Preordered forests, packed words and contraction algebras

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Laboratoire de Mathématiques, Université de Reims
Moulin de la Housse - BP 1039 - 51687 REIMS Cedex 2, France
e-mail : anthony.mansuy@univ-reims.fr

Abstract. We introduce the notions of preordered and heap-preordered forests, generalizing the construction of ordered and heap-ordered forests. We prove that the algebras of preordered and heap-preordered forests are Hopf for the cut coproduct, and we construct a Hopf morphism to the Hopf algebra of packed words. Moreover, we define another coproduct on the preordered forests given by the contraction of edges. Finally, we give a combinatorial description of morphims defined on Hopf algebras of forests with values in the Hopf algebras of shuffes or quasi-shuffles.

Résumé. Nous introduisons les notions de forêts préordonnées et préordonnées en tas, généralisant les constructions des forêts ordonnées et ordonnées en tas. On démontre que les algèbres des forêts préordonnées et préordonnées en tas sont des algèbres de Hopf pour le coproduit de coupes et on construit un morphisme d'algèbres de Hopf dans l'algèbre des mots tassés. D'autre part, nous définissons un autre coproduit sur les forêts préordonnées donné par la contraction d'arêtes. Enfin, nous donnons une description combinatoire de morphismes définis sur des algèbres de Hopf de forêts et à valeurs dans les algèbres de Hopf de battages et de battages contractants.

Keywords. Algebraic combinatorics, planar rooted trees, Hopf algebra of ordered forests, quasi-shuffle algebra.

AMS Classification. 05E99, 16W30, 05C05.

Contents

Introduction

The Connes-Kreimer Hopf algebra of rooted forests $\mathbf{H}_{CK}$ is introduced and studied in [CK98, Moe01]. This commutative, noncocommutative Hopf algebra is used to study a problem of Renormalisation in Quantum Field Theory, as explained in [CK00, CK01]. The coproduct is given by admissible cuts. We denote by $\mathbf{H}_{CK}^{\mathcal{D}}$ the Hopf algebra of rooted trees, where the vertices are decorated by decorations belonging to the set $\mathcal{D}$. A noncommutative version, the Hopf algebra $\mathbf{H}_{NCK}$ of planar rooted forests, is introduced in [Foi02a, Hol03]. When the vertices are given a total order, we obtain the Hopf algebra of ordered forests $\mathbf{H}_o$ and, adding an increasing condition, we obtain the Hopf subalgebra of heap-ordered forests $\mathbf{H}_{ho}$ (see [FU10, GL90]).

Moreover, the Hopf algebra **FQSym** of free quasi-symmetric functions is introduced in [DHT02, MR95]. L. Foissy and J. Unterberger prove in [FU10] that there exists a Hopf algebra morphism from $\mathbf{H}_o$ to **FQSym** and that its restriction to $\mathbf{H}_{ho}$ is an isomorphism of Hopf algebras.

In this text, we introduce the notion of preordered forests. A preorder is a binary reflexive and transitive relation. A preordered forest is a rooted forest with a total preorder on its vertices. We prove that the algebra of preordered forests $\mathbf{H}_{po}$ is a Hopf algebra for the cut coproduct. With an increasing condition, we define the algebra of heap-preordered trees $\mathbf{H}_{hpo}$ and we prove that $\mathbf{H}_{hpo}$ is a Hopf subalgebra of $\mathbf{H}_{po}$.

In [NT06], J.-C. Novelli and J.-Y. Thibon construct a generalization of **FQSym**: the Hopf algebra $\mathbf{WQSym}^*$ of free packed words. We prove a result similar to that of L. Foissy and J. Unterberger, by substituting the ordered forests by the preordered forests and the quasi-symmetric functions by the packed words. More precisely, we prove that there exists a Hopf algebra morphism from $\mathbf{H}_{po}$ to $\mathbf{WQSym}^*$. In addition, we prove that its restriction to $\mathbf{H}_{hpo}$ is an injection of Hopf algebras.

Afterwards, we study a another coproduct called in this paper the contraction coproduct. In [CEFM11], D. Calaque, K. Ebrahimi-Fard and D. Manchon define this coproduct in a commutative case, on a quotient $\mathbf{C}_{CK}$ of $\mathbf{H}_{CK}$ (see also [MS11]). We give a decorated version $\mathbf{C}_{CK}^{\mathcal{D}}$ of $\mathbf{C}_{CK}$. We define two operations $\curlyvee$ and $\rhd$ on the vector space $\mathbf{T}_{CK}^{\mathcal{D}}$ spanned by the trees of $\mathbf{C}_{CK}^{\mathcal{D}}$. We prove that $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \rhd)$ is a commutative prelie algebra, that is to say that $(A, \curlyvee)$ is a commutative algebra, $(A, \rhd)$ is a prelie algebra and with the following relation: for all $x, y, z \in \mathbf{T}_{CK}^{\mathcal{D}}$,

$$x \rhd (y \curlyvee z) = (x \rhd y) \curlyvee z + (x \rhd z) \curlyvee y.$$

We prove that $(\mathbf{T}_{CK}^{\mathcal{D}}, \curlyvee, \rhd)$ is generated as commutative prelie algebra by the trees $\uparrow^d$, $d \in \mathcal{D}$.

We construct a noncommutative version of $\mathbf{C}_{CK}$. For this, we consider quotients of $\mathbf{H}_{NCK}$, $\mathbf{H}_{ho}$, $\mathbf{H}_o$, $\mathbf{H}_{hpo}$, $\mathbf{H}_{po}$, denoted respectively $\mathbf{C}_{NCK}$, $\mathbf{C}_{ho}$, $\mathbf{C}_o$, $\mathbf{C}_{hpo}$, $\mathbf{C}_{po}$, and we define on these quotients a contraction coproduct. We prove that $\mathbf{C}_{ho}, \mathbf{C}_o, \mathbf{C}_{hpo}, \mathbf{C}_{po}$ are Hopf algebras and that $\mathbf{C}_{NCK}$ is a left comodule of the Hopf algebra $\mathbf{C}_{ho}$.

Finally, we study the Hopf algebra morphisms from $\mathbf{H}_{CK}^{\mathcal{D}}$ or $\mathbf{C}_{CK}^{\mathcal{D}}$ to the Hopf algebra $\mathbf{Sh}^{\mathcal{D}}$ of shuffles or the Hopf algebra $\mathbf{Csh}^{\mathcal{D}}$ of quasi-shuffles (see [Hof00]). We give a combinatorial description of these morphisms in each case. In particular, we note that, in the description of the morphism from $\mathbf{H}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$ or $\mathbf{Csh}^{\mathcal{D}}$, the contraction coproduct and the preordered forests appear naturally.

This text is organized as follows: the first section is devoted to recalls about the Hopf algebras, for the cut coproduct, of rooted forests, planar forests and ordered and heap-ordered forests. We give recalls on the Hopf algebras of words in the second section. We define the Hopf algebra of permutations and packed words and we deduce the construction of $\mathbf{Sh}^{\mathcal{D}}$ and $\mathbf{Csh}^{\mathcal{D}}$. In section three, we define the algebras $\mathbf{H}_{po}$ and $\mathbf{H}_{hpo}$ of preordered and heap-preordered forests and we prove that these are Hopf algebras. The contraction coproduct, is introduced in the section four. We describe a commutative case and we study an insertion operation. We give a noncommutative version using ordered and preordered forests. The last section deals with Hopf algebra morphisms from $\mathbf{H}_{CK}^{\mathcal{D}}$ or $\mathbf{C}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$ or $\mathbf{Csh}^{\mathcal{D}}$. We give a combinatorial description of these morphisms in each case.

Acknowledgment. I would like to thank my advisor Loïc Foissy for stimulating discussions and his support during my research.

Notations.

1. We shall denote by $\mathbb{K}$ a commutative field of characteristic zero. Every vector space, algebra, coalgebra, etc, will be taken over $\mathbb{K}$. Given a set X, we denote by $\mathbb{K}(X)$ the vector space spanned by X.
2. Let n be an integer. We denote Σ_n the symmetric group of order n ($\Sigma_0 = \{1\}$) and Σ the disjoint union of Σ_n for all $n \geq 0$.
3. Let (A, Δ, ε) be a counitary coalgebra. Let $1 \in A$, non zero, such that $\Delta(1) = 1 \otimes 1$. We then define the noncounitary coproduct:

$$\tilde{\Delta} : \left\{ \begin{array}{rcl} Ker(\varepsilon) & \rightarrow & Ker(\varepsilon) \otimes Ker(\varepsilon), \\ a & \mapsto & \Delta(a) - a \otimes 1 - 1 \otimes a. \end{array} \right.$$

1 Recalls on the Hopf algebras of forests

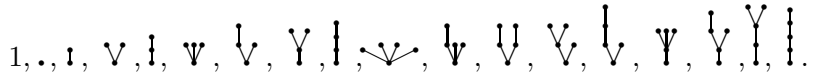
1.1 The Connes-Kreimer Hopf algebra of rooted trees

We briefly recall the construction of the Connes-Kreimer Hopf algebra of rooted trees [CK98]. A *rooted tree* is a finite graph, connected, without loops, with a distinguished vertex called the *root* [Sta02]. We denote by 1 the empty rooted tree. If T is a rooted tree, we denote by R_T the root of T. A *rooted forest* is a finite graph F such that any connected component of F is a rooted tree. The *length* of a forest F, denoted $l(F)$, is the number of connected components of F. The set of vertices of the rooted forest F is denoted by $V(F)$. The *vertices degree* of a forest F, denoted $|F|_v$, is the number of its vertices. The set of edges of the rooted forest F is denoted by $E(F)$. The *edges degree* of a forest F, denoted $|F|_e$, is the number of its edges.

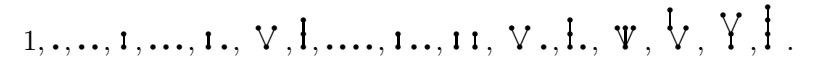
Remark. Let F be a rooted forest. Then $|F|_v = |F|_e + l(F)$.

Examples.

1. Rooted trees of vertices degree ≤ 5:



2. Rooted forests of vertices degree ≤ 4:



Let $\mathcal{D}$ be a nonempty set. A rooted forest with its vertices decorated by $\mathcal{D}$ is a couple (F, d) with F a rooted forest and $d : V(F) \to \mathcal{D}$ a map.

Examples. Rooted trees with their vertices decorated by $\mathcal{D}$ of vertices degree smaller than 4:

$$\bullet_a, a \in \mathcal{D}, \qquad \overset{b}{\underset{a}{\bullet}}, (a,b) \in \mathcal{D}^2, \qquad \overset{b}{\underset{a}{\vee}}{}^{c}, \overset{c}{\underset{a}{\overset{b}{\bullet}}}, (a,b,c) \in \mathcal{D}^3$$

$$\overset{b}{\underset{a}{\curlyvee}}{}^{c}{}_{d}, \overset{c}{\underset{a}{\overset{b}{\vee}}}{}_{d}, \overset{b}{\underset{a}{\overset{d}{\bullet}}}{}_{c}, \overset{c}{\underset{a}{\overset{d}{\vee}}}{}_{b}, \overset{d}{\underset{a}{\overset{c}{\overset{b}{\bullet}}}}, (a,b,c,d) \in \mathcal{D}^4.$$

Let $\mathbb{F}_{\mathbf{H}_{CK}}$ be the set of rooted forests and $\mathbb{F}^{\mathcal{D}}_{\mathbf{H}_{CK}}$ the set of rooted forests with their vertices decorated by $\mathcal{D}$. We will denote by $\mathbf{H}_{CK}$ the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_{CK}}$ and by $\mathbf{H}^{\mathcal{D}}_{CK}$ the $\mathbb{K}$-vector space generated by $\mathbb{F}^{\mathcal{D}}_{\mathbf{H}_{CK}}$. The set of nonempty rooted trees will be denoted $\mathbb{T}_{\mathbf{H}_{CK}}$ and the set of nonempty rooted trees with their vertices decorated by $\mathcal{D}$ will be denoted $\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}$. $\mathbf{H}_{CK}$ and $\mathbf{H}^{\mathcal{D}}_{CK}$ are algebras: the product is given by the concatenation of rooted forests.

Let F be a nonempty rooted forest. A *subtree* T of F is a nonempty connected subgraph of F. A *subforest* $T_1 \dots T_k$ of F is a product of disjoint subtrees $T_1, \dots, T_k$ of F. We can give the same definition in the decorated case.

Examples. Consider the tree $T = \overset{\bullet}{\underset{\bullet}{\vee}}$. Then:

- The subtrees of T are $\bullet$ (which appears 4 times), $\overset{\bullet}{\bullet}$ (which appears 3 times), $\underset{\bullet}{\overset{\bullet}{\vee}}$, $\overset{\bullet}{\overset{\bullet}{\bullet}}$ and $\overset{\bullet}{\underset{\bullet}{\overset{\bullet}{\vee}}}$ (which appear once).
- The subforests of T are $\bullet\bullet$, $\overset{\bullet}{\bullet}\bullet$ (which appear 6 times), $\bullet, \bullet\bullet\bullet$ (which appear 4 times), $\overset{\bullet}{\bullet}, \overset{\bullet}{\bullet}\bullet\bullet$ (which appear 3 times) and $\underset{\bullet}{\overset{\bullet}{\vee}}, \overset{\bullet}{\overset{\bullet}{\bullet}}, \bullet\bullet\bullet\bullet, \underset{\bullet}{\overset{\bullet}{\vee}}\bullet, \overset{\bullet}{\overset{\bullet}{\bullet}}\bullet, \overset{\bullet}{\underset{\bullet}{\overset{\bullet}{\vee}}}$ (which appear once).

Let F be a rooted forest. The edges of F are oriented downwards (from the leaves to the roots). If $v, w \in V(F)$, we shall note $v \to w$ if there is an edge in F from v to w and $v \twoheadrightarrow w$ if there is an oriented path from v to w in F. By convention, $v \twoheadrightarrow v$ for any $v \in V(F)$.

Let $\boldsymbol{v}$ be a subset of $V(F)$. We shall say that $\boldsymbol{v}$ is an admissible cut of F, and we shall write $\boldsymbol{v} \models V(F)$, if $\boldsymbol{v}$ is totally disconnected, that is to say that $v \not\twoheadrightarrow w$ for any couple (v, w) of two different elements of $\boldsymbol{v}$. If $\boldsymbol{v} \models V(F)$, we denote by $Lea_{\boldsymbol{v}}(F)$ the rooted subforest of F obtained by keeping only the vertices above $\boldsymbol{v}$, that is to say $\{w \in V(F),\ \exists v \in \boldsymbol{v},\ w \twoheadrightarrow v\}$, and the edges between these vertices. Note that $\boldsymbol{v} \subseteq Lea_{\boldsymbol{v}}(F)$. We denote by $Roo_{\boldsymbol{v}}(F)$ the rooted subforest obtained by keeping the other vertices and the edges between these vertices.

In particular, if $\boldsymbol{v} = \emptyset$, then $Lea_{\boldsymbol{v}}(F) = 1$ and $Roo_{\boldsymbol{v}}(F) = F$: this is the *empty cut* of F. If $\boldsymbol{v}$ contains all the roots of F, then it contains only the roots of F, $Lea_{\boldsymbol{v}}(F) = F$ and $Roo_{\boldsymbol{v}}(F) = 1$: this is the *total cut* of F. We shall write $\boldsymbol{v} \models\!\!\!| V(F)$ if $\boldsymbol{v}$ is a nontotal, nonempty admissible cut of F.

Connes and Kreimer proved in [CK98] that $\mathbf{H}_{CK}$ is a Hopf algebra. The coproduct is the cut coproduct defined for any rooted forest F by:

$$\Delta_{\mathbf{H}_{CK}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \models\!\!\!| V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example:

$$\Delta_{\mathbf{H}_{CK}}\left(\underset{\bullet}{\overset{\bullet}{\vee}}\right) = \underset{\bullet}{\overset{\bullet}{\vee}} \otimes 1 + 1 \otimes \underset{\bullet}{\overset{\bullet}{\vee}} + \bullet \otimes \overset{\bullet}{\bullet} + \overset{\bullet}{\bullet} \otimes \overset{\bullet}{\bullet} + \bullet \otimes \overset{\bullet}{\overset{\bullet}{\bullet}} + \bullet\bullet \otimes \overset{\bullet}{\bullet} + \overset{\bullet}{\bullet}\bullet \otimes \bullet\bullet.$$

In the same way, we can define a cut coproduct on $\mathbf{H}_{CK}^{\mathcal{D}}$. With this coproduct, $\mathbf{H}_{CK}^{\mathcal{D}}$ is also a Hopf algebra. For example:

$\Delta_{\mathbf{H}_{CK}^{\mathcal{D}}}($  $) =$ $\otimes 1 + 1 \otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$.

$\mathbf{H}_{CK}$ is graded by the number of vertices. We give some values of the number $f_n^{\mathbf{H}_{CK}}$ of rooted forests of vertices degree n:

n	0	1	2	3	4	5	6	7	8	9	10
$f_n^{\mathbf{H}_{CK}}$	1	1	2	4	9	20	48	115	286	719	1842

These is the sequence A000081 in [Slo].

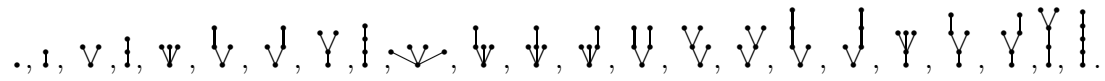
1.2 Hopf algebras of planar trees

We now recall the construction of the noncommutative generalization of the Connes-Kreimer Hopf algebra [Foi02a, Hol03].

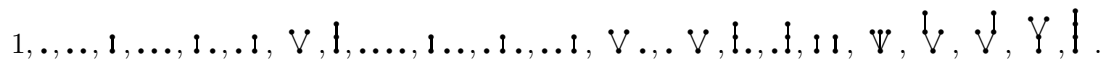
A *planar forest* is a rooted forest F such that the set of the roots of F is totally ordered and, for any vertex $v \in V(F)$, the set $\{w \in V(F) \mid w \to v\}$ is totally ordered. Planar forests are represented such that the total orders on the set of roots and the sets $\{w \in V(F) \mid w \to v\}$ for any $v \in V(F)$ is given from left to right. We denote by $\mathbb{T}_{\mathbf{H}_{NCK}}$ the set of nonempty planar trees and $\mathbb{F}_{\mathbf{H}_{NCK}}$ the set of planar forests.

Examples.

1. Planar rooted trees of vertices degree ≤ 5:



2. Planar rooted forests of vertices degree ≤ 4:



If $\boldsymbol{v} \models V(F)$, then $Lea_{\boldsymbol{v}}(F)$ and $Roo_{\boldsymbol{v}}(F)$ are naturally planar forests. It is proved in [Foi02a] that the space $\mathbf{H}_{NCK}$ generated by planar forests is a Hopf algebra. Its product is given by the concatenation of planar forests and its coproduct is defined for any rooted forest F by:

$$\Delta_{\mathbf{H}_{NCK}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F) = F \otimes 1 + 1 \otimes F + \sum_{\boldsymbol{v} \Vdash V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example:

$\Delta_{\mathbf{H}_{NCK}}($  $) =$ $\otimes 1 + 1 \otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$,

$\Delta_{\mathbf{H}_{NCK}}($  $) =$ $\otimes 1 + 1 \otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$ $+$ $\otimes$.

As in the nonplanar case, there is a decorated version $\mathbf{H}_{NCK}^{\mathcal{D}}$ of $\mathbf{H}_{NCK}$. Moreover, $\mathbf{H}_{NCK}$ is a Hopf algebra graded by the number of vertices. The number $f_n^{\mathbf{H}_{NCK}}$ of planar forests of vertices degree n (equal to the number of planar trees of vertices degree $n+1$) is the n-Catalan number $\frac{(2n)!}{n!(n+1)!}$, see sequence A000108 of [Slo]. We have:

$$T_{\mathbf{H}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2}, \qquad F_{\mathbf{H}_{NCK}}(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1}$$

This gives:

n	0	1	2	3	4	5	6	7	8	9	10
$f_n^{\mathbf{H}_{NCK}}$	1	1	2	5	14	42	132	429	1430	4862	16796

1.3 Ordered and heap-ordered forests

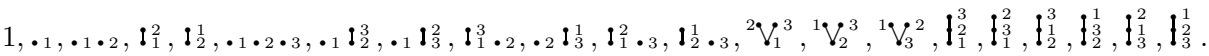
Definition 1 *An ordered forest F is a rooted forest F with a total order on $V(F)$. The set of ordered forests is denoted by $\mathbb{F}_{\mathbf{H}_o}$ and the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_o}$ is denoted by $\mathbf{H}_o$.*

Remarks and notations. If F is an ordered forest, then there exits a unique increasing bijection $\sigma : V(F) \to \{1, \dots, |F|_v\}$ for the total order on $V(F)$.

Reciprocally, if F is a rooted forest and if $\sigma : V(F) \to \{1, \dots, |F|_v\}$ is a bijection, then σ defines a total order on $V(F)$ and F is an ordered forest.

Depending on the case, we shall denote an ordered forest by F or (F, σ).

Examples. Ordered forests of vertices degree ≤ 3:



Let (F, σ^F) and (G, σ^G) be two ordered forests. Then the rooted forest FG is also an ordered forest (FG, σ^{FG}) where

$$\sigma^{FG} = \sigma^F \otimes \sigma^G : \begin{cases} V(F) \bigcup V(G) & \to & \{1, \dots, |F|_v + |G|_v\} \\ a \in V(F) & \mapsto & \sigma^F(a) \\ a \in V(G) & \mapsto & \sigma^G(a) + |F|_v \, . \end{cases} \tag{2}$$

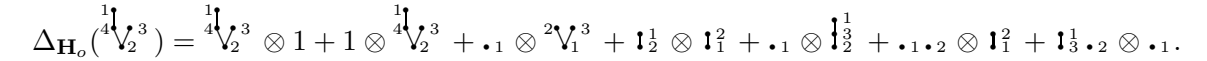
In other terms, we keep the initial order on the vertices of F and G and we assume that the vertices of F are smaller than the vertices of G. This defines a noncommutative product on the set of ordered forests. For example, the product of $\bullet_1$ and $\mathbb{I}_1^2$ gives $\bullet_1 \mathbb{I}_2^3 = \mathbb{I}_2^3 \bullet_1$, whereas the product of $\mathbb{I}_1^2$ and $\bullet_1$ gives $\mathbb{I}_1^2 \bullet_3 = \bullet_3 \mathbb{I}_1^2$. This product is linearly extended to $\mathbf{H}_o$, which in this way becomes an algebra.

$\mathbf{H}_o$ is graded by the number of vertices and there is $(n+1)^{n-1}$ ordered forests in vertices degree n, see sequence A000272 of [Slo].

If F is an ordered forest, then any subforest G of F is also ordered: the total order on $V(G)$ is the restriction of the total order of $V(F)$. So we can define a coproduct $\Delta_{\mathbf{H}_o} : \mathbf{H}_o \to \mathbf{H}_o \otimes \mathbf{H}_o$ on $\mathbf{H}_o$ in the following way: for all $F \in \mathbb{F}_{\mathbf{H}_o}$,

$$\Delta_{\mathbf{H}_o}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

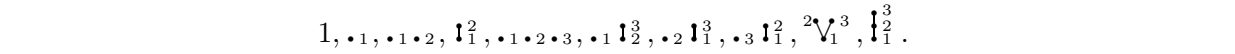
For example,



With this coproduct, $\mathbf{H}_o$ is a Hopf algebra.

