\{n > 0, \, \mathbb{P}(S_n = 0) > 0\}$. If S is aperiodic (i.e. has period 1), the strong ratio theorem for recurrent aperiodic random walks, see Theorem T1 p49 of [22], gives that, for $\ell \in \mathbb{Z}$:

$$\text{(22)} \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(S_n = \ell)}{\mathbb{P}(S_n = 0)} = \lim_{n \to +\infty} \frac{\mathbb{P}(S_n = 0)}{\mathbb{P}(S_{n+1} = 0)} = 1.$$

If S has period d, then for all $k \in \{1, \ldots, d\}$, there exist $j_k \in \mathbb{Z}$ and $n_k \in \mathbb{N}^*$ such that

$$\text{(23)} \qquad \forall n \geq n_k, \quad \mathbb{P}(S_{nd+k} = j_k) > 0.$$

The strong ratio theorem can then easily be adapted (using arguments similar to those used in the proof of Lemma 5.6) to get that, for $\ell \in \mathbb{Z}$, $k \in \{1, \ldots, d\}$:

$$\text{(24)} \qquad \lim_{m \to +\infty} \frac{\mathbb{P}(S_{md+k} = \ell d + j_k)}{\mathbb{P}(S_{md} = 0)} = 1.$$

Proposition 4.3. *Let τ be a critical GW tree with offspring distribution p satisfying Assumption (5). Let d be the span of $\mathrm{Card}\,(\tau) - 1$ (that is the span of the set $\{k > 0,\ p(k) > 0\}$). Then we have*

$$
\text{(25)} \qquad \qquad \mathrm{dist}\,(\tau | \mathrm{Card}\,(\tau) = nd + 1) \underset{n \to +\infty}{\longrightarrow} \mathrm{dist}\,(\tau^*)
$$

and

$$
\text{(26)} \qquad \qquad \mathrm{dist}\,(\tau | \mathrm{Card}\,(\tau) \geq n) \underset{n \to +\infty}{\longrightarrow} \mathrm{dist}\,(\tau^*).
$$

Remark 4.4. If we consider $A(\mathbf{t}) = \mathrm{Card}\,(\mathbf{t})$ and $n_0 = +\infty$ that is $\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T},\ \mathrm{Card}\,(\mathbf{t}) \geq n\}$, the converse of Theorem 3.1 gives

$$
\text{(27)} \qquad \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(\mathrm{Card}\,(\tau) \geq n + 1)}{\mathbb{P}(\mathrm{Card}\,(\tau) \geq n)} = 1.
$$

Notice it is not clear how we can obtain (27) directly.

Proof of Proposition 4.3. Consider $A(\mathbf{t}) = \mathrm{Card}\,(\mathbf{t})$ and $n_0 = d$. Then we have

$$
\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T};\ \mathrm{Card}\,(\mathbf{t}) \in [n, n + d)\}.
$$

We have for every $\mathbf{t} \in \mathbb{T}$, without any additional assumption,

$$
\text{(28)} \qquad \qquad A(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = A(\tilde{\mathbf{t}}) + A(\mathbf{t}),
$$

so Condition (9) holds. Again, it therefore suffices to prove

$$
\text{(29)} \qquad \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(\mathrm{Card}\,(\tau) \in [n + 1, n + 1 + d))}{\mathbb{P}(\mathrm{Card}\,(\tau) \in [n, n + d))} = 1
$$

to get (25). By the definition of d, a.s. we have $A(\tau) \in d\mathbb{N} + 1$. We consider an integer valued random variable X distributed according to p and we set $Y = X - 1$ so that $\mathbb{E}[Y] = 0$ since we supposed that $m = 1$. The random walk defined by (21) has period d and we can choose $j_1 = -1$ in (23) as $\mathbb{P}(Y = -1) > 0$. Dwass formula (20) implies that, for $m = \lfloor (n - 1)/d \rfloor$:

$$
\mathbb{P}(\tau \in \mathbb{A}_n) = \mathbb{P}(A(\tau) \in [n, n + d)) = \mathbb{P}(A(\tau) = md + 1) = \frac{1}{md + 1} \mathbb{P}(S_{md+1} = -1).
$$

Using (24), we deduce that:

$$
\lim_{n \to +\infty} \frac{\mathbb{P}(\tau \in \mathbb{A}_{n+1})}{\mathbb{P}(\tau \in \mathbb{A}_n)} = \lim_{m \to +\infty} \frac{\mathbb{P}(S_{(m+1)d+1} = -1)}{\mathbb{P}(S_{md+1} = -1)} = 1
$$

which readily implies (29).

The second assertion (26) is then a straightforward consequence of (29). $\qquad \square$

Remark 4.5. Notice that the local limit theorem gives asymptotics for $\mathbb{P}(S_n = -1)$ when the distribution of X belongs to the domain of attraction of a stable law, see Theorem 4.2.1 of [10] or Theorem 1.10 in [16]. This gives asymptotics for $\mathbb{P}(\tau \in \mathbb{A}_n)$ which in turns allow to recover Condition (10).

