



**HAL**  
open science

# Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations

Jean Dolbeault, Maria J. Esteban

► **To cite this version:**

Jean Dolbeault, Maria J. Esteban. Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations. *Nonlinearity*, 2014, 27, p. 435-465. <hal-00812996v2>

**HAL Id: hal-00812996**

**<https://hal.science/hal-00812996v2>**

Submitted on 2 Nov 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations

Jean Dolbeault, Maria J. Esteban

Ceremade (UMR CNRS no. 7534), Univ. Paris-Dauphine, Pl. de Lattre de Tassigny,  
75775 Paris Cedex 16, France

E-mail: [dolbeaul@ceremade.dauphine.fr](mailto:dolbeaul@ceremade.dauphine.fr), [esteban@ceremade.dauphine.fr](mailto:esteban@ceremade.dauphine.fr)

**Abstract.** In this paper we study the bifurcation of branches of non-symmetric solutions from the symmetric branch of solutions to the Euler-Lagrange equations satisfied by optimal functions in functional inequalities of Caffarelli-Kohn-Nirenberg type. We establish the asymptotic behavior of the branches for large values of the bifurcation parameter. We also perform an expansion in a neighborhood of the first bifurcation point on the branch of symmetric solutions, that characterizes the local behavior of the non-symmetric branch. These results are compatible with earlier numerical and theoretical observations. Further numerical results allow us to distinguish two global scenarios. This sheds a new light on the symmetry breaking phenomenon.

AMS classification scheme numbers: (MSC 2010) 35C20; 35J60; 26D10; 46E35; 58E35

*Keywords:* Hardy-Sobolev inequality; Caffarelli-Kohn-Nirenberg inequality; extremal functions; Emden-Fowler transformation; radial symmetry; symmetry breaking; ground state; Pöschl-Teller operator; bifurcation; elliptic equations; branches of solutions.

## 1. Introduction

In this paper we investigate how symmetry can be broken in some variational problems. Symmetry breaking occurs when antagonistic effects are competing, like weights or potentials (or coupling with other fields) on the one hand and nonlinearities on the other hand. An archetypal example for such issues is the question of symmetry of optimal functions in Caffarelli-Kohn-Nirenberg inequalities. While all terms are invariant under rotation around the origin, it is known that optimizers are not always radially symmetric. Caffarelli-Kohn-Nirenberg inequalities, also known as Hardy-Sobolev inequalities, is a particularly simple setting for the study of symmetry breaking because weights and nonlinear terms have simple homogeneity properties, so that Euler-Lagrange equations inherit scaling properties that allow to further simplify the study of the symmetry issues.

Still, ranges of parameters for which optimizers are radially symmetric have not been completely determined yet.

Symmetry breaking issues are present in many areas of physics involving partial differential equations: quantum mechanics, mean field models, equations for phase transition, ferromagnetism, mechanics, *etc.* Various mathematical methods are available either for proving symmetry (uniqueness, comparison techniques based for instance on moving plane methods, symmetrization: see for instance [1, 2, 3]) or for proving symmetry breaking (multiplicity and bifurcation, energy, spectral methods). However, threshold cases are not characterized even in the simplest cases.

A simple mechanism which can break symmetry is the instability of the *symmetric extremals*, that is, the case where the extremals among radially symmetric functions are not local minima in the larger space of functions with no symmetry assumption. In the case of the Caffarelli-Kohn-Nirenberg inequalities, this instability has been studied in several papers (see [4, 5, 6]) and the corresponding region of symmetry breaking is delimited by an explicit curve. However, it has been proved in [3] that symmetry breaking can occur even in a range of parameters for which the *symmetric extremals* are stable, that is, in cases where they are strict local minima. In order to understand this phenomenon, and symmetry breaking in general, we study the solution set of the Euler-Lagrange equations corresponding to a minimization problem associated with the Caffarelli-Kohn-Nirenberg inequalities. For those equations, we investigate the bifurcation of non-radially symmetric solutions from radially symmetric ones. The two theoretical contributions of the present paper are *an asymptotic analysis of the branches for large values of the bifurcation parameter*, in Section 2, and a detailed *expansion of the non-radial solutions in a neighborhood of the bifurcation point* on the branch of radial extremals, in Section 3. Both results are consistent with known and new numerical results presented in Section 4 and give a significant insight into the local behaviour of the solutions, either around the bifurcation point or asymptotically.

We shall consider a family of Caffarelli-Kohn-Nirenberg inequalities which, for a given dimension  $d \geq 3$ , depend on two exponents,  $p \in (2, 2^*]$  with  $2^* := 2d/(d-2)$  and  $\theta \in (0, 1]$ , and on a parameter  $\Lambda > 0$ .

For any dimension  $d \geq 3$ , let us consider the set  $\mathcal{D}$  of all smooth functions which are compactly supported in  $\mathbb{R}^d$ . Define the numbers

$$\vartheta(p, d) := d \frac{p-2}{2p}, \quad a_c := \frac{d-2}{2}, \quad p(a, b) := \frac{2d}{d-2+2(b-a)}.$$

For any  $a < a_c$ , we consider the following *Caffarelli-Kohn-Nirenberg inequalities*, which have been introduced in [7] (also see [6]):

*Let  $d \geq 3$ ,  $a < a_c$ ,  $b \in [a, a+1]$  and assume that  $p = p(a, b)$ . Then, there exists a*

finite positive constant  $\mathcal{K}_{\text{CKN}}$  depending on  $\theta$ ,  $a$  and  $p$  such that, for any  $w \in \mathcal{D}$ ,

$$\| |x|^{-b} w \|_{L^p(\mathbb{R}^d)}^2 \leq \mathcal{K}_{\text{CKN}} \| |x|^{-a} \nabla w \|_{L^2(\mathbb{R}^d)}^{2\theta} \| |x|^{-(a+1)} w \|_{L^2(\mathbb{R}^d)}^{2(1-\theta)}. \quad (1)$$

According to [5], the Caffarelli-Kohn-Nirenberg inequalities on  $\mathbb{R}^d$  can be rewritten in cylindrical variables using the Emden-Fowler transformation

$$s = \log |x|, \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}, \quad u(s, \omega) = |x|^{a_c - a} w(x).$$

The above inequalities are then equivalent to Gagliardo-Nirenberg-Sobolev inequalities on the cylinder  $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$  that can be written as

$$\|u\|_{L^p(\mathcal{C})}^2 \leq \mathcal{K}_{\text{CKN}}(\theta, \Lambda, p) \left( \|\nabla u\|_{L^2(\mathcal{C})}^2 + \Lambda \|u\|_{L^2(\mathcal{C})}^2 \right)^\theta \|u\|_{L^2(\mathcal{C})}^{2(1-\theta)}, \quad (2)$$

for any  $u \in H^1(\mathcal{C})$ , where  $a$  and  $\Lambda$  are related by  $\Lambda = (a - a_c)^2$ . Here we adopt the convention that the measure on  $\mathbb{S}^{d-1}$  is the uniform probability measure. Let us define

$$\mathcal{Q}_\Lambda^\theta[u] := \frac{\left( \|\nabla u\|_{L^2(\mathcal{C})}^2 + \Lambda \|u\|_{L^2(\mathcal{C})}^2 \right)^\theta \|u\|_{L^2(\mathcal{C})}^{2(1-\theta)}}{\|u\|_{L^p(\mathcal{C})}^2}. \quad (3)$$

In the case  $\theta = 1$ , we shall simply write  $\mathcal{Q}_\mu$  instead of  $\mathcal{Q}_\mu^1$ . We are interested in the map  $\Lambda \mapsto \mathcal{K}_{\text{CKN}}(\theta, \Lambda, p)$ , what amounts to study the dependence of the minimum of  $\mathcal{Q}_\Lambda^\theta$  on  $\Lambda$ . The corresponding Euler-Lagrange equation is

$$-\theta \Delta u + \left[ (1 - \theta) t[u] + \Lambda \right] u = u^{p-1}, \quad (4)$$

with

$$t[u] := \frac{\int_{\mathcal{C}} |\nabla u|^2 dy}{\int_{\mathcal{C}} u^2 dy}.$$

Let us introduce the parameter  $\mu = ((1 - \theta) t[u] + \Lambda)/\theta$ . Up to multiplication by a constant, the solutions of (4) are solutions of

$$-\Delta u + \mu u = u^{p-1}. \quad (5)$$

If  $\theta = 1$ , we may notice that  $\Lambda = \mu$ . Hence we may solve (5), denote by  $u_\mu$  the corresponding solution which minimizes  $\mathcal{Q}_\mu$ , compute  $\Lambda$  and then parametrize the solutions of (4) in terms of  $\mu$ . Let us give some details. With

$$\tau(\mu) := t[u_\mu] \quad \text{and} \quad \nu(\mu) := \frac{\|u_\mu\|_{L^2(\mathcal{C})}^2}{\|u_\mu\|_{L^p(\mathcal{C})}^2},$$

we can describe the set of solutions to (4) in parametric form as  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  where

$$\begin{aligned}\Lambda^\theta(\mu) &= \theta \mu - (1 - \theta) \tau(\mu), \\ J^\theta(\mu) &:= \mathcal{Q}_\Lambda^\theta[u_\mu] = \nu(\mu) \theta^\theta (\mu + \tau(\mu))^\theta.\end{aligned}$$

The uniqueness of  $u_\mu$  is not obvious and details will be provided in this paper. We shall denote by  $\tau_*(\mu)$ ,  $\nu_*(\mu)$ ,  $\Lambda_*^\theta(\mu)$  and  $J_*^\theta(\mu)$  the corresponding quantities for the symmetric solution  $u_{\mu,*}$  of (5).

With  $\vartheta = \vartheta(p, d) = d \frac{p-2}{2p}$ , denote by  $\mathbf{K}_{\text{GN}} = \mathbf{K}_{\text{GN}}(p, d)$  the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^p(\mathbb{R}^d)}^2 \leq \frac{\mathbf{K}_{\text{GN}}}{|\mathbb{S}^{d-1}|^{\frac{p-2}{p}}} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{2\vartheta} \|u\|_{L^2(\mathbb{R}^d)}^{2(1-\vartheta)}, \quad \forall u \in H^1(\mathbb{R}^d). \quad (6)$$

Our first result is a direct consequence of the above parametrization and deals with the asymptotics of  $\mathcal{Q}_\Lambda^\theta$  for large values of  $\Lambda$ .

**Theorem 1** *With the previous notations, for all  $\theta > \vartheta = \vartheta(p, d)$ , we have*

$$\lim_{\mu \rightarrow \infty} \mu^{\vartheta-\theta} J^\theta(\mu) = \frac{\theta^\theta}{\vartheta^\vartheta} (1 - \vartheta)^{\vartheta-\theta} \frac{1}{\mathbf{K}_{\text{GN}}}.$$

Moreover, the parametric curve  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  is asymptotic to the curve

$$\Lambda \mapsto \frac{\theta^\theta}{\vartheta(p, d)^{\vartheta(p, d)} (\theta - \vartheta(p, d))^{\theta - \vartheta(p, d)}} \frac{\Lambda^{\theta - \vartheta(p, d)}}{\mathbf{K}_{\text{GN}}},$$

for large values of  $\mu$  or, equivalently, for large values of  $\Lambda = \Lambda^\theta(\mu)$ .

The case  $\theta = 1$  has been established in [5, Theorem 1.2] and here we generalize it to the case  $\theta < 1$ . The proof will be given in Section 2.

Next we denote by  $\mathbf{K}_{\text{CKN}}^*(\theta, \Lambda, p)$  the optimal constant in (2) when the set of admissible functions is restricted to all symmetric functions in  $\mathcal{D}$ , *i.e.* functions which depend only on  $s$  and achieve their extremum at  $s = 0$ . It is achieved by an explicit function  $u_{\mu,*}$  with  $\mu$  such that  $\Lambda_*^\theta(\mu) = \Lambda$ . We recall that  $\mathbf{K}_{\text{CKN}}^*(\theta, \Lambda, p)$  is explicit (see [6, Lemma 3]). Let us define

$$\Lambda_{\text{FS}}(p, \theta) := 4 \frac{d-1}{p^2-4} \frac{(2\theta-1)p+2}{p+2}. \quad (7)$$

Symmetry breaking means  $\mathbf{K}_{\text{CKN}}^*(\theta, \Lambda, p) < \mathbf{K}_{\text{CKN}}(\theta, \Lambda, p)$ . It is known that

- (i)  $\mathbf{K}_{\text{CKN}}^*(\theta, \Lambda, p) = \mathbf{K}_{\text{CKN}}(\theta, \Lambda, p)$  for  $\Lambda > 0$  small (see [8]).

- (ii)  $\mathbf{K}_{\text{CKN}}^*(\theta, \Lambda, p) < \mathbf{K}_{\text{CKN}}(\theta, \Lambda, p)$  for  $\Lambda > \Lambda_{\text{FS}}(p, \theta)$  (see [4, 5]).
- (iii) The map  $\Lambda \mapsto 1/\mathbf{K}_{\text{CKN}}(\theta, \Lambda, p)$  is increasing and concave when  $\theta = 1$ .

The value  $\Lambda = \Lambda_{\text{FS}}(p, \theta)$  corresponds to the threshold of instability of the symmetric minimizers of (3). More estimates will be given in Section 2. Our next purpose is to study the bifurcation of non-symmetric minimizers from the symmetric ones. Let us start with  $\theta = 1$  and define

$$\mu_{\text{FS}} := \Lambda_{\text{FS}}(p, 1).$$

**Theorem 2** *Assume that  $\theta = 1$ ,  $d \geq 3$  and  $p \in (2, 2^*]$ . Under assumption (H), there exist a constant  $c_{p,d}$  and*

$$u_{(\mu)} := u_{\mu,*} + \sqrt{c_{p,d}(\mu - \mu_{\text{FS}})} \varphi + c_{p,d}(\mu - \mu_{\text{FS}}) \psi, \quad (8)$$

where  $\varphi$  and  $\psi$  are two smooth functions with exponential decay as  $|s| \rightarrow \infty$  such that, for  $c_{p,d}(\mu - \mu_{\text{FS}}) > 0$ ,

$$\mathcal{Q}_{\mu}[u_{(\mu)}] = \mathcal{Q}_{\mu}[u_{\mu,*}] \left( 1 - \frac{p^2 - 4}{8} c_{p,d}(\mu - \mu_{\text{FS}})^2 + o((\mu - \mu_{\text{FS}})^2) \right).$$

Moreover, if  $c_{p,d}$  is positive, then for  $\mu > \mu_{\text{FS}}$ ,  $\mathcal{Q}_{\mu}[u_{(\mu)}]$  minimizes  $\mathcal{Q}_{\mu}$  in a neighborhood of  $u_{\mu,*}$  among smooth functions with exponential decay as  $|s| \rightarrow \infty$ , up to terms of order  $o((\mu - \mu_{\text{FS}})^2)$ .

The assumption (H) is rather technical but explicit and will be stated only in Section 3.4. For a given  $d$ , it is a condition on  $p$ , which ensures the existence on  $c_{p,d}$ . Notice that the condition that  $c_{p,d}$  is positive is stronger than (H). We are not able to fully characterize the positivity of  $c_{p,d}$ , but at least we will give a sufficient condition in Theorem 7. The corresponding range of  $p$  is not expected to be optimal.

In Section 3 we shall perform an expansion of the energy  $\mathcal{Q}_{\mu}$  in a neighborhood of the first bifurcation point on the symmetric branch, by minimizing  $\mathcal{Q}_{\mu}$  among a special class of smooth functions with exponential decay, which is expected to contain all minimizers in  $H^1(\mathcal{C})$ . However we did not prove that such a regularity result holds order by order in the expansion. Anyway, our expansion provides us with an approximate, local minimizer under the condition that  $c_{p,d}$  is positive.

The function  $\varphi$  in Theorem 2 is explicit,  $\psi$  is given by a linear elliptic equation with an explicit source term and  $c_{p,d}$  is given by an identity involving  $\varphi$  and  $\psi$ . Numerically,  $c_{p,d}$  is positive in all cases considered in Section 4. Our last result is also written for  $u_{(\mu)}$ . With a slight abuse of notations, we may still use  $\tau$  and  $\nu$  for the approximated

solution, that is

$$(A) \quad \tau(\mu) := t[u_{(\mu)}] \quad \text{and} \quad \nu(\mu) := \frac{\|u_{(\mu)}\|_{L^2(C)}^2}{\|u_{(\mu)}\|_{L^p(C)}^2},$$

and then redefine  $\Lambda^\theta(\mu)$  and  $J^\theta(\mu)$  accordingly. Of course it is to be expected that the new versions of  $\tau$  and  $\nu$  differ from the former ones only by higher order terms but mathematically this is an open question. As we shall see in Section 4 and with the above definition (A),  $\tau'(\mu_{\text{FS}})$  can be computed in terms of  $c_{p,d}$ . Let us define

$$\vartheta_2(p, d) := \frac{\tau'(\mu_{\text{FS}})}{1 + \tau'(\mu_{\text{FS}})}.$$

One can expect that the value of  $\vartheta_2(p, d)$  is the same if  $\tau$  is computed on the basis of  $u_\mu$ , the solution to the Euler-Lagrange equation (5), or of  $u_{(\mu)}$ , the approximation defined by (8). The relative values of  $\vartheta_2(p, d)$  and  $\vartheta(p, d)$  determine the behavior of the non-symmetric branch close to the bifurcation point on the symmetric branch. More precisely, we have the following local result.