Definition 2 *[GL90] A heap-ordered forest is an ordered forest F such that if $a, b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$, then a is greater than b for the total order on $V(F)$. The set of heap-ordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{ho}}$.*

Examples. Heap-ordered forests of vertices degree ≤ 3:



Definition 3 *A linear order on a nonempty rooted forest F is a bijective map $\sigma : V(F) \to \{1, \dots, |F|_v\}$ such that if $a, b \in V(F)$ and $a \twoheadrightarrow b$, then $\sigma(a) \geq \sigma(b)$. We denote by $\mathcal{O}(F)$ the set of linear orders on the nonempty rooted forest F.*

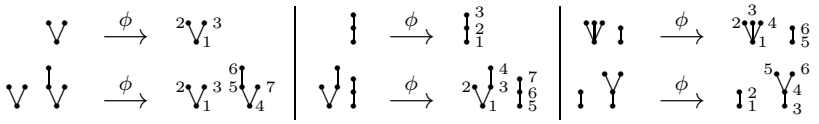
Remarks. If (F, σ) is a heap-ordered forest, then the increasing bijection $\sigma : V(F) \to \{1, \dots, |F|_v\}$ is a linear order on F. Reciprocally, if F is a rooted forest and $\sigma \in \mathcal{O}(F)$, then σ defines a total order on $V(F)$ such that (F, σ) is a heap-ordered forest.

If F and G are two heap-ordered forests, then FG is an ordered forest with (2) and also a heap-ordered forest. Moreover, any subforest G of a heap-ordered forest F is also a heap-ordered forest by restriction on $V(G)$ of the total order of $V(F)$. So the subspace $\mathbf{H}_{ho}$ of $\mathbf{H}_o$ generated by the heap-ordered forests is a graded Hopf subalgebra of $\mathbf{H}_o$.

The number of heap-ordered forests of vertices degree n is $n!$, see sequence A000142 of [Slo].

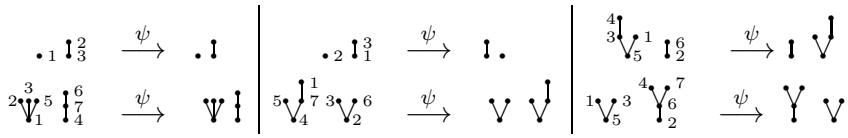
Remarks.

1. A planar forest can be considered as an ordered forest by ordering its vertices in the "north-west" direction (this is the order defined in [Foi02a] or given by the Depth First Search algorithm). This defines an algebra morphism $\phi : \mathbf{H}_{NCK} \to \mathbf{H}_o$. For example:



(3)

2. Reciprocally, an ordered forest is also planar, by restriction of the total order to the subsets of vertices formed by the roots or $\{w \in V(\mathbb{F}) \mid w \to v\}$. This defines an algebra morphism $\psi : \mathbf{H}_o \to \mathbf{H}_{NCK}$. For example:



(4)

Note that $\psi \circ \phi = Id_{\mathbf{H}_{NCK}}$ therefore ψ is surjective and ϕ is injective. ψ and ϕ are not bijective (by considering the dimensions).

Moreover ϕ is a Hopf algebra morphism and its image is included in $\mathbf{H}_{ho}$. ψ is not a Hopf algebra morphism: in the expression of $(\psi \otimes \psi) \circ \Delta_{\mathbf{H}_o}(\bullet_3 \begin{smallmatrix}2\\1\\4\end{smallmatrix})$ we have the tensor $\bullet \otimes \bullet$ and in the expression of $\Delta_{\mathbf{H}_{NCK}} \circ \psi(\bullet_3 \begin{smallmatrix}2\\1\\4\end{smallmatrix})$ we have the different tensor $\bullet \otimes \bullet$. By cons, the restriction of ψ of $\mathbf{H}_{ho}$ is a Hopf algebra morphism.

In the following, we consider $\mathbf{H}_{NCK}$ as a Hopf subalgebra of $\mathbf{H}_{ho}$ and $\mathbf{H}_o$.

2 Recalls on the Hopf algebras of words

2.1 Hopf algebra of permutations and shuffles

Notations.

1. Let k, l be integers. A (k,l)-*shuffle* is a permutation ζ of $\{1, \dots, k+l\}$, such that $\zeta(1) < \dots < \zeta(k)$ and $\zeta(k+1) < \dots < \zeta(k+l)$. The set of (k,l)-shuffles will be denoted by $Sh(k,l)$.

2. We represent a permutation $\sigma \in \Sigma_n$ by the word $(\sigma(1) \dots \sigma(n))$. For example, $Sh(2,1) = \{(123), (132), (231)\}$.

Remark. For any integer k, l, any permutation $\sigma \in \Sigma_{k+l}$ can be uniquely written as $\epsilon \circ (\sigma_1 \otimes \sigma_2)$, where $\sigma_1 \in \Sigma_k$, $\sigma_2 \in \Sigma_l$, and $\epsilon \in Sh(k,l)$. Similarly, considering the inverses, any permutation $\tau \in \Sigma_{k+l}$ can be uniquely written as $(\tau_1 \otimes \tau_2) \circ \zeta^{-1}$, where $\tau_1 \in \Sigma_k$, $\tau_2 \in \Sigma_l$, and $\zeta \in Sh(k,l)$. Note that, whereas ϵ renames the numbers of each lists $(\sigma(1), \dots, \sigma(k)), (\sigma(k+1), \dots, \sigma(k+l))$ without changing their orderings, ζ^{-1} shuffles the lists $(\tau(1), \dots, \tau(k)), (\tau(k+1), \dots, \tau(k+l))$. For instance, if $k = 4$, $l = 3$ and $\sigma = (5172436)$ then

- $\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2)$, with $\epsilon = (1257346) \in Sh(4,3)$, $\sigma_1 = (3142) \in \Sigma_4$ and $\sigma_2 = (213) \in \Sigma_3$,
- $\sigma = (\tau_1 \otimes \tau_2) \circ \zeta^{-1}$, with $\tau_1 = (1243) \in \Sigma_4$, $\tau_2 = (132) \in \Sigma_3$ and $\zeta = (2456137) \in Sh(4,3)$.

We here briefly recall the construction of the Hopf algebra **FQSym** of free quasi-symmetric functions, also called the Malvenuto-Reutenauer Hopf algebra [DHT02, MR95]. As a vector space, a basis of **FQSym** is given by the disjoint union of the symmetric groups Σ_n, for all $n \geq 0$. By convention, the unique element of Σ_0 is denoted by 1. The product of **FQSym** is given, for $\sigma \in \Sigma_k$, $\tau \in \Sigma_l$, by:

$$\sigma.\tau = \sum_{\zeta \in Sh(k,l)} (\sigma \otimes \tau) \circ \zeta^{-1}.$$

In other words, the product of σ and τ is given by shifting the letters of the word representing τ by k, and then summing all the possible shufflings of this word and of the word representing σ. For example:

$$\begin{aligned}(123)(21) &= (12354) + (12534) + (15234) + (51234) + (12543) \\ &\quad +(15243) + (51243) + (15423) + (51423) + (54123).\end{aligned}$$

Let $\sigma \in \Sigma_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \epsilon_k\right) \in \Sigma_k \times \Sigma_{n-k} \times Sh(k, n-k)$ such that $\sigma = \epsilon_k \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of **FQSym** is then defined by:

$$\Delta_{\mathbf{FQSym}}(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2) \\ \epsilon \in Sh(k,n-k), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_{n-k}}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k+1)$-th letter, and then *standardizing* the two obtained words by the following process. If v is a words of length n whose the letters are distinct integers, then the standardizing of v, denoted $Std(v)$, is the word obtained by applying to the letters of v the unique increasing bijection to $\{1, \dots, n\}$. For example:

$$\begin{aligned}\Delta_{\mathbf{FQSym}}((41325)) &= 1 \otimes (41325) + Std(4) \otimes Std(1325) + Std(41) \otimes Std(325) \\ &\quad +Std(413) \otimes Std(25) + Std(4132) \otimes Std(5) + (41325) \otimes 1 \\ &= 1 \otimes (41325) + (1) \otimes (1324) + (21) \otimes (213) \\ &\quad +(312) \otimes (12) + (4132) \otimes (1) + (41325) \otimes 1.\end{aligned}$$

Then **FQSym** is a Hopf algebra. It is graded, with $\mathbf{FQSym}(n) = Vect(\Sigma_n)$ for all $n \geq 0$.

It is also possible to give a decorated version of **FQSym**. Let $\mathcal{D}$ be a nonempty set. A $\mathcal{D}$-decorated permutation is a couple (σ, d), where $\sigma \in \Sigma_n$ and d is a map from $\{1, \dots, n\}$ to $\mathcal{D}$. A $\mathcal{D}$-decorated permutation is represented by two superposed words $\binom{a_1 \dots a_n}{v_1 \dots v_n}$, where $(a_1 \dots a_n)$ is the word representing σ and for all i, $v_i = d(a_i)$. The vector space $\mathbf{FQSym}^{\mathcal{D}}$ generated by the set of $\mathcal{D}$-decorated permutations is a Hopf algebra. For example, if $x, y, z, t \in \mathcal{D}$:

$$\begin{aligned}
\tbinom{213}{yxz} \cdot \tbinom{1}{t} &= \tbinom{2134}{yxzt} + \tbinom{2143}{yxtz} + \tbinom{2413}{ytxz} + \tbinom{4213}{tyxz}, \\
\Delta_{\mathbf{FQSym}^{\mathcal{D}}}\left(\tbinom{4321}{tzyx}\right) &= \tbinom{4321}{tzyx} \otimes 1 + \tbinom{321}{tzy} \otimes \tbinom{1}{x} + \tbinom{21}{tz} \otimes \tbinom{21}{yx} + \tbinom{1}{t} \otimes \tbinom{321}{zyx} + 1 \otimes \tbinom{4321}{tzyx}.
\end{aligned}$$

In other words, if (σ, d) and (τ, d') are decorated permutations of respective degrees k and l:

$$(\sigma, d).(\tau, d') = \sum_{\zeta \in Sh(k,l)} ((\sigma \otimes \tau) \circ \zeta^{-1}, d \otimes d'),$$

where $d \otimes d'$ is defined by $(d \otimes d')(i) = d(i)$ if $1 \leq i \leq k$ and $(d \otimes d')(k + j) = d'(j)$ if $1 \leq j \leq l$. If (σ, d) is a decorated permutation of degree n:

$$\Delta_{\mathbf{FQSym}^{\mathcal{D}}}((\sigma, d)) = \sum_{k=0}^{n} \sum_{\substack{\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2) \\ \epsilon \in Sh(k,l), \sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_l}} (\sigma_1, d') \otimes (\sigma_2, d''),$$

where $d = (d' \otimes d'') \circ \epsilon^{-1}$.

In some sense, a $\mathcal{D}$-decorated permutation can be seen as a word with a total order on the set of its letters.

We can now define the shuffle Hopf algebra $\mathbf{Sh}^{\mathcal{D}}$ (see [Hof00, Reu93]). A $\mathcal{D}$-word is a finite sequence of elements taken in $\mathcal{D}$. It is graded by the degree of words, that is to say the number of their letters. As a vector space, $\mathbf{Sh}^{\mathcal{D}}$ is generated by the set of $\mathcal{D}$-words.

The surjective linear map from $\mathbf{FQSym}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$, sending the decorated permutation $\binom{a_1 \dots a_n}{v_1 \dots v_n}$ to the $\mathcal{D}$-word $(v_1 \dots v_n)$, define a Hopf algebra structure on $\mathbf{Sh}^{\mathcal{D}}$:

- The product $\sqcup\!\sqcup$ of $\mathbf{Sh}^{\mathcal{D}}$ is given in the following way: if $(v_1 \dots v_k)$ is a $\mathcal{D}$-word of degree k, $(v_{k+1} \dots v_{k+l})$ is a $\mathcal{D}$-word of degree l, then

$$(v_1 \dots v_k) \sqcup\!\sqcup (v_{k+1} \dots v_{k+l}) = \sum_{\zeta \in Sh(k,l)} v_{\zeta^{-1}(1)} \dots v_{\zeta^{-1}(k+l)}.$$

- The coproduct $\Delta_{\mathbf{Sh}^{\mathcal{D}}}$ of $\mathbf{Sh}^{\mathcal{D}}$ is given on any $\mathcal{D}$-word $w = (v_1 \dots v_n)$ by

$$\Delta_{\mathbf{Sh}^{\mathcal{D}}}(w) = \sum_{i=0}^{n} (v_1 \dots v_i) \otimes (v_{i+1} \dots v_n).$$

Examples.

1. If $(v_1 v_2 v_3)$ and $(v_4 v_5)$ are two $\mathcal{D}$-words,

$$\begin{aligned}
(v_1 v_2 v_3) \sqcup\!\sqcup (v_4 v_5) &= (v_1 v_2 v_3 v_4 v_5) + (v_1 v_2 v_4 v_3 v_5) + (v_1 v_2 v_4 v_5 v_3) + (v_1 v_4 v_2 v_3 v_5) \\
&\quad + (v_1 v_4 v_2 v_5 v_3) + (v_1 v_4 v_5 v_2 v_3) + (v_4 v_1 v_2 v_3 v_5) + (v_4 v_1 v_2 v_5 v_3) \\
&\quad + (v_4 v_1 v_5 v_2 v_3) + (v_4 v_5 v_1 v_2 v_3).
\end{aligned}$$

2. If $(v_1 v_2 v_3 v_4)$ is a $\mathcal{D}$-word,

$$\begin{aligned}
\Delta_{\mathbf{Sh}^{\mathcal{D}}}((v_1 v_2 v_3 v_4)) &= (v_1 v_2 v_3 v_4) \otimes 1 + (v_1 v_2 v_3) \otimes (v_4) + (v_1 v_2) \otimes (v_3 v_4) \\
&\quad + (v_1) \otimes (v_2 v_3 v_4) + 1 \otimes (v_1 v_2 v_3 v_4).
\end{aligned}$$

There is a link between the algebras $\mathbf{H}_o$, $\mathbf{H}_{ho}$ and **FQSym** given by the following result (see [FU10]):

Proposition 4 *1. Let $n \geq 0$. For all $(F, \sigma) \in \mathbb{F}_{\mathbf{H}_o}$, let $\mathbb{S}_F$ be the set of permutations $\theta \in \Sigma_n$ such that for all $a, b \in V(F)$, $(a \twoheadrightarrow b) \Rightarrow (\theta^{-1}(\sigma(a)) \leq \theta^{-1}(\sigma(b)))$. Let us define:*

$$\Theta : \left\{ \begin{array}{rcl} \mathbf{H}_o & \to & \mathbf{FQSym} \\ F \in \mathbb{F}_{\mathbf{H}_o} & \mapsto & \displaystyle\sum_{\theta \in \mathbb{S}_F} \theta. \end{array} \right.$$

Then $\Theta : \mathbf{H}_o \to \mathbf{FQSym}$ is a Hopf algebra morphism, homogeneous of degree 0.

2. The restriction of Θ to $\mathbf{H}_{ho}$ is an isomorphism of graded Hopf algebras.

2.2 Hopf algebra of packed words and quasi-shuffles

Recall the construction of the Hopf algebra $\mathbf{WQSym}^*$ of free packed words (see [NT06]).

Notations.

1. Let $n \geq 0$. We denote by $Surj_n$ the set of maps $\sigma : \{1, \ldots, n\} \to \mathbb{N}^*$, such that $\sigma(\{1, \ldots, n\}) = \{1, \ldots, k\}$ for a certain $k \in \mathbb{N}$. In this case, k is the maximum of σ and is denoted by $\max(\sigma)$ and n is the length of σ. We represent the element σ of $Surj_n$ by the packed word $(\sigma(1) \ldots \sigma(n))$.

2. Let k, l be two integers. A (k,l)-*surjective shuffle* is an element ϵ of $Surj_{k+l}$ such that $\epsilon(1) < \ldots < \epsilon(k)$ and $\epsilon(k+1) < \ldots < \epsilon(k+l)$. The set of (k,l)-surjective shuffles will be denoted by $SjSh(k,l)$. For example, $SjSh(2,1) = \{(121), (122), (123), (132), (231)\}$.

Let v be a word such that the letters occuring in v are integers $a_1 < a_2 < \ldots < a_k$. The *packing* of v, denoted by $pack(v)$, is the image of letters of v by the application $a_i \mapsto i$. For example, $pack((22539)) = (11324)$, $pack((831535)) = (421323)$.

Remark. Let k, l be two integers and $\sigma \in Surj_{k+l}$. We set $p_k = \max(pack((\sigma(1) \ldots \sigma(k))))$ and $q_k = \max(pack((\sigma(k+1) \ldots \sigma(k+l))))$. Then σ can be uniquely written as $\epsilon \circ (\sigma_1 \otimes \sigma_2)$, where $\sigma_1 \in Surj_k$, $\sigma_2 \in Surj_l$, and $\epsilon \in SjSh(p_k, q_k)$. For instance, if $k = 4$, $l = 3$ and $\sigma = (2311223)$ then $p_4 = 3$, $q_4 = 2$ and $\sigma = \epsilon \circ (\sigma_1 \otimes \sigma_2)$ with $\epsilon = (12323) \in SjSh(3,2)$, $\sigma_1 = (2311) \in Surj_4$ and $\sigma_2 = (112) \in Surj_3$.

As a vector space, a basis of $\mathbf{WQSym}^*$ is given by the disjoint union of the sets $Surj_n$, for all $n \geq 0$. By convention, the unique element of $Surj_0$ is denoted by 1. The product of $\mathbf{WQSym}^*$ is given, for $\sigma \in Surj_k$ and $\tau \in Surj_l$, by:

$$\sigma.\tau = \sum_{\zeta \in Sh(k,l)} (\sigma \otimes \tau) \circ \zeta^{-1}.$$

In other words, as in the **FQSym** case, the product of σ and τ is given by shifting the letters of the word representing τ by k, and summing all the possible shuffings of this word and of the word representing σ. For example:

$$\begin{array}{rcl} (112)(21) & = & (11243) + (11423) + (14123) + (41123) + (11432) + (14132) \\ & & +(41132) + (14312) + (41312) + (43112) \end{array}$$

Let $\sigma \in Surj_n$. For all $0 \leq k \leq n$, there exists a unique triple $\left(\sigma_1^{(k)}, \sigma_2^{(k)}, \epsilon_k\right) \in Surj_k \times Surj_{n-k} \times SjSh(p_k, q_k)$ such that $\sigma = \epsilon_k \circ \left(\sigma_1^{(k)} \otimes \sigma_2^{(k)}\right)$. The coproduct of $\mathbf{WQSym}^*$ is then

given by:

$$\Delta_{\mathbf{WQSym}^*}(\sigma) = \sum_{k=0}^{n} \sigma_1^{(k)} \otimes \sigma_2^{(k)} = \sum_{k=0}^{n} \sum_{\substack{\sigma=\epsilon\circ(\sigma_1\otimes\sigma_2) \\ \epsilon\in SjSh(p_k,q_k),\sigma_1\in Surj_k,\sigma_2\in Surj_{n-k}}} \sigma_1 \otimes \sigma_2.$$

Note that $\sigma_1^{(k)}$ and $\sigma_2^{(k)}$ are obtained by cutting the word representing σ between the k-th and the $(k+1)$-th letter, and then packing the two obtained words. For example:

$$\begin{aligned}\Delta_{\mathbf{WQSym}^*}((21132)) &= 1\otimes(21132) + pack((2))\otimes pack((1132)) + pack((21))\otimes pack((132)) \\ &\quad + pack((211))\otimes pack((32)) + pack((2113))\otimes pack((2)) + (21132)\otimes 1 \\ &= 1\otimes(21132) + (1)\otimes(1132) + (21)\otimes(132) + (211)\otimes(21) \\ &\quad + (2113)\otimes(1) + (21132)\otimes 1.\end{aligned}$$

Then $\mathbf{WQSym}^*$ is a graded Hopf algebra, with $\mathbf{WQSym}^*(n) = Surj_n$ for all $n \geq 0$. We give some numerical values: if $f_n^{\mathbf{WQSym}^*}$ is the number of packed words of length n, then

n	0	1	2	3	4	5	6	7
$f_n^{\mathbf{WQSym}^*}$	1	1	3	13	75	541	4683	47293

These is the sequence A000670 in [Slo].

$\mathbf{WQSym}^*$ is the gradued dual of $\mathbf{WQSym}$, described as follows. A basis of $\mathbf{WQSym}$ is given by the disjoint union of the sets $Surj_n$. The product of $\mathbf{WQSym}$ is given, for $\sigma \in Surj_k$, $\tau \in Surj_l$ by:

$$\sigma.\tau = \sum_{\substack{\gamma=\gamma_1\dots\gamma_{k+l} \\ pack(\gamma_1\dots\gamma_k)=\sigma,\ pack(\gamma_{k+1}\dots\gamma_{k+l})=\tau}} \gamma$$

In other terms, the product of σ and τ is given by shifting certain letters of the words representing σ and τ and then summing all concatenations of obtained words. For example:

$$\begin{aligned}(112)(21) &= (11221) + (11321) + (22321) + (33421) + (11231) + (22331) + (22431) \\ &\quad + (11232) + (11332) + (11432) + (22341) + (11342) + (11243)\end{aligned}$$

If $\sigma \in Surj_n$, then the coproduct of $\mathbf{WQSym}$ is given by:

$$\Delta_{\mathbf{WQSym}}(\sigma) = \sum_{0\leq k\leq max(\sigma)} \sigma_{|[1,k]} \otimes pack(\sigma_{|[k+1,max(\sigma)]}),$$

where $\sigma_{|\mathcal{A}}$ is the subword obtained by tacking in σ the letters from the subset $\mathcal{A}$ of $[1, max(\sigma)]$. For example:

$$\begin{aligned}\Delta_{\mathbf{WQSym}}((21312245)) &= 1\otimes(21312245) + (11)\otimes pack((232245)) + (21122)\otimes pack((345)) \\ &\quad + (213122)\otimes pack((45)) + (2131224)\otimes pack((5)) + (21312245)\otimes 1 \\ &= 1\otimes(21312245) + (11)\otimes(121134) + (21122)\otimes(123) \\ &\quad + (213122)\otimes(12) + (2131224)\otimes(1) + (21312245)\otimes 1.\end{aligned}$$

Then $\mathbf{WQSym}$ is a Hopf algebra.

We give a decorated version of $\mathbf{WQSym}$. Let $\mathcal{D}$ be a nonempty set. A $\mathcal{D}$-decorated surjection is a couple (σ, d), where $\sigma \in Surj_n$ and d is a map from $\{1,\dots,n\}$ to $\mathcal{D}$. As in the $\mathbf{FQSym}^{\mathcal{D}}$ case, we represent a $\mathcal{D}$-decorated surjection by two superposed words $\binom{a_1\dots a_n}{v_1\dots v_n}$, where $(a_1\dots a_n)$ is

the packed word representing σ and for all i, $v_i = d(a_i)$. The vector space $\mathbf{WQSym}^{\mathcal{D}}$ generated by the set of $\mathcal{D}$-decorated surjections is a Hopf algebra. For example, if $x, y, z, t \in \mathcal{D}$:

$$\begin{array}{rcl} \binom{211}{yxz}.\binom{1}{t} & = & \binom{2111}{yxzt} + \binom{2112}{yxzt} + \binom{2113}{yxzt} + \binom{3221}{yxzt} + \binom{3112}{yxzt}. \\ \Delta_{\mathbf{WQSym}^{\mathcal{D}}}\left(\binom{2113}{yxzt}\right) & = & \binom{2113}{yxzt} \otimes 1 + \binom{11}{xz} \otimes \binom{12}{yt} + \binom{211}{yxz} \otimes \binom{1}{t} + 1 \otimes \binom{2113}{yxzt}. \end{array}$$

In other words, if (σ, d) and (τ, d') are decorated surjections of respective degrees k and l:

$$(\sigma, d).(\tau, d') = \sum_{\substack{\gamma = \gamma_1 \dots \gamma_{k+l} \\ pack(\gamma_1 \dots \gamma_k) = \sigma,\ pack(\gamma_{k+1} \dots \gamma_{k+l}) = \tau}} (\gamma, d \otimes d'),$$

where $d \otimes d'$ is defined by $(d \otimes d')(i) = d(i)$ if $1 \leq i \leq k$ and $(d \otimes d')(k+j) = d'(j)$ if $1 \leq j \leq l$. If (σ, d) is a decorated surjection of degree n:

$$\Delta_{\mathbf{WQSym}^{\mathcal{D}}}((\sigma, d)) = \sum_{0 \leq k \leq \max(\sigma)} (\sigma_{|[1,k]}, d') \otimes (\sigma_{|[k+1, \max(\sigma)]}, d'').$$

where d' and d take the same values on $\sigma^{-1}(\{1, \dots, k\})$ and d'' and d take the same values on $\sigma^{-1}(\{k+1, \dots, \max(\sigma)\})$.