4.4. **Conditioning on the number of leaves, critical case.** For a finite tree $\mathbf{t} \in \mathbb{T}_0$, we denote by $L_0(\mathbf{t}) = \operatorname{Card}(\mathcal{L}_0(\mathbf{t}))$ the number of leaves of $\mathbf{t}$. The next proposition (which seems to be a new result) is in fact a particular case of the proposition of the next section. However, we prove it separately for pedagogical purpose as its proof and in particular the construction of the GW tree that codes $\mathcal{L}_0(\mathbf{t})$ of Remark 4.8 are much simpler in that particular case.

Proposition 4.6. *Let τ be a critical GW tree with offspring distribution p satisfying Assumption (5). Let d_0 be the span of the random variable $L_0(\tau) - 1$. Then we have*

$$\text{(30)} \qquad \operatorname{dist}(\tau | L_0(\tau) = nd_0 + 1) \underset{n \to +\infty}{\longrightarrow} \operatorname{dist}(\tau^*).$$

and

$$\text{(31)} \qquad \operatorname{dist}(\tau | L_0(\tau) \geq n) \underset{n \to +\infty}{\longrightarrow} \operatorname{dist}(\tau^*).$$

Proof. We consider $A(\mathbf{t}) = L_0(\mathbf{t})$ and $n_0 = d_0$ which yields $\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}; \ L_0(\mathbf{t}) \in [n, n+d_0)\}$. We have for every trees $\mathbf{t}, \tilde{\mathbf{t}} \in \mathbb{T}_0$ and every $x \in \mathcal{L}_0(\mathbf{t})$

$$\text{(32)} \qquad A(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = A(\tilde{\mathbf{t}}) + A(\mathbf{t}) - 1.$$

According to [19], see also Remark 4.8 below, $L_0(\tau)$ is distributed as the total size of a critical GW tree τ_0 with offspring distribution given by the distribution of:

$$\text{(33)} \qquad X_0 = \sum_{k=1}^{N-1} Z_k,$$

with $(Z_k, k \in \mathbb{N}^*)$ and N independent random variables such that $(Z_k, k \in \mathbb{N}^*)$ are independent and distributed as $X - 1$ conditionally on $\{X \geq 1\}$ (where X is a random variable distributed according to p) and N has a geometric distribution with parameter $p(0)$. As $\mathbb{E}[X_0] = 1$, we get that τ_0 is critical. Notice that d_0 is also the span of the random variable X_0.

It follows from (29) that:

$$\text{(34)} \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(L_0(\tau) \in [n+1, n+1+d_0))}{\mathbb{P}(L_0(\tau) \in [n, n+d_0))} = 1.$$

Then use Theorem 3.1 to get that (30) holds.

If we consider $n_0 = +\infty$ that is:

$$\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}_0; \ L_0(\mathbf{t}) \geq n\},$$

arguing as in the proof of the second part of Proposition 4.3, we get (31). $\qquad \square$

Remark 4.7. We deduce from Remark 4.4 that (31) implies

$$\lim_{n \to +\infty} \frac{\mathbb{P}(L_0(\tau) \geq n+1)}{\mathbb{P}(L_0(\tau) \geq n)} = 1.$$

Remark 4.8. We shall briefly recall how one can prove that $L_0(\tau)$ is distributed as the total size of a GW process by mapping the set of leaves $\mathcal{L}_0(\tau)$ onto a GW tree, see [19, 21] for details.

Let $\mathbf{t}$ be a tree. For $u \in \mathbf{t}$, we define the left branch starting from u as:

$$B_g^{\mathbf{t}}(u) = \{uv; \ |v| \geq 1 \quad \text{and} \quad v = \{1\}^{|v|}\} \cap \mathbf{t}.$$

We also define the left leaf $G(u)$ of u and the left ancestors $A_g(v)$ of a leaf v as:

$$G^{\mathbf{t}}(u) = B_g^{\mathbf{t}}(u) \cap \mathcal{L}_0(\mathbf{t}) \quad \text{and} \quad A_g^{\mathbf{t}}(v) = \{u \in A_v; \ G^{\mathbf{t}}(u) = v\}.$$

For a leaf $v \in \mathcal{L}_0(\mathbf{t})$, we define its leaf-children as:

$$C^{\mathbf{t}}(v) = \{G^{\mathbf{t}}(ui); \, u \in A_g^{\mathbf{t}}(v), 1 < i \leq k_u(\mathbf{t})\},$$

labeled according to the following order: $G^{\mathbf{t}}(ui) < G^{\mathbf{t}}(u'i')$ if $u < u'$ in the lexicographic order or if $u = u'$ and $i < i'$. This defines a tree, obtained from the leaves of $\mathbf{t}$, denoted by $\mathbf{t}_{\{0\}} = F_{\{0\}}(\mathbf{t})$. And we have Card $(\mathbf{t}_{\{0\}}) = L_0(\mathbf{t})$.

If τ is a GW tree then $\tau_{\{0\}} = F_{\{0\}}(\tau)$ is also a GW tree with offspring distribution given by the distribution of X_0 in (33).
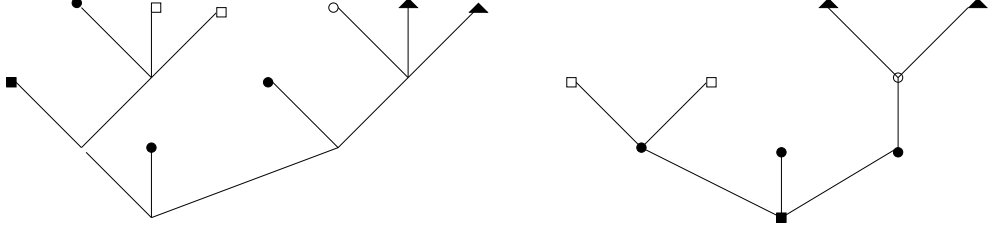

FIGURE 1. A tree $\mathbf{t}$ on the left and the coding of $\mathcal{L}_0(\mathbf{t})$ by a tree $\mathbf{t}_0 = F(\mathbf{t})$ tree on the right.