**Theorem 3** *Under the assumptions of Theorem 2 and definition (A), if  $c_{p,d}$  is positive, if  $u_{(\mu)}$  is given by (8) and if  $\mu$  is taken in a right neighborhood of  $\mu_{\text{FS}}$ , then we have the following alternative.*

- *Either  $\vartheta_2(p, d) \leq \vartheta(p, d)$  and then for all  $\theta \in (\vartheta(p, d), 1]$ , the branch  $(\Lambda^\theta(\mu), J^\theta(\mu))$  is concave, nondecreasing in  $\mu$  and it is below the symmetric branch  $(\Lambda_*^\theta(\mu), J_*^\theta(\mu))$ .*
- *Or, on the contrary,  $\vartheta_2(p, d) > \vartheta(p, d)$  and then we find two different behaviors:*
  - *if  $\theta \in (\vartheta_2(p, d), 1]$ , the branch is concave, nondecreasing in  $\mu$  and below the symmetric branch,*
  - *if  $\theta \in (\vartheta(p, d), \vartheta_2(p, d))$ , then the branch  $(\Lambda^\theta(\mu), J^\theta(\mu))$  is above the symmetric branch  $(\Lambda_*^\theta(\mu), J_*^\theta(\mu))$  and  $\frac{d}{d\mu} \Lambda^\theta(\mu_{\text{FS}}) < 0$ .*

In the last case, when  $\theta \in (\vartheta(p, d), \vartheta_2(p, d))$ , the branch in the  $(\Lambda^\theta, J^\theta)$  representation is on the left of the bifurcation point and above the curve corresponding to symmetric solutions. The case  $\theta = \vartheta(p, d)$  is of particular interest and will be discussed in details from a numerical point of view in Section 4.

## 2. Preliminaries observations and proof of Theorem 1

### 2.1. Caffarelli-Kohn-Nirenberg inequalities: more details on symmetry breaking

Recall that the threshold value for the stability of symmetric optimal functions is given by  $\Lambda_{\text{FS}}(p, \theta)$  defined in (7): symmetry breaking occurs for any  $\Lambda > \Lambda_{\text{FS}}(p, \theta)$  according

to [4, 6] (also see [5] for previous results and [9] if  $d = 2$  and  $\theta = 1$ ). As shown in [9, 10, 3], there is a continuous curve  $p \mapsto \Lambda_s(p)$  with  $\lim_{p \rightarrow 2^+} \Lambda_s(p) = \infty$  and  $\Lambda_s(p) > a_c^2$  for any  $p \in (2, 2^*)$  such that symmetry holds for any  $\Lambda \leq \Lambda_s$  and there is symmetry breaking if  $\Lambda > \Lambda_s$ . As proved in [8], for all  $p, d$  in the considered range,  $\theta = 1$ ,

$$\frac{(d-1)(6-p)}{4(p-2)} < \Lambda_s(p) \leq \Lambda_{\text{FS}}(p, 1).$$

Despite the fact that the exact shape of  $\Lambda_s$  is not known, it can be proved that we have  $\lim_{p \rightarrow 2^*} \Lambda_s(p) = a_c^2$  if  $d \geq 3$  and, if  $d = 2$ ,  $\lim_{p \rightarrow \infty} \Lambda_s(p) = 0$ , or more precisely,  $\lim_{p \rightarrow \infty} p^2 \Lambda_s(p) = 4$ . Moreover, we also know from [11, Theorem 3.1] that symmetry holds if  $\Lambda \leq d^2/p^2$ .

According to [12], it is known that an optimal function exists for any  $\theta \in (\vartheta(p, d), 1)$ , but only if  $\mathbf{K}_{\text{CKN}}(\theta, \Lambda, p) > \mathbf{K}_{\text{GN}}(p, d)$  when  $\theta = \vartheta(p, d)$ , where  $\mathbf{K}_{\text{GN}}(p, d)$  is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality (6). The case  $\mathbf{K}_{\text{CKN}}(\vartheta(p, d), \Lambda, p) = \mathbf{K}_{\text{GN}}(p, d)$  has not been studied yet. A sufficient condition for the existence of extremals can be deduced, by comparison with symmetric functions, namely  $\mathbf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda, p) > \mathbf{K}_{\text{GN}}(p, d)$ , which can be rephrased in terms of  $\Lambda$  as  $\Lambda < \Lambda_{\text{GN}}^*(p, d)$  for some non-explicit (but easy to compute numerically) function  $p \mapsto \Lambda_{\text{GN}}^*(p, d)$ . When  $\theta = \vartheta(p, d)$  and  $\Lambda > \Lambda_{\text{GN}}^*(p, d)$ , extremal functions (if they exist) cannot be symmetric and in the asymptotic regime  $p \rightarrow 2^+$ , this condition is weaker than  $\Lambda > \Lambda_{\text{FS}}(p, \vartheta(p, d))$ . One can indeed prove that  $\lim_{p \rightarrow 2^+} \Lambda_{\text{FS}}(p, \vartheta(p, d)) > \lim_{p \rightarrow 2^+} \Lambda_{\text{GN}}^*(p, d)$ . Hence, for  $\theta \in (\vartheta(p, d), 1]$ , close enough to  $\vartheta(p, d)$  and  $p - 2 > 0$ , small (but numerically not so small, actually, as shown in [13]; also see [3, Section 5] for estimates and Section 4.3 for some plots), optimal functions exist and are not symmetric if  $\Lambda > \Lambda_{\text{GN}}^*(p, d)$ , which is again a less restrictive condition than  $\Lambda > \Lambda_{\text{FS}}(p, \theta)$ . See [3] for proofs and [13] for a more detailed overview of known results.

In this paper we study perturbatively the non-symmetric solutions lying in the first branch bifurcating from the branch of *symmetric extremals* and show that they explain all phenomena of symmetry breaking known or observed so far, including cases in which the symmetric extremals are stable. Of course, it is not clear that all extremals for Caffarelli-Kohn-Nirenberg inequalities lie in those branches, even if probably that is the case. In Section 3 we will provide a complete description of the branch around the bifurcation point, based on an asymptotic expansion. This clarifies the local behavior of the branch and accounts for all phenomena numerically observed in [14]. Before doing so, let us study the branch of symmetric solutions and the asymptotic behavior of the branch of optimal functions (proof of Theorem 1).

## 2.2. The case of symmetric extremals

We start with the symmetric case for  $\theta = 1$  and adapt the computations that can be found in [6] (also see [8] and the Appendix). Consider the equation

$$-(p-2)^2 w'' + 4w - 2p|w|^{p-2}w = 0 \quad \text{in } \mathbb{R}. \quad (9)$$

The function  $w(s) := (\cosh s)^{-\frac{2}{p-2}}$  is, up to translations, the unique positive solution of (9). As a consequence, the function  $u(s) = (\frac{1}{2}p\mu)^{1/(p-2)} w(\frac{p-2}{2}\sqrt{\mu}s)$  is the unique solution of

$$-u'' + \mu u = |u|^{p-2}u \quad \text{in } \mathbb{R}. \quad (10)$$

The symmetric optimal function  $u_*$  for  $\theta < 1$  can be explicitly computed. Up to multiplication by a constant,  $u_*$  solves

$$-\theta u_*'' + \eta u_* = u_*^{p-1},$$

with  $\eta = (1-\theta)t[u_*] + \Lambda$ . After multiplying the above equation by  $u_*$ , integrating with respect to  $s \in \mathbb{R}$  and dividing by  $\int_{\mathbb{R}} u_*^2 ds$ , we find

$$t[u_*] + \Lambda = \frac{\int_{\mathbb{R}} u_*^p ds}{\int_{\mathbb{R}} u_*^2 ds},$$

where  $u_*(s) = Aw(Bs)$ , for all  $s \in \mathbb{R}$ ,  $w$  solves (9),  $A = (\frac{p\eta}{2})^{\frac{1}{p-2}}$  and  $B = \frac{p-2}{2}\sqrt{\frac{\eta}{\theta}}$ . From this expression, as in [8], we deduce that

$$t[u_*] = B^2 \frac{J_2}{l_2} = \frac{p-2}{p+2} \frac{\eta}{\theta} \quad \text{and} \quad \frac{\int_{\mathbb{R}} u_*^p ds}{\int_{\mathbb{R}} u_*^2 ds} = A^{p-2} \frac{l_p}{l_2} = \frac{2p\eta}{p+2},$$

where for all  $q \geq 2$ ,  $l_q := \int_{\mathbb{R}} |w(s)|^q ds$ , and  $J_2 := \int_{\mathbb{R}} |w'(s)|^2 ds$  (see Appendix A.1 for details). This provides the identity

$$\frac{p-2}{p+2} \frac{\eta}{\theta} + \Lambda = \frac{2p\eta}{p+2}$$

and uniquely determines  $\eta = \frac{(p+2)\theta}{(2\theta-1)p+2} \Lambda$ . As a consequence, we have

$$t[u_*] = \frac{p-2}{(2\theta-1)p+2} \Lambda.$$

### 2.3. Gagliardo-Nirenberg inequalities and the corresponding asymptotic regime

Now we investigate the asymptotic regimes corresponding to  $\Lambda \rightarrow \infty$  and prove Theorem 1. Let

$$\mathbf{S}_p(\mathbb{R}^d) := \inf_{u \in \mathbf{H}^1(\mathbb{R}^d) \setminus \{0\}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} |u|^2 dx}{|\mathbb{S}^{d-1}|^{\frac{p-2}{p}} \left( \int_{\mathbb{R}^d} |u|^p dx \right)^{\frac{2}{p}}}.$$

An optimization of the quotient in the expression of  $\mathbf{S}_p(\mathbb{R}^d)$  allows to relate this constant with  $\mathbf{K}_{\text{GN}}$ . Indeed, if we optimize  $\mathcal{N}[u] := \int_{\mathbb{R}^d} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^d} |u|^2 dx$  under the scaling  $\lambda \mapsto u_\lambda(x) := \lambda^{\frac{d}{p}} u(\lambda x)$ , we find that

$$\mathcal{N}[u_\lambda] = \lambda^{2(1-\vartheta)} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \lambda^{-2\vartheta} \mu \int_{\mathbb{R}^d} |u|^2 dx$$

achieves its minimum at  $\lambda_\star = \sqrt{\frac{\vartheta \mu}{1-\vartheta}} \frac{\|u\|_{\mathbf{L}^2(\mathbb{R}^d)}}{\|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^d)}}$ , so that

$$\mathcal{N}[u_{\lambda_\star}] = \vartheta^{-\vartheta} (1-\vartheta)^{-(1-\vartheta)} \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^d)}^{2\vartheta} \|u\|_{\mathbf{L}^2(\mathbb{R}^d)}^{2(1-\vartheta)} \mu^{1-\vartheta},$$

thus proving that, with the choice  $\mu = 1$ ,  $\mathbf{K}_{\text{GN}}^{-1} = \vartheta^\vartheta (1-\vartheta)^{1-\vartheta} \mathbf{S}_p(\mathbb{R}^d)$ . For any  $\mu > 0$ , if  $u_\mu$  is the solution of (5) and if it is a minimizer of  $1/\mathbf{K}_{\text{CKN}}(1, \Lambda, p)$ , we know from [5, Theorem 1.2] that as  $\mu \rightarrow \infty$ ,

$$(\tau(\mu) + \mu) \nu(\mu) = \mathcal{Q}_\mu[u_\mu] \sim \mathbf{S}_p(\mathbb{R}^d) \mu^{1-\vartheta}.$$

If  $u$  is an optimal function for  $\mathbf{S}_p(\mathbb{R}^d)$ , we also know from the above computations that  $\lambda_\star = 1$ , that is,

$$1 = \lambda_\star^2 = \frac{\vartheta \mu}{1-\vartheta} \frac{1}{\tau(\mu)} \quad \text{and so,} \quad \frac{\tau(\mu)}{\mu} = \frac{\vartheta}{1-\vartheta}.$$

Hence,

$$\nu(\mu) \sim (1-\vartheta) \mathbf{S}_p(\mathbb{R}^d) \mu^{-\vartheta} \quad \text{as} \quad \mu \rightarrow \infty.$$

Consider now the case  $\theta > \vartheta(p, d)$ . According to the parametrization of Section 1, that is, by definition of  $J^\theta$  and  $\Lambda^\theta$ , we obtain that

$$\Lambda^\theta(\mu) = \theta \mu - (1-\theta) \tau(\mu) = \frac{\theta - \vartheta(p, d)}{1 - \vartheta(p, d)} \mu,$$

$$J^\theta(\mu) = \nu(\mu) \theta^\theta (\mu + \tau(\mu))^\theta \sim \theta^\theta (1 - \vartheta(p, d))^{1-\theta} \mathbf{S}_p(\mathbb{R}^d) \mu^{\theta - \vartheta(p, d)},$$

as  $\mu \rightarrow \infty$ . Hence the parametric curve  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  is asymptotic to the curve

$$\Lambda \mapsto \frac{\theta^\theta (1 - \vartheta(p, d))^{1-\vartheta(p, d)}}{(\theta - \vartheta(p, d))^{\theta-\vartheta(p, d)}} \mathcal{S}_p(\mathbb{R}^d) \Lambda^{\theta-\vartheta(p, d)},$$

for large values of  $\mu$ . This completes the proof of Theorem 1. See Figs. 1–3 for some plots of the curves  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for various values of  $\theta$  and how these curves can be compared with the ones corresponding to the asymptotic regime as described in Theorem 1.  $\square$

The limit case  $\theta = \vartheta = \vartheta(p, d)$  is of particular interest. Indeed, according to [12], Gagliardo-Nirenberg inequalities play a special role. See Fig. 6. First of all, since  $1/\mathcal{K}_{\text{CKN}}(\vartheta, \Lambda^\vartheta(\mu), p) \leq J^\vartheta(\mu)$  and using the fact that  $\Lambda \mapsto \mathcal{K}_{\text{CKN}}(\vartheta, \Lambda, p)$  is a non-increasing function of  $\Lambda$ , we recover the known result that

$$\mathcal{K}_{\text{GN}} \leq \mathcal{K}_{\text{CKN}}(\vartheta(p, d), \Lambda, p) \quad \forall \Lambda > 0.$$

Such an inequality has deep implications on the existence of an optimal function (see [12] and in particular [12, Theorem 1.4]): either the inequality is strict and there exists a non-trivial optimal function for (1), or there is equality and a non-trivial optimal function may exist only if  $\Lambda = \inf\{\lambda > 0 : \mathcal{K}_{\text{GN}} \geq \mathcal{K}_{\text{CKN}}(\vartheta(p, d), \lambda, p)\}$ , but certainly not for any larger value of  $\Lambda$ , if the above infimum is finite.

In our setting, we can define  $\mu_{\text{GN}} := \inf\{\mu > 0 : J^\vartheta(\mu) \leq \mathcal{K}_{\text{GN}}\}$ , with  $\vartheta = \vartheta(p, d)$ . It is granted that  $\mu_{\text{GN}} > 0$ . Either  $\mu_{\text{GN}} = \infty$  and there is always a minimizer, or

$$\mathcal{K}_{\text{CKN}}(\vartheta(p, d), \Lambda^\vartheta(\mu), p) = J^\vartheta(\mu) \quad \forall \mu \in (0, \mu_{\text{GN}}]$$

and there exists a non-trivial optimal function for (1) if  $\mu < \mu_{\text{GN}}$ , while

$$\mathcal{K}_{\text{CKN}}(\vartheta(p, d), \Lambda^\vartheta(\mu), p) = \mathcal{K}_{\text{GN}} \quad \forall \mu \in [\mu_{\text{GN}}, \infty)$$

and there is no optimal function for (1) if  $\mu > \mu_{\text{GN}}$ .

### 3. An expansion at the bifurcation point: proof of Theorems 2 and 3

In this section, we determine the behavior of the branch of non-symmetric positive solutions that bifurcates from the branch of the symmetric ones in a neighborhood of the first bifurcation point  $\mu = \mu_{\text{FS}}$ . Consider the case  $\theta = 1$  and denote by  $u_{\mu,*}$  the positive symmetric solution of

$$-\Delta u + \mu u = u^{p-1}, \tag{11}$$

so that  $\mathcal{Q}_\mu[u_{\mu,*}] = \|u_{\mu,*}\|_{L^p(\mathcal{C})}^{p-2} = 1/\mathbf{K}_{\text{CKN}}^*(1, \mu, p)$ . Notice that if  $u$  is a solution to (11), we still have  $\mathcal{Q}_\mu[u] = \|u\|_{L^p(\mathcal{C})}^{p-2}$  even if  $u$  is not symmetric.