In some sense, a $\mathcal{D}$-decorated surjection can be seen as a packed word with a preorder on the set of its letters.

Suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot, \cdot] : (a, b) \in \mathcal{D}^2 \to [ab] \in \mathcal{D}$. We define by induction $[\cdot, \cdot]^{(k)}$:

$$[\cdot, \cdot]^{(0)} = Id,\ [\cdot, \cdot]^{(1)} = [\cdot, \cdot] \text{ and } [\cdot, \cdot]^{(k)} = \left[\cdot, [\cdot, \cdot]^{(k-1)}\right].$$

We can now define the quasi-shuffle Hopf algebra $\mathbf{Csh}^{\mathcal{D}}$ (see [Hof00]). $\mathbf{Csh}^{\mathcal{D}}$ is, as a vector space, generated by the set of $\mathcal{D}$-words.

Let φ be the surjective linear map from $\mathbf{WQSym}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$ defined, for (σ, d) a decorated surjection of maximum k, by $\varphi((\sigma, d)) = (w_1 \dots w_k)$ where $w_j = [d(i_1) \dots d(i_p)]^{(p)}$ with $\sigma^{-1}(j) = \{i_1, \dots, i_p\}$. For instance,

$$\varphi\left(\binom{2114324}{yxztvwu}\right) = ([xz]\ [yw]\ v\ [tu])$$

Notations. Let k, l be integers. A (k, l)-quasi-shuffle of type r is a surjective map $\zeta : \{1, \dots, k+l\} \twoheadrightarrow \{1, \dots, k+l-r\}$ such that

$$\begin{cases} \zeta(1) < \dots < \zeta(k), \\ \zeta(k+1) < \dots < \zeta(k+l). \end{cases}$$

Remark that $\zeta^{-1}(j)$ contains 1 or 2 elements for all $1 \leq j \leq k+l-r$. The set of (k, l)-quasi-shuffles of type r is denoted by $Csh(p, q, r)$. Remark that $Csh(k, l, 0) = Sh(k, l)$.

φ define a Hopf algebra structure on $\mathbf{Csh}^{\mathcal{D}}$:

- The product ⧢ of $\mathbf{Csh}^{\mathcal{D}}$ is given in the following way: if $(v_1 \dots v_k)$ is a $\mathcal{D}$-word of degree k, $(v_{k+1} \dots v_{k+l})$ is a $\mathcal{D}$-word of degree l, then

$$(v_1 \dots v_k) \text{⧢} (v_{k+1} \dots v_{k+l}) = \sum_{r \geq 0} \sum_{\zeta \in Csh(k,l,r)} (w_1 \dots w_{k+l-r}),$$

  where $w_j = v_i$ if $\zeta^{-1}(j) = \{i\}$ and $w_j = [v_{i_1} v_{i_2}]$ if $\zeta^{-1}(j) = \{i_1, i_2\}$.

- The coproduct $\Delta_{\mathbf{Csh}^{\mathcal{D}}}$ of $\mathbf{Csh}^{\mathcal{D}}$ is given on any $\mathcal{D}$-word $v = (v_1 \dots v_n)$ by

$$\Delta_{\mathbf{Csh}^{\mathcal{D}}}(v) = \sum_{i=0}^{n} (v_1 \dots v_i) \otimes (v_{i+1} \dots v_n).$$

  This is the same coproduct as for $\mathbf{Sh}^{\mathcal{D}}$.

Example. If $(v_1 v_2)$ and $(v_3 v_4)$ are two $\mathcal{D}$-words,

$$\begin{aligned}(v_1v_2)\text{⧢}(v_3v_4) \;=\; & (v_1v_2v_3v_4) + (v_1v_3v_2v_4) + (v_3v_1v_2v_4) + (v_1v_3v_4v_2) \\ & +(v_3v_1v_4v_2) + (v_3v_4v_1v_2) + (v_1\,[v_2v_3]\,v_4) + ([v_1v_3]\,v_2v_4) \\ & +(v_1v_3\,[v_2v_4]) + (v_3\,[v_1v_4]\,v_2) + ([v_1v_3]\,[v_2v_4])\end{aligned}$$

3 Preordered forests

3.1 Preordered and heap-preordered forests

A preorder is a binary, reflexive and transitive relation. In particular, an antisymmetric preorder is an order. A preorder is total if two elements are always comparable. We introduce another version of ordered forests, the preordered forests.

Definition 5 *A preordered forest F is a rooted forest F with a total preorder on $V(F)$. The set of preordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{po}}$ and the $\mathbb{K}$-vector space generated by $\mathbb{F}_{\mathbf{H}_{po}}$ is denoted by $\mathbf{H}_{po}$.*

Remarks and notations.

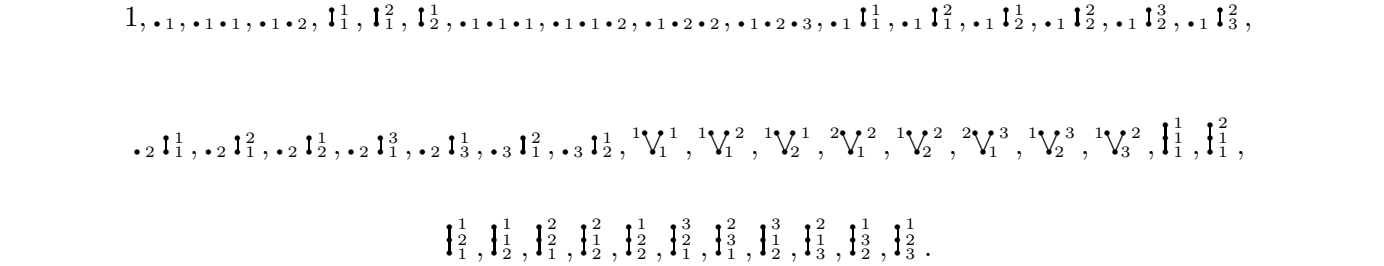
1. Let F be a preordered forest. We denote by $\leq$ the total preorder on $V(F)$. Remark that the antisymmetric relation "$x \leq y$ and $y \leq x$" is an equivalence relation denoted by $\mathcal{R}$ and the quotient set $V(F)/\mathcal{R}$ is totally ordered. We denote by q the cardinality of this quotient set. Let $\overline{\sigma}$ be the unique increasing map from $V(F)/\mathcal{R}$ to $\{1, \dots, q\}$. There exists a unique surjection $\sigma : V(F) \to \{1, \dots, q\}$, compatible with the equivalence $\mathcal{R}$, such that the induced map on $V(F)/\mathcal{R}$ is $\overline{\sigma}$. In the sequel, we shall note $q = \max(F)$ (and we have always $q \leq |F|_v$).

   Reciprocally, if F is a rooted forest and if $\sigma : V(F) \to \{1, \dots, q\}$ is a surjection, $q \leq |F|_v$, then σ define a total preorder on $V(F)$ and F is a preordered forest.

   As in the ordered case, we shall denote a preordered forest by F or (F, σ).

2. An ordered forest is also a preordered forest. Conversely, a preordered forest (F, σ) is an ordered forest if $|F|_v = \max(F)$.

Examples. Preordered forests of vertices degree ≤ 3:



Let (F, σ^F) and (G, σ^G) be two preordered forests with $\sigma^F : V(F) \to \{1, \dots, q\}$, $\sigma^G : V(G) \to \{1, \dots, r\}$, $q = max(F)$ and $r = max(G)$. Then FG is also a preordered forest (FG, σ^{FG}) where

$$\sigma^{FG} = \sigma^F \otimes \sigma^G : \begin{cases} V(F) \bigcup V(G) & \to & \{1, \dots, q+r\} \\ a \in V(F) & \mapsto & \sigma^F(a) \\ a \in V(G) & \mapsto & \sigma^G(a) + q. \end{cases} \tag{5}$$

In other terms, we keep the initial preorder on the vertices of F and G and we assume that the vertices of F are smaller than the vertices of G. In this way, we define a noncommutative product on the set of preordered forests. For example, the product of $\mathfrak{i}_1^3 \bullet_2$ and ${}^1\mathrm{V}_1^2$ gives $\mathfrak{i}_1^3 \bullet_2 {}^4\mathrm{V}_4^5$, whereas the product of ${}^1\mathrm{V}_1^2$ and $\mathfrak{i}_1^3 \bullet_2$ gives ${}^1\mathrm{V}_1^2 \, \mathfrak{i}_3^5 \bullet_4$. Remark that, if F and G are two preordered forests, $\max(FG) = \max(F) + \max(G)$. This product is linearly extended to $\mathbf{H}_{po}$, which in this way becomes an algebra, gradued by the number of vertices.

Remark. The formula (5) on the preordered forests extends the formula (2) on the ordered forests.

We give some numerical values: if $f_n^{\mathbf{H}_{po}}$ is the number of preordered forests of vertices degree n,

n	0	1	2	3	4	5
$f_n^{\mathbf{H}_{po}}$	1	1	5	38	424	6284

If F is a preordered forest, then any subforest G of F is also preordered: the total preorder on $V(G)$ is the restriction of the total preorder of $V(F)$. So we can define a coproduct $\Delta_{\mathbf{H}_{po}} : \mathbf{H}_{po} \to \mathbf{H}_{po} \otimes \mathbf{H}_{po}$ on $\mathbf{H}_{po}$ in the following way: for all $F \in \mathbb{F}_{\mathbf{H}_{po}}$,

$$\Delta_{\mathbf{H}_{po}}(F) = \sum_{\boldsymbol{v} \models V(F)} Lea_{\boldsymbol{v}}(F) \otimes Roo_{\boldsymbol{v}}(F).$$

For example,

$$\Delta_{\mathbf{H}_{po}}({}^{\overset{1}{\mid}}_{2}\mathrm{V}_1^3) = {}^{\overset{1}{\mid}}_{2}\mathrm{V}_1^3 \otimes 1 + 1 \otimes {}^{\overset{1}{\mid}}_{2}\mathrm{V}_1^3 + \bullet_1 \otimes {}^2\mathrm{V}_1^3 + \mathfrak{i}_2^1 \otimes \mathfrak{i}_1^2 + \bullet_1 \otimes \mathfrak{i}_2^1 + \bullet_1 \bullet_2 \otimes \mathfrak{i}_1^2 + \mathfrak{i}_2^1 \bullet_3 \otimes \bullet_1.$$

With this coproduct, $\mathbf{H}_{po}$ is a Hopf algebra. Remark that $\mathbf{H}_o$ is a Hopf subalgebra of $\mathbf{H}_{po}$.

Definition 6 *A heap-preordered forest is a preordered forest F such that if $a, b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$, then a is strictly greater than b for the total preorder on $V(F)$. The set of heap-preordered forests is denoted by $\mathbb{F}_{\mathbf{H}_{hpo}}$*

Examples. Heap-preordered forests of vertices degree ≤ 3:

$$1, \bullet_1, \bullet_1 \bullet_1, \bullet_1 \bullet_2, \mathfrak{i}_1^2, \bullet_1 \bullet_1 \bullet_1, \bullet_1 \bullet_1 \bullet_2, \bullet_1 \bullet_2 \bullet_2, \bullet_1 \bullet_2 \bullet_3, \bullet_1 \mathfrak{i}_1^2, \bullet_1 \mathfrak{i}_2^3, \bullet_2 \mathfrak{i}_1^2, \bullet_2 \mathfrak{i}_1^3, \bullet_3 \mathfrak{i}_1^2, {}^2\mathrm{V}_1^2, {}^2\mathrm{V}_1^3, {}^{\overset{3}{\mid}}_{2}\mathrm{V}_1^3.$$

Definition 7 *Let F be a nonempty rooted forest and q an integer $\leq |F|_v$. A linear preorder is a surjection $\sigma : V(F) \to \{1, \dots, q\}$ such that if $a, b \in V(F)$, $a \neq b$ and $a \twoheadrightarrow b$ then $\sigma(a) > \sigma(b)$. We denote by $\mathcal{O}_p(F)$ the set of linear preorders on the nonempty rooted forest F.*

Remarks. If (F, σ) is a heap-preordered forest, the surjection $\sigma : V(F) \to \{1, \dots, \max(F)\}$ is a linear preorder on F. Reciprocally, if F is a rooted forest and $\sigma \in \mathcal{O}_p(F)$, then σ define a total preorder on $V(F)$ such that (F, σ) is a heap-preordered forest.

If F and G are two heap-preordered forests, then FG is also heap-preordered. Moreover, any subforest G of a heap-preordered forest F is also a heap-preordered forest by restriction on $V(G)$ of the total preorder of $V(F)$. So the subspace $\mathbf{H}_{hpo}$ of $\mathbf{H}_{po}$ generated by the heap-preordered forests is a graded Hopf subalgebra of $\mathbf{H}_{po}$.

We give some numerical values: if $f_n^{\mathbf{H}_{hpo}}$ is the number of preordered forests of vertices degree n,

n	0	1	2	3	4	5
$f_n^{\mathbf{H}_{hpo}}$	1	1	3	12	64	428

We have the following diagram

$$\begin{array}{ccccc} \mathbf{H}_{NCK} & \hookrightarrow & \mathbf{H}_{ho} & \hookrightarrow & \mathbf{H}_{o} \\ & & \downarrow & & \downarrow \\ & & \mathbf{H}_{hpo} & \hookrightarrow & \mathbf{H}_{po} \end{array} \tag{6}$$

where the arrows $\hookrightarrow$ are injective morphisms of Hopf algebras (for the cut coproduct).

3.2 A morphism from $\mathbf{H}_{po}$ to WQSym*

In this section, we give a similar result of proposition 4 in the preordered case.

Definition 8 *Let (F, σ) be a nonempty preordered forest of vertices degree n and $\tau \in Surj_n$. Then we denote by $\mathbb{S}^{\tau}_{(F,\sigma)}$ the set of bijective maps $\varphi : V(F) \to \{1, \dots, n\}$ such that:*

1. *if $v \in V(F)$, then $\sigma(v) = \tau(\varphi(v))$,*
2. *if $v, v' \in V(F)$, $v' \twoheadrightarrow v$, then $\varphi(v) \geq \varphi(v')$.*

Remark.

1. If $\max(F) \neq \max(\tau)$, then $\mathbb{S}^{\tau}_{(F,\sigma)} = \emptyset$.
2. Let $F, G \in \mathbb{F}_{\mathbf{H}_{po}}$, $|F|_v = k$, $|G|_v = l$. If $\varphi_1 : V(F) \to \{1, \dots, k\}$ and $\varphi_2 : V(G) \to \{1, \dots, l\}$ are two bijective maps and $\zeta \in Sh(k, l)$, then $\zeta \circ (\varphi_1 \otimes \varphi_2) : V(FG) \to \{1, \dots, k+l\}$, where $\varphi_1 \otimes \varphi_2$ is defined in formula (2), is also a bijective map. Similary, considering a bijective map $\varphi : V(FG) \to \{1, \dots, k+l\}$ and $\zeta \in Sh(k, l)$. Then φ can be uniquely written as $\zeta \circ (\varphi_1 \otimes \varphi_2)$, where $\varphi_1 : V(F) \to \{1, \dots, k\}$ and $\varphi_2 : V(G) \to \{1, \dots, l\}$ are two bijective maps.

Theorem 9 *Let us define:*

$$\Phi : \left\{ \begin{array}{rcl} \mathbf{H}_{po} & \to & \mathbf{WQSym}^* \\ (F, \sigma) \in \mathbb{F}_{\mathbf{H}_{po}} & \mapsto & \displaystyle\sum_{\tau \in Surj_{|F|_v}} \mathrm{card}(\mathbb{S}^{\tau}_{(F,\sigma)})\ \tau. \end{array} \right. \tag{7}$$

Then $\Phi : \mathbf{H}_{po} \to \mathbf{WQSym}^*$ is a Hopf algebra morphism, homogeneous of degree 0.

Examples.

- In vertices degree 1: $\Phi(\bullet_1) = (1)$.
- In vertices degree 2:

$$\Phi(\bullet_1\bullet_1) = 2(11), \quad \Phi(\bullet_1\bullet_2) = (12)+(21), \quad \Phi(\overset{a}{\underset{b}{\bullet}}) = (ab).$$

- In vertices degree 3:

$$\begin{array}{rclrcl}
\Phi(\bullet_1\bullet_1\bullet_1) &=& 6(111) & \Phi(\overset{2}{\underset{1}{\bullet}}\bullet_2) &=& (212)+2(221)\\
\Phi(\overset{2}{\vee}{}^{1}_{2}) &=& (122)+(212) & \Phi(\bullet_2\overset{1}{\underset{3}{\bullet}}) &=& (213)+(123)+(132)\\
\Phi(\overset{\bullet}{\underset{1}{\overset{2}{\bullet}}}\overset{}{\underset{3}{\bullet}}) &=& (231) & \Phi(\bullet_1\overset{2}{\underset{3}{\bullet}}) &=& (123)+(213)+(231)\\
\Phi(\overset{2}{\vee}{}^{2}_{1}) &=& 2(221) & \Phi(\bullet_1\bullet_1\bullet_2) &=& 2\left[(112)+(121)+(211)\right]
\end{array}$$

Proof. Obviously, Φ is homogeneous of degree 0. Let $(F,\sigma^F),(G,\sigma^G) \in \mathbb{F}_{\mathbf{H}_{po}}$, $|F|_v = k$, $|G|_v = l$ and $\tau \in Surj_{k+l}$. τ can be uniquely written as $\tau = (\tau_1 \otimes \tau_2)\circ\zeta^{-1}$ with $\tau_1 \in Surj_k$, $\tau_2 \in Surj_l$ and $\zeta \in Sh(k,l)$.

Let $\varphi \in \mathbb{S}^{\tau}_{(FG,\sigma^{FG})}$. Then φ can be uniquely written as $\zeta\circ(\varphi_1\otimes\varphi_2)$ with $\varphi_1 : V(F) \to \{1,\dots,k\}$ and $\varphi_2 : V(G) \to \{1,\dots,l\}$ two bijective maps.

1. (a) If $v \in V(F)$, then

$$\sigma^F(v) = \sigma^{FG}(v) = \tau(\varphi(v)) = (\tau_1\otimes\tau_2)\circ\zeta^{-1}\circ\zeta\circ(\varphi_1\otimes\varphi_2)(v) = \tau_1(\varphi_1(v)).$$

   Note that with this equality, we also have that $\max(F) = \max(\tau_1)$.

   (b) If $v \in V(G)$, then

$$\begin{array}{rcl}
\sigma^G(v)+\max(F) &=& \sigma^{FG}(v) = \tau(\varphi(v)) = (\tau_1\otimes\tau_2)\circ\zeta^{-1}\circ\zeta\circ(\varphi_1\otimes\varphi_2)(v)\\
&=& \tau_2(\varphi_2(v))+\max(\tau_1).
\end{array}$$

   As $\max(F) = \max(\tau_1)$, $\sigma^G(v) = \tau_2(\varphi_2(v))$.

2. (a) If $v' \twoheadrightarrow v$ in F, then $v' \twoheadrightarrow v$ in FG, so:

$$\begin{array}{rcl}
\varphi(v) &\geq& \varphi(v')\\
\zeta\circ(\varphi_1\otimes\varphi_2)(v) &\geq& \zeta\circ(\varphi_1\otimes\varphi_2)(v')\\
\zeta\left(\varphi_1(v)\right) &\geq& \zeta\left(\varphi_1(v')\right)\\
\varphi_1(v) &\geq& \varphi_1(v'),
\end{array}$$

   as ζ is increasing on $\{1,\dots,k\}$.

   (b) If $v' \twoheadrightarrow v$ in G, then $v' \twoheadrightarrow v$ in FG, so:

$$\begin{array}{rcl}
\varphi(v) &\geq& \varphi(v')\\
\zeta\circ(\varphi_1\otimes\varphi_2)(v) &\geq& \zeta\circ(\varphi_1\otimes\varphi_2)(v')\\
\zeta\left(k+\varphi_2(v)\right) &\geq& \zeta\left(k+\varphi_2(v')\right)\\
\varphi_2(v) &\geq& \varphi_2(v'),
\end{array}$$

   as ζ is increasing on $\{k+1,\dots,k+l\}$.

So $\varphi_1 \in \mathbb{S}^{\tau_1}_{(F,\sigma^F)}$ and $\varphi_2 \in \mathbb{S}^{\tau_2}_{(G,\sigma^G)}$.

Conversely, if $\varphi = \zeta \circ (\varphi_1 \otimes \varphi_2)$, with $\varphi_1 \in \mathbb{S}^{\tau_1}_{(F,\sigma^F)}$ and $\varphi_2 \in \mathbb{S}^{\tau_2}_{(G,\sigma^G)}$, the same computations shows that $\varphi \in \mathbb{S}^{(\tau_1\otimes\tau_2)\circ\zeta^{-1}}_{(FG,\sigma^{FG})}$.

So

$$\mathrm{card}(\mathbb{S}^{\tau}_{(FG,\sigma^{FG})}) = \mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)}) \times \mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})$$

and

$$\begin{aligned}
\Phi((FG,\sigma^{FG})) &= \sum_{\tau\in Surj_{k+l}} \mathrm{card}(\mathbb{S}^{\tau}_{(FG,\sigma^{FG})})\ \tau \\
&= \sum_{\zeta\in Sh(k,l)} \sum_{\tau_1\in Surj_k} \sum_{\tau_2\in Surj_l} \mathrm{card}(\mathbb{S}^{(\tau_1\otimes\tau_2)\circ\zeta^{-1}}_{(FG,\sigma^{FG})})\ (\tau_1\otimes\tau_2)\circ\zeta^{-1} \\
&= \sum_{\zeta\in Sh(k,l)} \sum_{\tau_1\in Surj_k} \sum_{\tau_2\in Surj_l} \mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)}) \times \mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})\ (\tau_1\otimes\tau_2)\circ\zeta^{-1} \\
&= \left(\sum_{\tau_1\in Surj_k} \mathrm{card}(\mathbb{S}^{\tau_1}_{(F,\sigma^F)})\ \tau_1\right)\left(\sum_{\tau_2\in Surj_l} \mathrm{card}(\mathbb{S}^{\tau_2}_{(G,\sigma^G)})\ \tau_2\right) \\
&= \Phi((F,\sigma^F))\Phi((G,\sigma^G)).
\end{aligned}$$

So Φ is an algebra morphism.