5. CONDITIONING ON THE NUMBER OF INDIVIDUALS HAVING A GIVEN NUMBER OF CHILDREN

Let $\mathcal{A}$ be a non-empty subset of $\mathbb{N}$. For a tree $\mathbf{t} \in \mathbb{T}$, we write $\mathcal{L}_{\mathcal{A}}(\mathbf{t}) = \{u \in \mathbf{t}; \, k_u(\mathbf{t}) \in \mathcal{A}\}$ the set of individuals whose number of children belongs to $\mathcal{A}$ and $L_{\mathcal{A}}(\mathbf{t}) = \text{Card}\,(\mathcal{L}_{\mathcal{A}}(\mathbf{t}))$ its cardinal. The case $\mathcal{A} = \{0\}$ represents the set of leaves of $\mathbf{t}$ and has been treated in Section 4.4. We can also have $L_{\mathcal{A}}(\mathbf{t}) = \text{Card}\,(\mathbf{t})$ by taking $\mathcal{A} = \mathbb{N}$ or $L_{\mathcal{A}}$ can also be the number of internal nodes by taking $\mathcal{A} = \mathbb{N}^*$.

Let us first remark that for every $\mathbf{t} \in \mathbb{T}_0$, every $x \in \mathcal{L}_0(\mathbf{t})$ and every $\tilde{\mathbf{t}} \in \mathbb{T}$

$$L_{\mathcal{A}}(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = \begin{cases} L_{\mathcal{A}}(\mathbf{t}) + L_{\mathcal{A}}(\tilde{\mathbf{t}}) - 1 & \text{if } 0 \in \mathcal{A}, \\ L_{\mathcal{A}}(\mathbf{t}) + L_{\mathcal{A}}(\tilde{\mathbf{t}}) & \text{if } 0 \notin \mathcal{A}, \end{cases}$$

and hence $L_{\mathcal{A}}$ satisfies the additive property (9) with $D(\mathbf{t}, x) = L_{\mathcal{A}}(\mathbf{t}) - \mathbf{1}_{\{0 \in \mathcal{A}\}}$.

Theorem 5.1. *Let τ be a critical GW tree with offspring distribution p satisfying Assumption (5). Let $d_{\mathcal{A}}$ be the span of the random variable $L_{\mathcal{A}}(\tau) - 1$. Then we have*

$$(35) \qquad\qquad \text{dist}\,(\tau | L_{\mathcal{A}}(\tau) = n d_{\mathcal{A}} + 1) \xrightarrow[n \to +\infty]{} \text{dist}\,(\tau^*).$$

and

$$(36) \qquad\qquad\qquad \text{dist}\,(\tau | L_{\mathcal{A}}(\tau) \geq n) \xrightarrow[n \to +\infty]{} \text{dist}\,(\tau^*).$$

Remark 5.2. It is interesting to note that previous works [21, 16] studying conditioned GW trees involving $L_{\mathcal{A}}$ required additional assumptions on the moments of p or on $\mathcal{A}$ (finite variance offspring distribution and $0 \in \mathcal{A}$ in [21], and offspring distribution p in the domain of attraction of a stable law with either $\mathcal{A}$ or $\mathbb{N} \setminus \mathcal{A}$ finite in the case of infinite variance offspring distribution in [16].

Remark 5.3. In the proof of Theorem 5.1, we will see that if $0 \notin \mathcal{A}$, then $d_{\mathcal{A}} = 1$.

Remark 5.4. As a corollary, we get the following result, which is proven using the same technique as in Remark 4.4:

$$(37) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(L_{\mathcal{A}}(\tau) \geq n+1)}{\mathbb{P}(L_{\mathcal{A}}(\tau) \geq n)} = 1.$$

In what follows, we denote by X a random variable distributed according to p. To avoid triviality, we assume that $0 < \mathbb{P}(X \in \mathcal{A}) \leq 1$. The proofs differ whether $0 \in \mathcal{A}$ or not, the next two subsections are devoted each to one case. Then, we consider the same problem for the sub-critical case, $m = \mathbb{E}[X] < 1$, in a third subsection.

5.1. **Proof of Theorem 5.1 when** $0 \in \mathcal{A}$. We consider only $\mathbb{P}(X \in \mathcal{A}) < 1$, as the case $\mathbb{P}(X \in \mathcal{A}) = 1$ corresponds to the critical case with $\mathcal{A} = \mathbb{N}$ of Section 4.3. In this case, extending the decomposition of [19], it is easy to see that $L_{\mathcal{A}}(\tau)$ is distributed as the total size of a critical GW tree $\tau_{\mathcal{A}}$ whose offspring distribution is given by the distribution of:

$$(38) \qquad X_{\mathcal{A}} = \sum_{k=1}^{N-1} Y_k' + Y'',$$

where $(Y_k', k \in \mathbb{N}^*)$, N and Y'' are independent random variables such that $(Y_k', k \in \mathbb{N}^*)$ are independent and distributed as $X - 1$ conditionally on $\{X \notin \mathcal{A}\}$, N is geometric with parameter $\mathbb{P}(X \in \mathcal{A})$ and Y'' is distributed as X conditionally on $\{X \in \mathcal{A}\}$. See Remark 5.5 below for a construction of the corresponding GW tree.