We will search for minimizers of  $\mathcal{Q}_\mu$  in a restricted class of functions depending only on the variable  $s$  (see Section 2.1) along the axis of the cylinder and on the azimuthal angle  $\zeta$  of the sphere because of the result on *Schwarz foliated symmetry* of [15]. This guarantees that we are in the right class for minimizers when  $\theta = 1$ . For  $\theta < 1$ , no such result has been established in the literature but we will work in the same framework. It is indeed straightforward to check that the same result holds.

Let  $f_1$  be the first non-constant spherical harmonic function, *i.e.* the eigenfunction of the Laplace-Beltrami operator on the sphere  $\mathbb{S}^{d-1}$  corresponding to the eigenvalue  $d - 1$  and denote by  $f_2$  the next one (among the ones depending only on the azimuthal angle  $\zeta$ ), with corresponding eigenvalue equal to  $2d$ . See Appendix A.4 for details.

### 3.1. Expansion of $\mathcal{Q}_\mu$ at order two

Let us consider a solution of (11) that can be written as

$$u_\mu = u_{\mu,*} + \varepsilon \varphi + o(\varepsilon)$$

in a neighborhood of  $u_{\mu,*}$ . In the limiting regime corresponding to  $\varepsilon \rightarrow 0$ , an expansion at order two in  $\varepsilon$  gives

$$\frac{\mathcal{Q}_\mu[u_{\mu,*} + \varepsilon \varphi]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 = \varepsilon^2 \frac{q[\mu, \varphi]}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} + \varepsilon^2 (p-2) \frac{(\int_{\mathcal{C}} u_{\mu,*}^{p-1} \varphi \, dy)^2}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^{2p}} + o(\varepsilon^2),$$

where  $q[\mu, \varphi] := \int_{\mathcal{C}} (|\nabla \varphi|^2 + \mu |\varphi|^2 - (p-1) u_{\mu,*}^{p-2} |\varphi|^2) \, dy$ . By minimizing the term of order two, we find that

$$\frac{\mathcal{Q}_\mu[u_{\mu,*} + \varepsilon \varphi]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 \sim \varepsilon^2 \frac{(\varphi_1, \mathcal{H}_\mu \varphi_1)_{L^2(\mathcal{C})}}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} \quad \text{as } \varepsilon \rightarrow 0,$$

where  $\mathcal{H}_\mu := -\frac{d^2}{ds^2} + \mu + d - 1 - (p-1) u_{\mu,*}^{p-2}$  is a Pöschl-Teller operator whose lowest eigenvalue is given by  $\lambda_1(\mu) = d - 1 + \mu - \frac{1}{4} \mu p^2$ , and such that  $\varphi = \varphi_1 f_1$  is the corresponding eigenfunction (see Appendix A.2 for details). Since  $\varepsilon$  has not been specified yet, we can normalize  $\varphi$  by the condition

$$\|\varphi\|_{L^2(\mathcal{C})}^2 = \|\varphi_1\|_{L^2(\mathbb{R})}^2 = \|u_{\mu,*}\|_{L^p(\mathcal{C})}^p,$$

which slightly simplifies some computations below. This shows in particular that

$$\frac{q[\mu, \varphi]}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} = \lambda_1(\mu).$$

See (A.2) for an expression of  $\varphi_1$ , which is smooth and decays exponentially as  $|s| \rightarrow \infty$ .

As in [4], let  $\mu_{\text{FS}}$  be such that  $\lambda_1(\mu_{\text{FS}}) = 0$ , that is

$$\mu_{\text{FS}} = 4 \frac{d-1}{p^2-4}$$

(see Appendix A.2 for details). For any  $\mu > \mu_{\text{FS}}$ , we have

$$\lambda_1(\mu) = -\frac{1}{4}(p^2-4)(\mu-\mu_{\text{FS}}). \quad (12)$$

This determines the  $O(\varepsilon^2)$  term. Now we want to investigate the behavior of  $\mathcal{Q}_\mu$  in a neighborhood of  $\mu = \mu_{\text{FS}}$  and therefore need an expansion at higher order.

### 3.2. Expansion of $\mathcal{Q}_\mu$ at order four

Our purpose is to build an expansion  $(u_\mu)_{\mu > \mu_{\text{FS}}}$  of the branch of positive solutions of (11) that bifurcates from the branch  $\mu \mapsto u_{\mu,*}$  at  $u_{\text{FS},*} =: u_{\text{FS}}$  and satisfies  $\mathcal{Q}_\mu[u_{(\mu)}] < \mathcal{Q}_\mu[u_{\mu,*}]$ . For  $\mu$  in a right-neighborhood of  $\mu_{\text{FS}}$ , we look for solutions of (11) of the form  $u_\mu = u_{(\mu)}$ , up to higher order terms, where

$$u_{(\mu)} = u_{\mu,*} + \varepsilon \varphi + \eta \psi, \quad (13)$$

with  $\varepsilon > 0$  and  $\eta = o(\varepsilon)$ . The fact that an expansion starting with the above expression can be built is standard. From now on, we will assume that the solutions are given by the above expression and that  $\tau, \nu, \Lambda^\theta$  and  $J^\theta$  are defined according to definition (A). Some of our computations are formal, but can be justified by technical estimates that will be only sketched. Here  $\varphi = \varphi_1 f_1$  has been determined above. Recall that  $\varphi_1$  is a function depending on  $s$  only. For convenience, let us define

$$k_\psi := \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1} \psi \, dy}{\int_{\mathcal{C}} u_{\mu,*}^p \, dy}.$$

Since we are interested in functions depending only on the azimuthal angle  $\zeta$ , we indifferently use  $\omega \in \mathbb{S}^{d-1}$  or  $\zeta \in [0, \pi]$  with a slight abuse of notation. We consider the sequence  $(f_k)_{k \in \mathbb{N}}$  of spherical harmonics depending only on  $\zeta$ . See Appendix A.4 for details. We denote by  $\psi_k$  the decomposition of  $\psi$  in spherical harmonics:

$$\psi = k_\psi u_{\mu,*} + \sum_{k \geq 0} \psi_k f_k,$$

$$\text{with } \psi_k(r) := \int_{\mathbb{S}^{d-1}} \psi(r, \omega) f_k(\omega) \, d\nu(\omega) \quad \forall r \in [0, \infty),$$

where  $d\nu(\omega)$  is the uniform probability measure on the sphere. Here we have chosen  $\psi_0$  in such a way that  $\int_{\mathcal{C}} u_{\mu,*}^{p-1} \psi_0 dy = 0$  because

$$k_\psi = \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1} (k_\psi u_{\mu,*} + \psi_0) dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} = k_\psi + \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1} \psi_0 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy}.$$

We know that  $q[\mu, \psi] = \sum_{k \geq 0} q[\mu, \psi_k f_k]$ , where

$$q[\mu, \psi_k f_k] = \int_{\mathcal{C}} |\nabla(\psi_k f_k)|^2 dy + \mu \int_{\mathcal{C}} \psi_k^2 dy - (p-1) \int_{\mathcal{C}} u_{\mu,*}^{p-2} \psi_k^2 dy.$$

With  $\mu_k := \mu + k(k+d-2)$ , for any  $k \geq 2$ , we get that

$$q[\mu, \psi_k f_k] = \int_{\mathcal{C}} |\psi_k'|^2 dy + \mu_k \int_{\mathcal{C}} \psi_k^2 dy - (p-1) \int_{\mathcal{C}} u_{\mu,*}^{p-2} \psi_k^2 dy$$

is nonnegative for  $\mu - \mu_{\text{FS}} > 0$ , small enough, and positive unless  $\psi_k \equiv 0$ . Lengthy but straightforward computations show that

$$\frac{\mathcal{Q}_\mu[u(\mu)]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 = a \varepsilon^2 + b \varepsilon^4 + c \varepsilon^2 \eta + d \eta^2 + e \varepsilon \eta + o(\varepsilon^4 + \eta^2 + |a| \varepsilon^2 + \eta \varepsilon^2)$$

when the function  $\psi$  is smooth and has exponential decay as  $|s| \rightarrow \infty$ . The coefficients in the above expansion are given by

$$a(\mu) = \frac{q[\mu, \varphi]}{\int_{\mathcal{C}} u_{\mu,*}^p dy} = \lambda_1(\mu),$$

$$\begin{aligned} \frac{b(\mu)}{p-1} = & -\lambda_1(\mu) \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} + \frac{1}{4} (p-1)(p-2) \left[ \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} \right]^2 \\ & - \frac{1}{12} (p-2)(p-3) \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-4} \varphi^4 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy}, \end{aligned}$$

$$\begin{aligned} c(\mu) = & -2\lambda_1(\mu) k_\psi + (p-1)(p-2) \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} k_\psi \\ & - (p-1)(p-2) \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \psi dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy}, \end{aligned}$$

$$d(\mu) = \frac{q[\mu, \psi]}{\int_{\mathcal{C}} u_{\mu,*}^p dy} + (p-2) k_\psi^2,$$

$$e(\mu) = 2\lambda_1(\mu) \frac{\int_{\mathcal{C}} \varphi \psi dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy}.$$

With no restriction, we may require that  $\varphi$  is optimal in the direction  $f_1$ , that is

$$\int_{\mathcal{C}} \varphi \psi dy = 0. \tag{14}$$

In other words, this amounts to require that  $e(\mu) = 0$  for any  $\mu > \mu_{\text{FS}}$ .

According to (12), we get

$$a(\mu) = -\frac{1}{4}(p^2 - 4)(\mu - \mu_{\text{FS}}).$$

Using  $\int_0^\pi |f_1|^4 d\nu = \frac{3d}{d+2}$  (see Appendix A.4), we obtain

$$\frac{4b(\mu_{\text{FS}})}{(p-1)(p-2)} = (p-1) \left[ \frac{\int_{\mathbb{R}} u_{\text{FS}}^{p-2} \varphi_1^2 ds}{\int_{\mathbb{R}} u_{\text{FS}}^p ds} \right]^2 - \frac{d(p-3)}{d+2} \frac{\int_{\mathbb{R}} u_{\text{FS}}^{p-4} \varphi_1^4 ds}{\int_{\mathbb{R}} u_{\text{FS}}^p ds}.$$

All above integrals are computed in Appendix A.3 and allow to express  $b(\mu_{\text{FS}})$  as

$$b(\mu_{\text{FS}}) = \frac{4(d-1)^2 p^3 (p-1)^2 [2p(5p-6) - d(p^2 - 16p + 12)]}{(d+2)(p+2)^2 (p-2)(3p-2)^2 (5p-6)}.$$

As for the terms which depend on  $\eta$ , we observe that they sum as

$$\begin{aligned} & \varepsilon^2 \eta c(\mu) + \eta^2 d(\mu) + \varepsilon \eta e(\mu) \\ &= \eta^2 \left[ \frac{q[\mu, \psi]}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} + (p-2) k_\psi^2 \right] - 2 \lambda_1(\mu) k_\psi \varepsilon^2 \eta \\ & \quad + \varepsilon^2 \eta (p-1)(p-2) \left[ \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} k_\psi - \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \psi dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} \right]. \end{aligned}$$

Using the fact that  $f_1^2 = f_0 + \kappa_{(d)} f_2$  (see Appendix A.4), it is straightforward to observe that the optimal function  $\psi$  is given by

$$\psi = k_\psi u_{\mu,*} f_0 + \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2 \quad (15)$$

while  $\psi_k \equiv 0$  for any  $k > 2$  and solves the Euler-Lagrange equation

$$\begin{aligned} & -\Delta \psi + \mu \psi - (p-1) u_{\mu,*}^{p-2} \psi + (p-2) k_\psi u_{\mu,*}^{p-1} f_0 \\ & \quad + \frac{\varepsilon^2}{2\eta} \left[ \left( \frac{p^2(p-1)(p-2)}{3p-2} \mu - 2\lambda_1(\mu) \right) u_{\mu,*}^{p-1} f_0 \right. \\ & \quad \left. - (p-1)(p-2) u_{\mu,*}^{p-3} \varphi_1^2 (f_0 + \kappa_{(d)} f_2) \right] + \mathcal{L} \varphi_1 f_1 = 0. \end{aligned}$$

Here we used the fact that  $\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy / \|u_{\mu,*}\|_{L^p(\mathcal{C})}^p = \frac{p^2 \mu}{3p-2}$  (see Appendix A.3 for details). Constraint (14) is taken into account through the Lagrange multiplier  $\mathcal{L}$ . Here higher order terms have been omitted: see Remark 4.

The three components  $\psi_0$ ,  $\psi_1$  and  $\psi_2$  satisfy the equations

$$-\psi_0'' + \mu \psi_0 - \frac{p(p-1)\mu}{2[\cosh(\beta s)]^2} \psi_0 + \frac{\beta^2 A_0}{[\cosh(\beta s)]^2 \frac{p-1}{p-2}} - \frac{\beta^2 B_0}{[\cosh(\beta s)]^2 \frac{2p-3}{p-2}} = 0,$$

$$\begin{aligned}
& -\psi_1'' + \mu_1 \psi_1 - \frac{p(p-1)\mu}{2[\cosh(\beta s)]^2} \psi_1 + \mathcal{L} \varphi_1 = 0, \\
& -\psi_2'' + \mu_2 \psi_2 - \frac{p(p-1)\mu}{2[\cosh(\beta s)]^2} \psi_2 - \frac{\beta^2 B_2}{[\cosh(\beta s)]^{\frac{2p-3}{p-2}}} = 0,
\end{aligned}$$

with  $\mu_1 = \mu + d - 1$ ,  $\mu_2 = \mu + 2d$  and

$$\begin{aligned}
A_0 &= \frac{\alpha^{p-1}}{\beta^2} \frac{\varepsilon^2}{2\eta} \left( \frac{p^2(p-1)(p-2)}{3p-2} \mu - 2\lambda_1(\mu) \right), \\
B_0 &= \frac{\varepsilon^2}{2\eta} (p-1)(p-2) \frac{\alpha^{2p-3}}{\beta^2}, \\
B_2 &= \frac{\varepsilon^2}{2\eta} (p-1)(p-2) \kappa_{(d)} \frac{\alpha^{2p-3}}{\beta^2} = \kappa_{(d)} B_0,
\end{aligned}$$

Recall that  $\beta^2 = \frac{1}{4}(p-2)^2 \mu$ .

Multiplying the equation for  $\psi_1$  by  $\varphi_1$  and integrating by parts we get

$$\lambda_1(\mu) \int_{\mathbb{R}} \varphi_1 \psi_1 ds + \mathcal{L} \int_{\mathbb{R}} |\varphi_1|^2 ds = 0.$$

Using Assumption (14), this proves that  $\mathcal{L} = 0$ . This implies that  $\psi_1$  is an eigenfunction of  $\mathcal{H}_\mu$ , with eigenvalue  $\lambda_1(\mu)$ . Since  $\lambda_1(\mu)$  is simple, we find that  $\psi_1 \equiv 0$  by (14).

