# The three-mass model for the classical guitar revisited

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Input admittance and sound-pressure response functions for the classical guitar show several prominent peaks in the low-frequency range (80-250 Hz). The lowest three peaks can be modelled very effectively using a coupled three-mass model describing the interaction between the lowest modes of the soundboard and back plate and the air-cavity resonance (the Helmholtz resonance). Whilst there has been considerable qualitative or speculative discussion of the frequency placement of these modes, there have been very few quantitative studies which have attempted to identify the important acoustical features of these peaks. It is known that the frequency placement of strong peaks influences the "local" response of the played instrument, but psychoacoustical studies have shown that the "residual response" of these peaks has a "global" influence at frequencies above the resonances. In this study, we are using a three-mass model for the guitar coupled to a lossy string. The model generates plucked-string sounds which can then used for psychoacoustical evaluation of the relative influence of parameters such as plate mass, stiffness, damping and radiativity on the perceived sound quality of the guitar. This work-in-progress discusses some of the theoretical aspects of the current study.

# 1 Introduction

Frequency response functions (input admittance and sound pressure response) of classical guitars show several prominent peaks in the low-frequency range (from 80 to 250 Hz). Peaks in the higher frequency ranges tend to centre around a substantially lower value and occur at a rate of about two peaks per 100 Hz until modal overlap obscures individual resonances.

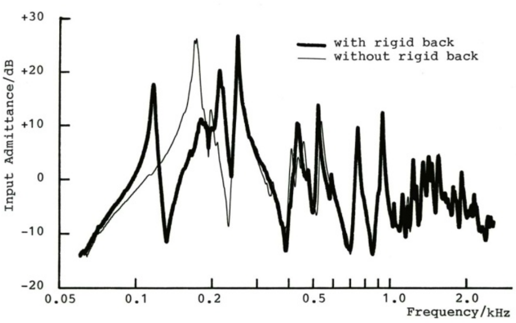
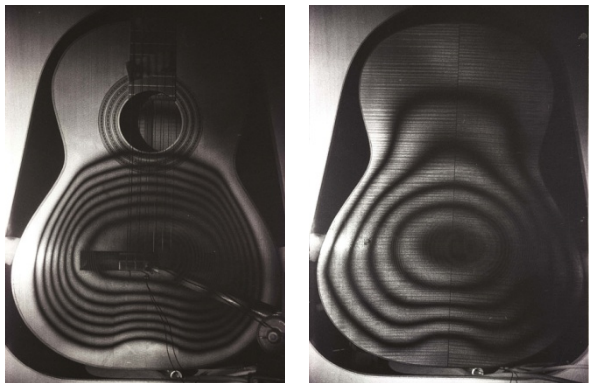



Figure 1: Input admittance of a guitar measured near the first string position on the bridge. The reference level is 1 s/kg. The figure compares the response of the top plate with and without a backing cavity. [1]

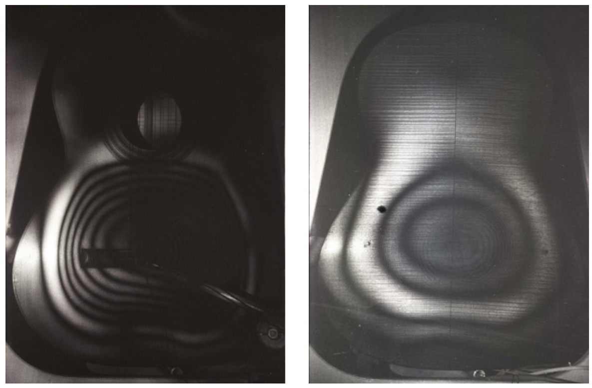
A typical response function is shown in Figure 1. These input admittance curves were measured on an experimental guitar with rigid ribs on which the back plate could be removed. The figure compares the response of the top plate (soundboard) alone with the same plate with added rigid back plate and air cavity. Coupling between the fundamental mode of the top plate and the Helmholtz resonance of the ported air cavity splits the single resonance of the top plate into the two peaks seen here at around 100 Hz and 230 Hz. Coupling between the top plate, a flexible back plate and other internal air-cavity modes are explored experimentally in [1] and theoretically in [2].