As $\mathbb{E}[X_{\mathcal{A}}] = 1$, we get that $\tau_{\mathcal{A}}$ is critical. Remark that $d_{\mathcal{A}}$ is also the span of $X_{\mathcal{A}}$. Consider $n_0 = d_{\mathcal{A}}$ which gives

$$\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}; \; L_{\mathcal{A}}(\mathbf{t}) \in [n, n+d_{\mathcal{A}})\}.$$

As $L_{\mathcal{A}}(\tau)$ is distributed as the total size of a critical GW tree, we deduce from Subsection 4.3 that

$$(39) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(L_{\mathcal{A}}(\tau) \in [n+1, n+1+d_{\mathcal{A}}))}{\mathbb{P}(L_{\mathcal{A}}(\tau) \in [n, n+d_{\mathcal{A}}))} = 1$$

and thus by Theorem 3.1 that (35) holds. $\square$

Remark 5.5. We shall briefly indicate how one can prove that $L_{\mathcal{A}}(\tau)$ is distributed as the total size of a GW process by mapping the set $\mathcal{L}_{\mathcal{A}}(\tau)$ onto a GW tree, generalizing the construction of Remark 4.8, see [21].

We use notations from Remark 4.8. For a tree $\mathbf{t}$, $u \in \mathbf{t}$ and $v \in \mathcal{L}_{\mathcal{A}}(\mathbf{t})$, we define the left descendant of u in $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$ and the left ancestors $A_g^{\mathbf{t}}(v)$ of v as:

$$G_{\mathcal{A}}^{\mathbf{t}}(u) = \operatorname{argmin}\{|w|; \; w \in B_g^{\mathbf{t}}(u) \cap \mathcal{L}_{\mathcal{A}}(\mathbf{t})\} \quad \text{and} \quad A_{g,\mathcal{A}}^{\mathbf{t}}(v) = \{u \in A_v; \; G_{\mathcal{A}}^{\mathbf{t}}(u) = v\}.$$

Remark that $G_{\mathcal{A}}^{\mathbf{t}}(u)$ exists since we supposed that $0 \in \mathcal{A}$.

For $v \in \mathcal{L}_{\mathcal{A}}(\mathbf{t})$, we define its children as:

$$C_{\mathcal{A}}^{\mathbf{t}}(v) = \begin{cases} \{G_{\mathcal{A}}^{\mathbf{t}}(ui); \; u \in A_{g,\mathcal{A}}^{\mathbf{t}}(v), 1 < i \leq k_u\} & \text{if } v \in \mathcal{L}(\mathbf{t}), \\ \{G_{\mathcal{A}}^{\mathbf{t}}(ui); \; u \in A_{g,\mathcal{A}}^{\mathbf{t}}(v), 1 < i \leq k_u\} \bigcup \{vi, 1 \leq i \leq k_v(\mathbf{t})\}, & \text{if } v \notin \mathcal{L}(\mathbf{t}), \end{cases}$$

labeled according to the following order: $G_{\mathcal{A}}^{\mathbf{t}}(ui) < G_{\mathcal{A}}^{\mathbf{t}}(u'i')$ if $u < u'$ in the lexicographic order or if $u = u'$ and $i < i'$; $G_{\mathcal{A}}^{\mathbf{t}}(ui) < vi'$ for $u \in A_{g,\mathcal{A}}^{\mathbf{t}}(v)$ and $1 \leq i' \leq k_v(\mathbf{t})$; and $vi < vi'$ for $1 \leq i < i' \leq k_v(\mathbf{t})$. This defines a tree, obtained from $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$, denoted by $\mathbf{t}_{\mathcal{A}} = F_{\mathcal{A}}(\mathbf{t})$. And we have Card $(\mathbf{t}_0) = L_{\mathcal{A}}(\mathbf{t})$.

If τ is a GW tree then $\tau_{\mathcal{A}} = F_{\mathcal{A}}(\tau)$ is also a GW tree with offspring distribution given by the distribution of $X_{\mathcal{A}}$ in (38).

5.2. **Proof of Theorem 5.1 when** $0 \notin \mathcal{A}$. Notice that in that case, $\mathbb{P}(X \in \mathcal{A}) < 1$. In that case, we will map the set $\mathcal{L}_{\mathcal{A}}(\tau)$ onto a forest of GW trees instead of a single tree as before.

For a tree $\mathbf{t}$ and $u \in \mathbf{t}$ we define $\mathcal{S}_u^{\mathcal{A}}(\mathbf{t})$ the sub-tree "above" u with no progeny in $\mathcal{A}$ by:

$$\mathcal{S}_u^{\mathcal{A}}(\mathbf{t}) = \{w \in u\mathcal{S}_u; A_w \cap A_u^c \cap \mathcal{L}_{\mathcal{A}}(\mathbf{t}) = \emptyset\}.$$

We define the children of u in $\mathcal{A}$ as:

$$C^{\mathbf{t}}(u) = \mathcal{L}_0(\mathcal{S}_u^{\mathcal{A}}(\mathbf{t})) \cap \mathcal{L}_{\mathcal{A}}(\mathbf{t}).$$

The nodes of $C^{\mathbf{t}}(\emptyset)$ will be the roots of the trees of the forest, then the children of a node $u \in \mathcal{L}_{\mathcal{A}}(\mathbf{t})$ in the forest are $C^{\mathbf{t}}(u)$.