We may next observe that by taking

$$\psi_0(s) = A_0 \chi_{0,p-1}(\beta s) + B_0 \chi_{0,2p-3}(\beta s) \text{ and } \psi_2(s) = B_2 \chi_{2,2p-3}(\beta s), \quad (16)$$

the problem is reduced to the set of equations

$$\begin{aligned}
& -\chi_{0,p-1}'' + \frac{4\chi_{0,p-1}}{(p-2)^2} - \frac{2p(p-1)\chi_{0,p-1}}{(p-2)^2(\cosh s)^2} + w^{p-1} = 0, \\
& -\chi_{0,2p-3}'' + \frac{4\chi_{0,2p-3}}{(p-2)^2} - \frac{2p(p-1)\chi_{0,2p-3}}{(p-2)^2(\cosh s)^2} - w^{2p-3} = 0, \\
& -\chi_{2,2p-3}'' + \frac{4\mu_2\chi_{2,2p-3}}{\mu(p-2)^2} - \frac{2p(p-1)\chi_{2,2p-3}}{(p-2)^2(\cosh s)^2} - w^{2p-3} = 0,
\end{aligned} \quad (17)$$

where  $w(s) = (\cosh s)^{-\frac{2}{p-2}}$ . Since

$$-w'' + \frac{4w}{(p-2)^2} - \frac{2p(p-1)w}{(p-2)^2(\cosh s)^2} + \frac{2p}{p-2} w^{p-1} = 0,$$

it follows that

$$\chi_{0,p-1} = \frac{p-2}{2p} w.$$

We may notice that the equations for  $\chi_{0,p-1}$ ,  $\chi_{0,2p-3}$  and  $\chi_{2,2p-3}$  are all independent of  $k_\psi$ . Moreover, since  $\int_{\mathbb{R}} u_{\mu,*}^{p-1} \psi_0 ds = 0$ , we get

$$\int_{\mathbb{R}} w^{p-1} (A_0 \chi_{0,p-1} + B_0 \chi_{0,2p-3}) ds = 0,$$

i. e.

$$0 = A_0 \int_{\mathbb{R}} \frac{\chi_{0,p-1}}{(\cosh s)^2 \frac{p-1}{p-2}} ds + B_0 \int_{\mathbb{R}} \frac{\chi_{0,2p-3}}{(\cosh s)^2 \frac{p-1}{p-2}} ds.$$

**Remark 4** The decomposition (15) of  $\psi$  is done up to higher order terms. Hence the above equality only holds for  $\mu = \mu_{\text{FS}}$ , as can be checked by computing

$$\int_{\mathbb{R}} \chi_{0,p-1} w^{p-1} ds = \frac{p-2}{2p} \int_{\mathbb{R}} w^p ds = \frac{p-2}{2p} \mathfrak{l}_p$$

and, using (17),

$$\mathfrak{b}_{0,p-1} := \int_{\mathbb{R}} \chi_{0,2p-3} w^{p-1} ds = - \int_{\mathbb{R}} \chi_{0,p-1} w^{2p-3} ds = - \frac{p-2}{3p-2} \mathfrak{l}_p.$$

(also see Appendix A.1). With  $A_0 = \frac{\alpha^{p-1}}{\beta^2} \frac{\varepsilon^2}{2\eta} \left( \frac{p^2(p-1)(p-2)}{3p-2} \mu - 2\lambda_1(\mu) \right)$  and  $B_0 = \frac{\varepsilon^2}{2\eta} (p-1)(p-2) \frac{\alpha^{2p-3}}{\beta^2}$ , we find that

$$A_0 \frac{p-2}{2p} \mathfrak{l}_p - B_0 \frac{p-2}{3p-2} \mathfrak{l}_p = 0$$

holds if and only if  $\lambda_1(\mu) = 0$ . As we shall see below, this is consistent with our expansion in terms of powers of  $\varepsilon$  and  $\eta$  because for  $\mu > \mu_{\text{FS}}$ , close enough to  $\mu_{\text{FS}}$ ,  $\lambda_1(\mu)$  corresponds to a term of higher order.

The reader is invited to check that

$$\chi_{0,2p-3} = - \frac{1}{4} \frac{p-2}{p-1} (2w - w^{p-1}).$$

As a consequence, one can compute

$$\mathfrak{b}_{0,1} := \int_{\mathbb{R}} \chi_{0,2p-3} w ds = - \frac{1}{4} \frac{p-2}{p-1} (2\mathfrak{l}_2 - \mathfrak{l}_p) = - \frac{p(p-2)}{2(p-1)(p+2)} \mathfrak{l}_2. \quad (18)$$

Altogether we have found that

$$\varepsilon^2 \eta c(\mu) + \eta^2 d(\mu) + \varepsilon \eta e(\mu) = \eta^2 Q[\psi] - \varepsilon^2 \eta L[\psi]$$

up to higher order terms in  $\varepsilon$ ,  $\eta$  and  $(\mu - \mu_{\text{FS}})$ , where

$$Q[\psi] := \frac{q[\mu, \psi]}{\|u_{\mu,*}\|_{\mathbb{L}^p(\mathcal{C})}^p} + (p-2) k_\psi^2$$

and

$$L[\psi] := -(p-1)(p-2) \left[ \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\|u_{\mu,*}\|_{\mathbb{L}^p(\mathcal{C})}^p} k_\psi - \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \psi dy}{\|u_{\mu,*}\|_{\mathbb{L}^p(\mathcal{C})}^p} \right]$$

are respectively quadratic and linear with respect to  $\psi$ . Since we can multiply  $\psi$  by any positive constant  $\nu$  and  $\eta$  by  $1/\nu$  simultaneously without changing the value of  $\eta^2 Q[\psi] - \varepsilon^2 \eta L[\psi]$ , the optimal choice of  $\eta$  in terms of  $\varepsilon$  is

$$\eta = \varepsilon^2, \quad (19)$$

thus making the sum of the two terms equal to  $\varepsilon^4 (Q[\psi] - L[\psi])$  and  $\psi$  independent of  $\varepsilon$ , at least at leading order. Moreover, if  $\psi$  is a minimizer of  $Q[\psi] - L[\psi]$ , then it is straightforward to check that  $2Q[\psi] - L[\psi] = 0$ , as follows by multiplying the Euler-Lagrange equation by  $\psi$  and integrating. Altogether, we have found that

$$\varepsilon^2 \eta c(\mu) + \eta^2 d(\mu) + \varepsilon \eta e(\mu) = -\frac{1}{2} \varepsilon^4 L[\psi] < 0$$

up to higher order terms in  $\varepsilon$  and for  $\mu - \mu_{\text{FS}}$  small enough, if  $\psi$  is a minimizer of  $Q[\psi] - L[\psi]$ , that is,

$$\frac{\mathcal{Q}_\mu[u(\mu)]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 - \lambda_1(\mu) \varepsilon^2 - b(\mu) \varepsilon^4 = -\frac{1}{2} L[\psi] \varepsilon^4 + o(\varepsilon^4). \quad (20)$$

At this point, we may notice that the function  $u(\mu)$  has not been normalized. Multiplying it by a constant would not change the value of  $\mathcal{Q}_\mu[u(\mu)]$ . If we want it to be a solution of (11) at leading order, then this implies that  $\mathcal{Q}_\mu[u(\mu)] = \|u(\mu)\|_{L^p(C)}^{p-2}$  and we may therefore impose the corresponding constraint, *i.e.*

$$\int_C |\nabla u(\mu)|^2 dy + \mu \int_C u(\mu)^2 dy = \int_C u(\mu)^p dy,$$

without changing the equations written order by order (in other words, the Lagrange multiplier associated with this constraint is zero). Written in terms of  $\varphi$  and  $\psi$ , at lowest order, that is at order  $\varepsilon^2$ , this constraint amounts to

$$\int_C \left( |\nabla \varphi|^2 + \mu |\varphi|^2 - \frac{p(p-1)}{2} u_{\mu,*}^{p-2} |\varphi|^2 \right) dy - (p-2) k_\psi \int_C u_{\mu,*}^p dy = 0.$$

Hence by taking the limit as  $\mu \rightarrow (\mu_{\text{FS}})_+$  and observing that  $\lambda_1(\mu) = O(\mu - \mu_{\text{FS}})$ , we find that, for  $\mu = \mu_{\text{FS}}$ ,

$$k_\psi = -\frac{1}{2} (p-1) \frac{\int_C u_{\mu,*}^{p-2} |\varphi|^2 dy}{\int_C u_{\mu,*}^p dy} = -\frac{2p^2(p-1)(d-1)}{(p-2)(p+2)(3p-2)}.$$

The explicit value of  $k_\psi$  will however not be needed later, because of cancellations that occur in all subsequent computations.

Next comes the observation that, as long as we are interested in computing  $L[\psi]$ , we do not even need to normalize  $u(\mu)$  nor to compute  $k_\psi$ . Indeed with  $\tilde{\psi} := \psi - k_\psi u_{\mu,*} =$

$\psi_0 f_0 + \psi_2 f_2$ , we see that

$$\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} k_\psi - \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \psi dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} = - \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \tilde{\psi} dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p},$$

where  $\tilde{\psi}$  is fully determined by the coefficients  $A_0$ ,  $B_0$ , and  $B_2$ , and by (17). This also determines

$$L[\psi] = (p-1)(p-2) \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \tilde{\psi} dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p},$$

which is not known explicitly but is independent of  $k_\psi$  and can be computed as

$$\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3} \varphi^2 \tilde{\psi} dy}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} = \frac{A_0}{\alpha} \frac{p-2}{2p} \mathbf{a}_0 + \frac{B_0}{\alpha} \alpha^{p-2} \frac{\mathbf{b}_{0,2p-3}}{l_p} + \frac{B_2}{\alpha} \alpha^{p-2} \kappa_{(d)} \frac{\mathbf{b}_{2,2p-3}}{l_p},$$

with

$$\mathbf{a}_0 := \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} = \frac{p^2 \mu}{3p-2}, \quad \mathbf{b}_{0,2p-3} := \int_{\mathbb{R}} \chi_{0,2p-3} w^{2p-3} ds,$$

$$\text{and } \mathbf{b}_{2,2p-3} := \int_{\mathbb{R}} \chi_{2,2p-3} w^{2p-3} ds.$$

Notice that here we have taken advantage of the facts that  $\chi_{0,p-1}(\beta s) = \frac{p-2}{2p} \frac{1}{\alpha} u(s)$  and  $\varphi^2 = \varphi_1^2 (f_0 + \kappa_{(d)} f_2)$ . Using the expression of  $\chi_{0,2p-3}$ , we can also compute

$$\begin{aligned} \mathbf{b}_{0,2p-3} &= - \frac{1}{4} \frac{p-2}{p-1} \left( 2 \int_{\mathbb{R}} w^{2(p-1)} ds - \int_{\mathbb{R}} w^{3p-4} ds \right) \\ &= - \frac{(p-2)p(3p-4)}{(p-1)(3p-2)(5p-6)} l_p. \end{aligned}$$

Altogether, with  $\mathbf{y} := \frac{\mathbf{b}_{2,2p-3}}{l_p}$ , we have found that

$$L[\psi] = 4(d-1)^2 \frac{(p-1)p^3}{(p+2)^2} \left[ \frac{(p-2)p}{(3p-2)^2(5p-6)} + 2 \frac{d-1}{d+2} \frac{p-1}{(p-2)^2} \mathbf{y} \right]. \quad (21)$$

We have not been able to find an explicit expression for  $\mathbf{b}_{2,2p-3}$ , but we can prove that this is a positive quantity and even give an upper bound. According to [16, p. 74], the lowest eigenvalue of the Pöschl-Teller operator  $-\frac{d^2}{ds^2} - U_0 (\cosh s)^{-2}$  is given by

$$\lambda_0 = \frac{1}{2} \sqrt{1+4U_0} - \frac{1}{2} - U_0,$$

if we assume that  $U_0$  is positive. Here we have that  $U_0$  is given by

$$U_0(p) := \frac{2p(p-1)}{(p-2)^2}$$

and the reader is invited to check that

$$\begin{aligned}\sigma(p, d) &:= \lambda_0 + \frac{4\mu_2}{\mu_{\text{FS}}(p-2)^2} \\ &= \frac{1}{2} \sqrt{1 + 4U_0(p)} - \frac{1}{2} - U_0(p) + \frac{2}{(p-2)^2} \left( 2 + \frac{d(p^2-4)}{d-1} \right)\end{aligned}$$

is larger than 1 for any  $p > 2$  and any  $d \geq 2$ . As a straightforward consequence of (17), we deduce that

$$\sigma(p, d) \|\chi_{2,2p-3}\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} \chi_{2,2p-3} w^{2p-3} ds \leq \|\chi_{2,2p-3}\|_{L^2(\mathbb{R})} \|w^{2p-3}\|_{L^2(\mathbb{R})}$$

and, finally,

$$\mathbf{b}_{2,2p-3} \leq \frac{1}{\sigma(p, d)} \int_{\mathbb{R}} w^{2(2p-3)} ds = \frac{16p(p-1)(3p-4)}{(3p-2)(5p-6)(7p-10)} \frac{\mathbf{l}_p}{\sigma(p, d)}.$$

### 3.3. Optimization and a technical statement

Collecting the above estimates and using (20), we get

$$\frac{\mathcal{Q}_\mu[u(\mu)]}{\mathcal{Q}_\mu[u_{\mu,*}]} = 1 - \frac{1}{4}(p^2-4)(\mu - \mu_{\text{FS}})\varepsilon^2 + \left[ b(\mu) - \frac{1}{2}L[\psi] \right] \varepsilon^4 + o(\varepsilon^4),$$

provided  $\mu - \mu_{\text{FS}} = O(\varepsilon^2)$ . With  $L[\psi]$  given by (21), assume that

$$(H) \quad b(\mu_{\text{FS}}) - \frac{1}{2}L[\psi] \neq 0.$$

Let

$$c_{p,d} := \frac{1}{8}(p^2-4) \left[ b(\mu) - \frac{1}{2}L[\psi] \right]_{|\mu=\mu_{\text{FS}}}^{-1}.$$

If  $c_{p,d}$  is positive, in a neighborhood of  $\mu = \mu_{\text{FS}}$ ,

$$\frac{\mathcal{Q}_\mu[u(\mu)]}{\mathcal{Q}_\mu[u_{\mu,*}]} = 1 - \frac{1}{4}(p^2-4) \left( (\mu - \mu_{\text{FS}})\varepsilon^2 - \frac{\varepsilon^4}{2c_{p,d}} \right) + o(\varepsilon^4)$$

is optimized, up to higher order terms, by taking

$$\varepsilon^2 = \varepsilon^2(\mu) \sim c_{p,d}(\mu - \mu_{\text{FS}}) \quad \text{as } \mu \rightarrow \mu_{\text{FS}+}. \quad (22)$$

**Remark 5** We may also consider the case  $c_{p,d} < 0$ , which then requires that  $\mu < \mu_{\text{FS}}$ . We will not emphasize it because we are interested in minimizers and  $c_{p,d} < 0$  means that we deal with local maximizers of  $\mathcal{Q}_\mu$ . Moreover, we have no example of such a case for specific values of  $p$  and  $d$ .

Altogether, if  $c_{p,d} > 0$ , we have found that

$$\frac{\mathcal{Q}_\mu[u_{(\mu)}]}{\mathcal{Q}_\mu[u_{\mu,*}]} = 1 - \frac{p^2 - 4}{8} c_{p,d} (\mu - \mu_{\text{FS}})^2 + o((\mu - \mu_{\text{FS}})^2) \quad (23)$$

which ends the proof of Theorem 2. A more detailed statement goes as follows.

**Theorem 6** *In a neighborhood of  $\mu = \mu_{\text{FS}}$ , if  $u_{(\mu)}$  is given by (8),*

$$\begin{aligned} \text{(i)} \quad & \frac{\mathcal{Q}_\mu^1[u_{(\mu)}]}{\mathcal{Q}_\mu^1[u_{\mu,*}]} = 1 - \frac{1}{4} (p^2 - 4) \left( (\mu - \mu_{\text{FS}}) \varepsilon^2 - \frac{\varepsilon^4}{2 c_{p,d}} \right) + o(\varepsilon^4), \\ \text{(ii)} \quad & \tau'(\mu_{\text{FS}}) = \frac{p-2}{p+2} + \frac{16 p^2 (d-1)^2}{(p-2)(p+2)^3} c_{p,d}, \\ \text{(iii)} \quad & \frac{\nu'(\mu_{\text{FS}})}{\nu_*(\mu_{\text{FS}})} = -\frac{p-2}{2 p \mu_{\text{FS}}} + c_{p,d} \left[ p \mu_{\text{FS}} \left( \frac{2}{p+2} - \frac{p(p-1)}{3p-2} \right) + 2 \frac{B_0}{\alpha} \left( \frac{\mathbf{b}_{0,1}}{\mathbf{l}_2} + \frac{p-2}{3p-2} \right) \right], \end{aligned}$$

where  $c_{p,d}$ ,  $B_0$  and  $\mathbf{b}_{0,1}$  are explicit constants that have been defined above and that can be computed numerically. For (ii) and (iii), we assume that  $c_{p,d}$  is positive.

Property (i) has already been established. Before proving (ii) and (iii), let us discuss the positivity of  $c_{p,d}$ .

### 3.4. A sufficient condition for the positivity of $c_{p,d}$

All above computations are valid under the assumption that  $c_{p,d}$  is positive, but this is not *a priori* guaranteed. With the estimate of  $\mathbf{b}_{2,2p-3}$  that has been found at the end of Section 3.2, we can *a posteriori* give a sufficient condition for the consistency of the method. Since

$$\begin{aligned} L[\psi] &\leq L_{\text{approx}}[\psi] \\ &:= 4(d-1)^2 \frac{(p-1)p^3}{(p+2)^2} \left[ \frac{(p-2)p}{(3p-2)^2(5p-6)} + 2 \frac{d-1}{d+2} \frac{p-1}{(p-2)^2} \mathbf{y} \right]. \end{aligned}$$

with  $\mathbf{y} := \frac{16 p (p-1) (3p-4)}{(3p-2)(5p-6)(7p-10)\sigma(p,d)}$ , we know that  $c_{p,d}$  is well defined and positive if  $L_{\text{approx}}[\psi] < 2b(\mu_{\text{FS}})$ . Moreover, we have

$$c_{p,d} \leq c_{p,d}^{\text{approx}} := \frac{1}{8} (p^2 - 4) \left[ b(\mu) - \frac{1}{2} L_{\text{approx}}[\psi] \right]_{|\mu=\mu_{\text{FS}}}^{-1},$$

at least as long as  $L_{\text{approx}}[\psi] \leq 2b(\mu_{\text{FS}})$ . Hence we have shown the following result.