Let $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_{po}}$ be a preordered forest such that $|F|_v = n$ and let $\boldsymbol{v}$ be an admissible cut of F. We obtain two preordered forests $(Lea_{\boldsymbol{v}}(F),\sigma_1)$ and $(Roo_{\boldsymbol{v}}(F),\sigma_2)$. We set $k = |Lea_{\boldsymbol{v}}(F)|_v$ and $l = |Roo_{\boldsymbol{v}}(F)|_v$.

Let $\tau_1 \in Surj_k$, $\tau_2 \in Surj_l$ and $\varphi_1 \in \mathbb{S}^{\tau_1}_{(Lea_{\boldsymbol{v}}(F),\sigma_1)}$, $\varphi_2 \in \mathbb{S}^{\tau_2}_{(Roo_{\boldsymbol{v}}(F),\sigma_2)}$. We set $\varphi = \varphi_1 \otimes \varphi_2$ and we define τ by $\tau = \sigma \circ \varphi^{-1}$. $\tau \in Surj_n$ and $\max(\tau) = \max(F)$. Let us show that $\varphi \in \mathbb{S}^{\tau}_{(F,\sigma)}$.

1. By definition, $\tau = \sigma \circ \varphi^{-1}$. So $\sigma(v) = \tau(\varphi(v))$ for all $v \in V(F)$.
2. If $v' \twoheadrightarrow v$ in F, then three cases are possible:
   (a) v and v' belong to $V(Lea_{\boldsymbol{v}}(F))$. As $\varphi_1 \in \mathbb{S}^{\tau_1}_{(Lea_{\boldsymbol{v}}(F),\sigma_1)}$, $\varphi_1(v) \geq \varphi_1(v')$. Then $\varphi(v) = (\varphi_1\otimes\varphi_2)(v) = \varphi_1(v) \geq \varphi_1(v') = (\varphi_1\otimes\varphi_2)(v') = \varphi(v')$.
   (b) v and v' belong to $V(Roo_{\boldsymbol{v}}(F))$. As $\varphi_2 \in \mathbb{S}^{\tau_2}_{(Lea_{\boldsymbol{v}}(F),\sigma_2)}$, $\varphi_2(v) \geq \varphi_2(v')$. Then $\varphi(v) = (\varphi_1\otimes\varphi_2)(v) = \varphi_2(v) + k \geq \varphi_2(v') + k = (\varphi_1\otimes\varphi_2)(v') = \varphi(v')$.
   (c) v' belong to $V(Lea_{\boldsymbol{v}}(F))$ and v belong to $V(Roo_{\boldsymbol{v}}(F))$. Then $\varphi(v') = (\varphi_1\otimes\varphi_2)(v') = \varphi_1(v') \in \{1,\dots,k\}$ and $\varphi(v) = (\varphi_1\otimes\varphi_2)(v) = \varphi_2(v) + k \in \{k+1,\dots,k+l\}$. So $\varphi(v) > \varphi(v')$.

   In any case, $\varphi(v) \geq \varphi(v')$.

Conversely, let $(F,\sigma) \in \mathbb{F}_{\mathbf{H}_{po}}$ be a preordered forest of vertices degree n, $\tau \in Surj_n$ and $\varphi \in \mathbb{S}^{\tau}_{(F,\sigma)}$. Let $k \in \{0,\dots,n\}$ be an integer.

We set $\tau_1^{(k)}$ and $\tau_2^{(k)}$ the words obtained by cutting the word representing τ between the k-th and the $(k+1)$-th letter, and then packing the two obtained words.

Moreover, we define $\boldsymbol{v}$ a subset of $\varphi^{-1}(\{1,\dots,k\})$ such that $v \not\twoheadrightarrow w$ for any couple (v,w) of two different elements of $\boldsymbol{v}$. Then $\boldsymbol{v} \models V(F)$ and we considere the two preordered forests $(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})$ and $(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})$. Remark that, with the second point of definition 8, $V(Lea_{\boldsymbol{v}}(F)) = \varphi^{-1}(\{1,\dots,k\})$ and $V(Roo_{\boldsymbol{v}}(F)) = \varphi^{-1}(\{k+1,\dots,n\})$.

We set $\varphi_1^{(k)} : v \in V(Lea_{\boldsymbol{v}}(F)) \to \varphi(v) \in \{1,\dots,k\}$ and $\varphi_2^{(k)} : v \in V(Roo_{\boldsymbol{v}}(F)) \to \varphi(v) - k \in \{1,\dots,n-k\}$. Thus $\varphi = \varphi_1^{(k)} \otimes \varphi_2^{(k)}$.

Let us prove that $\varphi_1^{(k)} \in \mathbb{S}^{\tau_1^{(k)}}_{(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})}$ and $\varphi_2^{(k)} \in \mathbb{S}^{\tau_2^{(k)}}_{(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})}$.

1. (a) If $v \in V(Lea_{\boldsymbol{v}}(F))$, $\varphi(v) = \varphi_1^{(k)}(v) \in \{1,\dots,k\}$ and then
$$\sigma_1^{(k)}(v) = pack \circ \sigma(v) = pack \circ \tau \circ \varphi(v) = \tau_1^{(k)} \circ \varphi_1^{(k)}(v).$$
(b) If $v \in V(Roo_{\boldsymbol{v}}(F))$, $\varphi(v) = \varphi_2^{(k)}(v) + k \in \{k+1,\dots,n\}$ and then
$$\sigma_2^{(k)}(v) = pack \circ \sigma(v) = pack \circ \tau \circ \varphi(v) = \tau_2^{(k)} \circ \varphi_2^{(k)}(v).$$

2. (a) If $v' \twoheadrightarrow v$ in $Lea_{\boldsymbol{v}}(F)$, then $v' \twoheadrightarrow v$ in F and $\varphi_1^{(k)}(v) = \varphi(v) \geq \varphi(v') = \varphi_2^{(k)}(v')$.
(b) If $v' \twoheadrightarrow v$ in $Roo_{\boldsymbol{v}}(F)$, then $v' \twoheadrightarrow v$ in F and $\varphi_2^{(k)}(v) = \varphi(v) - k \geq \varphi(v') - k = \varphi_2^{(k)}(v')$.

Hence, there is a bijection:
$$\begin{cases} \mathbb{S}^{\tau}_{(F,\sigma)} \times \{0,\dots,|F|_v\} & \to & \displaystyle\bigsqcup_{\boldsymbol{v} \models V(F)} \mathbb{S}^{\tau_1^{(k)}}_{(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})} \times \mathbb{S}^{\tau_2^{(k)}}_{(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})} \\ (\varphi,k) & \mapsto & \left(\varphi_1^{(k)}, \varphi_2^{(k)}\right). \end{cases}$$

Finally,
$$\begin{aligned} & \Delta_{\mathbf{WQSym}^*} \circ \Phi((F,\sigma)) \\ = & \sum_{\tau \in Surj_{|F|_v}} \sum_{0 \leq k \leq n} \mathrm{card}(\mathbb{S}^{\tau}_{(F,\sigma)})\ \tau_1^{(k)} \otimes \tau_2^{(k)} \\ = & \sum_{\boldsymbol{v} \models V(F)} \sum_{\tau_1 \in Surj_{|Lea_{\boldsymbol{v}}(F)|_v}} \sum_{\tau_2 \in Surj_{|Roo_{\boldsymbol{v}}(F)|_v}} \mathrm{card}(\mathbb{S}^{\tau_1}_{(Lea_{\boldsymbol{v}}(F),\sigma_1^{(k)})})\ \tau_1 \otimes \mathrm{card}(\mathbb{S}^{\tau_2}_{(Roo_{\boldsymbol{v}}(F),\sigma_2^{(k)})})\ \tau_2 \\ = & (\Phi \otimes \Phi) \circ \Delta_{\mathbf{H}_{po}}. \end{aligned}$$

So Φ is a coalgebra morphism. □

Theorem 10 *The restriction of Φ defined in formula (7) to $\mathbf{H}_{hpo}$ is an injection of graded Hopf algebras.*

Proof. We introduce a lexicographic order on the words with letters $\in \mathbb{N}^*$. Let $u = (u_1 \dots u_k)$ and $v = (v_1 \dots v_l)$ be two words. Then

- if $u_k = v_k, u_{k-1} = v_{k-1}, \dots, u_{i+1} = v_{i+1}$ and $u_i > v_i$ (resp. $u_i < v_i$) with $i \in \{1,\dots,\min(k,l)\}$, then $u > v$ (resp. $u < v$),
- if $u_i = v_i$ for all $i \in \{1,\dots,\min(k,l)\}$ and if $k > l$ (resp. $k < l$) then $u > v$ (resp. $u < v$).

For example,
$$(541) < (22), \quad (433) < (533), \quad (5362) < (72), \quad (8225) < (1327), \quad (215) < (1215).$$

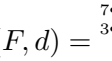
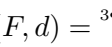
If u and v are two words, we denote by uv the concatenation of u and v.

In this proof, if (F,σ) is a preordered forest, we consider F as a decorated forest where the vertices are decorated by integers. Consider
$$\mathbb{F} = \{(F,d) \ \mid \ F \in \mathbb{F}_{\mathbf{H}_{CK}}, d : V(F) \to \mathbb{N}^* \text{ such that if } v \to w \text{ then } d(v) > d(w)\}$$
the set of forests with their vertices decorated by nonzero integers and with an increasing condition.

Let $(F,d) \in \mathbb{F}$ be a forest of vertices degree n and if $u = (u_1 \dots u_n)$ is a word of length n with $u_i \in \mathbb{N}^*$. In the same way that definition 8, we define $\mathbb{S}^{u}_{(F,d)}$ as the set of bijective maps $\varphi : V(F) \to \{1,\dots,n\}$ such that:

1. if $v \in V(F)$, then $d(v) = u_{\varphi(v)}$,
2. if $v, v' \in V(F)$, $v' \twoheadrightarrow v$, then $\varphi(v) \geq \varphi(v')$.

For example,

- if $(F,d) = $  $\in \mathbb{F}$, then the words u such that $\mathbb{S}^u_{(F,d)} \neq \emptyset$ are $(7342), (7432), (4732)$.
- if $(F,d) = $  $\in \mathbb{F}$, then the words u such that $\mathbb{S}^u_{(F,d)} \neq \emptyset$ are

$$(43163), (43613), (46313), (64313), (43631), (46331), (64331), (63431),$$
$$(34163), (34613), (36413), (63413), (34631), (36431), (36341), (63341).$$

Let (F,d) be a forest of $\mathbb{F}$. Then we set

$$m((F,d)) = \max\left(\{u \mid \mathbb{S}^u_{(F,d)} \neq \emptyset\}\right).$$

For example, for $(F,d) = $ $\in \mathbb{F}$, $m((F,d)) = (7342)$ and for $(F,d) = $ $\in \mathbb{F}$, $m((F,d)) = (34163)$.

If $(F,d) \in \mathbb{F}$ is the empty tree, $m((F,\sigma)) = 1$. Let $(F,d) \in \mathbb{F}$ be a nonempty tree of vertices degree n. We denote by (G,d') the forest of $\mathbb{F}$ obtained by deleting the root of F. Then, if $m((F,d)) = (u_1 \dots u_n)$, we have $m((G,d')) = (u_1 \dots u_{n-1})$ and $u_n = d(R_F)$ the decoration of the root of F. Let (F,d) be a forest of vertices degree n, (F,d) is the disjoint union of trees with their vertices decorated by nonzero integers $(F_1,d_1), \dots, (F_k,d_k)$ ordered such that $m((F_1,d_1)) \leq \dots \leq m((F_k,d_k))$. Then $m((F,d)) = m((F_1,d_1)) \dots m((F_k,d_k))$:

- By definition, $\mathbb{S}^{m((F_i,d_i))}_{(F_i,d_i)} \neq \emptyset$ and if $\varphi_i \in \mathbb{S}^{m((F_i,d_i))}_{(F_i,d_i)}$ then $\varphi : V(F) \to \{1, \dots, n\}$, defined for all $1 \leq i \leq k$ and $v \in V(F_i)$ by $\varphi(v) = \varphi_i(v)$, is an element of $\mathbb{S}^{m((F_1,d_1)) \dots m((F_k,d_k))}_{(F,d)}$ and $\mathbb{S}^{m((F_1,d_1)) \dots m((F_k,d_k))}_{(F,d)} \neq \emptyset$. So $m((F,d)) \geq m((F_1,d_1)) \dots m((F_k,d_k))$.
- If $\mathbb{S}^u_{(F,d)} \neq \emptyset$, u is the shuffle of $u_1, \dots, u_k$ such that $\mathbb{S}^{u_i}_{(F_i,d_i)} \neq \emptyset$ (see the proof of theorem 9). In particular, $u_i \leq m((F_i,d_i))$, so $u \leq m((F_1,d_1)) \dots m((F_k,d_k))$ and $m((F,d)) \leq m((F_1,d_1)) \dots m((F_k,d_k))$.

Let $(F,d) \in \mathbb{F}$ be a forest of vertices degree n and $m((F,d)) = (u_1 \dots u_n)$. Let i_1 be the smallest index such that $u_1, \dots, u_{i_1 - 1} > u_{i_1}$ and, for all $j > i_1$, $u_{i_1} \leq u_j$. By construction, there exists a connected component (F_1,d_1) of (F,d) such that $m((F_1,d_1)) = (u_1 \dots u_{i_1})$. Consider the word $(u_{i_1+1} \dots u_n)$. Let $i_2 > i_1$ be the smallest index such that $u_{i_1+1}, \dots, u_{i_2-1} > u_{i_2}$ and, for all $j > i_2$, $u_{i_2} \leq u_j$. Then there exists a connected component (F_2,d_2) (different from (F_1,d_1)) such that $m((F_2,d_2)) = (u_{i_1+1} \dots u_{i_2})$. In the same way, we construct $i_3, \dots, i_k$ and $(F_3,d_3), \dots, (F_k,d_k)$. Then we have $m((F,d)) = m((F_1,d_1)) \dots m((F_k,d_k))$

Let us prove that m is injective on $\mathbb{F}$ by induction on the vertices degree. If (F,d) is the empty tree, it is obvious. Let (F,d) be a nonempty forest of $\mathbb{F}$ of vertices degree n.

- If (F,d) is a tree, $m((F,d)) = (u_1 \dots u_{n-1} u_n)$ with $u_n = d(R_F)$ the decoration of the root of F. Let (G,d') be the forest of $\mathbb{F}$ obtained by deleting the root of F. Then $m((G,d')) = (u_1 \dots u_{n-1})$. By induction hypothesis, (G,d') is the unique forest of $\mathbb{F}$ such that $m((G,d')) = (u_1 \dots u_{n-1})$. So (F,d) is also the unique forest of $\mathbb{F}$ such that $m((F,\sigma)) = (u_1 \dots u_{n-1} d(R_F))$.

- If (F,d) is not a tree, then (F,d) is the product of trees $(F_1,d_1),\dots,(F_k,d_k)$ of $\mathbb{F}$ ordered such that $m((F_1,d_1)) \leq \dots \leq m((F_k,d_k))$. So $m((F,d)) = m((F_1,d_1))\dots m((F_k,d_k))$. By the induction hypothesis, for all $1 \leq i \leq k$, (F_i,d_i) is the unique tree of $\mathbb{F}$ such that its image by m is $m((F_i,d_i))$. So the product (F,d) of (F_i,d_i)'s is the unique forest of $\mathbb{F}$ such that its image by m is $m((F,d))$.

So m is injective on $\mathbb{F}$. By triangularity, m is injective on $\mathbb{F}_{\mathbf{H}_{hpo}}$ and we deduce that the restriction of Φ to $\mathbf{H}_{hpo}$ is an injection of graded Hopf algebras. □

4 Hopf algebras of contractions

4.1 Commutative case

In [CEFM11], D. Calaque, K. Ebrahimi-Fard and D. Manchon introduce a new coproduct, called in this paper the contraction coproduct, on the augmentation ideal of $\mathbf{H}_{CK}$ (see also [MS11]).

Definition 11 *Let F be a nonempty rooted forest and $\boldsymbol{e}$ a subset of $E(F)$. Then we denote by*

1. *$Part_{\boldsymbol{e}}(F)$ the subforest of F obtained by keeping all the vertices of F and the edges of $\boldsymbol{e}$,*
2. *$Cont_{\boldsymbol{e}}(F)$ the subforest of F obtained by contracting each edge of $\boldsymbol{e}$ and identifying the two extremities of each edge of $\boldsymbol{e}$.*

We shall say that $\boldsymbol{e}$ is a contraction of F, $Part_{\boldsymbol{e}}(F)$ is the partition of F by $\boldsymbol{e}$ and $Cont_{\boldsymbol{e}}(F)$ is the contracted of F by $\boldsymbol{e}$. Each vertex of $Cont_{\boldsymbol{e}}(F)$ can be identified to a connected component of $Part_{\boldsymbol{e}}(F)$.

Remarks.

- If $\boldsymbol{e} = \emptyset$, then $Part_{\boldsymbol{e}}(F) = \underbrace{\bullet \dots \bullet}_{|F|_v \times}$ and $Cont_{\boldsymbol{e}}(F) = F$: this is the *empty contraction* of F.
- If $\boldsymbol{e} = E(F)$, then $Part_{\boldsymbol{e}}(F) = F$ and $Cont_{\boldsymbol{e}}(F) = \bullet$: this is the *total contraction* of F.

Notations. We shall write $\boldsymbol{e} \models E(F)$ if $\boldsymbol{e}$ is a contraction of F and $\boldsymbol{e} \Vdash E(F)$ if $\boldsymbol{e}$ is a nonempty, nontotal contraction of F.

Example. Let $T =$  be a rooted tree. Then

contraction $\boldsymbol{e}$								
$Part_{\boldsymbol{e}}(T)$								
$Cont_{\boldsymbol{e}}(T)$								

where, in the first line, the edges not belonging to $\boldsymbol{e}$ are striked out.

Remarks. Let F be a nonempty rooted forest and $\boldsymbol{e} \models E(F)$.

1. We have the following relation on the vertices degrees:

$$|F|_v = |Cont_{\boldsymbol{e}}(F)|_v + |Part_{\boldsymbol{e}}(F)|_v - l(Part_{\boldsymbol{e}}(F)).$$

2. Note $\overline{\boldsymbol{e}}$ the complementary to $\boldsymbol{e}$ in $E(F)$. Then $E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$ and

$$|F|_e = |Cont_{\boldsymbol{e}}(F)|_e + |Part_{\boldsymbol{e}}(F)|_e \,. \tag{8}$$

Let $\mathbf{C}'_{CK}$ be the (nonunitary) $\mathbb{K}$-algebra spanned by the rooted forests and $\mathbf{C}_{CK}$ the quotient algebra $\mathbf{C}'_{CK}/I_{CK}$ where I_{CK} is the ideal spanned by $\bullet - 1$. In others terms, one identifies the unit 1 (for the concatenation) with the tree $\bullet$. We denote in the same way a rooted forest and his class in $\mathbf{C}_{CK}$. Then we define on $\mathbf{C}_{CK}$ a contraction coproduct on each forest $F \in \mathbf{C}_{CK}$:

$$\begin{aligned} \Delta_{\mathbf{C}_{CK}}(F) &= \sum_{\boldsymbol{e} \models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F), \\ &= F \otimes \bullet + \bullet \otimes F + \sum_{\boldsymbol{e} \Vdash E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F). \end{aligned}$$

In particular, $\Delta_{\mathbf{C}_{CK}}(\bullet) = \bullet \otimes \bullet$.

Example.

$\Delta_{\mathbf{C}_{CK}}($  $) = \bullet \otimes$  $+$  $\otimes \bullet + 2\,\bullet \otimes$  $+ \bullet \otimes$  $+$  $\otimes \bullet +$ $\otimes \bullet + \bullet\bullet \otimes \bullet$.

We define an algebra morphism ε:

$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{C}_{CK} & \to & \mathbb{K} \\ F \text{ forest} & \mapsto & \delta_{F,\bullet}. \end{array} \right.$$

Then $(\mathbf{C}_{CK}, \Delta_{\mathbf{C}_{CK}}, \varepsilon)$ is a commutative Hopf algebra graded by the number of edges. $\mathbf{C}_{CK}$ is non cocommutative (see for example the coproduct of).

Remark. We define inductively:

$$\Delta^{(0)}_{\mathbf{C}_{CK}} = Id, \qquad \Delta^{(1)}_{\mathbf{C}_{CK}} = \Delta_{\mathbf{C}_{CK}}, \qquad \Delta^{(k)}_{\mathbf{C}_{CK}} = (\Delta_{\mathbf{C}_{CK}} \otimes Id^{\otimes(k-1)}) \circ \Delta^{(k-1)}_{\mathbf{C}_{CK}}.$$

For all $k \in \mathbb{N}$, $\Delta^{(k)}_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \to \mathbf{C}_{CK}^{\otimes(k+1)}$. If F is a rooted forest with n edges, there are $(k+1)^n$ terms in the expression of $\Delta^{(k)}_{\mathbf{C}_{CK}}(F)$:

- If $k = 0$, this is obvious.
- If $k > 0$, we have $\binom{n}{l}$ tensors $F^{(1)} \otimes F^{(2)}$ in $\Delta_{\mathbf{C}_{CK}}(F)$ such that the left term $F^{(1)}$ have l edges. By the induction hypothesis, there are k^l terms in $\Delta^{(k-1)}_{\mathbf{C}_{CK}}(F^{(1)})$. So there are $\sum_{0 \leq l \leq n} \binom{n}{l} k^l = (k+1)^n$ terms in the expression of $\Delta^{(k)}_{\mathbf{C}_{CK}}(F)$.

We give the first numbers of trees $t_n^{\mathbf{C}_{CK}}$ and forests $f_n^{\mathbf{C}_{CK}}$:

n	0	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{C}_{CK}}$	1	1	2	4	9	20	48	115	286	719	1842
$f_n^{\mathbf{C}_{CK}}$	1	1	3	7	19	47	127	330	889	2378	6450

The first sequence is the sequence A000081 in [Slo].

We recall a combinatorial description of the antipode $S_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \to \mathbf{C}_{CK}$ (see [CEFM11]):

Proposition 12 *The antipode $S_{\mathbf{C}_{CK}} : \mathbf{C}_{CK} \to \mathbf{C}_{CK}$ of the Hopf algebra $(\mathbf{C}_{CK}, \Delta_{\mathbf{C}_{CK}}, \varepsilon)$ is given (recursively with respect to number of edges) by the following formulas: for all forest $F \in \mathbf{C}_{CK}$,*

$$\begin{aligned} S_{\mathbf{C}_{CK}}(F) &= -F - \sum_{\boldsymbol{e} \Vdash E(F)} S_{\mathbf{C}_{CK}}(Part_{\boldsymbol{e}}(F)) Cont_{\boldsymbol{e}}(F) \\ &= -F - \sum_{\boldsymbol{e} \Vdash E(F)} Part_{\boldsymbol{e}}(F) S_{\mathbf{C}_{CK}}(Cont_{\boldsymbol{e}}(F)). \end{aligned}$$

Examples.

$$\begin{aligned} S_{\mathbf{C}_{CK}}(\bullet) &= \bullet, \\ S_{\mathbf{C}_{CK}}(\uparrow) &= -\uparrow - \bullet, \\ S_{\mathbf{C}_{CK}}(\vee) &= -\vee + 2\uparrow\uparrow + 2\uparrow, \\ S_{\mathbf{C}_{CK}}(\uparrow\uparrow) &= -\uparrow\uparrow + 2\uparrow\uparrow + 2\uparrow, \\ S_{\mathbf{C}_{CK}}(\curlyvee) &= -\curlyvee + 3\uparrow\vee + 2\vee + 2\uparrow\uparrow\uparrow + \uparrow\uparrow - 5\uparrow\uparrow\uparrow - 6\uparrow\uparrow - \uparrow. \end{aligned}$$

We now give a decorated version of $\mathbf{C}_{CK}$. Let $\mathcal{D}$ be a nonempty set. A rooted forest with their edges decorated by $\mathcal{D}$ is a couple (F, d) where F is a forest of $\mathbf{C}_{CK}$ and $d : E(F) \to \mathcal{D}$ is a map. We denote by $\mathbf{C}_{CK}^{\mathcal{D}}$ the $\mathbb{K}$-vector space spanned by rooted forests with edges decorated by $\mathcal{D}$.