The forest associated with $\mathcal{L}_{\mathcal{A}}(\tau)$ hence has a random number of trees given by $\tilde{N}_\emptyset := $ Card $(C^\tau(\emptyset))$. Its distribution is given as follows. We set $p(\mathcal{A}) = \mathbb{P}(X \in \mathcal{A})$. We define $\tilde{p}(0) = p(0) + \sum_{k \in \mathcal{A}} p(k)$ and, for $k \in \mathbb{N}^*$, $\tilde{p}(k) = p(k)\mathbf{1}_{\{k \notin \mathcal{A}\}}$. Let $\tilde{X}$ be a random variable with distribution $\tilde{p}$, so that $\mathbb{E}[\tilde{X}] = \mathbb{E}[X\mathbf{1}_{\{X \in \mathcal{A}^c\}}] < 1$. Notice that $S_\emptyset^{\mathcal{A}}(\tau)$ is a sub-critical GW tree with offspring distribution $\tilde{p}$. We denote by $\tilde{L}$ be the number of leaves of $S_\emptyset^{\mathcal{A}}(\tau)$. Elementary computations yield:

$$(40) \qquad \mathbb{E}[\tilde{L}] = \frac{\tilde{p}(0)}{1 - \mathbb{E}[\tilde{X}]} = \frac{p(0) + p(\mathcal{A})}{\mathbb{E}\left[X\mathbf{1}_{\{X \in \mathcal{A}\}}\right]},$$

where we used that $\mathbb{E}[X] = 1$ for the last equality. Then conditionally given $\tilde{L}$, $\tilde{N}_\emptyset$ has a binomial distribution with parameter $(\tilde{L}, p(\mathcal{A})/\tilde{p}(0))$.

Then the trees of the forest associated with $\mathcal{L}_{\mathcal{A}}(\tau)$ are independent GW trees, also independent of $\tilde{N}_\emptyset$. Their offspring distribution is the distribution of the random variable

$$(41) \qquad X_{\mathcal{A}} = \sum_{k=1}^{Z'} \tilde{N}_k,$$

where Z' is a random variable distributed as X conditionally on $\{X \in \mathcal{A}\}$, and $(\tilde{N}_k, k \in \mathbb{N}^*)$ are independent random variables distributed as $\tilde{N}_\emptyset$ and independent of Z'. We therefore have, using (40):

$$\mathbb{E}[X_{\mathcal{A}}] = \mathbb{E}[Z']\mathbb{E}[\tilde{N}_k] = \frac{1}{p(\mathcal{A})} \mathbb{E}\left[X\mathbf{1}_{\{X \in \mathcal{A}\}}\right] \mathbb{E}[\tilde{L}] \frac{p(\mathcal{A})}{p(0) + p(\mathcal{A})} = 1.$$

Let τ' be a GW tree with offspring distribution given by the law of $X_{\mathcal{A}}$ and initial size 1. Let $(W'_k, k \in \mathbb{N}^*)$ be independent random variables distributed as Card (τ') and independent of $\tilde{N}_\emptyset$. We deduce that $L_{\mathcal{A}}(\tau)$ is distributed as $\sum_{k=1}^{\tilde{N}_\emptyset} W'_k$. Notice that $\mathcal{L}_{\mathcal{A}}(\tau)$ is empty if $\tilde{N}_\emptyset = 0$. By construction, notice that the span of $X_{\mathcal{A}}$ is 1. The proof of the next lemma is postponed to the end of the section.

Lemma 5.6. *We have:*

$$(42) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(L_{\mathcal{A}}(\tau) = n)}{\mathbb{P}(W'_1 = n)} = \mathbb{E}[\tilde{N}_\emptyset].$$

Then, by (20) and (22), we have

$$(43) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(W'_1 = n)}{\mathbb{P}(W'_1 = n + 1)} = 1,$$

and thus we deduce that:

$$(44) \qquad \lim_{n \to +\infty} \frac{\mathbb{P}(L_\mathcal{A}(\tau) = n)}{\mathbb{P}(L_\mathcal{A}(\tau) = n + 1)} = 1.$$

As the span of $X_\mathcal{A}$ is 1, we consider $n_0 = 1$ that is:

$$\mathbb{A}_n = \{\mathbf{t} \in \mathbb{T}; \ L_\mathcal{A}(\mathbf{t}) = n\}.$$

Since τ is a.s. finite, the first part of Condition (39) is satisfied; and (44) readily implies the second part of Condition (39). As usual, we deduce that (35) holds. $\square$

Proof of Lemma 5.6. Let $S'_n = \sum_{k=1}^n (X'_k - 1)$, where $(X'_k, k \in \mathbb{N}^*)$ are independent random variables distributed as $X_\mathcal{A}$. By construction and using Dwass formula (20), we have:

$$\mathbb{P}(L_\mathcal{A}(\tau) = n) = \sum_{k=1}^{+\infty} \mathbb{P}(\tilde{N}_\emptyset = k) \mathbb{P}\left(\sum_{j=1}^k W'_j = n\right) = \frac{1}{n} \sum_{k=1}^{+\infty} k \mathbb{P}(\tilde{N}_\emptyset = k) \mathbb{P}(S'_n = -k).$$