**Theorem 7** *The constant  $c_{p,d}$  is positive if  $p$  is contained in a non empty interval  $(2, p_{\text{approx}}) \subset (2, 2^*)$ , where  $p_{\text{approx}}$  is defined as the largest root of the fourth order polynomial  $p \mapsto \frac{1}{5} (54 - 227d + 103d^2) p^4 - 4(18 - 37d + 25d^2) p^3 + \frac{8}{3} (63 - 67d + 46d^2) p^2 + 16(d+3)(5d-3)p - 240d(d+1)$ .*

In practice, for all  $d \geq 3$ ,  $p_{\text{approx}}(d)$  is close to  $2d/(d-2)$  and converges to 2 as  $d \rightarrow \infty$ . See Fig. 11 for an illustration.

### 3.5. Expansion of $\tau(\mu)$ around $\mu_{\text{FS}}$

Let  $\tau(\mu) = t[u(\mu)] = \frac{\int_{\mathcal{C}} |\nabla u(\mu)|^2 dy}{\int_{\mathcal{C}} u(\mu)^2 dy}$  and

$$\tau_*(\mu) := t[u_{\mu,*}] = \frac{p-2}{p+2} \mu. \quad (24)$$

We can notice that  $\tau(\mu_{\text{FS}}) = \beta^2 J_2/I_2$  (see Appendix A.1 and Appendix A.2) so that

$$\tau(\mu_{\text{FS}}) = \frac{p-2}{p+2} \mu_{\text{FS}} = \frac{4(d-1)}{(p+2)^2}.$$

With the above expressions in hand, we can now compute the derivative

$$\tau'(\mu_{\text{FS}}) = \frac{d}{d\mu} t[u(\mu)]|_{\mu=\mu_{\text{FS}}}.$$

From (24) we know that  $t[u_{\mu,*}] - t[u_{\text{FS}}] = \frac{p-2}{p+2} (\mu - \mu_{\text{FS}})$ . By expanding the expression  $\tau(\mu) - \tau(\mu_{\text{FS}}) = t[u(\mu)] - t[u_{\text{FS}}] = t[u_{\mu,*}] - t[u_{\text{FS}}] + t[u(\mu)] - t[u_{\mu,*}]$  in powers of  $\varepsilon$  with  $u(\mu) = u_{\mu,*} + \varepsilon \varphi_1 f_1 + \varepsilon^2 \psi + o(\varepsilon^2)$  where  $\varphi, \psi$  have been chosen in Section 3.2, we get, up to higher order terms,

$$\begin{aligned} t[u(\mu)] - t[u_{\mu,*}] \sim \varepsilon^2 \left[ (\lambda_1(\mu) - \mu - t[u_{\mu,*}]) \frac{\int_{\mathcal{C}} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^2 dy} + (p-1) \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^2 dy} \right. \\ \left. - 2(\mu + t[u_{\mu,*}]) \frac{\int_{\mathcal{C}} u_{\mu,*} \psi dy}{\int_{\mathcal{C}} u_{\mu,*}^2 dy} + 2 \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1} \psi dy}{\int_{\mathcal{C}} u_{\mu,*}^2 dy} \right], \end{aligned}$$

and, by computing as above, we find that

$$\begin{aligned} \tau'(\mu_{\text{FS}}) = \frac{p-2}{p+2} + c_{p,d} \left[ -\frac{2p\mu_{\text{FS}}}{p+2} \frac{\int_{\mathcal{C}} \varphi_1^2 dy}{\int_{\mathcal{C}} u_{\text{FS}}^2 dy} + (p-1) \frac{\int_{\mathcal{C}} u_{\text{FS}}^{p-2} \varphi_1^2 dy}{\int_{\mathcal{C}} u_{\text{FS}}^2 dy} \right. \\ \left. - \frac{4p\mu_{\text{FS}}}{p+2} \frac{\int_{\mathcal{C}} u_{\text{FS}} \psi_0 dy}{\int_{\mathcal{C}} u_{\text{FS}}^2 dy} \right], \end{aligned}$$

because we notice that the terms involving  $k_\psi$  cancel. Hence, using (15) and (16), we have found that

$$\begin{aligned} \tau'(\mu_{\text{FS}}) = \frac{p-2}{p+2} + c_{p,d} \left[ -\frac{4p^2 \mu_{\text{FS}}^2}{(p+2)^2} + \frac{2(p-1)p^3 \mu_{\text{FS}}^2}{(p+2)(3p-2)} \right. \\ \left. - \frac{2(p-2)}{p+2} \mu_{\text{FS}} \frac{A_0}{\alpha} - \frac{4p\mu_{\text{FS}} B_0}{p+2} \frac{\mathbf{b}_{0,1}}{\mathbf{l}_2} \right] \\ = \frac{p-2}{p+2} - c_{p,d} \left[ \frac{8p(d-1)}{(p-2)(p+2)^2} \right]^2 \left[ 1 + \frac{(p-1)p(p+2)}{2(p-2)} \frac{\mathbf{b}_{0,1}}{\mathbf{l}_2} \right]. \end{aligned}$$

Here the coefficient  $\mathbf{b}_{0,1}$  is given by  $\mathbf{b}_{0,1} := \int_{\mathbb{R}} \chi_{0,2p-3} w ds$ . Using (18), this proves part (ii) of Theorem 6.

3.6. Expansion of  $\nu(\mu)$  around  $\mu_{\text{FS}}$ 

Let us consider  $\nu(\mu) := \|u_{(\mu)}\|_{L^2(C)}^2 / \|u_{(\mu)}\|_{L^p(C)}^2$ . Again we can write

$$\nu(\mu) - \nu(\mu_{\text{FS}}) = \left( \frac{\|u_{\mu,*}\|_{L^2(C)}^2}{\|u_{\mu,*}\|_{L^p(C)}^2} - \frac{\|u_{\text{FS}}\|_{L^2(C)}^2}{\|u_{\text{FS}}\|_{L^p(C)}^2} \right) + \left( \nu(\mu) - \frac{\|u_{\mu,*}\|_{L^2(C)}^2}{\|u_{\mu,*}\|_{L^p(C)}^2} \right).$$

With  $\beta = \frac{p-2}{2} \sqrt{\mu}$ , using expressions that can be found in Appendix A.1, we see that

$$\nu_*(\mu) := \frac{\|u_{\mu,*}\|_{L^2(C)}^2}{\|u_{\mu,*}\|_{L^p(C)}^2} = \beta^{-\frac{p-2}{p}} \frac{l_2}{l_p^{2/p}} = \kappa_p \mu^{-\frac{p-2}{2p}}, \quad (25)$$

with  $\kappa_p = \left(\frac{p+2}{4}\right)^{\frac{2}{p}} \left(\frac{2\sqrt{\pi}}{p-2} \frac{\Gamma(\frac{2}{p-2})}{\Gamma(\frac{2}{p-2} + \frac{1}{2})}\right)^{\frac{p-2}{p}}$ , and hence

$$\nu'_*(\mu_{\text{FS}}) = \lim_{\mu \rightarrow \mu_{\text{FS}}} \frac{1}{\mu - \mu_{\text{FS}}} \left( \frac{\|u_{\mu,*}\|_{L^2(C)}^2}{\|u_{\mu,*}\|_{L^p(C)}^2} - \frac{\|u_{\text{FS}}\|_{L^2(C)}^2}{\|u_{\text{FS}}\|_{L^p(C)}^2} \right) = -\frac{p-2}{2p} \frac{\nu_*(\mu_{\text{FS}})}{\mu_{\text{FS}}}.$$

If  $u_{(\mu)} = u_{\mu,*} + \varepsilon \varphi + \varepsilon^2 \psi$ , where  $\varphi, \psi$  have been chosen in Section 3.2, after a Taylor expansion we find that

$$\begin{aligned} \nu(\mu) = \nu_*(\mu) & \left[ 1 + \varepsilon^2 \left( \frac{\int_C \varphi^2 dy}{\int_C u_{\mu,*}^2 dy} - (p-1) \frac{\int_C u_{\mu,*}^{p-2} \varphi^2 dy}{\int_C u_{\mu,*}^p dy} \right) \right. \\ & \left. + 2\varepsilon^2 \frac{\int_C u_{\mu,*} \psi dy}{\int_C u_{\mu,*}^2 dy} - 2\varepsilon^2 \frac{\int_C u_{\mu,*}^{p-1} \psi dy}{\int_C u_{\mu,*}^p dy} + o(\varepsilon^2) \right]. \end{aligned}$$

Again we may notice that the terms involving  $k_\psi$  cancel and, based on (15) and (16), we arrive at

$$\begin{aligned} \frac{\nu'(\mu_{\text{FS}})}{\nu_*(\mu_{\text{FS}})} &= -\frac{p-2}{2p\mu_{\text{FS}}} + c_{p,d} \left[ p\mu_{\text{FS}} \left( \frac{2}{p+2} - \frac{p(p-1)}{3p-2} \right) \right. \\ & \quad \left. + 2 \left( \frac{B_0}{\alpha} \frac{\mathbf{b}_{0,1}}{l_2} - \frac{B_0}{\alpha} \frac{\mathbf{b}_{0,p-1}}{l_p} \right) \right] \\ &= -\frac{p-2}{2p\mu_{\text{FS}}} + c_{p,d} \left[ p\mu_{\text{FS}} \left( \frac{2}{p+2} - \frac{p(p-1)}{3p-2} \right) \right. \\ & \quad \left. + 2 \frac{B_0}{\alpha} \left( \frac{\mathbf{b}_{0,1}}{l_2} + \frac{p-2}{3p-2} \right) \right]. \end{aligned}$$

**Lemma 8** *At the bifurcation point  $\mu = \mu_{\text{FS}}$  we get the following.*

$$\frac{\nu'_*(\mu_{\text{FS}})}{\nu_*(\mu_{\text{FS}})} + \frac{\tau'_*(\mu_{\text{FS}})}{\mu_{\text{FS}} + \tau_*(\mu_{\text{FS}})} = \frac{\nu'(\mu_{\text{FS}})}{\nu_*(\mu_{\text{FS}})} + \frac{\tau'(\mu_{\text{FS}})}{\mu_{\text{FS}} + \tau_*(\mu_{\text{FS}})} = 0.$$

*Proof.* Recall that  $\mathcal{Q}_\mu[u_{\mu,*}] = \nu_*(\mu) (\tau_*(\mu) + \mu)$  and  $\mathcal{Q}_\mu[u(\mu)] = \nu(\mu) (\tau(\mu) + \mu)$ , so that

$$\frac{1}{\mathcal{Q}_{\mu_{\text{FS}}}[u_{\text{FS}}]} \frac{d}{d\mu} \mathcal{Q}_\mu[u_{\mu,*}]|_{\mu=\mu_{\text{FS}}} = \frac{\nu'_*(\mu_{\text{FS}})}{\nu_*(\mu_{\text{FS}})} + \frac{1 + \tau'_*(\mu_{\text{FS}})}{\mu_{\text{FS}} + \tau_*(\mu_{\text{FS}})} = 0$$

according to (24) and (25), and

$$\frac{1}{\mathcal{Q}_{\mu_{\text{FS}}}[u_{\text{FS}}]} \frac{d}{d\mu} \mathcal{Q}_\mu[u(\mu)]|_{\mu=\mu_{\text{FS}}} = \frac{\nu'(\mu_{\text{FS}})}{\nu(\mu_{\text{FS}})} + \frac{1 + \tau'(\mu_{\text{FS}})}{\mu_{\text{FS}} + \tau(\mu_{\text{FS}})}.$$

According to (23), these two quantities are equal, thus proving the result. Alternatively, the identity can be proved directly using the expressions of  $\tau'$  and  $\nu'$  established in Sections 3.5 and 3.6. This ends the proof of Lemma 8 and of Theorem 6, (iii).  $\square$

### 3.7. Reparametrization of the branch for $\theta < 1$ and proof of Theorem 3

Now we are in position to study the local behavior of the branch of the solutions to (4) parametrized by  $\mu$  close to the bifurcation point, that is, for  $\mu$  in a neighborhood of  $\mu_{\text{FS}}$ . More precisely, we are interested in the monotonicity of  $\mu \mapsto \Lambda^\theta(\mu)$  and the behavior of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  in a neighborhood of  $\mu = \mu_{\text{FS}}$ . According to the parametrization of Section 1, we know that  $\Lambda^\theta(\mu) = \theta \mu - (1 - \theta) \tau(\mu)$ , so that

$$(\Lambda^\theta)' = \theta - (1 - \theta) \tau'$$

can be computed at  $\mu = \mu_{\text{FS}}$  using the expression of  $\tau'(\mu_{\text{FS}})$ , that has been computed in Section 3.5. Hence we find that

$$(\Lambda^\theta)'(\mu_{\text{FS}}) = \theta (1 + \tau'(\mu_{\text{FS}})) - \tau'(\mu_{\text{FS}}).$$

**Lemma 9** *If  $\tau'(\mu_{\text{FS}})$  is positive, then we have that  $\frac{d}{d\mu} \Lambda^\theta(\mu_{\text{FS}}) < 0$  if and only*

$$\theta < \vartheta_2(p, d) := \frac{\tau'(\mu_{\text{FS}})}{1 + \tau'(\mu_{\text{FS}})}.$$

Notice that with this definition,  $\vartheta_2(p, d)$  is defined for any  $p \in (2, 2^*)$  and any  $d \geq 2$ . As long as  $\vartheta(p, d) < \vartheta_2(p, d)$ ,  $(\Lambda^\theta)'(\mu_{\text{FS}})$  is negative if  $\theta \in (\vartheta(p, d), \vartheta_2(p, d))$ . In all numerical examples that are under consideration in this paper, we find that  $\tau'(\mu_{\text{FS}})$  is positive. This is of course automatically the case if  $c_{p,d}$  itself is positive, because of (ii) in Theorem 6.

We recall that  $J^\theta(\mu) := \theta^\theta (\mu + \tau(\mu))^\theta \nu(\mu)$  and  $\Lambda^\theta(\mu) = \theta \mu - (1 - \theta) \tau(\mu)$ . Hence

$$(\log J^\theta)' = \frac{\nu'}{\nu} + \theta \frac{1 + \tau'}{\mu + \tau}$$

and a similar formula holds for  $J_*^\theta$ . At  $\mu = \mu_{\text{FS}}$ , we can use Lemma 8 and get

$$(\log J^\theta)'(\mu_{\text{FS}}) = \frac{\theta(1 + \tau'(\mu_{\text{FS}})) - \tau'(\mu_{\text{FS}})}{\mu_{\text{FS}} + \tau(\mu_{\text{FS}})} = \frac{(\Lambda^\theta)'(\mu_{\text{FS}})}{\mu_{\text{FS}} + \tau(\mu_{\text{FS}})}.$$

Let us define

$$\delta^\theta := \frac{(J^\theta)'}{(\Lambda^\theta)'} - \frac{(J_*^\theta)'}{(\Lambda_*^\theta)'}$$

**Lemma 10** *Assuming that  $c_{p,d}$  is positive, we have*

$$\delta^\theta(\mu_{\text{FS}}) = 0.$$

In other words, the parametric curves  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  and  $\mu \mapsto (\Lambda_*^\theta(\mu), J_*^\theta(\mu))$  are tangent at  $\mu = \mu_{\text{FS}}$ . It remains to decide the relative position of the two branches in a neighborhood of the bifurcation point. In order to do so, let us define the function

$$\xi^\theta := \frac{((\Lambda^\theta)')^2}{J^\theta} \left[ \frac{1}{(\Lambda^\theta)'} \frac{d}{d\mu} \left( \frac{(J^\theta)'}{(\Lambda^\theta)'} \right) - \frac{1}{(\Lambda_*^\theta)'} \frac{d}{d\mu} \left( \frac{(J_*^\theta)'}{(\Lambda_*^\theta)'} \right) \right]$$

and discuss the range of positivity of  $\xi^\theta$ . An elementary computation shows that

$$\xi^\theta = \frac{(J^\theta)''}{J^\theta} - \frac{(J^\theta)'}{(\Lambda^\theta)'} \frac{(\Lambda^\theta)''}{J^\theta} - \frac{(J_*^\theta)''}{J_*^\theta} \left( \frac{(\Lambda^\theta)'}{(\Lambda_*^\theta)'} \right)^2$$

because  $(\Lambda_*^\theta)'' = 0$ . By definition of  $\Lambda^\theta(\mu) = \theta\mu - (1 - \theta)\tau(\mu)$  and  $\vartheta_2(p, d)$ , we get that  $(\Lambda^\theta)''(\mu_{\text{FS}}) = -(1 - \theta)\tau''(\mu_{\text{FS}})$  and  $(\Lambda^\theta)'(\mu_{\text{FS}}) = \frac{\theta - \vartheta_2(p, d)}{1 - \vartheta_2(p, d)}$ . From (24) and  $\Lambda_*^\theta(\mu) = \theta\mu - (1 - \theta)\tau_*(\mu)$ , we get that  $(\Lambda_*^\theta)' = \frac{2p\theta - (p-2)}{p+2}$ . From (24) and (25), we know also that  $J_*^\theta(\mu) = \kappa_p (2p\theta/(p+2))^\theta \mu^{\theta - \frac{p-2}{2p}}$ , so that  $\frac{(J_*^\theta)'}{J_*^\theta} = \frac{2p\theta - (p-2)}{2p\mu}$  and  $(\log J_*^\theta)'' = -\frac{2p\theta - (p-2)}{2p\mu^2}$ . As a consequence of Lemma 10, we can write that

$$\frac{(J^\theta)'(\mu_{\text{FS}})}{J^\theta(\mu_{\text{FS}})} = \frac{(J_*^\theta)'(\mu_{\text{FS}})}{J_*^\theta(\mu_{\text{FS}})} \frac{(\Lambda^\theta)'(\mu_{\text{FS}})}{(\Lambda_*^\theta)'(\mu_{\text{FS}})}.$$