The action described above is akin to that of the well-documented bass-reflex action in ported loudspeakers. A number of models for the guitar's low-frequency action for a two-mass system (top plate and air cavity) or three-mass system (top plate, back plate and air cavity) were developed thirty years ago [3,4,5] and more recently the subject has been revisited with the development of four-mass models [6], the latter including elements which account in part for mobility of the ribs of the instrument.

The three-mass model predicts three resonances of the guitar with similar operational deflection shapes but with various phase relationships between the displacements of the top plate, back plate and "air plug" in the sound hole.



(a)



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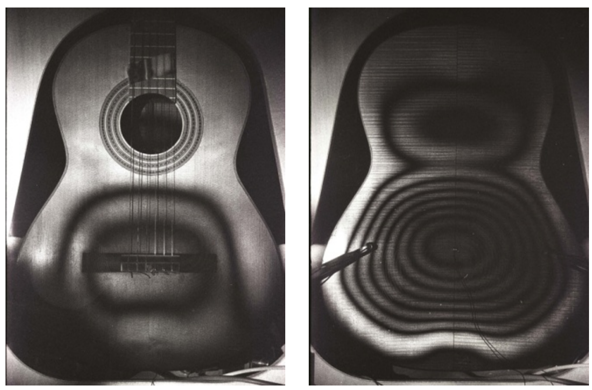


(c)

Figure 2: The first three resonances (coupled modes) of a guitar visualised using holographic interferometry. The very bright fringes are nodes.

Figure 2 shows a visualisation of the vibration amplitude of the plates at the three lowest resonances of a classical guitar. In each pair of these figures the same excitation force was used when recording the motion of the top plate and back plate. In (a) the two plates vibrate in phase (swell outward together) but out of phase with the air motion in the sound hole, i.e. as the plates swell out together air rushes into the body cavity. In (c) the plates again vibrate in phase but there is a phase reversal of the air motion in the sound hole, and in (b) the plates vibrate out of phase. The relative amplitudes of the motions of the two plates (as observed in Figure 2) and that of the "plug of air" in the sound hole and their relative phases determine the overall volume velocity of the instrument as a whole from which the monopole radiation can be estimated. It will be apparent from Figure 2 that each of these resonances couple readily to the strings (the bridge lies on an antinodal area) and that the coupled motion tends to induce large volume changes (hence a large monopole contribution). Strong coupling, however, comes at a price, and that is that the body is "over-coupled" to the strings generating "wolf notes" [7]. Hill *et al.* [8,9] have studied structural modes and their associated sound fields (including multipole radiation) up to about 600 Hz in a number of guitars. These, and other studies, stress the important contribution of low-order modes to sound radiation in higher frequency ranges (i.e. above their resonance frequencies) as well as the "local" influence they have on inharmonicity in plucked-string sounds due to strong coupling between the body and strings.

Historically there has been a preoccupation with the frequency placement of modes, and in the case of the guitar, particularly the placement of the two or three prominent resonance peaks which result from coupling between the top plate, back plate and air cavity. Our own research of the physics and psychophysics of the guitar [10,11] suggest that whilst frequency placement can have an important influence on the "localised" determinants of sound quality of an instrument that it is other features, such as the peak heights and Q-values of resonances, which appear to have a much more overarching effect on quality – what we have referred to as the "global" properties of the instrument. The motivation to revisit the three-mass model was stimulated by discussions with makers and an observation that the topic of "mode placement" comes up for much discussion between makers on Internet forums. It is true, of course, that with a PC, soundcard and microphone (or even a well-tuned ear), frequency placement is easy to measure and document, whereas the acoustical and mechanical parameters which we consider important can only be derived from rather more subtle analysis or more-complex measurements. It might also be that frequency placement acts as an excellent "indicator" of those parameters considered more important, but that can only be established by suitable physical studies in conjunction with psychoacoustical listening tests, such as those reported by Wright [10] and Richardson *et al.* [11]. Work such as this inevitably requires some sort of model of the system sufficiently accurate to generate test tones which can be related to features of construction.

## 2 The three-mass model with strings

Full details of two- and three-mass models are given elsewhere [4,5]. Because of significant transverse motion of the ribs in some resonances (cf. Figure 2b), there is an argument for using a four-mass system, but this seems an unnecessary complication and nor does the four-mass model adequately describe the structural mechanics of the complete guitar body.