We set:

$$\delta_n(k) = \frac{\mathbb{P}(S'_n = -k)}{\mathbb{P}(S'_n = -1)}.$$

The random walk $(S'_n, n \in \mathbb{N})$ is recurrent and aperiodic as the span of $X_\mathcal{A}$ is 1. We deduce from the strong ratio theorem (22) that:

$$(45) \qquad \lim_{n \to +\infty} \delta_n(k) = 1.$$

Using Fatou's Lemma, we get:

$$\liminf_{n \to +\infty} \frac{\mathbb{P}(L_\mathcal{A}(\tau) = n)}{\mathbb{P}(W'_1 = n)} \geq \sum_{k=1}^{+\infty} k \mathbb{P}(\tilde{N}_\emptyset = k) \liminf_{n \to +\infty} \delta_n(k) = \mathbb{E}[\tilde{N}_\emptyset].$$

As W'_1 is the total size of a GW process with offspring distribution given by $X_\mathcal{A}$, we have that W'_1 has the same distribution as:

$$1 + \sum_{k=1}^{X_\mathcal{A}} W'_k.$$

Using Dwass formula (20), we have:

$$\frac{\mathbb{P}(W'_1 = n + 1)}{\mathbb{P}(W'_1 = n)} = \sum_{k=1}^{+\infty} \mathbb{P}(X_\mathcal{A} = k) \frac{\mathbb{P}\left(\sum_{j=1}^k W'_j = n\right)}{\mathbb{P}(W'_1 = n)}$$

$$= \sum_{k=1}^{+\infty} k \mathbb{P}(X_\mathcal{A} = k) \delta_n(k).$$

Let $m_0 \geq 1$ such that $\mathbb{P}(Z' = m_0) > 0$. We deduce from (41) that:

$$\{\tilde{N}_1 = k, \tilde{N}_2 = 0, \ldots, \tilde{N}_{m_0} = 0, Z' = m_0\} \subset \{X_\mathcal{A} = k\},$$

and thus, with $C^{-1} = \mathbb{P}(\tilde{N}_\emptyset = 0)^{m_0 - 1} \mathbb{P}(Z' = m_0) > 0$ we have for all $k \in \mathbb{N}$:

$$\mathbb{P}(\tilde{N}_\emptyset = k) \leq C \mathbb{P}(X_\mathcal{A} = k).$$

We get for any $K \in \mathbb{N}^*$:

$$
\begin{aligned}
\frac{\mathbb{P}(L_{\mathcal{A}}(\tau) = n)}{\mathbb{P}(W_1' = n)} &= \sum_{k=1}^{K} k\mathbb{P}(\tilde{N}_{\emptyset} = k)\delta_n(k) + \sum_{k=K+1}^{+\infty} k\mathbb{P}(\tilde{N}_{\emptyset} = k)\delta_n(k) \\
&\leq \sum_{k=1}^{K} k\mathbb{P}(\tilde{N}_{\emptyset} = k)\delta_n(k) + C \sum_{k=K+1}^{+\infty} k\mathbb{P}(X_{\mathcal{A}} = k)\delta_n(k) \\
&= \sum_{k=1}^{K} k\mathbb{P}(\tilde{N}_{\emptyset} = k)\delta_n(k) - C \sum_{k=1}^{K} k\mathbb{P}(X_{\mathcal{A}} = k)\delta_n(k) + C\frac{\mathbb{P}(W_1' = n+1)}{\mathbb{P}(W_1' = n)}.
\end{aligned}
$$

Using (45), we get:

$$
\limsup_{n \to +\infty} \frac{\mathbb{P}(L_{\mathcal{A}}(\tau) = n)}{\mathbb{P}(W_1' = n)} \leq \sum_{k=1}^{K} k\mathbb{P}(\tilde{N}_{\emptyset} = k) - C \sum_{k=1}^{K} k\mathbb{P}(X_{\mathcal{A}} = k) + C.
$$

Letting K goes to infinity and using that $\mathbb{E}[X_{\mathcal{A}}] = 1$, we get:

$$
\limsup_{n \to +\infty} \frac{\mathbb{P}(L_{\mathcal{A}}(\tau) = n)}{\mathbb{P}(W_1' = n)} \leq \mathbb{E}[\tilde{N}_{\emptyset}].
$$

This implies (42).   □

5.3. **The sub-critical case.** Let p be an offspring distribution. Let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. For every $\theta > 0$ such that $\sum_{k \in \mathbb{N}} \theta^k p(k)$ is finite, we define on $\mathbb{N}$ the function p_θ by

$$
\forall k \geq 0, \ p_\theta(k) = \begin{cases} c_{\mathcal{A}}(\theta)\theta^k p(k) & \text{if } k \in \mathcal{A}, \\ \theta^{k-1} p(k) & \text{if } k \notin \mathcal{A} \end{cases}
$$

where the normalizing constant $c_{\mathcal{A}}(\theta)$ is given by:

$$
c_{\mathcal{A}}(\theta) = \frac{1 - \sum_{k \notin \mathcal{A}} \theta^{k-1} p(k)}{\sum_{k \in \mathcal{A}} \theta^k p(k)}.
$$

We denote by I the set of θ such that p_θ defines a probability distribution on $\mathbb{N}$. Notice that I is an interval with bounds $\theta_0 < 1 \leq \theta_1$. We have the special cases $\theta_0 = 0$ if $0 \in \mathcal{A}$ and $\theta_0 = p(0)$ if $\mathcal{A} = \mathbb{N}^*$.