According to (23), we have the identity

$$-\frac{1}{4}(p^2 - 4)c_{p,d} = \left( \log \frac{J^\theta}{J_*^\theta} \right)''(\mu_{\text{FS}}) + (1 - \theta) \left( \log \frac{\mu + \tau(\mu)}{\mu + \tau_*(\mu)} \right)'' \Big|_{\mu = \mu_{\text{FS}}},$$

which allows us to compute

$$\begin{aligned} & \frac{(J^\theta)''(\mu_{\text{FS}})}{J^\theta(\mu_{\text{FS}})} \\ &= -\frac{1}{4}(p^2 - 4)c_{p,d} + \left( \frac{(J^\theta)'(\mu_{\text{FS}})}{J^\theta(\mu_{\text{FS}})} \right)^2 + (\log J_*^\theta)''(\mu_{\text{FS}}) \end{aligned}$$

$$\begin{aligned}
& - (1 - \theta) \left[ \frac{p+2}{2p\mu_{\text{FS}}} \tau''(\mu_{\text{FS}}) - \left( \frac{p+2}{2p\mu_{\text{FS}}} (1 + \tau'(\mu_{\text{FS}})) \right)^2 + \frac{1}{\mu_{\text{FS}}^2} \right] \\
& = -\frac{1}{4} (p^2 - 4) c_{p,d} + \left( \frac{(J_*^\theta)'(\mu_{\text{FS}})}{J_*^\theta(\mu_{\text{FS}})} \right)^2 \left( \frac{(\Lambda^\theta)'(\mu_{\text{FS}})}{(\Lambda_*^\theta)'(\mu_{\text{FS}})} \right)^2 + (\log J_*^\theta)''(\mu_{\text{FS}}) \\
& - (1 - \theta) \left[ \frac{p+2}{2p\mu_{\text{FS}}} \tau''(\mu_{\text{FS}}) - \left( \frac{p+2}{2p\mu_{\text{FS}}} (1 + \tau'(\mu_{\text{FS}})) \right)^2 + \frac{1}{\mu_{\text{FS}}^2} \right].
\end{aligned}$$

Because of Lemma 10, we also have

$$\begin{aligned}
-\frac{(J^\theta)'(\mu_{\text{FS}})}{(\Lambda^\theta)'(\mu_{\text{FS}})} \frac{(\Lambda^\theta)''(\mu_{\text{FS}})}{J^\theta(\mu_{\text{FS}})} &= -\frac{(J_*^\theta)'(\mu_{\text{FS}})}{J_*^\theta(\mu_{\text{FS}})} \frac{(\Lambda^\theta)''(\mu_{\text{FS}})}{(\Lambda_*^\theta)'(\mu_{\text{FS}})} \\
&= \frac{p+2}{2p\mu_{\text{FS}}} (1 - \theta) \tau''(\mu_{\text{FS}}).
\end{aligned}$$

Collecting the above identities, we can compute the value of  $\xi^\theta(\mu_{\text{FS}})$  as

$$\begin{aligned}
\xi^\theta(\mu_{\text{FS}}) &= -\frac{1}{4} (p^2 - 4) c_{p,d} \\
&+ (\log J_*^\theta)''(\mu_{\text{FS}}) \left[ 1 - \left( \frac{(\Lambda^\theta)'(\mu_{\text{FS}})}{(\Lambda_*^\theta)'(\mu_{\text{FS}})} \right)^2 \right] \\
&- \frac{1 - \theta}{\mu_{\text{FS}}^2} \left[ 1 - \left( \frac{p+2}{2p} (1 + \tau'(\mu_{\text{FS}})) \right)^2 \right].
\end{aligned}$$

The cancellation of the terms involving  $\tau''(\mu_{\text{FS}})$  is a remarkable fact. By definition of  $\vartheta_2(p, d)$ , we get

$$\begin{aligned}
\xi^\theta(\mu_{\text{FS}}) &= -\frac{1}{4} (p^2 - 4) c_{p,d} \\
&- \frac{2p\theta - (p-2)}{2p\mu_{\text{FS}}^2} \left[ 1 - \left( \frac{p+2}{2p\theta - (p-2)} \frac{\theta - \vartheta_2(p, d)}{1 - \vartheta_2(p, d)} \right)^2 \right] \\
&- \frac{1 - \theta}{\mu_{\text{FS}}^2} \left[ 1 - \left( \frac{p+2}{2p} \frac{1}{1 - \vartheta_2(p, d)} \right)^2 \right]
\end{aligned}$$

and finally arrive at

$$\xi^\theta(\mu_{\text{FS}}) = -\frac{1}{4} (p^2 - 4) c_{p,d} + \frac{p+2}{4p^2\mu_{\text{FS}}^2} \frac{(1 - \theta) (2p\vartheta_2(p, d) - (p-2))^2}{(1 - \vartheta_2(p, d))^2 (2p\theta - (p-2))}.$$

At this point, we can observe that  $\vartheta(p, d) \geq \frac{p-2}{2p}$ . The reader is then invited to check that the function  $\theta \mapsto \xi^\theta(\mu_{\text{FS}})$  is nonincreasing on  $[\vartheta(p, d), 1]$  and

$$\xi^{\vartheta_2(p, d)}(\mu_{\text{FS}}) = -\frac{1}{4} (p^2 - 4) c_{p,d} + \frac{p+2}{4p^2\mu_{\text{FS}}^2} \frac{2p\vartheta_2(p, d) - (p-2)}{1 - \vartheta_2(p, d)} = 0$$

because of (ii) in Theorem 6. Recall that the positivity of  $c_{p,d}$  is required in (22).

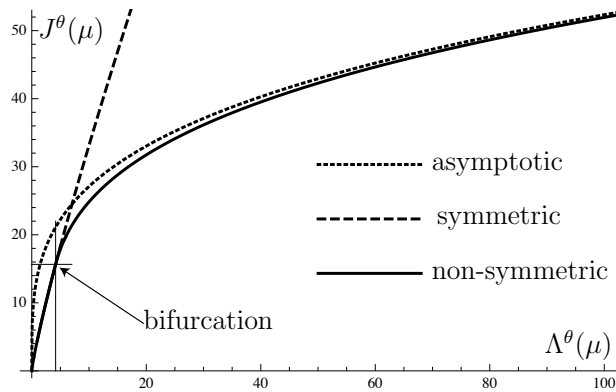
We have then proved that if  $\vartheta_2(p, d) \leq \vartheta(p, d)$ ,  $\xi^\theta(\mu_{\text{FS}})$  is negative for any  $\theta \in (\vartheta(p, d), 1]$ . Otherwise, if  $c_{p,d}$  is positive,  $\xi^\theta(\mu_{\text{FS}})$  is positive if  $\theta \in [\vartheta(p, d), \vartheta_2(p, d))$  and negative if  $\theta \in (\vartheta_2(p, d), 1]$ .

The expansion (13) and the subsequent computations are valid, and make sense for the approximation of a local minimizer of  $\mathcal{Q}_\Lambda^\theta$ , as soon as the coefficient  $c_{p,d}$ , whose expression is established in Section 3.3, is positive. Then for any  $\theta \in (\max\{\vartheta(p, d), \vartheta_2(p, d)\}, 1)$ , the curve of the energies of the non-symmetric solutions of the Euler-Lagrange equations is concave, nondecreasing as a function of  $\Lambda$  in a neighborhood of the bifurcation point, and below the energies of the symmetric functions. If  $\vartheta_2(p, d) > \vartheta(p, d)$ , then the curve of the energies of the non-symmetric solutions is above the energies of symmetric functions in a neighborhood of the bifurcation point if  $\theta \in [\vartheta(p, d), \vartheta_2(p, d))$ . Practically, whether  $c_{p,d}$  is positive or not relies either on the sufficient condition given in Theorem 7 or on numerical computations. However, the estimate of Section 3.4 shows that this occurs at least in a large subinterval of  $(2, 2^*)$ . This completes the proof of Theorem 3. □

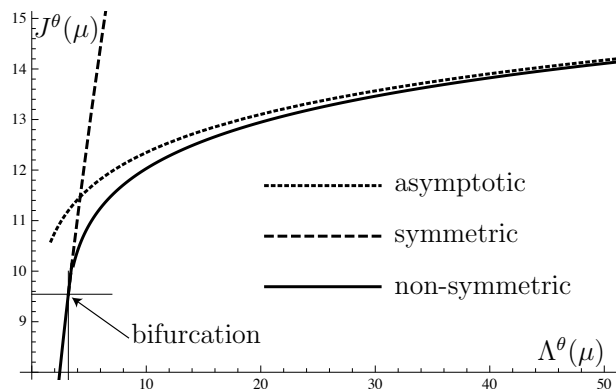
## 4. Numerical results and the two scenarios

### 4.1. Symmetric and non-symmetric branches, and their asymptotic behavior

In [14] the branches of solutions which bifurcate from the branches of symmetric solutions at the smallest possible value of  $\Lambda$  have been computed numerically. For completeness, we start by presenting some of these numerical results, which are the main motivation of the present paper. The branch of symmetric solutions is explicit. The branch  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  bifurcates from the symmetric ones at  $\mu = \mu_{\text{FS}}$  and is computed numerically. The algorithm is based on descent techniques and on an iteration scheme which allows us to compute the branches of solutions by continuation. We carried out the computations for dimension  $d = 5$  and various values of  $p$  and  $\theta$ . We have of course no guarantee that the solutions that we have computed are the optimal ones, but at least the values that we have found are fully compatible with what is expected for theoretical reasons. In particular, the curve of the computed estimates of the best constant is an increasing function of  $\Lambda$  with the right convexity properties, which can reasonably be expected to coincide with  $\Lambda \mapsto \mathbf{K}_{\text{CKN}}(\theta, \Lambda, p)$ . Moreover, when  $\theta$  approaches  $\vartheta(p, d)$  from above, the curve  $\Lambda \mapsto \mathbf{K}_{\text{CKN}}(\theta, \Lambda, p)$  approaches  $\Lambda \mapsto \max\{\mathbf{K}_{\text{CKN}}^*(\theta, \Lambda, p), \mathbf{K}_{\text{GN}}\}$  when  $\mathbf{K}_{\text{GN}} > \mathbf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p)$ . Last but not least, the asymptotics predicted in Theorem 1 are not only correct (dotted lines in Figs 1–3) but provide a good upper estimate of the curve in the whole range  $\Lambda > 0$ .



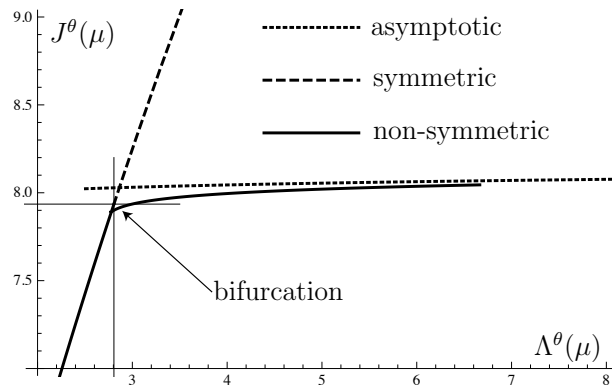
**Figure 1.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 2.8$ ,  $d = 5$ ,  $\theta = 1$ . Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point  $\mu = \mu_{\text{FS}}$  computed by V. Felli and M. Schneider. The branch behaves for large values of  $\Lambda$  as predicted by F. Catrina and Z.-Q. Wang.



**Figure 2.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 2.8$ ,  $d = 5$ ,  $\theta = 0.8$ . The behavior is similar to the case  $\theta = 1$  up to the reparametrization described in Section 1, while the asymptotic behavior of the branch for large values of  $\Lambda$  is in agreement with the results of Section 2.3.

#### 4.2. Two scenarios

The branch of symmetric minimal solutions,  $(\Lambda, u_\Lambda^*)$ , is known explicitly and is monotone in energy, that is, the function  $\Lambda \mapsto \mathcal{Q}_\Lambda^\theta(u_\Lambda^*)$  is monotone increasing in  $(0, +\infty)$ . In the computations described in [14] we observe that the branch of non-symmetric solutions,  $(\Lambda^\theta(\mu), \mathcal{Q}_\mu^\theta[u_\mu])$ , is monotone for some values of  $\theta$  (for instance for  $\theta = 1$ ), but not always. More concretely, for certain values of  $p$  and  $d$ , the numerical results show that there exists an exponent  $\vartheta_1 = \vartheta_1(p, d) \in (\vartheta(p, d), 1)$  such that for any  $\theta \in [\vartheta_1, 1]$  the branch  $(\Lambda^\theta(\mu), \mathcal{Q}_\mu^\theta[u_\mu])$  is monotone increasing. But when  $\vartheta(p, d) < \theta < \vartheta_1(p, d)$ , a dramatic change occurs: see Figs. 5 and 6. For the values of  $p$  and  $d$  that have been considered numerically in those figures, the branch is not monotone anymore for  $\mu > \mu_{\text{FS}}$ , thus producing non-symmetric solutions and candidates for optimal functions



**Figure 3.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 2.8$ ,  $d = 5$ ,  $\theta = 0.72 > \vartheta(2.8, 5) \approx 0.7143$ . The asymptotic behavior of the branch for large values of  $\Lambda$  is in agreement with the results of Section 2.3. We may however notice that the bifurcation seems not to occur where expected. A more detailed computation in the Section 4.3 sheds light on this phenomenon.

in the inequalities for values of  $\Lambda < \Lambda^\theta(\mu_{\text{FS}})$ . This phenomenon provides an explanation for the results proved in [3] using rigorous *a priori* estimates.

The limiting case  $\theta = \vartheta(p, d)$  is very interesting: see Figs. 6 and 9. Let us define  $\Lambda_{\text{GN}}^*(p, d)$  and  $\mu_{\text{GN}}$  such that

$$\mathsf{K}_{\text{GN}} = \mathsf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{GN}}^*(p, d), p) \quad \text{and} \quad \Lambda_*^{\vartheta(p, d)}(\mu_{\text{GN}}) = \Lambda_{\text{GN}}^*(p, d).$$

Whether  $\mathsf{K}_{\text{GN}}$  is larger or smaller than  $\mathsf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p)$  determines, at least in the framework of our computations, whether  $\theta$  is smaller than  $\vartheta_1(p, d)$  or not. This has been observed in [13] and theoretical consequences have been established in [3], in the limit regime  $p \rightarrow 2$ . Before going further, let us observe that  $\vartheta_1(p, d)$  is an exponent associated with a global property of the branch.

Based on our numerical computations, we are now in position to formulate the following alternative.

*Scenario 1.* If  $\mathsf{K}_{\text{GN}} \leq \mathsf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p)$ , then for all  $\theta \geq \vartheta(p, d)$ , the optimal functions are symmetric for any  $\Lambda \in (0, \Lambda_{\text{FS}}(p, \theta)]$  and the branch of non-symmetric solutions is optimal for any  $\Lambda > \Lambda_{\text{FS}}(p, \theta)$ . Such solutions exist for arbitrarily large values of  $\Lambda$  if  $\theta > \vartheta(p, d)$ : see Figs. 7 and 8, but may exist only for a finite range of  $\Lambda$  if  $\theta = \vartheta(p, d)$  and  $\mathsf{K}_{\text{GN}} < \mathsf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p)$ : see Fig. 9.