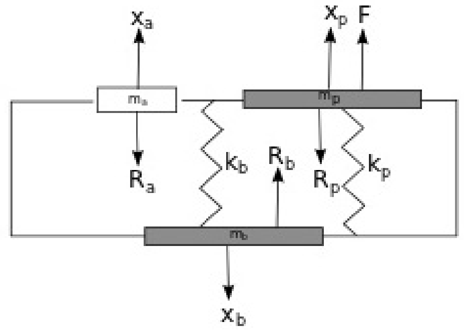



Figure 3: The three-mass model.

Figure 3 shows the system and its associated physical variables. The top plate and back plate are treated as rigid pistons of masses $m_p$ and $m_b$ (subscripts respectively for the top **p**late and **b**ack plate) whose effective areas $A_p$ and $A_b$ multiplied by their displacement amplitudes $x_p$ and $x_b$ give equivalent volume displacements as encountered in a real instrument with more complex mode shapes (cf. Figure 2). The volume displacement of the air in the sound hole is similarly given by $x_a A_a$. The $k$s represent the effective stiffnesses of the structural element and $R$s represent losses (including radiation damping, a quantity which will be discussed later).

Monopole sound radiation from the model can be determined readily from the volume velocities calculated at each frequency. The finite size of the source is also accounted for (see [8]) and a good estimation of the dipole radiation could also be made from this simple model; the latter is necessary for an accurate representation of the acoustical function of the instrument (see [8]), but in the comparative tests undertaken here it has not been included.

A scheme for coupling a lossy, dispersive string (single polarisation) to the structure is described elsewhere [11,12]. This can be used to calculate the transfer response function (TRF) describing the sound pressure radiated to an arbitrary point in free space in response to an input force applied to the string. An example of this TRF is shown in Figure 4.

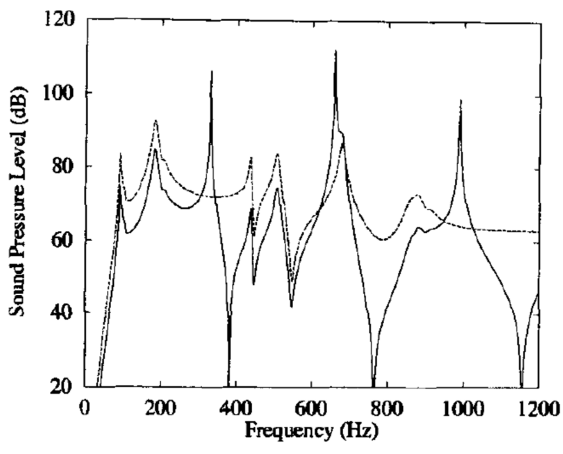



Figure 4: TRF from the model (amplitude only). Calculated SPL responses of guitar body with $E_4$ string attached (solid line) and body only (solid line). The tall, high-Q peaks in the solid line at approximately integer multiples of 330 Hz are what are normally referred to as the "string partials".

The Fourier transform of the complex TRF gives the sound radiated in response to a delta-function unit force applied at a chosen point along the string. The TRF can be readily modified to represent plucking by a step-function – as used here – and / or by a spatially extended force.

# 3 Adjusting model parameters

Although there are analytical schemes for determining resonance frequencies in these models, calculating peak heights and peak widths (damping) is facilitated using a numerical approach.

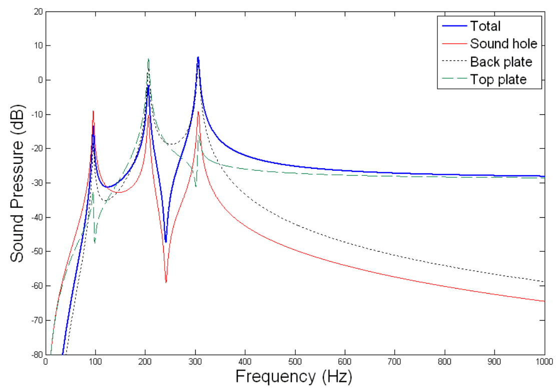



Figure 5: Monopole sound radiation from the three-mass model.