Proposition 5.7. *Let τ be a GW tree with offspring distribution p satisfying $p(0) > 0$ and $p(0) + p(1) < 1$. Let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. For every $\theta \in I$, let τ_θ be a GW tree with offspring distribution p_θ. Then the conditional distributions of τ given $\{\mathcal{L}_{\mathcal{A}}(\tau) = n\}$ and of τ_θ given $\{\mathcal{L}_{\mathcal{A}}(\tau_\theta) = n\}$ are the same.*

Remark 5.8. This proposition covers Kennedy's result [14] for $\mathcal{A} = \mathbb{N}$ and the pruning procedure of [2] for $\mathcal{A} = \{0\}$.

Proof. Let $\mathbf{t} \in \mathbb{T}_0$. Then we have, using the definition of p_θ and (2):

$$\mathbb{P}(\tau_\theta = \mathbf{t}) = \prod_{v \in \mathbf{t}} p_\theta(k_v(\mathbf{t}))$$

$$= \prod_{v \in \mathbf{t}, k_v(\mathbf{t}) \in \mathcal{A}} c_\mathcal{A}(\theta) \theta^{k_v(\mathbf{t})} p(k_v(\mathbf{t})) \prod_{v \in \mathbf{t}, k_v(\mathbf{t}) \notin \mathcal{A}} \theta^{k_v(\mathbf{t})-1} p(k_v(\mathbf{t}))$$

$$= c_\mathcal{A}(\theta)^{L_\mathcal{A}(\mathbf{t})} \theta^{\sum_{v \in \mathbf{t}} k_v(\mathbf{t}) - L_{\mathcal{A}^c}(\mathbf{t})} \mathbb{P}(\tau = \mathbf{t})$$

$$= c_\mathcal{A}(\theta)^{L_\mathcal{A}(\mathbf{t})} \theta^{\mathrm{Card}\,(\mathbf{t})-1-L_{\mathcal{A}^c}(\mathbf{t})} \mathbb{P}(\tau = \mathbf{t})$$

$$= \theta^{-1} (\theta c_\mathcal{A}(\theta))^{L_\mathcal{A}(\mathbf{t})} \mathbb{P}(\tau = \mathbf{t}).$$

We deduce that

$$\mathbb{P}(L_\mathcal{A}(\tau_\theta) = n) = \sum_{\mathbf{t} \in \mathbb{T}_0,\ L_\mathcal{A}(\mathbf{t})=n} \mathbb{P}(\tau_\theta = \mathbf{t})$$

$$= \theta^{-1}(\theta c_\mathcal{A}(\theta))^n \sum_{\mathbf{t} \in \mathbb{T}_0,\ L_\mathcal{A}(\mathbf{t})=n} \mathbb{P}(\tau = \mathbf{t})$$

$$= \theta^{-1}(\theta c_\mathcal{A}(\theta))^n \mathbb{P}(L_\mathcal{A}(\tau) = n)$$

and finally, for every $\mathbf{t} \in \mathbb{T}_0$ such that $L_\mathcal{A}(\mathbf{t}) = n$, we have

$$\mathbb{P}(\tau_\theta = \mathbf{t} \mid L_\mathcal{A}(\tau_\theta) = n) = \frac{\mathbb{P}(\tau_\theta = \mathbf{t})}{\mathbb{P}(L_\mathcal{A}(\tau_\theta) = n)}$$

$$= \frac{\theta^{-1}(\theta c_\mathcal{A}(\theta))^n \mathbb{P}(\tau = \mathbf{t})}{\theta^{-1}(\theta c_\mathcal{A}(\theta))^n \mathbb{P}(L_\mathcal{A}(\tau_\theta) = n)} = \mathbb{P}(\tau = \mathbf{t} \mid L_\mathcal{A}(\tau) = n).$$

$\square$

We shall say that the offspring distribution p is generic (with respect to $\mathcal{A}$) if there exists $\theta_c \in I$ such that p_{θ_c} is critical.

Corollary 5.9. *Let τ be a sub-critical GW tree with offspring distribution p satisfying Assumption (5). Let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. For every $\theta \in I$, let τ_θ be a GW tree with offspring distribution p_θ. If p is generic, that is there exists $\theta_c \in I$ such that p_{θ_c} is critical, then*

$$\mathrm{dist}\,(\tau | L_\mathcal{A}(\tau) = n d_\mathcal{A} + 1) \underset{n \to +\infty}{\longrightarrow} \mathrm{dist}\,(\tau_{\theta_c}^*).$$

and

$$\mathrm{dist}\,(\tau | L_\mathcal{A}(\tau) \geq n) \underset{n \to +\infty}{\longrightarrow} \mathrm{dist}\,(\tau_{\theta_c}^*).$$

Remark 5.10. The first convergence of the corollary remains valid for a super-critical offspring distribution but not the second one as the conditional distribution cannot be written as a mixture of the first one as the tree may be infinite.