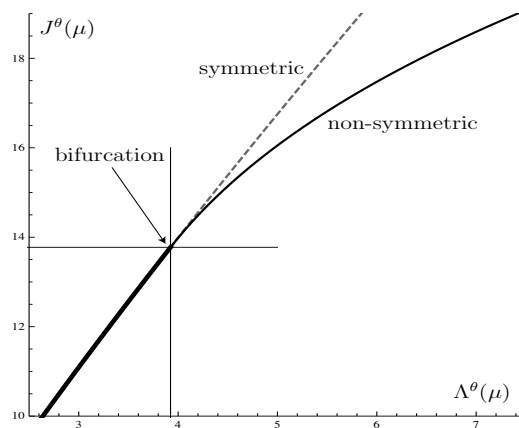
*Scenario 2.* If  $\mathsf{K}_{\text{GN}} > \mathsf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p)$ , there exists  $\vartheta_1 = \vartheta_1(p, d) \in (\vartheta(p, d), 1)$  such that for any  $\theta \in [\vartheta_1, 1]$  the branch is monotone increasing. We further observe numerically that  $\vartheta_1 = \vartheta_2$  (see Fig. 12), where  $\vartheta_2$  has been defined in Section 3.7: for any  $\theta \in [\vartheta(p, d), \vartheta_2(p, d)]$ , we know that the branch of non-symmetric functions is decreasing in a neighborhood of the bifurcation point, but also has a larger energy than the symmetric solutions of the Euler-Lagrange equation (for the same value of  $\Lambda$ ). Hence,

in the critical case  $\theta = \vartheta(p, d)$  we have  $\Lambda_{\text{GN}}^*(p, d) < \Lambda_{\text{FS}}(p, \vartheta(p, d))$  and for any  $\Lambda$  in the range  $(0, \Lambda_{\text{GN}}^*] := \{\Lambda_*^{\vartheta(p, d)}(\mu) : \mu \in (0, \mu_{\text{GN}}]\}$ , the optimal functions are symmetric and  $\mathbf{K}_{\text{CKN}}(\vartheta(p, d), \Lambda, p) = \mathbf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda, p)$ . See Fig. 6. Moreover, in the language of the concentration-compactness method, according to [12], for any  $\Lambda > \Lambda_{\text{GN}}^*(p, d)$  the optimal constant is determined by the *problem at infinity* and  $\mathbf{K}_{\text{CKN}}(\vartheta(p, d), \Lambda, p) = \mathbf{K}_{\text{GN}}$ . From the viewpoint of the reparametrized branch, we numerically observe that both  $\mu \mapsto \Lambda^{\vartheta(p, d)}(\mu)$  and  $\mu \mapsto J^{\vartheta(p, d)}(\mu)$  are decreasing for  $\mu > \mu_{\text{FS}}$ , at least for the values of  $p$  for which computations have been done. In the subcritical case corresponding to  $\vartheta(p, d) < \theta < \vartheta_2(p, d)$ , the reparametrized branch  $(\Lambda^\theta, J^\theta)$  is not monotone. Numerically we observe that it is monotone for  $\theta > \vartheta_2(p, d)$ , hence supporting the observed fact that  $\vartheta_1(p, d) = \vartheta_2(p, d)$  (see Fig. 12).

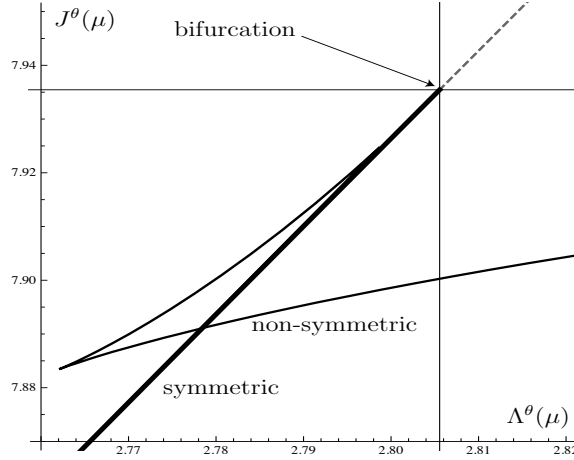
Altogether, based on our numerical observations, what decides between *Scenario 1* and *Scenario 2* is the relative value of  $\mathbf{K}_{\text{GN}}$  and  $\mathbf{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p)$ . Equality of these two optimal constants determines a value of  $p = p^*(d)$ . Numerically we find that  $p^*(5) \approx 3.001$  and  $\vartheta(p^*(5), 5) = \vartheta_1(p^*(5), 5) \approx 0.834$ . For  $p \in [p^*(d), 2^*)$ , only *Scenario 1* occurs (numerical observation). For  $p \in (2, p^*(d))$  we have  $\vartheta(p, d) > \vartheta_1(p, d)$  and *Scenario 2* occurs. More precisely, the fact that the branch cannot be globally monotone increasing if  $\theta < \vartheta_2(p, d)$  is a consequence of Section 3.7 while the fact that the branch is monotone increasing if  $\theta > \vartheta_2(p, d)$  is a numerical observation. These results are fully consistent with the ones of [3] and [13]. Now let us give some details.

### 4.3. Bifurcations and qualitative dependence in $\theta$

In Fig. 3, a careful inspection shows that the symmetric and the non-symmetric branches of solutions differ for values of  $\Lambda$  strictly less than  $\Lambda^\theta(\mu_{\text{FS}})$ . This is not the case for  $\theta$  close enough to 1: see Fig. 4, but very clear on Fig. 5. When  $\theta$  approaches  $\vartheta(p, d)$ , the

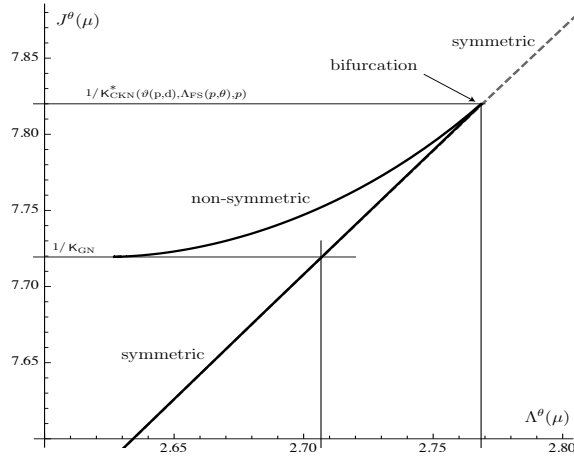


**Figure 4.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 2.8$ ,  $d = 5$ ,  $\theta = 0.95$  close to the bifurcation point. For  $\Lambda \leq \Lambda^\theta(\mu_{\text{FS}})$ , all solutions are symmetric, while for  $\Lambda > \Lambda^\theta(\mu_{\text{FS}})$ , non-symmetric solutions provide a better constant in the interpolation inequalities, exactly as for the case  $\theta = 1$ .



**Figure 5.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 2.8$ ,  $d = 5$ ,  $\theta = 0.72$  close to the bifurcation point. Non-symmetric solutions exist for  $\Lambda < \Lambda^\theta(\mu_{\text{FS}})$ . There exists a value  $\Lambda_{\text{GN}}^* < \Lambda^\theta(\mu_{\text{FS}})$  such that optimal functions are symmetric for any  $\Lambda \in (0, \Lambda_{\text{GN}}^*)$  and are non-symmetric for  $\Lambda > \Lambda_{\text{GN}}^*$ . When  $\Lambda = \Lambda_{\text{GN}}^*$ , symmetric and non-symmetric optimal functions co-exist.

branch (locally) converges to its limit: see Fig. 6. The figures 5 and 6 correspond to



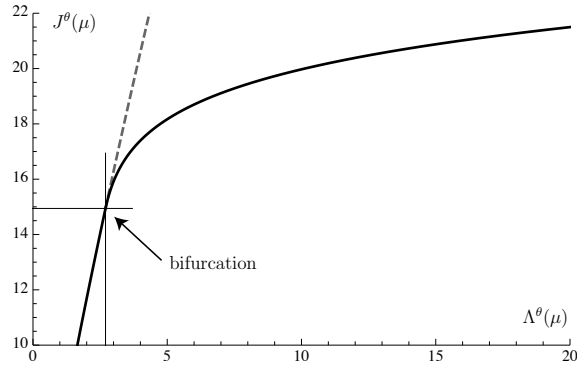
**Figure 6.** Critical case  $\theta = \vartheta(p, d)$ . Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 2.8$ ,  $d = 5$ ,  $\theta = \vartheta(2.8, 5) \approx 0.7143$  close to the bifurcation point. Non-symmetric solutions exist for  $\Lambda < \Lambda^\theta(\mu_{\text{FS}})$  but, at least for the chosen values of  $p$  and  $d$ , are never optimal functions for the interpolation inequalities. There exists a value  $\Lambda_{\text{GN}}^* < \Lambda^\theta(\mu_{\text{FS}})$  such that optimal functions exist and are symmetric for any  $\Lambda \in (0, \Lambda_{\text{GN}}^*]$  and do not exist for  $\Lambda > \Lambda_{\text{GN}}^*$ . Moreover,  $\Lambda_{\text{GN}}^* = \Lambda^\theta(\mu_{\text{GN}})$  with the notations of Section 2.3 and  $\mathcal{K}_{\text{CKN}}(\vartheta(p, d), \Lambda, p) = \mathcal{K}_{\text{GN}}$  for any  $\Lambda > \Lambda_{\text{GN}}^*$ .

*Scenario 2* (for  $\vartheta \leq \theta < \vartheta_1$ ), that is, to the case

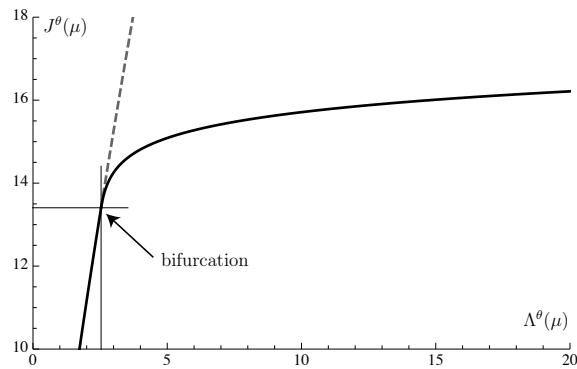
$$\mathcal{K}_{\text{GN}} > \mathcal{K}_{\text{CKN}}^*(\vartheta(p, d), \Lambda_{\text{FS}}(p, \vartheta(p, d)), p).$$

In other words, this means that  $p < p^*(d)$ .

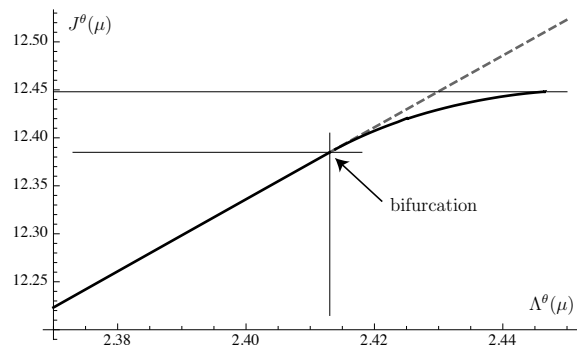
The case  $p > p^*(d)$ , *i.e.*, *Scenario 1*, also occurs and corresponding plots are shown in Figs. 7–9. There we take  $d = 5$ ,  $p = 3.15 \geq p^*(5) \approx 3.001$ .



**Figure 7.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 3.15$ ,  $d = 5$ ,  $\theta = 1$ . The behavior of the non-symmetric branch is similar to the one found in Fig. 1.



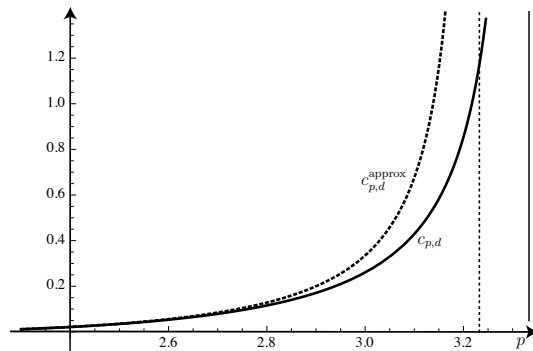
**Figure 8.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 3.15$ ,  $d = 5$ ,  $\theta = 0.95$ . The behavior of the non-symmetric branch is still similar to the one found in Fig. 4.



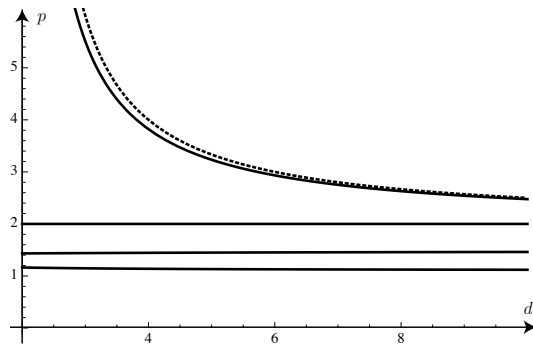
**Figure 9.** Parametric plot of  $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$  for  $p = 3.15$ ,  $d = 5$ ,  $\theta = \vartheta(3.15, 5) \approx 0.9127$ , that is for the critical value of  $\theta$ .

In Section 3 we proved that the symmetric and the non-symmetric branches of solutions are always tangent at  $\Lambda_{\text{FS}}$ . What happens in a neighborhood of the bifurcation point is therefore difficult to decide in view of the plots of the branches, especially when  $\theta$  is in a neighborhood of  $\vartheta_1(p, d)$ . To illustrate this difficulty, we may for instance observe that figure Fig. 5 is an enlargement of Fig. 3. Hence we have to discard the possibility of other scenarios than the ones described in Section 4.2 at least in a neighborhood of the bifurcation point.

The computations of Section 3 have been done for the approximation  $u_{(\mu)}$  of the solution  $u_\mu$ . We expect that the estimates converge as  $\mu \rightarrow (\mu_{\text{FS}})_+$  and this is what is observed numerically. Whether  $c_{p,d}$  is positive has been discussed in Section 3.4, but can be checked numerically: we know that  $c_{p,d}$  is positive and finite as long as  $c_{p,d}^{\text{approx}}$  is positive (see Fig. 10, and Fig. 11 for a discussion of the sign of  $c_{p,d}^{\text{approx}}$ ), and numerically we find that  $c_{p,d}$  is always positive.

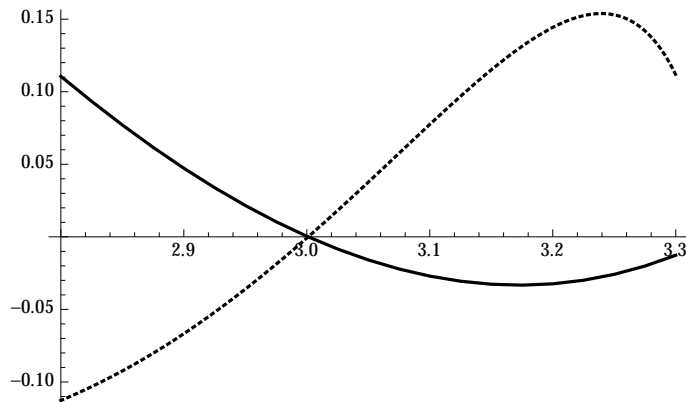


**Figure 10.** Computation of  $c_{p,d}$  with  $d = 5$  as a function of  $p$ . We observe that the numerical solution is positive for any  $p \in (2, 10/3)$  where  $10/3$  is the critical exponent (corresponding to the plain vertical line). The estimate of Section 3.4 corresponds to the dotted line and holds for  $p \lesssim 3.2323$  (corresponding to the dotted vertical line).



**Figure 11.** The plain curves are the zeros of  $c_{p,d}^{\text{approx}}$  as  $d$  varies in the interval  $(2, 10)$ , except  $p = 2$  which is a singularity. The dotted curve is given by  $d \mapsto 2d/(d-2) = 2^*$ . For a given  $d$ ,  $c_{p,d}^{\text{approx}}$  is therefore positive for a large subinterval in  $p$  of  $(2, 2^*)$ .

Under the above precautions, we know from Section 3 that there exists a number  $\vartheta_2(p, d)$  such that the behavior of the branch in a neighborhood of the bifurcation point  $\mu = \mu_{\text{FS}}$  discriminates between two regimes corresponding to  $\theta > \vartheta_2(p, d)$  and  $\theta < \vartheta_2(p, d)$ . When  $\theta < \vartheta_2(p, d)$ , we have  $(\Lambda^\theta)'(\mu_{\text{FS}}) < 0$  and the contrary happens when  $\theta > \vartheta_2(p, d)$ . So, locally, the reparametrized branch is on the right of the bifurcation point and  $J^\theta$  is a monotone increasing function of  $\Lambda$  (at least when  $\mu$  is in a right neighborhood of  $\mu_{\text{FS}}$ ) if and only if  $\theta > \vartheta_2(p, d)$ . Since global monotonicity implies local monotonicity near the bifurcation point, if the numerical computations of the branches are consistent with the study of the bifurcation carried out in Section 3, then we should have that  $\vartheta_1(p, d) = \vartheta_2(p, d)$ . It is not easy to establish a qualitative property such as the monotonicity, but at least we observe in Fig. 12 that for  $\theta = \vartheta(p, d)$  the range in  $p$  for which  $\vartheta_2(p, d) \geq \vartheta(p, d)$  corresponds to the range in  $p$  for which the Gagliardo-Nirenberg constant compares well with the energy at the bifurcation point.