Figure 5 shows the contribution to the monopole radiation (in this case treated as point sources) from the different elements of the system when the body is driven directly, i.e. no strings attached. Whilst different relative tunings of uncoupled top and back plates modify the detail in the low-frequency range, in all real systems investigated it is the radiation from the top plate which dominates at higher frequencies. Using Cramer's rule, an analytical expression can be obtained from the three-mass model for the total sound radiation, which is proportional to $A_p/m_p$ at higher frequencies (e.g. above 400 Hz in the above). As noted elsewhere [10,11] it seems sensible to conclude that it is the properties of the top plate which contribute most to the "global" playing qualities of the instrument and that the "tuning" of the plate's fundamental mode – beyond that which has an effect on either $A_p$ or $m_p$ – would seem to be unimportant. (Of course it is true that a low tuning would tend to produce a low value of $m_p$.) The "local" effects will be considered later.

Measurements on 10 guitars (data from some of which are shown in [8] and [9] and which are consistent with data discussed by Christensen in [4] and [5]) show a wide range in tuning of the three coupled modes ranging from 88-109 Hz for the lowest mode (the "air mode"), 172-248 Hz for the central mode (the one almost always dominated by the properties of the top plate and usually the most prominent peak in the input admittance and sound pressure response) and 212-289 Hz for the upper resonance. (The average values – if they have any significance – were 102 Hz, 207 Hz and 249 Hz.) In this work we modelled the system based on typical dimensions of a classical guitar with estimated modal parameters based on measured values (cf. [8]) as starting points. Our "base" instrument involved $m_p = 0.1$ kg and $m_b = 0.2$ kg with $Q_p = Q_b = 60$, and the stiffnesses of the plates were adjusted to give uncoupled tunings of $f_p = 200$ Hz with $f_b$ varying from 180-250 Hz. The uncoupled Helmholtz resonance was at 123 Hz with a Q-value of 50. Computed input admittances at the bridge are shown in Figure 6.

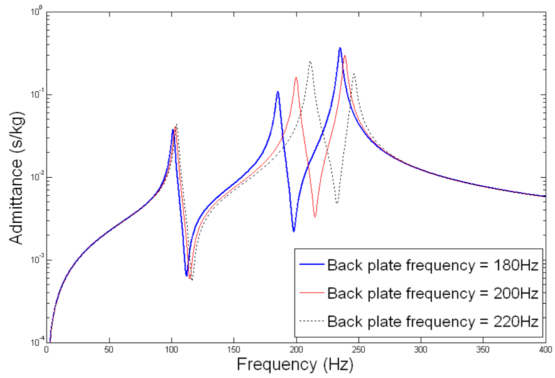



Figure 6: Comparison of input admittance with variable $f_b$.

Curve-fitting routines were used to extract the effective Q-values ($Q_{eff}$), frequencies ($f_{eff}$) and peak heights of the resonance peaks from these graphs, from which effective masses ($M_{eff}$) could be extracted. These data are shown in Table 1.

Table 1: Resonance triplet – extracted mode parameters.

| back plate freq / Hz | resonance triplet mode parameters | | |
|---|---|---|---|
| | $f_{eff}$ / Hz | $Q_{eff}$ | $M_{eff}$ / kg |
| $f_b$ = 180 Hz | 101<br>185<br>235 | 40<br>61<br>74 | 1.68<br>0.48<br>0.14 |
| $f_b$ = 200 Hz | 103<br>200<br>239 | 41<br>60<br>73 | 0.48<br>0.30<br>0.17 |
| $f_b$ = 220 Hz | 104<br>211<br>246 | 40<br>61<br>72 | 1.40<br>0.12<br>0.72 |

It might be assumed that tall admittance peaks of modes with strong radiativities would give rise to excessively strong sound radiation when string frequencies are tuned close to or coincidently with body resonances, but because of strong coupling between string and body there is a kind of "self-limiting" effect. This is best illustrated in an example. Figure 7 shows TRF data from the model incorporating strings. For simplicity a single string mode is shown tuning through the upper two resonances of the resonance triplet. When the frequencies of uncoupled body and string resonances are well separated, the sound signal comprises a long-lived "string" component (which contributes to the perceived pitch) and short-lived "body" components ("noise" components). When the (uncoupled) string and body resonances are tuned coincidently, a pair of relatively-short lived components are generated (with Q-values of approximately $2Q_{eff}$) which are separated in frequency by a small amount $2\Delta f$, neither of which are harmonically related to the higher string overtones. As is clear of Figure 7, the initial amplitudes of these components is much lower than generally anticipated. The "wolf notes" detract considerably from the uniform playing quality of a guitar, though they appear to be an unavoidable feature even of "better-quality" instruments. These sorts of effects lead to large perceived differences in sound quality, but these effects are "local" (as discussed extensively by Wright [10] and Richardson *et al.* [11]).