Examples.

1. Rooted trees decorated by $\mathcal{D}$ with edges degree smaller than 3:

$\uparrow^a, a \in \mathcal{D}, \quad a\vee b, \uparrow_a^b, (a,b) \in \mathcal{D}^2, \quad a\curlyvee c, {}^c a\vee b, a\vee^c b, b\vee_a c, \uparrow_a^{\uparrow_b^c}, (a,b,c) \in \mathcal{D}^3.$

2. Rooted forests decorated by $\mathcal{D}$ with edges degree smaller than 3:

$\uparrow^a, a \in \mathcal{D}, \quad \uparrow^a \uparrow^b, a\vee b, \uparrow_a^b, (a,b) \in \mathcal{D}^2,$

$\uparrow^a \uparrow^b \uparrow^c, \uparrow^a b\vee c, \uparrow^a \uparrow_b^c, a\curlyvee c, {}^c a\vee b, a\vee^c b, b\vee_a c, \uparrow_a^{\uparrow_b^c}, (a,b,c) \in \mathcal{D}^3.$

If $F \in \mathbf{C}_{CK}^{\mathcal{D}}$ $\boldsymbol{e} \models E(F)$, then $Part_{\boldsymbol{e}}(F)$ and $Cont_{\boldsymbol{e}}(F)$ are naturally rooted forests with their edges decorated by $\mathcal{D}$: we keep the decoration of each edges. The vector space $\mathbf{C}_{CK}^{\mathcal{D}}$ is a Hopf algebra. Its product is given by the concatenation and its coproduct is the contraction coproduct. For example: if $(a,b,c) \in \mathcal{D}^3$,

$$\begin{aligned} \Delta_{\mathbf{C}_{CK}^{\mathcal{D}}}({}^c a\vee b) &= {}^c a\vee b \otimes \bullet + \bullet \otimes {}^c a\vee b + \uparrow^c \otimes a\vee b + \uparrow^a \otimes c\vee b + \uparrow^b \otimes \uparrow_a^c + \uparrow^c \uparrow^b \otimes \uparrow^a \\ &\quad + a\vee b \otimes \uparrow^c + \uparrow_a^c \otimes \uparrow^b. \end{aligned}$$

Notation. The set of nonempty trees of $\mathbf{C}_{CK}$ (that is to say with at least one edge) will be denoted by $\mathbb{T}_{\mathbf{C}_{CK}}$. The set of nonempty trees with their edges decorated by $\mathcal{D}$ of $\mathbf{C}_{CK}^{\mathcal{D}}$ will be denoted by $\mathbb{T}_{\mathbf{C}_{CK}}^{\mathcal{D}}$.

4.2 Insertion operations

Let $\mathbf{T}^{\mathcal{D}}_{CK}$ be the $\mathbb{K}$-vector space having for basis $\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$. In this section, we prove that $\mathbf{T}^{\mathcal{D}}_{CK}$ is equiped with two operations $\curlyvee$ and $\rhd$ such that $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \rhd)$ is a commutative prelie algebra.

Definition 13 *1. A commutative prelie algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps $\curlyvee, \rhd : A \otimes A \to A$ such that $x \curlyvee y = y \curlyvee x$ for all $x, y \in A$ (that is to say $\curlyvee$ is commutative) and satisfying the following relations : for all $x, y, z \in A$,*

$$\begin{cases} (x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z), \\ x \rhd (y \rhd z) - (x \rhd y) \rhd z = y \rhd (x \rhd z) - (y \rhd x) \rhd z, \\ x \rhd (y \curlyvee z) = (x \rhd y) \curlyvee z + (x \rhd z) \curlyvee y. \end{cases} \tag{9}$$

In other words, $(A, \curlyvee, \rhd)$ is a commutative prelie algebra if $(A, \curlyvee)$ is a commutative algebra and $(A, \rhd)$ is a left prelie algebra with a relationship between $\curlyvee$ and $\rhd$.

2. The commutative prelie operad, denoted $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$, is the operad such that $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebras are commutative prelie algebras.

Remark. From this definition, it is clear that the operad $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$ is binary and quadratic (see [LV12] for a definition).

Notations.

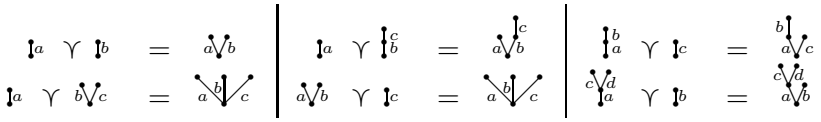
1. Let $T \in \mathbb{T}_{\mathbf{C}_{CK}}$ be a tree with at least one edge. We denote by $V^*(T) = V(T) \setminus \{R_T\}$ the set of vertices of T different from the root of T.
2. Let $T_1, T_2 \in \mathbb{T}_{\mathbf{C}_{CK}}$ and $v \in V(T_2)$. Then $T_1 \circ_v T_2$ is the tree obtained by identifying the root R_{T_1} of T_1 and the vertex v of T_2.

We define two $\mathbb{K}$-linear maps $\curlyvee : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$ and $\rhd : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$ as follow: if $T_1, T_2 \in \mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$,

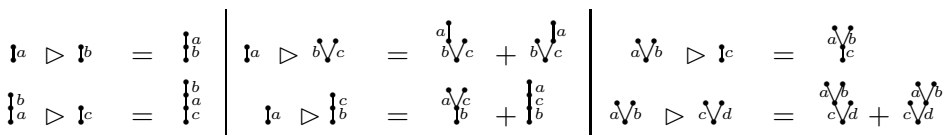
$$\begin{aligned} T_1 \curlyvee T_2 &= T_1 \circ_{R_{T_2}} T_2, \\ T_1 \rhd T_2 &= \sum_{s \in V^*(T_2)} T_1 \circ_s T_2. \end{aligned}$$

Examples.

1. For the map $\curlyvee : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$:



2. For the map $\rhd : \mathbf{T}^{\mathcal{D}}_{CK} \otimes \mathbf{T}^{\mathcal{D}}_{CK} \to \mathbf{T}^{\mathcal{D}}_{CK}$:



Proposition 14 *$(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \rhd)$ is a $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra.*

Proof. Let $T_1, T_2, T_3 \in \mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}$. Then

$$T_1 \curlyvee T_2 = T_1 \circ_{R_{T_2}} T_2 = T_2 \circ_{R_{T_1}} T_1 = T_2 \curlyvee T_1.$$

Moreover,

$$(T_1 \curlyvee T_2) \curlyvee T_3 = (T_1 \circ_{R_{T_2}} T_2) \circ_{R_{T_3}} T_3 = T_1 \circ_{R_{(T_2 \circ_{R_{T_3}} T_3)}} (T_2 \circ_{R_{T_3}} T_3) = T_1 \curlyvee (T_2 \curlyvee T_3).$$

Therefore $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee)$ is a commutative algebra.

$$\begin{aligned}
T_1 \rhd (T_2 \rhd T_3) &= \sum_{\substack{v \in V^*(T_3) \\ w \in V^*(T_2) \cup V^*(T_3)}} T_1 \circ_w (T_2 \circ_v T_3) \\
&= \sum_{v \in V^*(T_3), w \in V^*(T_2)} T_1 \circ_w (T_2 \circ_v T_3) + \sum_{v,w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3) \\
&= \sum_{v \in V^*(T_3), w \in V^*(T_2)} (T_1 \circ_w T_2) \circ_v T_3 + \sum_{v,w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3) \\
&= (T_1 \rhd T_2) \rhd T_3 + \sum_{v,w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3).
\end{aligned}$$

So

$$\begin{aligned}
T_1 \rhd (T_2 \rhd T_3) - (T_1 \rhd T_2) \rhd T_3 &= \sum_{v,w \in V^*(T_3)} T_1 \circ_w (T_2 \circ_v T_3) \\
&= \sum_{v,w \in V^*(T_3)} T_2 \circ_v (T_1 \circ_w T_3) \\
&= T_2 \rhd (T_1 \rhd T_3) - (T_2 \rhd T_1) \rhd T_3.
\end{aligned}$$

Therefore, $(\mathbf{T}^{\mathcal{D}}_{CK}, \rhd)$ is a left prelie algebra.

It remains to prove the last relation of (9):

$$\begin{aligned}
T_1 \rhd (T_2 \curlyvee T_3) &= \sum_{v \in V^*(T_2 \circ_{R_{T_3}} T_3)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) \\
&= \sum_{v \in V^*(T_2)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) + \sum_{v \in V^*(T_3)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) \\
&= \sum_{v \in V^*(T_2)} T_1 \circ_v (T_2 \circ_{R_{T_3}} T_3) + \sum_{v \in V^*(T_3)} T_1 \circ_v (T_3 \circ_{R_{T_2}} T_2) \\
&= \left(\sum_{v \in V^*(T_2)} T_1 \circ_v T_2 \right) \circ_{R_{T_3}} T_3 + \left(\sum_{v \in V^*(T_3)} T_1 \circ_v T_3 \right) \circ_{R_{T_2}} T_2 \\
&= (T_1 \rhd T_2) \curlyvee T_3 + (T_1 \rhd T_3) \curlyvee T_2.
\end{aligned}$$

□

Theorem 15 $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \rhd)$ *is generated as ComPreLie-algebra by* $\wr^d$, $d \in \mathcal{D}$.

Notation. To prove the previous proposition, we introduce a notation. Let $T_1, \dots, T_k$ are trees (possibly empty) of $\mathbf{C}^{\mathcal{D}}_{CK}$ and $d_1, \dots, d_k \in \mathcal{D}$. Then $B_{d_1 \otimes \dots \otimes d_k}(T_1 \otimes \dots \otimes T_k)$ is the tree

obtained by grafting each T_i on a common root with an edge decorated by d_i. For examples, if $a, b, c, d \in \mathcal{D}$,

$B_a(\bullet) = \uparrow_a$ | $B_{a\otimes b}(\bullet \otimes \bullet) = {}^{a}\!\vee_{b}$ | $B_a(\uparrow_b) = \begin{smallmatrix}\bullet^{b}\\ \bullet_{a}\end{smallmatrix}$

$B_a(\uparrow_b^c) = \begin{smallmatrix}\bullet^{c}\\ \bullet_{b}\\ \bullet_{a}\end{smallmatrix}$ | $B_{a\otimes b}(\uparrow_c \otimes \bullet) = {}^{c}\!\uparrow_{a}\!\vee_{b}$ | $B_{a\otimes b}(\bullet \otimes \uparrow_c) = {}^{\bullet^{c}}\!\vee_{a\ \bullet_{b}}$

$B_{a\otimes b\otimes c}(\bullet \otimes \bullet \otimes \bullet) = {}_{a}\!\searrow\!{}_{b}\!\swarrow_{c}$ | $B_a({}_{b}\!\vee_{c}) = {}^{b}\!\vee_{a}^{c}$ | $B_{a\otimes b}({}_{c}\!\vee_{d} \otimes \bullet) = {}^{c}\!\vee_{d\ a}\!\vee_{b}$

Proof. Let us prove that $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \rhd)$ is generated as $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra by $\uparrow_d$, $d \in \mathcal{D}$ by induction on the edges degree n. If $n = 1$, this is obvious. Let $T \in \mathbf{T}^{\mathcal{D}}_{CK}$ be a tree of edges degree $n \geq 2$. Let k be an integer such that $T = B_{d_1\otimes\ldots\otimes d_k}(T_1 \otimes \ldots \otimes T_k)$ with $d_1, \ldots, d_k \in \mathcal{D}$ and $T_1, \ldots, T_k$ trees (possibly empty) of $\mathbf{C}^{\mathcal{D}}_{CK}$. Then:

1. If $k = 1$, $T = B_{d_1}(T_1)$ with $|T_1|_e = n - 1 \geq 1$. By induction hypothesis, T_1 can be constructed from trees $\uparrow_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\rhd$. So $T = T_1 \rhd \uparrow_{d_1}$ can be also constructed from trees $\uparrow_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\rhd$.

2. Suppose that $k \geq 2$. Then, for all i, $1 \leq |B_{d_i}(T_i)|_e \leq n - 1$. By induction hypothesis, the trees $B_{d_i}(T_i)$ can be constructed from trees $\uparrow_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\rhd$. So $T = B_{d_1}(T_1) \curlyvee \ldots \curlyvee B_{d_k}(T_k)$ can be also constructed from trees $\uparrow_d$, $d \in \mathcal{D}$, with the operations $\curlyvee$ and $\rhd$.

We conclude with the induction principle. □

Remarks.

1. $(\mathbf{T}^{\mathcal{D}}_{CK}, \curlyvee, \rhd)$ is not the free $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra generated by $\uparrow_d$, $d \in \mathcal{D}$. For example,

$$\uparrow_a \rhd (\uparrow_b \rhd \uparrow_c) = {}^{a}\!\vee_{b\ \bullet_{c}} + \begin{smallmatrix}\bullet^{a}\\ \bullet_{b}\\ \bullet_{c}\end{smallmatrix} = (\uparrow_a \curlyvee \uparrow_b) \rhd \uparrow_c + (\uparrow_a \rhd \uparrow_b) \rhd \uparrow_c .$$

2. A description of the free $\mathcal{C}om\mathcal{P}re\mathcal{L}ie$-algebra is given in [Foi13].

4.3 Noncommutative case

We give a noncommutative version of $\mathbf{C}_{CK}$. To do this, we work on the augmentation ideal of $\mathbf{H}_{po}$.

Definition 16 *Let (F, σ^F) be a nonempty preordered forest. In particular, F is a nonempty rooted forest. Let $\boldsymbol{e}$ be a contraction of F, $Part_{\boldsymbol{e}}(F)$ the partition of F by $\boldsymbol{e}$ and $Cont_{\boldsymbol{e}}(F)$ the contracted of F by $\boldsymbol{e}$ (see definition 11). Then:*

1. *$Part_{\boldsymbol{e}}(F)$ is a preordered forest $(Part_{\boldsymbol{e}}(F), \sigma^P)$ where $\sigma^P : v \in V(Part_{\boldsymbol{e}}(F)) \mapsto \sigma^F(v)$. In other words, we keep the initial preorder of the vertices of F in $Part_{\boldsymbol{e}}(F)$.*

2. *$Cont_{\boldsymbol{e}}(F)$ is also a preordered forest $(Cont_{\boldsymbol{e}}(F), \sigma^C)$ where $\sigma^C : V(Cont_{\boldsymbol{e}}(F)) \to \{1, \ldots, p\}$ is the surjection ($p \leq |Cont_{\boldsymbol{e}}(F)|_v$) such that if A, B are two connected components of $Part_{\boldsymbol{e}}(F)$, if a (resp. b) is the vertex obtained by contracting A (resp. B) in F, then*

$$\begin{cases} \sigma^F(R_A) < \sigma^F(R_B) & \Longrightarrow \quad \sigma^C(a) < \sigma^C(b), \\ \sigma^F(R_A) = \sigma^F(R_B) & \Longrightarrow \quad \sigma^C(a) = \sigma^C(b), \\ \sigma^F(R_A) > \sigma^F(R_B) & \Longrightarrow \quad \sigma^C(a) > \sigma^C(b). \end{cases} \tag{10}$$

In other words, we contract each connected component of $Part_{\boldsymbol{e}}(F)$ to its root and we keep the initial preorder of the roots.

Example. Let $T =$  be a preordered tree. Then

contraction $\boldsymbol{e}$								
$Part_{\boldsymbol{e}}(T)$								
$Cont_{\boldsymbol{e}}(T)$								

where, in the first line, the edges not belonging to $\boldsymbol{e}$ are striked out.

Let $\mathbf{C}'_{po}$ be the $\mathbb{K}$-algebra spanned by nonempty preordered forests. Let I_{po} be the ideal of $\mathbf{C}'_{po}$ generated by the elements $F_{\bullet_i} - \tilde{F}$ with $F_{\bullet_i} \in \mathbf{C}'_{po}$ and $\tilde{F}$ the forest contructed from $F_{\bullet_i}$ by deleting the vertex $\bullet_i$ and keeping the same preorder on $V(F)$. For example,

- if $F_{\bullet_i} =$  then $\tilde{F} =$,
- if $F_{\bullet_i} =$  then $\tilde{F} =$.

Let $\mathbf{C}_{po}$ be the quotient algebra $\mathbf{C}'_{po}/I_{po}$. So one identifies the unit 1 (for the concatenation) with the tree $\bullet_1$. $\mathbf{C}_{po}$ is a graded algebra by the number of edges. We denote in the same way a forest and his class in $\mathbf{C}_{po}$. We define on $\mathbf{C}_{po}$ a contraction coproduct on each preordered forest $F \in \mathbf{C}_{po}$:

$$\begin{aligned}\Delta_{\mathbf{C}_{po}}(F) &= \sum_{\boldsymbol{e} \models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F),\\ &= F \otimes \bullet_1 + \bullet_1 \otimes F + \sum_{\boldsymbol{e} \Vert \models E(F)} Part_{\boldsymbol{e}}(F) \otimes Cont_{\boldsymbol{e}}(F).\end{aligned}$$

Examples.

$$\begin{aligned}\Delta_{\mathbf{C}_{po}}(\bullet_1) &= \bullet_1 \otimes \bullet_1\\ \Delta_{\mathbf{C}_{po}}(\mathbb{I}^1_2) &= \mathbb{I}^1_2 \otimes \bullet_1 + \bullet_1 \otimes \mathbb{I}^1_2\\ \Delta_{\mathbf{C}_{po}}(\mathbb{V}^{1\ 2}_{\ 2}) &= \mathbb{V}^{1\ 2}_{\ 2} \otimes \bullet_1 + \bullet_1 \otimes \mathbb{V}^{1\ 2}_{\ 2} + \mathbb{I}^1_2 \otimes \mathbb{I}^1_1 + \mathbb{I}^1_1 \otimes \mathbb{I}^1_2\\ \Delta_{\mathbf{C}_{po}}(\mathbb{I}^4_2\,\mathbb{I}^1_3) &= \mathbb{I}^4_2\,\mathbb{I}^1_3 \otimes \bullet_1 + \bullet_1 \otimes \mathbb{I}^4_2\,\mathbb{I}^1_3 + \mathbb{I}^2_1 \otimes \mathbb{I}^1_2 + \mathbb{I}^1_2 \otimes \mathbb{I}^2_1\\ \Delta_{\mathbf{C}_{po}}(\mathbb{Y}^{1\ 3}_{3\ 2}) &= \mathbb{Y}^{1\ 3}_{3\ 2} \otimes \bullet_1 + \bullet_1 \otimes \mathbb{Y}^{1\ 3}_{3\ 2} + \mathbb{I}^3_2\,\mathbb{I}^1_3 \otimes \mathbb{I}^2_1 + \mathbb{V}^{2\ 2}_{1\ } \otimes \mathbb{I}^1_2 + \mathbb{I}^{1}_{3\ 2} \otimes \mathbb{I}^2_1 + \mathbb{I}^2_1 \otimes \mathbb{I}^{1}_{3\ 2}\\ &\quad + \mathbb{I}^1_2 \otimes \mathbb{V}^{2\ 2}_{1\ } + \mathbb{I}^2_1 \otimes \mathbb{V}^{1\ 3}_{2\ }\\ \Delta_{\mathbf{C}_{po}}(\mathbb{V}^{2\ 5}_{3\ }\,\mathbb{I}^1_4) &= \mathbb{V}^{2\ 5}_{3\ }\,\mathbb{I}^1_4 \otimes \bullet_1 + \bullet_1 \otimes \mathbb{V}^{2\ 5}_{3\ }\,\mathbb{I}^1_4 + \mathbb{I}^1_2 \otimes \mathbb{I}^4_2\,\mathbb{I}^1_3 + \mathbb{I}^2_1 \otimes \mathbb{I}^2_3\,\mathbb{I}^1_4 + \mathbb{I}^1_2 \otimes \mathbb{V}^{1\ 3}_{2\ }\\ &\quad + \mathbb{I}^2_3\,\mathbb{I}^1_4 \otimes \mathbb{I}^2_1 + \mathbb{I}^4_2\,\mathbb{I}^1_3 \otimes \mathbb{I}^1_2 + \mathbb{V}^{1\ 3}_{2\ } \otimes \mathbb{I}^1_2\end{aligned}$$

Remark. $\Delta_{\mathbf{C}_{po}}$ is non cocommutative (see for example the coproduct of). In particular, if T is a preordered tree and $\boldsymbol{e} \models E(T)$, $Cont_{\boldsymbol{e}}(T)$ is a preordered tree and $Part_{\boldsymbol{e}}(T)$ can be disconnected. The second component of the coproduct is linear: a tree instead of a polynomial in trees. This is a right combinatorial Hopf algebra (see [LR10]).

Proposition 17 *1. $\Delta_{\mathbf{C}_{po}}$ is a graded algebra morphism.*

2. $\Delta_{\mathbf{C}_{po}}$ is coassociative.

Proof.

1. Let F, G be two preordered forests. Then

$$\begin{aligned}\Delta_{\mathbf{C}_{po}}(FG) &= \sum_{\boldsymbol{e}\models E(FG)} Part_{\boldsymbol{e}}(FG)\otimes Cont_{\boldsymbol{e}}(FG)\\ &= \sum_{\boldsymbol{e}\models E(F),\boldsymbol{f}\models E(G)} (Part_{\boldsymbol{e}}(F)Part_{\boldsymbol{f}}(G))\otimes(Cont_{\boldsymbol{e}}(F)Cont_{\boldsymbol{f}}(G))\\ &= \left(\sum_{\boldsymbol{e}\models E(F)} Part_{\boldsymbol{e}}(F)\otimes Cont_{\boldsymbol{e}}(F)\right)\left(\sum_{\boldsymbol{f}\models E(G)} Part_{\boldsymbol{f}}(G)\otimes Cont_{\boldsymbol{f}}(G)\right)\\ &= \Delta_{\mathbf{C}_{po}}(F)\Delta_{\mathbf{C}_{po}}(G),\end{aligned}$$

   and $\Delta_{\mathbf{C}_{po}}$ is an algebra morphism. It is a graded algebra morphism with (8).