**Figure 12.** Comparison of the local and asymptotic criteria in the critical case  $\theta = \vartheta(p, d)$  when  $d = 5$ . The dotted curve corresponds to the function  $p \mapsto 1/\mathcal{K}_{\text{GN}}(p, d) - J^{\vartheta(p, d)}(\mu_{\text{FS}})$ , that is, the difference of the asymptotic energy of the branch and the energy at the bifurcation point: when it is negative, this means that  $\vartheta_1(p, d)$  is defined and larger than  $\vartheta(p, d)$ , so that *Scenario 2* takes place. When it is positive, this means that *Scenario 1* can be expected. The exponent  $\vartheta_2(p, d)$  can be defined for any  $p \in (2, 2^*)$ . The plain curve represents  $p \mapsto 5(\vartheta_2(p, d) - \vartheta(p, d))$  and positivity indicates that, at least locally around the bifurcation point, *Scenario 2* takes place. Hence the local (around the bifurcation point) and asymptotic (as  $\Lambda \rightarrow +\infty$ ) criteria coincide.

## 5. Concluding remarks

In this paper we have established the asymptotic behavior of the branches for all  $\theta \in (\vartheta(p, d), 1]$ . There is a good agreement between this behavior and that of the numerical branches, which reinforces the conjecture that the computed branches contain the extremals for the Caffarelli-Kohn-Nirenberg inequalities.

We have also studied the precise behavior of the branches of non-symmetric solutions near the first possible bifurcation point on the symmetric branch within the framework of a particular *Ansatz* defined by (13). By doing so, we have obtained the existence of a critical exponent  $\vartheta_2(p, d)$  above which the branch is monotone, increasing and potentially optimal, and below which it is certainly not optimal in a neighborhood of the bifurcation point.

The final result of this paper is the comparison of the above criterion based on the local behavior of the branch near the bifurcation point and the criterion based on the asymptotic energy of the branch in the critical case  $\theta = \vartheta(p, d)$ , using numerical methods. They coincide, which gives solid grounds to the alternative that has been numerically observed:

*Scenario 1.* The non-symmetric branch is monotone increasing for any  $\theta \in [\vartheta(p, d), 1]$ .  
*Scenario 2.* The non-symmetric branch is monotone increasing for any  $\theta \in (\vartheta_2(p, d), 1]$  but it is not optimal near the bifurcation point if  $\theta \in [\vartheta(p, d), \vartheta_2(p, d))$ .

This also suggests that no other *scenario* can take place, consistently with our numerical computations. Hence we arrive at the conclusion that *Scenario 1* takes place when  $\vartheta_2(p, d) < \vartheta(p, d)$  and *Scenario 2* holds if  $\vartheta_2(p, d) > \vartheta(p, d)$ . The branches of solutions that we have computed are likely to be optimal for the Caffarelli-Kohn-Nirenberg inequalities.

## Appendix A. Some useful quantities

### Appendix A.1. Computing integrals

We recall that  $f(q) := \int_{\mathbb{R}} \frac{ds}{(\cosh s)^q}$  can be explicitly computed:  $f(q) = \frac{\sqrt{\pi} \Gamma(\frac{q}{2})}{\Gamma(\frac{q+1}{2})}$ . An integration by parts shows that  $f(q+2) = \frac{q}{q+1} f(q)$ . The following formulae are reproduced with no change from [6] (also see [8]). As in [6], with  $w(s) = (\cosh s)^{-\frac{2}{p-2}}$ , we can define

$$I_q := \int_{\mathbb{R}} |w(s)|^q ds \quad \text{and} \quad J_2 := \int_{\mathbb{R}} |w'(s)|^2 ds.$$

Using the function  $f$ , we can compute  $I_2 = f(\frac{4}{p-2})$ ,  $I_p = f(\frac{2p}{p-2}) = f(\frac{4}{p-2} + 2)$  and get the relations

$$I_2 = \frac{\sqrt{\pi} \Gamma(\frac{2}{p-2})}{\Gamma(\frac{p+2}{2(p-2)})}, \quad I_p = \frac{4 I_2}{p+2}, \quad J_2 = \frac{4 I_2}{(p+2)(p-2)}.$$

As a special case, we have

$$I_p := \int_{\mathbb{R}} (\cosh s)^{-\frac{2p}{p-2}} ds = \frac{4\sqrt{\pi} \Gamma\left(\frac{2}{p-2}\right)}{(p+2) \Gamma\left(\frac{p+2}{2(p-2)}\right)}.$$

### Appendix A.2. Symmetric extremals and linearization

Consider  $w(s) = (\cosh s)^{-\frac{2}{p-2}}$ , which is the unique positive solution of

$$-(p-2)^2 w'' + 4w - 2pw^{p-1} = 0$$

on  $\mathbb{R}$ , up to translations. The function  $u(s) := \alpha w(\beta s)$  solves

$$-u'' + \frac{4\beta^2}{(p-2)^2} u - \frac{2p\beta^2}{(p-2)^2} \alpha^{2-p} u^{p-1} = 0. \quad (\text{A.1})$$

With  $\beta = \frac{p-2}{2} \sqrt{\mu}$  and  $\alpha = \left(\frac{p}{2}\mu\right)^{\frac{1}{p-2}}$ ,  $u = u_{\mu,*}$  is given by

$$u_{\mu,*}(s) = \left(\frac{p}{2}\mu\right)^{\frac{1}{p-2}} \left[ \cosh\left(\frac{p-2}{2} \sqrt{\mu} s\right) \right]^{-\frac{2}{p-2}} \quad \forall s \in \mathbb{R}$$

and solves (10).

Next we are interested in computing the ground state energy of the Pöschl-Teller operator  $\mathcal{H}_\mu = -\frac{d^2}{ds^2} + d - 1 + \mu - (p-1)u_{\mu,*}^{p-2}$ , that is the lowest eigenvalue  $\lambda_1(\mu)$  in the eigenvalue problem

$$\mathcal{H}_\mu \varphi_1 = \lambda_1(\mu) \varphi_1.$$

See [16, 17] for further references. The function

$$\varphi_1(s) := \alpha^{\frac{p}{2}} (\cosh(\beta s))^{-\frac{p}{p-2}} = u_{\mu,*}^{p/2} \quad (\text{A.2})$$

solves

$$-\varphi_1'' + \frac{1}{4} \mu p^2 \varphi_1 - (p-1) u_{\mu,*}^{p-2} \varphi_1 = 0$$

and provides a solution with

$$\lambda_1(\mu) = d - 1 + \mu - \frac{1}{4} \mu p^2.$$

The Sturm-Liouville theory guarantees that  $\varphi_1$  generates the ground state and

$$\inf_{\varphi \in \mathbf{H}^1(\mathbb{R}^d)} \frac{q[\mu, \varphi]}{\|u_{\mu,*}\|_{L^p(\mathcal{C})}^p} = \lambda_1(\mu) = \frac{\int_{\mathcal{C}} \varphi_1 \mathcal{H}_\mu \varphi_1 dy}{\|\varphi\|_{L^2(\mathcal{C})}^2}.$$

$$\|\varphi\|_{L^2(\mathcal{C})}^2 = \|u_{\mu,*}\|_{L^p(\mathcal{C})}^p$$

Notice that the condition  $\lambda_1(\mu_{\text{FS}}) = 0$  determines

$$\mu_{\text{FS}} = 4 \frac{d-1}{p^2-4}.$$

Moreover, this shows that

$$\lambda_1(\mu) = -\frac{p^2-4}{4} (\mu - \mu_{\text{FS}}).$$

Other eigenvalues of  $\mathcal{H}_\mu$  can also be computed using classical transformations and special functions: see [16, p. 74]. Notice that in Section 3, we normalize the function  $\varphi = \varphi_1 f_1$  in the expansion (13) by the condition  $\|\varphi\|_{L^2(\mathcal{C})} = \|u_{\mu,*}\|_{L^p(\mathcal{C})}$  consistently with (A.2).

### Appendix A.3. Useful quantities

Collecting results of Sections Appendix A.1 and Appendix A.2, with  $\alpha^{p-2} = \frac{p}{2} \mu$ , we find that

$$\frac{\int_{\mathcal{C}} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} = 1,$$

$$\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2} \varphi^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} = \frac{f(q+2)}{f(q)} \Big|_{q=\frac{2p}{p-2}} \alpha^{p-2} = \frac{p^2 \mu}{3p-2},$$

$$\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-4} \varphi^4 dy}{\int_{\mathcal{C}} u_{\mu,*}^p dy} = \frac{f(q+4)}{f(q)} \Big|_{q=\frac{2p}{p-2}} \alpha^{2(p-2)} = \frac{2p^3(p-1)\mu^2}{(3p-2)(5p-6)},$$

and

$$\frac{\int_{\mathcal{C}} |\nabla \varphi|^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^2 dy} = \alpha^{p-2} \frac{l_p}{l_2} \left[ (d-1) + \left( \frac{p}{p-2} \right)^2 \beta^2 \left( 1 - \frac{f(q+2)}{f(q)} \right) \Big|_{q=\frac{2p}{p-2}} \right]$$

$$= \frac{1}{2} \frac{p\mu}{p+2} \left[ 4(d-1) + \frac{p^2(p-2)\mu}{3p-2} \right].$$

If  $\mu = \mu_{\text{FS}}$ , then we find that  $\frac{\int_{\mathcal{C}} |\nabla \varphi|^2 dy}{\int_{\mathcal{C}} u_{\mu,*}^2 dy} = \frac{2p(p-2)(p^2+p-1)}{(p+2)(3p-2)} \mu_{\text{FS}}^2$ .

### Appendix A.4. First spherical harmonics

Denote by  $\zeta \in [0, \pi]$  the azimuthal angle and consider the Laplace-Beltrami operator  $\mathcal{L}$  on the sphere  $\mathbb{S}^{d-1}$ . When  $\mathcal{L}$  is restricted to functions on  $\mathbb{S}^{d-1}$  depending only on  $\zeta$ , it

takes the form

$$\mathcal{L} f = \sin^{2-d} \zeta \frac{d}{d\zeta} \left( \sin^{d-2} \zeta \frac{df}{d\zeta} \right)$$

and  $\mathcal{L}$  is unitarily equivalent to  $\mathbf{L}$  defined by

$$\mathbf{L} g = (1 - x^2) g'' - (d - 1) x g' \quad x \in [-1, 1]$$

whose eigenvalues are the *Gegenbauer polynomials* or *ultra-spherical polynomials*. The correspondence between the operators is simply given by

$$f(\zeta) = g(\cos \zeta)$$

and one can check that

$$\int_0^\pi |f(\zeta)|^2 \sin^{d-2} \zeta d\zeta = \int_{-1}^1 |g(x)|^2 (1 - x^2)^{\frac{d}{2}-1} dx .$$

It is also not difficult to check that the first Gegenbauer polynomials are

$$g_0(x) = 1, \quad g_1(x) = x, \quad g_2(x) = dx^2 - 1, \quad g_3(x) = (d + 2)x^3 - 3x,$$

with eigenvalues respectively equal to  $\Lambda_0 = 0$ ,  $\Lambda_1 = d - 1$ ,  $\Lambda_2 = 2d$  and  $\Lambda_3 = 3(d + 1)$ :

$$-\mathbf{L} g_0 = 0, \quad -\mathbf{L} g_1 = (d - 1) g_1, \quad -\mathbf{L} g_2 = 2d g_2, \quad -\mathbf{L} g_3 = 3(d + 1) g_3 .$$

On  $[0, \pi]$ , we consider the probability measure

$$d\nu(\zeta) = \frac{1}{Z_d} \sin^{d-2} \zeta d\zeta$$

where  $Z_d = \int_0^\pi \sin^{d-2} \zeta d\zeta = \sqrt{\pi} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}$ . Then

$$f_0(\zeta) = 1, \quad f_1(\zeta) = \sqrt{d} \cos \zeta, \quad f_2(\zeta) = \sqrt{\frac{d+2}{2(d-1)}} (d \cos^2 \zeta - 1)$$

are normalized eigenfunctions in  $L^2([0, \pi], d\nu(\zeta))$ :

$$\int_0^\pi |f_i|^2 d\nu = 1 \quad \forall i = 0, 1, 2,$$

with eigenvalues  $\Lambda_0 = 0$ ,  $\Lambda_1 = d - 1$  and  $\Lambda_2 = 2d$ :

$$-\mathcal{L} f_0 = 0, \quad -\mathcal{L} f_1 = (d - 1) f_1, \quad -\mathcal{L} f_2 = 2d f_2 .$$

We also have the useful formulae

$$\int_0^\pi |f_1|^4 d\nu = \frac{3d}{d+2}, \quad \kappa_{(d)} := \int_0^\pi |f_1|^2 f_2 d\nu = \sqrt{\frac{2(d-1)}{d+2}},$$

$$f_1^2 = f_0 + \sqrt{\frac{2(d-1)}{d+2}} f_2 = f_0 + \kappa_{(d)} f_2.$$

## Acknowledgments

This work has been supported by the ANR project NoNAP. J.D. also acknowledges support from the ANR project STAB. We thank Frédéric Hecht and Olivier Pironneau for their help for the implementation of our numerical scheme.

© 2013 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

## References

- [1] Kai Seng Chou and Chiu Wing Chu. On the best constant for a weighted Sobolev-Hardy inequality. *J. London Math. Soc. (2)*, 48(1):137–151, 1993.
- [2] Toshio Horiuchi. Best constant in weighted Sobolev inequality with weights being powers of distance from the origin. *J. Inequal. Appl.*, 1(3):275–292, 1997.
- [3] Jean Dolbeault, Maria Esteban, Gabriella Tarantello, and Achilles Tertikas. Radial symmetry and symmetry breaking for some interpolation inequalities. *Calculus of Variations and Partial Differential Equations*, 42:461–485, 2011.
- [4] Veronica Felli and Matthias Schneider. Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type. *J. Differential Equations*, 191(1):121–142, 2003.
- [5] Florin Catrina and Zhi-Qiang Wang. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2):229–258, 2001.
- [6] Manuel del Pino, Jean Dolbeault, Stathis Filippas, and Achilles Tertikas. A logarithmic Hardy inequality. *J. Funct. Anal.*, 259(8):2045–2072, 2010.
- [7] Luis Caffarelli, Robert Kohn, and Louis Nirenberg. First order interpolation inequalities with weights. *Compositio Math.*, 53(3):259–275, 1984.
- [8] Jean Dolbeault, Maria J. Esteban, and Michael Loss. Symmetry of extremals of functional inequalities via spectral estimates for linear operators. *J. Math. Phys.*, 53(P):095204, 2012.
- [9] Jean Dolbeault, Maria J. Esteban, and Gabriella Tarantello. The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 7(2):313–341, 2008.
- [10] Jean Dolbeault, Maria J. Esteban, Michael Loss, and Gabriella Tarantello. On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities. *Adv. Nonlinear Stud.*, 9(4):713–726, 2009.
- [11] Maria Francesca Betta, Friedemann Brock, Anna Mercaldo, and Maria Rosaria Posteraro. A weighted isoperimetric inequality and applications to symmetrization. *J. Inequal. Appl.*, 4(3):215–240, 1999.
- [12] Jean Dolbeault and Maria J. Esteban. Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, 142(04):745–767, 2012.

- [13] Jean Dolbeault and Maria J. Esteban. About existence, symmetry and symmetry breaking for extremal functions of some interpolation functional inequalities. In Helge Holden and Kenneth H. Karlsen, editors, *Nonlinear Partial Differential Equations*, volume 7 of *Abel Symposia*, pages 117–130. Springer Berlin Heidelberg, 2012. 10.1007/978-3-642-25361-4-6.
- [14] Jean Dolbeault and Maria J. Esteban. A scenario for symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities. *Journal of Numerical Mathematics*, 20(3-4):233–249, March 2013.
- [15] Vincent Bouchez and Michel Willem. Extremal functions for the Caffarelli-Kohn-Nirenberg inequalities: a simple proof of the symmetry. *J. Math. Anal. Appl.*, 352(1):293–300, 2009.
- [16] Lev Davidovich Landau and E. Lifschitz. *Quantum Mechanics. Non-relativistic theory*. Third edition. Translated from the russian by J.B. Sykes and J.S. Bell. Pergamon, New York, 1977.
- [17] G Pöschl and E Teller. Bemerkungen zur quantenmechanik des anharmonischen oszillators. *Zeitschrift für Physik A Hadrons and Nuclei*, 83(3):143–151, 1933.