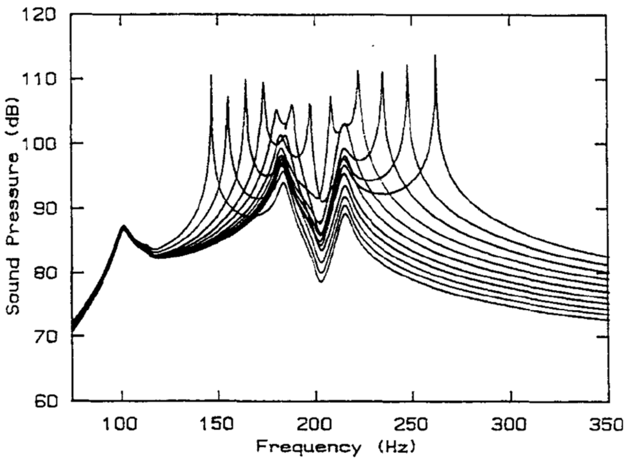



Figure 7: Calculated TRF of a strongly-coupled string as it is tuned through the upper two resonances of the low-frequency resonance triplet.

Gough [13] introduced a useful measure of this coupling strength through a coupling constant $K$. For $K > 1$ the coupling is strong and leads to the sort of mode-splitting described above. The value of $K$ can be determined from the following:

$$K = \frac{2Q_{eff}\left(2m_s/M_{eff}\right)^{1/2}}{n\pi}, \tag{1}$$

where $m_s$ is the vibrating mass of the string and $n$ is the string harmonic. At coincident tuning the coupled modes are each shifted from their uncoupled frequencies by an amount

$$\Delta f = \pm f_{eff}K/4Q_{eff}. \tag{2}$$

To put these equations into context, for the middle resonance of the $f_b = 220$ Hz case above (Table 1), strong coupling would occur for G#$_3$ on the third and fourth strings (1st and 6th frets respectively) with $K$ values of 3.6 and 4.9 and $2\Delta f$ values of 6 Hz and 8 Hz, both sufficient to create disturbing "wolf notes". It is clear from Figure 6 and Table 1 that different relative tunings of $f_p$ and $f_b$ have a significant effect on the value of $K$; we would emphasise, however, that it is the peak heights which are important rather than their frequencies. Adjustment of mode parameters which keep $K$ under control whilst maintaining large $A_p/m_p$ would seem to offer the best compromise in producing an instrument with strong projection and more uniform playing qualities.

# 4 Discussion

## 4.1 Acoustical merit

We have stressed previously the importance of the ratio of $A_p/m_p$ (for which we coined the expression "acoustical merit" [14]), but this review of the three-mass model highlights the important distinction to be made between the uncoupled effective mass of the top plate ($m_p$) and the effective masses extracted from real input admittance data. It also demonstrates how tuning of the modes may well be a very positive indicator of establishing the appropriate balance of peak heights to ensure a strong "global" response whilst helping to reduce the over-coupling conditions which result in bad "wolf notes".

Models such as these should be treated with considerable caution, however. One observation made from practical experiments on guitars is the wide range of Q-values measured for these low-frequency "air-pumping" modes. Many authors refer to these as "monopole" modes, but measurements by Hill *et al.* [8] clearly indicate that these modes can also have sizeable dipole components. As the radiativity of modes increase, their Q-values decrease because of increased radiation damping, a factor not built into any of these models; it is not uncommon to find the typical mid-range-mode Q-value of 60 drop by a factor of two or more with a consequential reduction in $K$ (see Equation 1).

We have argued elsewhere [14] that the acoustical merit of the fundamental top-plate mode is strongly influenced by the design and materials of the plate itself (including bridge design and fan-strutting arrangements) and is dependent on the explicit form of the mode shapes. Again, these are not factors which can be built into such a simple model. It suggests, therefore, that one person's experience of mode tuning may well not provide a universal indicator of quality to be used equally successfully by others.