2. Let F be a nonempty preordered forest. Then

$$\begin{aligned}&(\Delta_{\mathbf{C}_{po}}\otimes Id)\circ\Delta_{\mathbf{C}_{po}}(F)\\ =&\sum_{\boldsymbol{e}\models E(F)} \Delta_{\mathbf{C}_{po}}(Part_{\boldsymbol{e}}(F))\otimes Cont_{\boldsymbol{e}}(F)\\ =&\sum_{\boldsymbol{e}\models E(F)}\sum_{\boldsymbol{f}\models E(Part_{\boldsymbol{e}}(F))} Part_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))\otimes Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))\otimes Cont_{\boldsymbol{e}}(F)\\ =&\sum_{\boldsymbol{f}\subseteq\boldsymbol{e}\subseteq E(F)} Part_{\boldsymbol{f}}(F)\otimes Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))\otimes Cont_{\boldsymbol{e}}(F),\end{aligned}$$

   and

$$\begin{aligned}&(Id\otimes\Delta_{\mathbf{C}_{po}})\circ\Delta_{\mathbf{C}_{po}}(F)\\ =&\sum_{\boldsymbol{f}\models E(F)} Part_{\boldsymbol{f}}(F)\otimes\Delta_{\mathbf{C}_{po}}(Cont_{\boldsymbol{f}}(F))\\ =&\sum_{\boldsymbol{f}\models E(F)}\sum_{\boldsymbol{e}\models E(Cont_{\boldsymbol{f}}(F))} Part_{\boldsymbol{f}}(F)\otimes Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))\otimes Cont_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))\\ =&\sum_{\boldsymbol{f}\models E(F),\boldsymbol{e}\subseteq\overline{\boldsymbol{f}}} Part_{\boldsymbol{f}}(F)\otimes Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))\otimes Cont_{\boldsymbol{e}\cup\boldsymbol{f}}(F),\end{aligned}$$

   where to the last equality we use that $E(Cont_{\boldsymbol{f}}(F)) = \overline{\boldsymbol{f}}$ the complement of $\boldsymbol{f}$ in $E(F)$ and $Cont_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F)) = Cont_{\boldsymbol{e}\cup\boldsymbol{f}}(F)$.

   Remark that $\{(\boldsymbol{e},\boldsymbol{f}) \mid \boldsymbol{f}\subseteq\boldsymbol{e}\subseteq E(F)\}$ and $\{(\boldsymbol{e},\boldsymbol{f}) \mid \boldsymbol{f}\models E(F), \boldsymbol{e}\subseteq\overline{\boldsymbol{f}}\}$ are in bijection:

$$\begin{cases}\{(\boldsymbol{e},\boldsymbol{f}) \mid \boldsymbol{f}\subseteq\boldsymbol{e}\subseteq E(F)\} & \to & \{(\boldsymbol{e},\boldsymbol{f}) \mid \boldsymbol{f}\models E(F), \boldsymbol{e}\subseteq\overline{\boldsymbol{f}}\}\\ (\boldsymbol{e},\boldsymbol{f}) & \to & (\boldsymbol{e}\setminus\boldsymbol{f},\boldsymbol{f})\\ (\boldsymbol{e}\cup\boldsymbol{f},\boldsymbol{f}) & \leftarrow & (\boldsymbol{e},\boldsymbol{f}).\end{cases}$$

   Moreover,

   - in $Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))$ with $\boldsymbol{f}\subseteq\boldsymbol{e}\subseteq E(F)$: the edges belong to $\boldsymbol{e}\cap\overline{\boldsymbol{f}} = \boldsymbol{e}\setminus\boldsymbol{f}$; the vertices are the connected components of $Part_{\boldsymbol{e}\cap\boldsymbol{f}}(F) = Part_{\boldsymbol{f}}(F)$. The preorder on the vertices is given by the preorder on the roots of the connected components of $Part_{\boldsymbol{f}}(F)$.
   - in $Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))$ with $\boldsymbol{f}\models E(F), \boldsymbol{e}\subseteq\overline{\boldsymbol{f}}$: the edges belong to $\overline{\boldsymbol{f}}\cap\boldsymbol{e} = \boldsymbol{e}\setminus\boldsymbol{f} = \boldsymbol{e}$; the vertices are the connected components of $Part_{\boldsymbol{f}}(F)$. As in the precedent case, the preorder on the vertices is given by the preorder on the roots of the connected components of $Part_{\boldsymbol{f}}(F)$.

So $Cont_{\boldsymbol{f}}(Part_{\boldsymbol{e}}(F))$ and $Part_{\boldsymbol{e}}(Cont_{\boldsymbol{f}}(F))$ are the same forests with the same preorder on the vertices.

Therefore $(\Delta_{\mathbf{C}_{po}} \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) = (Id \otimes \Delta_{\mathbf{C}_{po}}) \circ \Delta_{\mathbf{C}_{po}}(F)$.

□

We now define

$$\varepsilon : \left\{ \begin{array}{rcl} \mathbf{C}_{po} & \to & \mathbb{K} \\ F \text{ forest} & \mapsto & \delta_{F,\bullet_1}. \end{array} \right.$$

ε is an algebra morphism.

Proposition 18 *ε is a counit for the coproduct $\Delta_{\mathbf{C}_{po}}$.*

Proof. Let F be a forest $\in \mathbf{C}_{po}$. We use the Sweedler notation:

$$\Delta_{\mathbf{C}_{po}}(F) = F \otimes \bullet_1 + \bullet_1 \otimes F + \sum_F F^{(1)} \otimes F^{(2)}.$$

Then

$$\begin{array}{rcl} (\varepsilon \otimes Id) \circ \Delta_{\mathbf{C}_{po}}(F) & = & \varepsilon(F)\bullet_1 + \varepsilon(\bullet_1)F + \sum_F \varepsilon(F^{(1)}) \otimes F^{(2)} = F, \\ (Id \otimes \varepsilon) \circ \Delta_{\mathbf{C}_{po}}(F) & = & F\varepsilon(\bullet_1) + \bullet_1\varepsilon(F) + \sum_F F^{(1)}\varepsilon(F^{(2)}) = F. \end{array}$$

Therefore ε is a counit for the coproduct $\Delta_{\mathbf{C}_{po}}$. □

As $(\mathbf{C}_{po}, \Delta_{\mathbf{C}_{po}}, \varepsilon)$ is gradued (by the number of edges) and connected, we have the following theorem:

Theorem 19 *$(\mathbf{C}_{po}, \Delta_{\mathbf{C}_{po}}, \varepsilon)$ is a Hopf algebra.*

We denote the antipode of the Hopf algebra $\mathbf{C}_{po}$ by $S_{\mathbf{C}_{po}}$. We have the same combinatorial description of $S_{\mathbf{C}_{po}}$ as the commutative case (see proposition 12). We give some values of $S_{\mathbf{C}_{po}}$:

- In edges degree 0, $S_{\mathbf{C}_{po}}(\bullet_1) = \bullet_1$.
- In edges degree 1, $S_{\mathbf{C}_{po}}(\mathbb{I}_1^1) = -\mathbb{I}_1^1 - \bullet_1$, $S_{\mathbf{C}_{po}}(\mathbb{I}_1^2) = -\mathbb{I}_1^2 - \bullet_1$ and $S_{\mathbf{C}_{po}}(\mathbb{I}_2^1) = -\mathbb{I}_2^1 - \bullet_1$.
- In edges degree 2,

$$\begin{array}{rcl} S_{\mathbf{C}_{po}}({}^{2}\mathbb{V}_{1}^{2}) & = & -{}^{2}\mathbb{V}_{1}^{2} + 2\mathbb{I}_1^2\,\mathbb{I}_3^4 + 2\mathbb{I}_1^2, \\ S_{\mathbf{C}_{po}}({}^{1}\mathbb{V}_{2}^{3}) & = & -{}^{1}\mathbb{V}_{2}^{3} + \mathbb{I}_2^1\,\mathbb{I}_3^4 + \mathbb{I}_1^2\,\mathbb{I}_3^4 + \mathbb{I}_1^2 + \mathbb{I}_2^1, \\ S_{\mathbf{C}_{po}}(\mathbb{I}_1^1\,\mathbb{I}_2^1) & = & -\mathbb{I}_1^1\,\mathbb{I}_2^1 + \mathbb{I}_1^2\,\mathbb{I}_3^3 + \mathbb{I}_2^1\,\mathbb{I}_3^4 + \mathbb{I}_1^1 + \mathbb{I}_1^2, \\ S_{\mathbf{C}_{po}}(\mathbb{I}_2^3\,\mathbb{I}_3^1) & = & -\mathbb{I}_2^3\,\mathbb{I}_3^1 + \mathbb{I}_1^2\,\mathbb{I}_4^3 + \mathbb{I}_2^1\,\mathbb{I}_3^4 + \mathbb{I}_2^1 + \mathbb{I}_1^2. \end{array}$$

- In edges degree 3,

$$\begin{array}{rcl} S_{\mathbf{C}_{po}}({}^{1}\mathbb{I}{}^{3}\mathbb{V}_{2}^{3}) & = & -{}^{1}\mathbb{I}{}^{3}\mathbb{V}_{2}^{3} + \mathbb{I}_2^3\,\mathbb{I}_3^1\,\mathbb{I}_4^5 - \mathbb{I}_1^2\,\mathbb{I}_4^3\,\mathbb{I}_5^6 - \mathbb{I}_2^1\,\mathbb{I}_3^4\,\mathbb{I}_5^6 - \mathbb{I}_2^1\,\mathbb{I}_3^4 - \mathbb{I}_1^2\,\mathbb{I}_3^4 + {}^{2}\mathbb{V}_{1}^{2}\,\mathbb{I}_4^3 \\ & & -2\mathbb{I}_1^2\,\mathbb{I}_3^4\,\mathbb{I}_6^5 - 2\mathbb{I}_1^2\,\mathbb{I}_4^3 + \mathbb{I}{}_2^3{}^{1}\,\mathbb{I}_3^4 - \mathbb{I}_2^1\,\mathbb{I}_3^4\,\mathbb{I}_5^6 - \mathbb{I}_1^2\,\mathbb{I}_3^4 - \mathbb{I}_1^2\,\mathbb{I}_4^3\,\mathbb{I}_5^6 - \mathbb{I}_2^1\,\mathbb{I}_3^4 \\ & & +\mathbb{I}_1^2\,\mathbb{I}{}_4^5{}^{3} + \mathbb{I}{}_3^2{}^{1} + \mathbb{I}_2^1\,{}^{4}\mathbb{V}_{3}^{4} + {}^{2}\mathbb{V}_{1}^{2} + \mathbb{I}_1^2\,{}^{3}\mathbb{V}_{4}^{5} + {}^{1}\mathbb{V}_{2}^{3}. \end{array}$$

Let $\mathbf{C}'_{hpo}$ be the $\mathbb{K}$-algebra spanned by nonempty heap-preordered forests, $\mathbf{C}'_o$ be the $\mathbb{K}$-algebra spanned by nonempty ordered forests, $\mathbf{C}'_{ho}$ be the $\mathbb{K}$-algebra spanned by nonempty heap-ordered forests and $\mathbf{C}'_{NCK}$ be the $\mathbb{K}$-algebra spanned by nonempty planar forests. We consider the quotients $\mathbf{C}_{hpo} = \mathbf{C}'_{hpo}/(I_{po} \cap \mathbf{C}'_{hpo})$, $\mathbf{C}_o = \mathbf{C}'_o/(I_{po} \cap \mathbf{C}'_o)$, $\mathbf{C}_{ho} = \mathbf{C}'_{ho}/(I_{po} \cap \mathbf{C}'_{ho})$ and $\mathbf{C}_{NCK} = \mathbf{C}'_{NCK}/(I_{po} \cap \mathbf{C}'_{NCK})$. We have in this case a similar diagram to (6):

$$\begin{array}{ccccc} \mathbf{C}_{NCK} & \hookrightarrow & \mathbf{C}_{ho} & \hookrightarrow & \mathbf{C}_o \\ & & \downarrow & & \downarrow \\ & & \mathbf{C}_{hpo} & \hookrightarrow & \mathbf{C}_{po} \end{array}$$

where the arrows $\hookrightarrow$ are injective morphisms of algebras. But they are not always morphisms of Hopf algebras (for the contraction coproduct):

Theorem 20 *1. $\mathbf{C}_{hpo}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$.*

2. $\mathbf{C}_o$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$.

3. $\mathbf{C}_{ho}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_o$ and of the Hopf algebra $\mathbf{C}_{hpo}$.

4. $\mathbf{C}_{NCK}$ is a left comodule of the Hopf algebra $\mathbf{C}_{ho}$.

Notations. We denote by $\Delta_{\mathbf{C}_{hpo}}, \Delta_{\mathbf{C}_o}, \Delta_{\mathbf{C}_{ho}}$ the restrictions of $\Delta_{\mathbf{C}_{po}}$ to $\mathbf{C}_{hpo}, \mathbf{C}_o, \mathbf{C}_{ho}$.

Remark. $\mathbf{C}_{NCK}$ is not a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{ho}$. For example, $\begin{smallmatrix} \bullet^3 \\ {}^2\!\!\bullet \quad \bullet^4 \\ {}_1 \end{smallmatrix} \in \mathbf{C}_{NCK}$ and

$$\begin{aligned} \Delta_{\mathbf{C}_{ho}}\left(\begin{smallmatrix} \bullet^3 \\ {}^2\!\!\bullet \quad \bullet^4 \\ {}_1 \end{smallmatrix}\right) &= \begin{smallmatrix} \bullet^3 \\ {}^2\!\!\bullet \quad \bullet^4 \\ {}_1 \end{smallmatrix} \otimes \bullet_1 + \bullet_1 \otimes \begin{smallmatrix} \bullet^3 \\ {}^2\!\!\bullet \quad \bullet^4 \\ {}_1 \end{smallmatrix} + \begin{smallmatrix} \bullet^3 \\ \bullet^2 \\ \bullet^1 \end{smallmatrix} \otimes \bullet^2_1 + \begin{smallmatrix} \bullet^2 \\ {}^1\!\!\bullet \quad \bullet^3 \\ {}_1 \end{smallmatrix} \otimes \bullet^2_1 + 2\bullet^2_1 \otimes \begin{smallmatrix} \bullet^2 \\ {}^1\!\!\bullet \quad \bullet^3 \\ {}_1 \end{smallmatrix} \\ &\quad + \bullet^2_1 \otimes \begin{smallmatrix} \bullet^3 \\ \bullet^2 \\ \bullet^1 \end{smallmatrix} + \bullet^4_1 \bullet^3_2 \otimes \bullet^2_1 . \end{aligned}$$

Then $\bullet^4_1 \bullet^3_2 \otimes \bullet^2_1 \notin \mathbf{C}_{NCK} \otimes \mathbf{C}_{NCK}$.

Proof.

1. $\mathbf{C}_{hpo}$ is a subalgebra of $\mathbf{C}_{po}$. Let us prove that if $(F, \sigma^F) \in \mathbf{C}_{hpo}$ and $\boldsymbol{e} \models E(F)$ then $(Cont_{\boldsymbol{e}}(F), \sigma^C)$ and $(Part_{\boldsymbol{e}}(F), \sigma^P) \in \mathbf{C}_{hpo}$.

   If $a, b \in V(Part_{\boldsymbol{e}}(F))$, $a \neq b$, such that $a \twoheadrightarrow b$ then a, b are the vertices of a subtree of $(F, \sigma^F) \in \mathbf{C}_{hpo}$ and $\sigma^F(a) > \sigma^F(b)$. With definition 16, $\sigma^P(a) > \sigma^P(b)$. So $(Part_{\boldsymbol{e}}(F), \sigma^P) \in \mathbf{C}_{hpo}$.

   If $a, b \in V(Cont_{\boldsymbol{e}}(F))$, $a \neq b$, such that $a \twoheadrightarrow b$, then a and b are the vertices obtained by contracting two connected components A and B of $Part_{\boldsymbol{e}}(F)$. As $a \twoheadrightarrow b$, $R_A \twoheadrightarrow R_B$ and as $(F, \sigma^F) \in \mathbf{C}_{hpo}$, $\sigma^F(R_A) > \sigma^F(R_B)$. Then, by definition 16, $\sigma^C(a) > \sigma^C(b)$. So $(Cont_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{hpo}$.

   Therefore if $(F, \sigma^F) \in \mathbf{C}_{hpo}$, $\Delta_{\mathbf{C}_{po}}(F) \in \mathbf{C}_{hpo} \otimes \mathbf{C}_{hpo}$ and $\mathbf{C}_{hpo}$ is a Hopf subalgebra of $\mathbf{C}_{po}$.

2. $\mathbf{C}_o$ is a subalgebra of $\mathbf{C}_{po}$. Let $(F, \sigma^F) \in \mathbf{C}_o$ and $\boldsymbol{e} \models E(F)$. Let us show that $(Cont_{\boldsymbol{e}}(F), \sigma^C)$ and $(Part_{\boldsymbol{e}}(F), \sigma^P) \in \mathbf{C}_o$, that is to say that σ^C and σ^P are bijective.

   By definition 16, σ^P is bijective because we keep the initial order of the vertices of F in $Part_{\boldsymbol{e}}(F)$. By definition, σ^C is a surjection. Let $a, b \in V(Cont_{\boldsymbol{e}}(F))$ such that $\sigma^C(a) = \sigma^C(b)$ and A and B be the two connected components of $Part_{\boldsymbol{e}}(F)$ associated with a and b. With (10), $\sigma^F(R_A) = \sigma^F(R_B)$ and $R_A = R_B$ because σ^F is bijective. So $A = B$, $a = b$ and σ^C is injective.

   Therefore σ^C and σ^P are bijective and $\mathbf{C}_o$ is a Hopf subalgebra of $\mathbf{C}_{ho}$.

3. As $\mathbf{C}_{hpo}$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$ and $\mathbf{C}_o$ is a Hopf subalgebra of the Hopf algebra $\mathbf{C}_{po}$, $\mathbf{C}_{ho} = \mathbf{C}_{hpo} \cap \mathbf{C}_o$ is a Hopf subalgebra of $\mathbf{C}_{hpo}$ and $\mathbf{C}_o$.

4. Let us prove that if $(F, \sigma^F) \in \mathbf{C}_{NCK}$ and $\boldsymbol{e} \models E(F)$ then $(Cont_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{NCK}$. As $\mathbf{C}_{ho}$ is a Hopf algebra, $(Cont_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{ho}$. So, if $a, b \in V(Cont_{\boldsymbol{e}}(F))$, such that $a \twoheadrightarrow b$ then $\sigma^C(a) \geq \sigma^C(b)$.

   Moreover, if $a, b, c \in V(Cont_{\boldsymbol{e}}(F))$ three distinct vertices such that $a \twoheadrightarrow c$, $b \twoheadrightarrow c$ and a is on the left of b. The vertices a, b and c are obtained by contracting of connected components A, B and C in F. As $a \twoheadrightarrow c$, $b \twoheadrightarrow c$ and a is on the left of b, $R_A \twoheadrightarrow R_C$, $R_B \twoheadrightarrow R_C$ and R_A is on the left of R_B. As $(F, \sigma^F) \in \mathbf{C}_{NCK}$, $\sigma^F(R_A) < \sigma^F(R_B)$. So $\sigma^C(a) < \sigma^C(b)$.

   Therefore if $(F, \sigma^F) \in \mathbf{C}_{NCK}$ and $\boldsymbol{e} \models E(F)$ then $(Cont_{\boldsymbol{e}}(F), \sigma^C) \in \mathbf{C}_{NCK}$. Consequently, $\Delta_{\mathbf{C}_{ho}}(\mathbf{C}_{NCK}) \subseteq \mathbf{C}_{ho} \otimes \mathbf{C}_{NCK}$.

□

4.4 Formal series

The algebras $\mathbf{C}_{po}$, $\mathbf{C}_{hpo}$, $\mathbf{C}_o$, $\mathbf{C}_{ho}$ and $\mathbf{C}_{NCK}$ are graded by the number of edges.

In the ordered case, we give some values in a small degrees :

n	1	2	3	4	5	6	7	8
$f_n^{\mathbf{C}_o}$	2	9	76	805	10626	167839	3091768	65127465

These is the sequence A105785 in [Slo].

Let us now study the heap-ordered case. We denote by $f_{n,l}^{\mathbf{C}_{ho}}$ the forests of $\mathbf{C}_{ho}$ of edges degree n and of length l, and by $f_n^{\mathbf{C}_{ho}}$ the forests of $\mathbf{C}_{ho}$ of edges degree n. In small degree, we have the following values:

$$\begin{cases} f_{0,0}^{\mathbf{C}_{ho}} = f_{1,1}^{\mathbf{C}_{ho}} = 1, \\ f_{0,l}^{\mathbf{C}_{ho}} = 0 \qquad \text{for all } l \geq 1, \\ f_{1,l}^{\mathbf{C}_{ho}} = 0 \qquad \text{for all } l \neq 1, \\ f_{n,0}^{\mathbf{C}_{ho}} = 0 \qquad \text{for all } n \neq 1. \end{cases}$$

Let n and l be two integers ≥ 1. To obtain a forest $F \in \mathbf{C}_{ho}$ of edges degree n and of length l (so $|F|_v = n + l$), we have two cases :

1. We consider a forest $G \in \mathbf{C}_{ho}$ of edges degree $n-1$ and of length l and we graft the vertex $n+l$ on the vertex i of G. For each forest G, we have $n + l - 1$ possibilities.

2. We consider a forest $G \in \mathbf{C}_{ho}$ of edges degree $n-1$ and of length $l-1$. Then, for all $i \in \{1, \ldots, n+l-1\}$, the forest $\tilde{G}\updownarrow_i^{n+l}$ of edges degree n and of vertices degree l is an element of $\mathbf{C}_{ho}$ (where $\tilde{G}$ is the same forest of G with for all $j \geq i$ the vertex j in G is the vertex $j+1$ in $\tilde{G}$). For each forest G, we have $n + l - 1$ possibilities.

So

$$f_{n,l}^{\mathbf{C}_{ho}} = (n + l - 1) f_{n-1,l}^{\mathbf{C}_{ho}} + (n + l - 1) f_{n-1,l-1}^{\mathbf{C}_{ho}}.$$

We give some values of $f_{n,l}^{\mathbf{C}_{ho}}$ in a small degrees and in a small lengths :

$n\backslash l$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	2	3	0	0	0
3	0	6	20	15	0	0
4	0	24	130	210	105	0
5	0	120	924	2380	2520	945

Note that $f_{n,1}^{\mathbf{C}_{ho}} = n!$ for all $n \geq 1$. With the formula $f_n^{\mathbf{C}_{ho}} = \sum_{l \geq 0} f_{n,l}^{\mathbf{C}_{ho}}$, we obtain the number of forests of edges degree n. This gives:

n	0	1	2	3	4	5	6
$f_n^{\mathbf{C}_{ho}}$	0	1	5	41	469	6889	123605

This is the sequence A032188 in [Slo].

Remark. Consider the map $\varphi : \mathbb{F}_{\mathbf{H}_{ho}} \to \Sigma$ defined by induction as follows. If $F = 1$, $\varphi(F) = 1$ and if $F = \bullet_1$, $\varphi(F) = (1)$. Let $F \in \mathbf{H}_{ho}$ be a forest of vertices degree n and v the vertex indexed by n. As F is a heap-ordered forest, two cases are possible:

- The vertex v is an isolated vertex. We denote by G the heap-ordered forest obtained by deleting the vertex v of F. Thus $\varphi(G) = \tau'$ is well defined by induction. Then $\varphi(F)$ is the permutation τ defined by

$$\begin{cases} \tau(i) &= \tau'(i) \qquad \text{if } i \neq n \\ \tau(n) &= n. \end{cases}$$

- The vertex v is a leaf and we denote by k the indexe of v' with $v \to v'$. Similarly, we denote by G the heap-ordered forest obtained by deleting the vertex v of F. $\varphi(G) = \tau'$ is well defined by induction and $\varphi(F)$ is the permutation τ defined by

$$\begin{cases} \tau(i) &= \tau'(i) \qquad \text{if } i \neq k \\ \tau(k) &= n \\ \tau(n) &= \tau'(k). \end{cases}$$

Then $\varphi : \mathbb{F}_{\mathbf{H}_{ho}} \to \Sigma$ is a bijective map. Remark that, if $F \in \mathbb{F}_{\mathbf{H}_{ho}}$, each connected component of F corresponds to one cycle in the writing of $\varphi(F)$ in product of disjoint cycles. Moreover, the restriction of φ to the forests of $\mathbf{C}_{ho}$ is a bijective map with values in the set of permutations without fixed point.