Remark 5.11. If the critical value θ_c of Corollary 5.9 does not exist, then we observe a condensation phenomenon: the limiting tree does not have an infinite spine, but exhibits a unique vertex with an infinite number of children, see [11] for $\mathcal{A} = \mathbb{N}$ and the forthcoming paper [1] for the general case.

6. Conditioning by the size of a high generation

We end this paper with a conditioning which does not enter into the framework of Theorem 3.1. However its proof can be easily adapted. For a tree $\mathbf{t}$, we denote by

$$\mathcal{G}_n(\mathbf{t}) = \text{Card}\left(\{u \in \mathbf{t}, \ |u| = n\}\right)$$

the size of the n-th generation of $\mathbf{t}$. Then we have

Proposition 6.1. *Let τ be a critical GW tree with offspring distribution p satisfying Assumption (5). Let $(\alpha_n, n \in \mathbb{N})$ be a sequence of positive integers. If for all $j \in \mathbb{N}^*$*

$$\tag{46} \lim_{n \to +\infty} \frac{\mathbb{P}(\mathcal{G}_{n-j}(\tau) = \alpha_n)}{\mathbb{P}(\mathcal{G}_n(\tau) = \alpha_n)} = 1,$$

then we have

$$\tag{47} \text{dist}\,(\tau|\mathcal{G}_n(\tau) = \alpha_n) \underset{n \to +\infty}{\longrightarrow} \text{dist}\,(\tau^*).$$

Proof. For every tree $\mathbf{t} \in \mathbb{T}_0$, every $x \in \mathcal{L}_0(\mathbf{t})$ and every tree $\tilde{\mathbf{t}} \in \mathbb{T}$, we have

$$\mathcal{G}_n(\mathbf{t} \circledast (\tilde{\mathbf{t}}, x)) = \mathcal{G}_n(\mathbf{t}) + \mathcal{G}_{n-|x|}(\tilde{\mathbf{t}})$$

which generalizes Assumption (9).

The same computations as in the proof of Theorem 3.1 give for $\mathbf{t} \in \mathbb{T}_0$, $x \in \mathcal{L}_0(\mathbf{t})$ and $n \geq H(\mathbf{t})$:

$$\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathcal{G}_n(\tau) = \alpha_n) = \frac{1}{p(0)}\mathbb{P}(\tau = \mathbf{t})\mathbb{P}(\mathcal{G}_{n-|x|}(\tau) = \alpha_n - \mathcal{G}_n(\mathbf{t}))$$

$$= \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x))\mathbb{P}(\mathcal{G}_{n-|x|}(\tau) = \alpha_n).$$

Therefore, we obtain by Assumption (46):

$$\lim_{n \to +\infty} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)|\mathcal{G}_n(\tau) = \alpha_n) = \lim_{n \to +\infty} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \frac{\mathbb{P}(\mathcal{G}_{n-|x|}(\tau) = \alpha_n)}{\mathbb{P}(\mathcal{G}_n(\tau) = \alpha_n)}$$

$$= \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)).$$

The result follows from Lemma 2.1. $\qquad \square$

Corollary 6.2. *Let τ be a critical GW tree with offspring distribution p given by a mixture of a geometric distribution with parameter $\rho \in (0, 1]$ and a Dirac mass at 0, i.e. $p(0) = 1 - \rho$ and $p(k) = \rho^2(1 - \rho)^{k-1}$ for $k \geq 1$. Let $(\alpha_n, n \in \mathbb{N})$ be a sequence of positive integers such that $\lim_{n \to +\infty} n^{-2}\alpha_n = 0$. Then we have:*

$$\text{dist}\,(\tau|\mathcal{G}_n(\tau) = \alpha_n) \underset{n \to +\infty}{\longrightarrow} \text{dist}(\tau^*).$$

Proof. In that particular case, the generating function φ_n of $\mathcal{G}_n(\tau)$ is explicitly known and we have for every $s \in [0, 1]$

$$\varphi_n(s) = \frac{nc - (nc - 1)s}{(nc + 1) - ncs}$$

with $c = (1 - \rho)/\rho$. Expending φ_n gives for every $k \geq 1$:

$$\mathbb{P}(\mathcal{G}_n(\tau) = k) = \frac{(nc)^{k-1}}{(nc + 1)^{k+1}},$$

and therefore for $j \geq 1$

$$\lim_{n \to +\infty} \frac{\mathbb{P}(\mathcal{G}_{n-j}(\tau) = \alpha_n)}{\mathbb{P}(\mathcal{G}_n(\tau) = \alpha_n)} = \lim_{n \to +\infty} \frac{n(nc+1)}{(n-j)((n-j)c+1)} \left(\frac{1 + \frac{1}{nc}}{1 + \frac{1}{(n-j)c}} \right)^{\alpha_n} = 1.$$

Then use Proposition 6.1 to conclude. $\qquad\square$

Remark 6.3. As for Theorem 3.1, we can obtain the converse of Proposition 6.1. We deduce that, in the geometric case of Corollary 6.2, the GW tree τ conditioned on $\{\mathcal{G}_n(\tau) = k\lfloor n^a \rfloor\}$, with $k \in \mathbb{N}^*$, converges in distribution to Kesten's tree if and only if $a \in [0, 2)$.

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