## 4.2 The mid frequency response

Whilst we have found this to have been a useful and informative re-visitation to the three-mass model, much of the psychoacoustical work which can be done with the model will, we believe, replicate the conclusions of Wright's work [10]. In order to break new ground we decided to add additional modes to the model to replicate the typical input admittance and sound pressure responses found in real instruments to gauge the relative importance of the low-order modes in relation to the mid-frequency response.

In previous work [8] we have determined all the mechanical and acoustical parameters required to reconstruct input admittance and sound pressure response curves up to about 500 Hz (radiativity data allows reconstruction at arbitrary positions in free space). Although we have some data on modal parameters beyond this range, we do not have radiativity data, so we have simply chosen frequencies, effective masses and Q-values and radiativities to closely resemble the broad details see in real data (from [8]).

Figure 8 shows the effect of adding about 20 additional modes. Strings were coupled to this system to generate the TRFs described earlier and test tones generated for open strings and some fretted notes for comparison of the system with and without the addition of these higher modes. Preliminary (informal) listening tests show very little difference when the detail is added unless the radiativities are made uncharacteristically large.

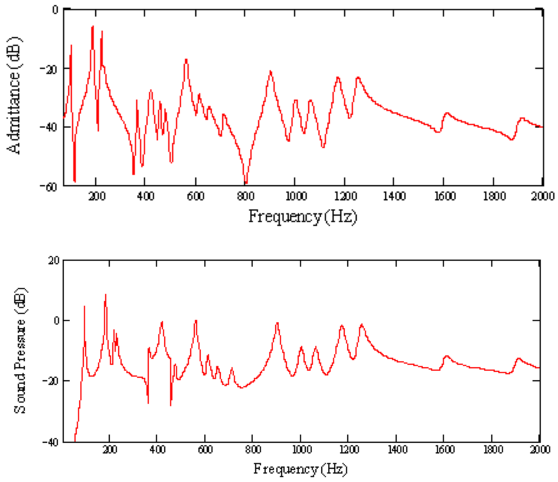
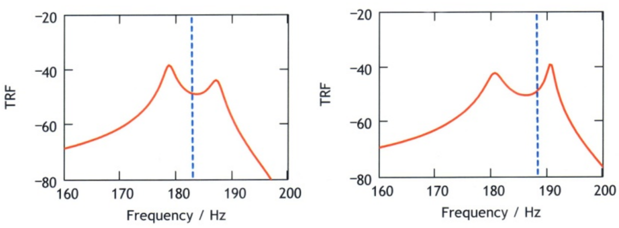



Figure 8: Modelled input admittance and sound pressure response in the range 100-2000 Hz.

We conclude therefore, as we have postulated previously, that the low-order modes really do have the major controlling influence on the playing qualities of the guitar. Looking at the symmetries of typical guitar modes (e.g. Figure 5 in [7]) it seems unlikely that the radiativities of these higher modes can be increased substantially without some major redesign of the instrument (assuming that increasing these radiativities were to be a desirable goal).

## 4.3 For the maker

Mode tuning is an attractive mechanism for the maker in ensuring some measure of consistency in the finished instrument, though it is probably reliant on the maker using similar timber and using consistent design and construction to be truly effective. Some makers tune strong body resonances off pitch, i.e. mid-way between notes of the equal-tempered scale, in an attempt to reduce strong coupling of strings and body. Figure 9 shows, however, that the "wolf-note" problem extends easily over half a semitone (data from [8] in this particular example with $K = 5.1$). This is a fairly extreme example, which perhaps highlights again the necessity to have some control on $K$ without compromising too much the acoustical merit of these important modes. One way to reduce the $K$ value is to aim for relatively high tuning of the top-plate mode. High tuning also tends to reduce the relative proportion of the body noise in the radiated sounds.




(a) (b)

Figure 9: TRF of coupled string and body modes: (a) coincident tuning, and (b) half-semitone mismatch. In each case the blue line shows the uncoupled string tuning.

The noise components generated particularly by these low-order modes when plucking the guitar have been shown to be an important perceptual element of the guitar sound [11]. It is possible that there is some preference for the tuning of what is in effect an added percussive element in the sound, though none of our studies have considered this formally.

# Acknowledgments

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