In the planar case, we can obtain the formal series. Let $t_n^{\mathbf{C}_{NCK}}$ be the number of trees $\in \mathbf{C}_{NCK}$ of edges degree n and $f_n^{\mathbf{C}_{NCK}}$ be the number of forests $\in \mathbf{C}_{NCK}$ of edges degree n. We put $T_{\mathbf{C}_{NCK}}(x) = \sum_{k \geq 0} t_k^{\mathbf{C}_{NCK}} x^k$ and $F_{\mathbf{C}_{NCK}}(x) = \sum_{k \geq 0} f_k^{\mathbf{C}_{NCK}} x^k$. Then:

Proposition 21 *The formal series $T_{\mathbf{C}_{NCK}}$ and $F_{\mathbf{C}_{NCK}}$ are given by:*

$$T_{\mathbf{C}_{NCK}}(x) = \frac{1 - 2x - \sqrt{1-4x}}{2x}, \qquad F_{\mathbf{C}_{NCK}}(x) = \frac{2x}{4x - 1 + \sqrt{1-4x}}.$$

Proof. With formula (1), we deduce that:

$$T_{\mathbf{C}_{NCK}}(x) = \frac{1 - \sqrt{1-4x}}{2x} - 1 = \frac{1 - 2x - \sqrt{1-4x}}{2x}.$$

$\mathbf{C}_{NCK}$ is freely generated by the trees, therefore

$$F_{\mathbf{C}_{NCK}}(x) = \frac{1}{1-T_{\mathbf{C}_{NCK}}(x)} = \frac{2x}{4x-1+\sqrt{1-4x}}$$

□

Then for all $n \geq 1$ $t_n^{\mathbf{C}_{NCK}} = \frac{1}{n+1}\binom{2n}{n}$ is the n-Catalan number, $f_n^{\mathbf{C}_{NCK}} = \binom{2n-1}{n}$ and this gives:

n	1	2	3	4	5	6	7	8	9	10
$t_n^{\mathbf{C}_{NCK}}$	1	2	5	14	42	132	429	1430	4862	16796
$f_n^{\mathbf{C}_{NCK}}$	1	3	10	35	126	462	1716	6435	24310	92378

These are the sequences A000108 and A088218 in [Slo].

5 Hopf algebra morphisms

Recall that the tensor algebra $T(V)$ over a $\mathbb{K}$-vector space V is the tensor module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots$$

equipped with the concatenation.

Dually, the tensor coalgebra $T^c(V)$ over a $\mathbb{K}$-vector space V is the tensor module (as above) equiped with the coassociative coproduct $\Delta_{\mathcal{A}ss}$ called deconcatenation:

$$\Delta_{\mathcal{A}ss}((v_1 \ldots v_n)) = \sum_{i=0}^{n} (v_1 \ldots v_i) \otimes (v_{i+1} \ldots v_n).$$

We will say that a bialgebra $\mathbf{H}$ is cofree if, as a coalgebra, it is isomorphic to $T^c(Prim(\mathbf{H}))$ (for more details, see [LR06]).

We give the useful following lemma:

Lemma 22 *Let (A, Δ, ε) be a cofree Hopf algebra. Then*

$$Ker(\tilde{\Delta} \otimes Id_A - Id_A \otimes \tilde{\Delta}) = Im(\tilde{\Delta}).$$

Proof. Indeed, if $x = \sum a_{w,w'} w \otimes w' \in Ker(\tilde{\Delta} \otimes Id_A - Id_A \otimes \tilde{\Delta})$,

$$\sum_{w_1 w_2 = w} a_{w,w'} w_1 \otimes w_2 \otimes w' = \sum_{w_1' w_2' = w'} a_{w,w'} w \otimes w_1' \otimes w_2'.$$

So $a_{w_1 w_2, w_3} = a_{w_1, w_2 w_3}$ for all words w_1, w_2, w_3 different from the unit. We put $b_{ww'} = a_{w,w'}$. Then

$$x = \sum b_w \left(\sum_{w_1 w_2 = w} w_1 \otimes w_2 \right) = \tilde{\Delta}\left(\sum b_w w \right) \in Im(\tilde{\Delta}).$$

The coassociativity of $\tilde{\Delta}$ implies the other inclusion. □

5.1 From $\mathbf{H}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map.

Theorem 23 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{H}_{CK}^{\mathcal{D}} \to \mathbf{Sh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{array}{ccc} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}\right) & \xrightarrow{\varphi} & \mathbb{K}(\mathcal{D}) \\ \Big\downarrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle \pi} \\ \mathbf{H}_{CK}^{\mathcal{D}} & \xrightarrow{\Phi} & \mathbf{Sh}^{\mathcal{D}} \end{array} \tag{11}$$

is commutative.

Proof. Existence: We define Φ by induction on the number of vertices. We put $\Phi(1) = 1 \otimes 1$ and $\Phi(\bullet_a) = \varphi(\bullet_a)$ for all $a \in \mathcal{D}$. Suppose that Φ is defined for all forest F of vertices degree $< n$ and satisfies the condition $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) = \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F)$. Let $F \in \mathbf{H}_{CK}^{\mathcal{D}}$ be a forest of vertices degree n. If $F = F_1F_2$, we put $\Phi(F) = \Phi(F_1)\Phi(F_2)$. Suppose that F is a tree. By induction hypothesis, $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F)$ is well defined. Moreover,

$$\begin{aligned} & (\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \otimes Id_{\mathbf{Sh}^{\mathcal{D}}} - Id_{\mathbf{Sh}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}) \circ (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \\ = \; & (\Phi \otimes \Phi \otimes \Phi) \circ (\tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}} \otimes Id_{\mathbf{H}_{CK}^{\mathcal{D}}} - Id_{\mathbf{H}_{CK}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \\ = \; & 0, \end{aligned}$$

using induction hypothesis in the first equality and the coassociativity in the second equality.

So $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \in Ker(\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \otimes Id_{\mathbf{Sh}^{\mathcal{D}}} - Id_{\mathbf{Sh}^{\mathcal{D}}} \otimes \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}})$. As $\mathbf{Sh}^{\mathcal{D}}$ is cofree, with lemma 22, $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) \in Im(\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}})$ and there exists $w \in \mathbf{Sh}^{\mathcal{D}}$ such that $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F) = \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w)$. We put $\Phi(F) = w - \pi(w) + \varphi(F)$. Then

$$\begin{aligned} \pi \circ \Phi(F) &= \pi(w) - \pi \circ \pi(w) + \pi \circ \varphi(F) = \varphi(F), \\ \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w) - \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(\pi(w)) + \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(\varphi(F)) \\ &= \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}}(w) \\ &= (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(F). \end{aligned}$$

By induction, the result is established.

Uniqueness: Let Φ_1 and Φ_2 be two Hopf algebra morphisms such that the diagram (11) is commutative. Let us prove that $\Phi_1(T) = \Phi_2(T)$ for all tree $T \in \mathbf{H}_{CK}^{\mathcal{D}}$ by induction on the vertices degree of T. If $n = 0$, $\Phi_1(1) = \Phi_2(1) = 1$. If $n = 1$, for $i = 1, 2$, $\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_i(\bullet_a) = (\Phi_i \otimes \Phi_i) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(\bullet_a) = 0$. So $\Phi_i(\bullet_a) \in Vect(\mathcal{D})$. As the diagram (11) is commutative, $\Phi_1(\bullet_a) = \Phi_2(\bullet_a) = \varphi(\bullet_a)$. Suppose that the result is true in vertices degree $< n$ and let T be a tree of vertices degree n. Using induction hypothesis in the second equality,

$$\begin{aligned} \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_1(T) &= (\Phi_1 \otimes \Phi_1) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T) \\ &= (\Phi_2 \otimes \Phi_2) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T) \\ &= \tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi_2(T). \end{aligned}$$

So $\Phi_1(T) - \Phi_2(T) \in Vect(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}})$ and $\Phi_1(T) - \Phi_2(T) = \pi(\Phi_1(T) - \Phi_2(T)) = \varphi(T) - \varphi(T) = 0$. □

Notation. We consider $F \in \mathbf{H}_{CK}$, $\boldsymbol{e} \models E(F)$ and $\sigma \in \mathcal{O}(Cont_{\boldsymbol{e}}(F))$ a linear order on $Cont_{\boldsymbol{e}}(F)$ (see definition 3). For all $i \in \{1, \dots, |Cont_{\boldsymbol{e}}(F)|_v\}$, $\sigma^{-1}(i)$ is the connected component of $Part_{\boldsymbol{e}}(F)$ such that her image by σ is equal to i.

The following proposition give a combinatorial description of the morphism Φ defined in theorem 23 :

Proposition 24 *Let T be a nonempty tree $\in \mathbf{H}_{CK}^{\mathcal{D}}$. Then*

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\sigma \in \mathcal{O}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(1)) \dots \varphi(\sigma^{-1}(|Cont_{\boldsymbol{e}}(F)|_v)) \right). \tag{12}$$

Proof. We use the following lemma:

Lemma 25 *Let T be a rooted tree of vertices degree n. We define :*

$$\begin{aligned} \mathbb{E}(T) &= \{(\boldsymbol{v}, \sigma_1, \sigma_2) \mid \boldsymbol{v} \Vdash V(T), \sigma_1 \in \mathcal{O}(Lea_{\boldsymbol{v}}(T)), \sigma_2 \in \mathcal{O}(Roo_{\boldsymbol{v}}(T))\}, \\ \mathbb{F}(T) &= \{(\sigma, p) \mid \sigma \in \mathcal{O}(T), p \in \{1, \dots, n-1\}\}. \end{aligned}$$

Then $\mathbb{E}(T)$ and $\mathbb{F}(T)$ are in bijection.

Proof. We define two maps f and g.

Let f be the map defined by

$$f : \left\{ \begin{array}{rcl} \mathbb{E}(T) & \to & \mathbb{F}(T) \\ (\boldsymbol{v}, \sigma_1, \sigma_2) & \mapsto & (\sigma, |Roo_{\boldsymbol{v}}(T)|_v) \end{array} \right.$$

where $\sigma : V(T) \to \{1, \dots, n\}$ is defined by $\sigma(v) = \sigma_2(v)$ for all $v \in V(Roo_{\boldsymbol{v}}(T))$ and $\sigma(v) = \sigma_1(v) + |Roo_{\boldsymbol{v}}(T)|_v$ for all $v \in V(Lea_{\boldsymbol{v}}(T))$. By definition, $\sigma \in \mathcal{O}(T)$.

Let g be the map defined by

$$g : \left\{ \begin{array}{rcl} \mathbb{F}(T) & \to & \mathbb{E}(T) \\ (\sigma, p) & \mapsto & (\boldsymbol{v}, \sigma_1, \sigma_2) \end{array} \right.$$

where

- $\sigma_1 : V(Lea_{\boldsymbol{v}}(T)) \to \{1, \dots, |Lea_{\boldsymbol{v}}(T)|_v\}$ is defined by $\sigma_1(v) = \sigma(v) - |Roo_{\boldsymbol{v}}(T)|_v$ for all $v \in V(Lea_{\boldsymbol{v}}(T))$. Then $\sigma_1 \in \mathcal{O}(Lea_{\boldsymbol{v}}(T))$
- $\sigma_2 : V(Roo_{\boldsymbol{v}}(T)) \to \{1, \dots, |Roo_{\boldsymbol{v}}(T)|_v\}$ is defined by $\sigma_2(v) = \sigma(v)$ for all $v \in V(Roo_{\boldsymbol{v}}(T))$. Then $\sigma_2 \in \mathcal{O}(Roo_{\boldsymbol{v}}(T))$
- $\boldsymbol{v}$ is the subset $\left\{ v \in \sigma^{-1}(\{k, \dots, n\}) \mid \text{if } w \in \sigma^{-1}(\{k, \dots, n\}) \text{ and } v \twoheadrightarrow w \text{ then } v = w \right\}$ of $V(T)$. We have $\boldsymbol{v} \Vdash V(T)$.

So f and g are well defined. Then we show easily that $f \circ g = Id_{\mathbb{F}(T)}$ and $g \circ f = Id_{\mathbb{E}(T)}$. □

Let us show formula (12) by induction on the number n of vertices. If $n = 1$, $T = \bullet_a$ with $a \in \mathcal{D}$. Then $\Phi(\bullet_a) = \varphi(\bullet_a)$ and formula (12) is true. If $n \geq 2$,

$$\begin{aligned}
&\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(T)\\
=\ & (\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)\\
=\ & \sum_{\boldsymbol{v} \models V(T)} \Phi(Lea_{\boldsymbol{v}}(T)) \otimes \Phi(Roo_{\boldsymbol{v}}(T))\\
=\ & \sum_{\boldsymbol{v} \models V(T)} \left(\sum_{\boldsymbol{e} \models E(Lea_{\boldsymbol{v}}(T))} \left(\sum_{\sigma_1 \in \mathcal{O}(Cont_{\boldsymbol{e}}(Lea_{\boldsymbol{v}}(T)))} \varphi(\sigma_1^{-1}(1)) \dots \varphi(\sigma_1^{-1}(|Cont_{\boldsymbol{e}}(Lea_{\boldsymbol{v}}(T))|_v)) \right) \right)\\
& \otimes \left(\sum_{\boldsymbol{f} \models E(Roo_{\boldsymbol{v}}(T))} \left(\sum_{\sigma_2 \in \mathcal{O}(Cont_{\boldsymbol{f}}(Roo_{\boldsymbol{v}}(T)))} \varphi(\sigma_2^{-1}(1)) \dots \varphi(\sigma_2^{-1}(|Cont_{\boldsymbol{f}}(Roo_{\boldsymbol{v}}(T))|_v)) \right) \right)\\
=\ & \sum_{\boldsymbol{e} \models E(T)} \sum_{(\boldsymbol{v},\sigma_1,\sigma_2) \in \mathbb{E}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma_1^{-1}(1)) \dots \varphi(\sigma_1^{-1}(|Lea_{\boldsymbol{v}}(Cont_{\boldsymbol{e}}(T))|_v))\\
& \otimes \varphi(\sigma_2^{-1}(1)) \dots \varphi(\sigma_2^{-1}(|Roo_{\boldsymbol{v}}(Cont_{\boldsymbol{e}}(T))|_v))\\
=\ & \sum_{\boldsymbol{e} \models E(T)} \sum_{(\sigma,p) \in \mathbb{F}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(1)) \dots \varphi(\sigma^{-1}(p)) \otimes \varphi(\sigma^{-1}(p+1)) \dots \varphi(\sigma^{-1}(|Cont_{\boldsymbol{e}}(T)|_v)).
\end{aligned}$$

So

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\sigma \in \mathcal{O}(Cont_{\boldsymbol{e}}(T))} \varphi(\sigma^{-1}(1)) \dots \varphi(\sigma^{-1}(|Cont_{\boldsymbol{e}}(F)|_v)) \right)$$

and by induction, we have the result. □

Examples.

- In vertices degree 1, $\Phi(\bullet_a) = \varphi(\bullet_a)$.
- In vertices degree 2,

$$\begin{aligned}
\Phi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix}) &= \varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix})\\
\Phi(\bullet_a \bullet_b) &= \varphi(\bullet_a)\varphi(\bullet_b) + \varphi(\bullet_b)\varphi(\bullet_a).
\end{aligned}$$

- In vertices degree 3,

$$\begin{aligned}
\Phi(\begin{smallmatrix}\bullet_b & \bullet_c\\ & \bullet_a\end{smallmatrix}) &= \varphi(\bullet_b)\varphi(\bullet_c)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_b)\varphi(\begin{smallmatrix}\bullet_c\\ \bullet_a\end{smallmatrix}) + \varphi(\bullet_c)\varphi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix}) + \varphi(\begin{smallmatrix}\bullet_b & \bullet_c\\ & \bullet_a\end{smallmatrix})\\
\Phi(\begin{smallmatrix}\bullet_c\\ \bullet_b\\ \bullet_a\end{smallmatrix}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix}) + \varphi(\begin{smallmatrix}\bullet_c\\ \bullet_b\end{smallmatrix})\varphi(\bullet_a) + \varphi(\begin{smallmatrix}\bullet_c\\ \bullet_b\\ \bullet_a\end{smallmatrix})\\
\Phi(\begin{smallmatrix}\bullet_c & \bullet_d\\ \bullet_b & \\ & \bullet_a\end{smallmatrix}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_d)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a)\\
&\quad + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\begin{smallmatrix}\bullet_d\\ \bullet_a\end{smallmatrix}) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix}) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix}) + \varphi(\begin{smallmatrix}\bullet_c\\ \bullet_b\end{smallmatrix})\varphi(\bullet_d)\varphi(\bullet_a)\\
&\quad + \varphi(\bullet_d)\varphi(\begin{smallmatrix}\bullet_c\\ \bullet_b\end{smallmatrix})\varphi(\bullet_a) + \varphi(\begin{smallmatrix}\bullet_c\\ \bullet_b\end{smallmatrix})\varphi(\begin{smallmatrix}\bullet_d\\ \bullet_a\end{smallmatrix}) + \varphi(\bullet_c)\varphi(\begin{smallmatrix}\bullet_b & \bullet_d\\ & \bullet_a\end{smallmatrix}) + \varphi(\bullet_d)\varphi(\begin{smallmatrix}\bullet_c\\ \bullet_b\\ \bullet_a\end{smallmatrix}) + \varphi(\begin{smallmatrix}\bullet_c & \bullet_d\\ \bullet_b & \\ & \bullet_a\end{smallmatrix}).
\end{aligned}$$

Particular case. If $\varphi(\bullet_a) = a$ for all $a \in \mathcal{D}$ and $\varphi(T) = 0$ if $|T|_v \geq 1$, then this is the particular case of arborification (see [EV04]). For example :

$$\begin{array}{lll|lll|lll}
\Phi(\bullet_a) &=& a & \Phi(\begin{smallmatrix}\bullet_b\\ \bullet_a\end{smallmatrix}) &=& ba & \Phi(\bullet_a \bullet_b) &=& ab + ba\\
\Phi(\begin{smallmatrix}\bullet_b & \bullet_c\\ & \bullet_a\end{smallmatrix}) &=& bca + cba & \Phi(\begin{smallmatrix}\bullet_c\\ \bullet_b\\ \bullet_a\end{smallmatrix}) &=& cba & \Phi(\begin{smallmatrix}\bullet_c & \bullet_d\\ \bullet_b & \\ & \bullet_a\end{smallmatrix}) &=& cbda + cdba + dcba.
\end{array}$$

5.2 From $\mathbf{H}_{CK}^{\mathcal{D}}$ to $\mathbf{Csh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map. We suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b) \in \mathcal{D}^2 \mapsto [ab] \in \mathcal{D}$.

Theorem 26 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{H}_{CK}^{\mathcal{D}} \to \mathbf{Csh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{array}{ccc} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{H}_{CK}}\right) & \xrightarrow{\varphi} & \mathbb{K}(\mathcal{D}) \\ \Big\downarrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle \pi} \\ \mathbf{H}_{CK}^{\mathcal{D}} & \xrightarrow{\Phi} & \mathbf{Csh}^{\mathcal{D}} \end{array} \tag{13}$$

is commutative.

Proof. Noting that $\mathbf{Csh}^{\mathcal{D}}$ is cofree, this is the same proof as for theorem 23. □

Notation. Let $F \in \mathbf{H}_{CK}$ be a nonempty rooted forest, $\boldsymbol{e} \models E(F)$ and $\sigma \in \mathcal{O}_p(Cont_{\boldsymbol{e}}(F))$ a lineair preorder on $Cont_{\boldsymbol{e}}(F)$ (see definition 7), $\sigma : V(Cont_{\boldsymbol{e}}(F)) \to \{1,\dots,q\}$ surjective. For all $i \in \{1,\dots,q\}$, $\sigma^{-1}(i)$ is the forest $T_1 \dots T_n$ of all connected components T_k such that $\sigma(T_k) = i$ for all $k \in \{1,\dots,n\}$. In this case, $\varphi(\sigma^{-1}(i))$ is the element $[\varphi(T_1)\dots\varphi(T_n)]^{(n)}$.

Now, we give a combinatorial description of the morphism Φ defined in theorem 26:

Proposition 27 *Let T be a nonempty tree $\in \mathbf{H}_{CK}^{\mathcal{D}}$. Then*

$$\Phi(T) = \sum_{\boldsymbol{e} \models E(T)} \left(\sum_{\substack{\sigma \in \mathcal{O}_p(Cont_{\boldsymbol{e}}(T)) \\ Im(\sigma) = \{1,\dots,q\}}} \varphi(\sigma^{-1}(1)) \dots \varphi(\sigma^{-1}(q)) \right). \tag{14}$$

Proof. It suffices to resume the proof of proposition 24. Note that, if T is a rooted tree and $\boldsymbol{v} \models V(T)$, $Roo_{\boldsymbol{v}}(T)$ is a tree and $Lea_{\boldsymbol{v}}(T)$ is a forest. So there is possibly contractions for the product $[\cdot,\cdot]$ to the left of $(\Phi \otimes \Phi) \circ \tilde{\Delta}_{\mathbf{H}_{CK}^{\mathcal{D}}}(T)$. We deduce formula (14). □

Examples.

- In vertices degree 1, $\Phi(\bullet_a) = \varphi(\bullet_a)$.
- In vertices degree 2,

$$\begin{aligned} \Phi(\bullet_a \bullet_b) &= \varphi(\bullet_a)\varphi(\bullet_b) + \varphi(\bullet_b)\varphi(\bullet_a) + [\varphi(\bullet_a)\varphi(\bullet_b)] \\ \Phi(\overset{b}{\underset{a}{\bullet}}) &= \varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\overset{b}{\underset{a}{\bullet}}). \end{aligned}$$

- In vertices degree 3,

$$\begin{aligned} \Phi(\overset{b}{\underset{a}{\bullet}}\!\!\vee\!\!\bullet_c) &= \varphi(\bullet_b)\varphi(\bullet_c)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + [\varphi(\bullet_c)\varphi(\bullet_b)]\,\varphi(\bullet_a) + \varphi(\bullet_b)\varphi(\overset{c}{\underset{a}{\bullet}}) \\ &\quad + \varphi(\bullet_c)\varphi(\overset{b}{\underset{a}{\bullet}}) + \varphi(\overset{b}{\underset{a}{\bullet}}\!\!\vee\!\!\bullet_c) \\ \Phi(\overset{c}{\underset{b}{\overset{|}{\bullet}}}\!\!\!\!\underset{a}{\bullet}) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\overset{b}{\underset{a}{\bullet}}) + \varphi(\overset{c}{\underset{b}{\bullet}})\varphi(\bullet_a) + \varphi(\overset{c}{\underset{b}{\overset{|}{\bullet}}}\!\!\!\!\underset{a}{\bullet}) \\ \Phi(\overset{c}{\underset{b}{\overset{|}{\bullet}}}\!\!\!\!\underset{a}{\vee}\!\!\bullet_d) &= \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_d)\varphi(\bullet_a) + \varphi(\bullet_c)\,[\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\bullet_b)\varphi(\bullet_a) \\ &\quad + [\varphi(\bullet_c)\varphi(\bullet_d)]\,\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\bullet_b)\varphi(\bullet_a) + \varphi(\bullet_c)\varphi(\bullet_b)\varphi(\overset{d}{\underset{a}{\bullet}}) \\ &\quad + \varphi(\bullet_c)\varphi(\bullet_d)\varphi(\overset{b}{\underset{a}{\bullet}}) + \varphi(\bullet_d)\varphi(\bullet_c)\varphi(\overset{b}{\underset{a}{\bullet}}) + \varphi(\overset{c}{\underset{b}{\bullet}})\varphi(\bullet_d)\varphi(\bullet_a) \\ &\quad + [\varphi(\overset{c}{\underset{b}{\bullet}})\varphi(\bullet_d)]\,\varphi(\bullet_a) + \varphi(\bullet_d)\varphi(\overset{c}{\underset{b}{\bullet}})\varphi(\bullet_a) + \varphi(\overset{c}{\underset{b}{\bullet}})\varphi(\overset{d}{\underset{a}{\bullet}}) + \varphi(\bullet_c)\varphi(\overset{b}{\underset{a}{\bullet}}\!\!\vee\!\!\bullet_d) \\ &\quad + \varphi(\bullet_d)\varphi(\overset{c}{\underset{b}{\overset{|}{\bullet}}}\!\!\!\!\underset{a}{\bullet}) + \varphi(\overset{c}{\underset{b}{\overset{|}{\bullet}}}\!\!\!\!\underset{a}{\vee}\!\!\bullet_d). \end{aligned}$$

5.3 From $\mathbf{C}_{CK}^{\mathcal{D}}$ to $\mathbf{Sh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map.

Theorem 28 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{C}_{CK}^{\mathcal{D}} \to \mathbf{Sh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{array}{ccc} \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) & \xrightarrow{\varphi} & \mathbb{K}(\mathcal{D}) \\ \Big\downarrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle \pi} \\ \mathbf{C}_{CK}^{\mathcal{D}} & \xrightarrow{\Phi} & \mathbf{Sh}^{\mathcal{D}} \end{array} \tag{15}$$

is commutative.

Proof. This is the same proof as for theorem 23. □

As in the sections 5.1 and 5.2, we give a combinatorial description of the morphism Φ defined in theorem 28. We need the following definition:

Definition 29 *Let F be a nonempty rooted forest of $\mathbf{C}_{CK}$. A generalized partition of F is a k-uplet $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_k)$ of subsets of $E(F)$, $1 \leq k \leq |F|_e$, such that:*

1. *$\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ if $i \neq j$ and $\cup_i \boldsymbol{e}_i = E(F)$,*
2. *the edges $\in \boldsymbol{e}_i$ are the edges of the same connected component of F,*
3. *if v and w are two vertices of $Part_{\boldsymbol{e}_i}(F)$ and if the shortest path in F between v and w contains an edge $\in \boldsymbol{e}_j$, then $j < i$.*

We shall denote by $\mathcal{P}(F)$ the set of generalized partitions of F.

Remark. If F is a nonempty rooted forest and if $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_k) \in \mathcal{P}(F)$, $Cont_{\overline{\boldsymbol{e}_i}}(F) = Part_{\boldsymbol{e}_i}(F)$ is a tree for all i (with the second point of the definition 29).

Proposition 30 *Let F be a nonempty forest $\in \mathbf{C}_{CK}^{\mathcal{D}}$. Then*

$$\Phi(F) = \sum_{(\boldsymbol{e}_1,\dots,\boldsymbol{e}_k)\in\mathcal{P}(F)} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F)) \dots \varphi(Cont_{\overline{\boldsymbol{e}_k}}(F)). \tag{16}$$

Proof. We use the following lemma:

Lemma 31 *If $F \in \mathbf{C}_{CK}^{\mathcal{D}}$ is a nonempty tree, then the sets*

$$\begin{array}{rcl} \mathbb{E}(F) & = & \{((\boldsymbol{e}_1, \dots, \boldsymbol{e}_k), p) \mid (\boldsymbol{e}_1, \dots, \boldsymbol{e}_k) \in \mathcal{P}(F), 1 \leq p \leq k-1\} \\ \mathbb{F}(F) & = & \{\big(\boldsymbol{e}, (\boldsymbol{f}_1, \dots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \dots, \boldsymbol{g}_r)\big) \mid \boldsymbol{e} \Vdash E(F), (\boldsymbol{f}_1, \dots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F)), \\ & & (\boldsymbol{g}_1, \dots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))\} \end{array}$$

are in bijections.

Proof. Consider the following two maps:

$$f : \left\{ \begin{array}{rcl} \mathbb{E}(F) & \to & \mathbb{F}(F) \\ ((\boldsymbol{e}_1, \dots, \boldsymbol{e}_k), p) & \mapsto & (\cup_{1\leq i\leq p}\boldsymbol{e}_i, (\boldsymbol{e}_1, \dots, \boldsymbol{e}_p), (\boldsymbol{e}_{p+1}, \dots, \boldsymbol{e}_k)) \end{array} \right.$$

and

$$g : \left\{ \begin{array}{rcl} \mathbb{F}(F) & \to & \mathbb{E}(F) \\ \big(\boldsymbol{e}, (\boldsymbol{f}_1, \dots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \dots, \boldsymbol{g}_r)\big) & \mapsto & \big((\boldsymbol{f}_1, \dots, \boldsymbol{f}_q, \boldsymbol{g}_1, \dots, \boldsymbol{g}_r), q\big). \end{array} \right.$$

f is well defined :

Let $((\boldsymbol{e}_1, \dots, \boldsymbol{e}_k), p) \in \mathbb{E}(F)$. Then $\boldsymbol{e} = \cup_{1\leq i\leq p}\boldsymbol{e}_i$ is a nonempty nontotal contraction of F.

1. (a) $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p)$ is a p-uplet of subsets of $E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$. By hypothesis, $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$. So $\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ and $\cup_{1 \leq i \leq p} \boldsymbol{e}_i = E(Part_{\boldsymbol{e}}(F))$.
   (b) The edges $\in \boldsymbol{e}_i$, $1 \leq i \leq p$, are the edges of the same connected component of F therefore of $Part_{\boldsymbol{e}}(F)$ because $\boldsymbol{e}_i \subseteq \boldsymbol{e}$.
   (c) Let v and w be two vertices of $Part_{\boldsymbol{e}_i}(Part_{\boldsymbol{e}}(F)) = Part_{\boldsymbol{e}_i}(F)$ (because $\boldsymbol{e}_i \subseteq \boldsymbol{e}$). If the shortest path in $Part_{\boldsymbol{e}}(F)$ between v and w contains an edge $\in \boldsymbol{e}_j$, then the shortest path in F between v and w contains also an edge $\in \boldsymbol{e}_j$. As $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$, we have $j < i$.

   So $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))$.

2. (a) $(\boldsymbol{e}_{p+1}, \ldots, \boldsymbol{e}_k)$ is a $(k-p)$-uplet of subsets of $E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$. By hypothesis, $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$. So $\boldsymbol{e}_i \neq \emptyset$, $\boldsymbol{e}_i \cap \boldsymbol{e}_j = \emptyset$ and $\cup_{p+1 \leq i \leq k} \boldsymbol{e}_i = E(Cont_{\boldsymbol{e}}(F))$.
   (b) The edges $\in \boldsymbol{e}_i$, $p+1 \leq i \leq k$, are the edges of the same connected component of F therefore of $Cont_{\boldsymbol{e}}(F)$ (we contract in F some connected components).
   (c) Let i be an integer $\in \{p+1, \ldots, k\}$ and v and w two vertices of $Part_{\boldsymbol{e}_i}(Cont_{\boldsymbol{e}}(F)) = Part_{\boldsymbol{e}_i}(F)$ (because $\boldsymbol{e}_i \cap \boldsymbol{e} = \emptyset$). If the shortest path in $Cont_{\boldsymbol{e}}(F)$ between v and w contains an edge $\in \boldsymbol{e}_j$ then the shortest path in F between v and w contains also an edge $\in \boldsymbol{e}_j$. As $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)$, we have $j < i$.

   Thus $(\boldsymbol{e}_{p+1}, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))$.

So $f\left(((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p)\right) \in \mathbb{F}(F)$.

g is well defined :

Let $\big(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\big) \in \mathbb{F}(F)$. Let us show that $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(F)$.

1. As $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))$ and $(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))$, $\boldsymbol{f}_i \neq \emptyset$, $\boldsymbol{g}_i \neq \emptyset$, $\boldsymbol{f}_i \cap \boldsymbol{f}_j = \emptyset$, $\boldsymbol{g}_i \cap \boldsymbol{g}_j = \emptyset$ and $(\cup_i \boldsymbol{f}_i) \bigcup (\cup_i \boldsymbol{g}_i) = E(Part_{\boldsymbol{e}}(F)) \bigcup E(Cont_{\boldsymbol{e}}(F)) = E(F)$. In addition, as $\boldsymbol{f}_i \subseteq E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $\boldsymbol{g}_j \subseteq E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$, $\boldsymbol{f}_i \cap \boldsymbol{g}_j = \emptyset$.

2. The edges $\in \boldsymbol{f}_i$ are the edges of the same connected component of $Part_{\boldsymbol{e}}(F)$. As all the trees of the forest $Part_{\boldsymbol{e}}(F)$ are subtrees of F, the edges $\in \boldsymbol{f}_i$ are the edges of the same connected component of F. Moreover if the edges $\in \boldsymbol{g}_i$ are the edges of the same connected component of $Cont_{\boldsymbol{e}}(F)$, it is also true in F.

3. (a) Let i be an integer $\in \{1, \ldots, q\}$ and v and w two vertices of $Part_{\boldsymbol{f}_i}(F)$. We have $\boldsymbol{f}_i \subseteq \boldsymbol{e}$ therefore $Part_{\boldsymbol{f}_i}(F) = Part_{\boldsymbol{f}_i}(Part_{\boldsymbol{e}}(F))$. If the shortest path in F between v and w contains:
      i. an edge $\in \boldsymbol{f}_j$. As $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))$, $j < i$.
      ii. an edge $\in \boldsymbol{g}_j$. Then the connected component of $Part_{\boldsymbol{e}}(F)$ containing v and w has an edge $\in \boldsymbol{g}_j$. This is impossible because $E(Part_{\boldsymbol{e}}(F)) = \boldsymbol{e}$ and $\boldsymbol{g}_j \subseteq E(Cont_{\boldsymbol{e}}(F)) = \overline{\boldsymbol{e}}$.
   (b) Let i be an integer $\in \{1, \ldots, r\}$ and v and w two vertices of $Part_{\boldsymbol{g}_i}(F)$. $\boldsymbol{g}_i \cap \boldsymbol{e} = \emptyset$ therefore $Part_{\boldsymbol{g}_i}(F) = Part_{\boldsymbol{g}_i}(Cont_{\boldsymbol{e}}(F))$. If the shortest path in F between v and w contains:
      i. an edge $\in \boldsymbol{g}_j$. As $(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))$, $j < i$.
      ii. an edge $\in \boldsymbol{f}_j$. It is good because $\boldsymbol{f}_j$ is before $\boldsymbol{g}_i$.

   Thus $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(F)$.

So $g\left(\big(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\big)\right) \in \mathbb{E}(F)$

Finally, we easily see that $f \circ g = Id_{\mathbb{F}(F)}$ and $g \circ f = Id_{\mathbb{E}(F)}$. □

We now prove proposition 30. By induction on the edges degree n of $F \in \mathbf{C}_{CK}^{\mathcal{D}}$. If $n = 1$, $F = \uparrow^a$ with $a \in \mathcal{D}$. Then $\Phi(\uparrow^a) = \varphi(\uparrow^a)$ and formula (16) is true. Suppose that $n \geq 2$ and that the property is true in degrees $k < n$. Then

$$\begin{aligned}
\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= (\Phi \circ \Phi) \circ \tilde{\Delta}_{\mathbf{C}_{CK}^{\mathcal{D}}}(F) \\
&= \sum_{\boldsymbol{e} \models E(F)} \Phi\left(Part_{\boldsymbol{e}}(F)\right) \otimes \Phi\left(Cont_{\boldsymbol{e}}(F)\right) \\
&= \sum_{\boldsymbol{e} \models E(F)} \left(\sum_{(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q) \in \mathcal{P}(Part_{\boldsymbol{e}}(F))} \varphi(Cont_{\overline{\boldsymbol{f}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{f}_q}}(F)) \right) \\
&\qquad \otimes \left(\sum_{(\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r) \in \mathcal{P}(Cont_{\boldsymbol{e}}(F))} \varphi(Cont_{\overline{\boldsymbol{g}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{g}_r}}(F)) \right)
\end{aligned}$$

using induction hypothesis in the last equality. So, with lemma 31,

$$\begin{aligned}
\tilde{\Delta}_{\mathbf{Sh}^{\mathcal{D}}} \circ \Phi(F) &= \sum_{\left(\boldsymbol{e}, (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_q), (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r)\right) \in \mathbb{F}(F)} \varphi(Cont_{\overline{\boldsymbol{f}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{f}_q}}(F)) \\
&\qquad \otimes \varphi(Cont_{\overline{\boldsymbol{g}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{g}_r}}(F)) \\
&= \sum_{((\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k), p) \in \mathbb{E}(F)} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{e}_p}}(F)) \\
&\qquad \otimes \varphi(Cont_{\overline{\boldsymbol{e}_{p+1}}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{e}_k}}(F)) \\
&= \sum_{(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)} \sum_{1 \leq p \leq k-1} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{e}_p}}(F)) \\
&\qquad \otimes \varphi(Cont_{\overline{\boldsymbol{e}_{p+1}}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{e}_k}}(F)).
\end{aligned}$$

Therefore

$$\Phi(F) = \sum_{(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k) \in \mathcal{P}(F)} \varphi(Cont_{\overline{\boldsymbol{e}_1}}(F)) \ldots \varphi(Cont_{\overline{\boldsymbol{e}_k}}(F))$$

and by induction, we have the result. □

Examples. We introduce a notation. If $w = w_1 \ldots w_n$ is a $\mathcal{D}$-word, we denote $\mathrm{Perm}(w)$ the sum of all $\mathcal{D}$-words whose letters are $w_1, \ldots, w_n$. For example, $\mathrm{Perm}(abc) = abc + acb + bac + bca + cab + cba$.

- In edges degree 1, $\Phi(\uparrow^a) = \varphi(\uparrow^a)$.
- In edges degree 2,

$$\begin{aligned}
\Phi(\ {}^{a}\!\vee^{b}\ ) &= \varphi(\ {}^{a}\!\vee^{b}\ ) + \varphi(\uparrow^a)\varphi(\uparrow^b) + \varphi(\uparrow^b)\varphi(\uparrow^a) \\
\Phi(\ {}^{b}_{a}\!\uparrow\ ) &= \varphi(\ {}^{b}_{a}\!\uparrow\ ) + \varphi(\uparrow^a)\varphi(\uparrow^b) + \varphi(\uparrow^b)\varphi(\uparrow^a).
\end{aligned}$$

- In edges degree 3,

$$\begin{aligned}
\Phi(\ {}^{a}\!\backslash^{b}\!\vee^{c}\ ) &= \varphi(\ {}^{a}\!\backslash^{b}\!\vee^{c}\ ) + \varphi(\uparrow^a)\varphi(\ {}^{b}\!\vee^{c}\ ) + \varphi(\ {}^{b}\!\vee^{c}\ )\varphi(\uparrow^a) + \varphi(\uparrow^b)\varphi(\ {}^{a}\!\vee^{c}\ ) + \varphi(\ {}^{a}\!\vee^{c}\ )\varphi(\uparrow^b) \\
&\quad + \varphi(\uparrow^c)\varphi(\ {}^{a}\!\vee^{b}\ ) + \varphi(\ {}^{a}\!\vee^{b}\ )\varphi(\uparrow^c) + \mathrm{Perm}(\varphi(\uparrow^a)\varphi(\uparrow^b)\varphi(\uparrow^c)) \\
\Phi(\ {}^{b}\!\vee^{c}_{a}\ ) &= \varphi(\ {}^{b}\!\vee^{c}_{a}\ ) + \varphi(\uparrow^a)\varphi(\ {}^{b}\!\vee^{c}\ ) + \varphi(\ {}^{b}\!\vee^{c}\ )\varphi(\uparrow^a) + \varphi(\uparrow^c)\varphi(\ {}^{b}_{a}\!\uparrow\ ) + \varphi(\ {}^{b}_{a}\!\uparrow\ )\varphi(\uparrow^c) \\
&\quad + \varphi(\uparrow^b)\varphi(\ {}^{c}_{a}\!\uparrow\ ) + \varphi(\ {}^{c}_{a}\!\uparrow\ )\varphi(\uparrow^b) + \mathrm{Perm}(\varphi(\uparrow^a)\varphi(\uparrow^b)\varphi(\uparrow^c)) \\
\Phi(\ {}^{c}\!\uparrow_{a}\!\vee^{b}\ ) &= \varphi(\ {}^{c}\!\uparrow_{a}\!\vee^{b}\ ) + \varphi(\uparrow^a)\varphi(\ {}^{a}\!\vee^{b}\ ) + \varphi(\uparrow^b)\varphi(\ {}^{c}_{a}\!\uparrow\ ) + \varphi(\ {}^{c}_{a}\!\uparrow\ )\varphi(\uparrow^b) + \varphi(\uparrow^c)\varphi(\ {}^{a}\!\vee^{b}\ ) \\
&\quad + \varphi(\ {}^{a}\!\vee^{b}\ )\varphi(\uparrow^c) + \mathrm{Perm}(\varphi(\uparrow^a)\varphi(\uparrow^b)\varphi(\uparrow^c)).
\end{aligned}$$

- Finally, in edges degree 4, with the tree  ,

$$\begin{aligned}\Phi(\,\cdot\!\vee\!\cdot\,) &= \varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)\\ &+\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)+\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)\\ &+\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\mathrm{Perm}(\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\,))\\ &+\mathrm{Perm}(\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\,))+\mathrm{Perm}(\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\,))\\ &+\mathrm{Perm}(\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\,))+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)\\ &+\varphi(\,\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)\varphi(\,\cdot\,)+\varphi(\,\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)\\ &+\varphi(\,\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\!\vee\!\cdot\,)+\mathrm{Perm}(\varphi(\,\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\,)\varphi(\,\cdot\,))\end{aligned}$$

5.4 From $\mathbf{C}^{\mathcal{D}}_{CK}$ to $\mathbf{Csh}^{\mathcal{D}}$

Let $\varphi : \mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) \to \mathbb{K}(\mathcal{D})$ be a $\mathbb{K}$-linear map. We suppose that $\mathcal{D}$ is equipped with an associative and commutative product $[\cdot,\cdot] : (a,b) \in \mathcal{D}^2 \mapsto [ab] \in \mathcal{D}$.

Theorem 32 *There exists a unique Hopf algebra morphism $\Phi : \mathbf{C}^{\mathcal{D}}_{CK} \to \mathbf{Csh}^{\mathcal{D}}$ such that the following diagram*

$$\begin{array}{ccc}\mathbb{K}\left(\mathbb{T}^{\mathcal{D}}_{\mathbf{C}_{CK}}\right) & \xrightarrow{\varphi} & \mathbb{K}(\mathcal{D}) \\ \Big\downarrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle \pi} \\ \mathbf{C}^{\mathcal{D}}_{CK} & \xrightarrow{\Phi} & \mathbf{Csh}^{\mathcal{D}}\end{array} \tag{17}$$

is commutative.

Proof. This is the same proof as for theorem 23. □

We give a combinatorial description of the morphism Φ defined in theorem 32. For this, we give the following definition:

Definition 33 *Let F be a nonempty rooted forest of $\mathbf{C}_{CK}$. A generalized and contracted partition of F is a l-uplet $(f_1,\dots,f_l)$ such that:*

1. *for all $1 \leq i \leq l$, $f_i = (\boldsymbol{e}^i_1,\dots,\boldsymbol{e}^i_{k_i})$ is a k_i-uplet of subsets of $E(F)$,*
2. *$(\boldsymbol{e}^1_1,\dots,\boldsymbol{e}^1_{k_1},\boldsymbol{e}^2_1,\dots,\boldsymbol{e}^l_{k_l}) \in \mathcal{P}(F)$,*
3. *if $Part_{\boldsymbol{e}^i_p}(F)$ and $Part_{\boldsymbol{e}^i_q}(F)$ are two disconnected components of F and if the shortest path in F between $Part_{\boldsymbol{e}^i_p}(F)$ and $Part_{\boldsymbol{e}^i_q}(F)$ contains an edge $\in \boldsymbol{e}^j_r$, then $j > i$.*

We shall denote by $\mathcal{P}_c(F)$ the set of generalized and contracted partitions of F.

Proposition 34 *Let F be a nonempty forest $\in \mathbf{C}^{\mathcal{D}}_{CK}$. Then*

$$\begin{aligned}\Phi(F) = \sum_{\substack{(f_1,\dots,f_l)\in\mathcal{P}_c(F)\\ f_i=(\boldsymbol{e}^i_1,\dots,\boldsymbol{e}^i_{k_i})}} \Bigg(&\left[\varphi(Cont_{\overline{\boldsymbol{e}^1_1}}(F))\dots\varphi(Cont_{\overline{\boldsymbol{e}^1_{k_1}}}(F))\right]^{(k_1)}\dots\\ &\dots\left[\varphi(Cont_{\overline{\boldsymbol{e}^l_1}}(F))\dots\varphi(Cont_{\overline{\boldsymbol{e}^l_{k_l}}}(F))\right]^{(k_l)}\Bigg).\end{aligned} \tag{18}$$

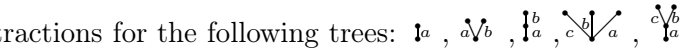
Proof. It suffices to resume the proof of proposition 30. Note that, if T is a rooted tree and $\boldsymbol{e} \models E(T)$, $Cont_{\boldsymbol{e}}(T)$ is a tree and $Part_{\boldsymbol{e}}(T)$ is a forest. So there is possibly contractions for the product $[\cdot,\cdot]$ to the left of $(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$. Remark that

- the trees of $Part_{\boldsymbol{e}}(T)$ are disconnected components of T and they appear to the left of $(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$,
- the edges of $\overline{\boldsymbol{e}}$ between two disconnected components of $Part_{\boldsymbol{e}}(T)$ in T are edges of $Cont_{\boldsymbol{e}}(T)$ and thus they appear to the right of $(\Phi\otimes\Phi)\circ\tilde{\Delta}_{\mathbf{H}^{\mathcal{D}}_{CK}}(T)$.

We deduce formula (18). □

Remark. In the expression of $\Phi(F)$ (formula (18)), we find the terms of (16) and other terms with contractions for the product $[\cdot,\cdot]$. Taking $[\cdot,\cdot]=0$, we obtain (16) again.

Examples. From the examples at the end of section 5.3, we give the other terms with contractions for the product $[\cdot,\cdot]$.

- There are no terms with contractions for the following trees:  .
- For the tree  ,

$$\Phi(\text{}) = \ldots + [\varphi(\bullet_b)\varphi(\bullet_c)]\,\varphi(\bullet_a)$$

- For the tree  ,

$$\begin{aligned}\Phi(\text{}) \;=\; & \ldots + [\varphi(\bullet_b)\varphi(\bullet_c)]\,\varphi(\text{  }) + [\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\text{}) + \Big[\varphi(\bullet_b)\varphi(\text{})\Big]\,\varphi(\bullet_a)\\ & + \varphi(\bullet_c)\,[\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\bullet_a) + [\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\bullet_a)\varphi(\bullet_c)\\ & + [\varphi(\bullet_b)\varphi(\bullet_d)]\,\varphi(\bullet_c)\varphi(\bullet_a) + [\varphi(\bullet_b)\varphi(\bullet_c)]\,\varphi(\bullet_a)\varphi(\bullet_d)\\ & + [\varphi(\bullet_b)\varphi(\bullet_c)]\,\varphi(\bullet_d)\varphi(\bullet_a) + \varphi(\bullet_d)\,[\varphi(\bullet_b)\varphi(\bullet_c)]\,\varphi(\bullet_a).\end{aligned}$$

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