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Large scale modulation of high frequency acoustics fields in porous media

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This paper dealt with the description of large modulation of high frequency acoustic fields in periodic porous media. High frequencies mean local dynamics at the pores scale, and therefore absence of scale separation in the usual sense of homogenization. However, meanwhile the pressure is fast varying in the pores (according to periodic eigenmodes), the mode amplitude can present large scale modulation, hence introduces another type of scale separation on which the asymptotic multi-scale procedure applies. The approach is first presented on a periodic network of inter-connected Helmholtz resonators. The equations governing the modulations carried by eigenmodes, at frequencies close to the eigenfrequency, are derived. Because of the local dynamic state, the number of cells on which the carrying periodic mode is defined becomes a parameter. In a second part, the asymptotic approach is developed for periodic porous media saturated by a perfect gas. With a strict use of the "multi-cells" periodic condition, one obtains the family of equations governing large modulation of high frequency waves. The significant difference between modulations of simple or multiple mode are evidenced. Thus, this theory extracts, from the comprehensive Floquet-Bloch modal space, the particular frequency range enabling large modulations, therefore large correlation, of high frequency acoustic waves.

1 Introduction

This paper investigates in periodic porous media, the phenomena of large modulation of high frequency acoustic waves. The interest lies in the fact that despite the short acoustic wavelengths, the acoustic field presents large correlation lengths.

High frequencies mean local dynamics at the pores scale, and therefore absence of scale separation in the usual sense of homogenization [9], [2],[3]. However, meanwhile the pressure is fast varying in the pores (according to periodic eigenmodes), the mode amplitude can present large scale modulation. This situation introduces another type of scale separation on which the multi-scale asymptotic method can be performed. This idea is in the same spirit than the study proposed by [7] in the context of composite elastic media with some difference in the theoretical implementation.

In the first part, the physical principle of this approach is introduced on a network of inter-connected Helmholtz resonators. Equations governing modulations carried by a given eigenmode, at frequencies close to the eigenfrequency, are derived. Because of the local dynamic state, the number of cells on which the carrying periodic mode is defined becomes a parameter.

In the second part, the same question is addressed for periodic porous media saturated by a perfect gas. The asymptotic approach enables to derive the governing equation of modulation. It is established that the situation of simple or double mode differs significantly. Further a strict use of the "multi" periodicity condition leads to describe the whole family of large modulation phenomena.

2 Periodic network of Helmholtz resonators

As an introduction of the method, consider an idealized periodic porous medium, the channels of which are made of interconnected Helmholtz resonators. Each resonator is made of a large "box" connected to a much smaller channel. For the sake of simplicity all the resonators are assumed identical and their connections are actually in a single direction. Hence, the medium consists in unidirectional identical and parallel channels of periodic shape, and each channel appears as a periodic chain of large pores ("box" of length l_0 and volume V) linked by small constricted pipes (of section s , volume v and negligible length compared to l_0). Then, to describe acoustic wave propagation through this medium, it is sufficient to focus on a single channel.

Acoustics of this medium can be described by the usual equivalent fluid (or dynamic permeability) approach of porous media, provided that the scale separation assumption is satisfied [1], [3]. This latter applies for large wavelengths, corresponding to frequencies significantly lower than the eigenfrequency of the resonator f_0 . In the sequel we focus on the frequency range $O(f_0)$, where the usual approach becomes irrelevant.

According to the classic simplified analysis of Helmholtz resonators, (i) thermal and viscous dissipation effects can be neglected, (ii) the gas in the box is compressed adiabatically and quasi-statically (limiting thus the investigated frequency range), and (iii) the gas in the pipe suffers a negligible compression. Conveniently the motion u of the gas at the aperture of the box (and in the pipe) is taken as acoustic variable.

With these assumptions a channel may be sketched by a line of mass-less spring k of length l_0 alternating with rigid point mass m . The resonator spring k is defined from the compressibility of the gas box with aperture s , and the actuator mass of the resonator is the mass of the gas in the pipe

$$k = \frac{\gamma P^e s^2}{V}, \quad m = \rho^e v$$

where P^e and ρ^e are the ambient equilibrium pressure and gas density and γ the adiabatic coefficient. Consequently

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{\gamma P^e}{\rho^e}} \sqrt{\frac{s^2}{Vv}} = c_{sound} \sqrt{\frac{s^2}{Vv}}$$

Notice that the dynamics of the resonator results from the interaction between box and pipe, the gas in both domains being in quasi-static regime.

2.1 Waves in 1D resonator network

Wave propagation in 1D spring-mass system has been widely studied in the literature and in this case exact solution can be established. This latter explicitly accounts for the periodicity of the system, however this condition can be applied either on the irreducible cell period Ω_0 (of size l_0 and constituted by one spring one mass), or double cells period $\Omega_2 = \Omega_0 \cup \Omega_0$ or multiple cells period $\Omega_M = \cup_M \Omega_0$.

2.1.1 Irreducible period Ω_0

Studying harmonic wave propagation at the frequency ω , the motion u_n of n^{th} mass (i.e. on the n^{th} cell Ω_0) is on the form :

$$u_n = U_0 e^{i(\omega t - \mathcal{K}_1 n l_0)} \quad (1)$$

where $\mathcal{K}_1(\omega)$ is the wave number. The balance of forces on the n^{th} mass, reads

$$(u_{n+1} - 2u_n + u_{n-1}) + \chi u_n = 0, \quad \chi = (\omega/\omega_0)^2 \quad (2)$$

Then replacing u_n by its expression Eq. (1) leads to the dispersion relation

$$\sin^2\left(\frac{\mathcal{K}_1(\omega) l_0}{2}\right) = \chi \quad (3)$$

From Eq. (1), large wave lengths correspond to $\mathcal{K}_1 l_0 \approx 0 [2\pi]$. The unique physical value is $\mathcal{K}_1 l_0 \ll 1$ (since springs have no mass) that belongs to the first Brillouin zone $\mathcal{K}_1 l_0 \in [-\pi, \pi]$. The others Brillouin zones are excluded. Developing the left member of Eq. (3) near zero imposes a low frequency range $\omega \ll \omega_0$, that gives the usual description of long wave propagation.

2.1.2 Double cells period Ω_2

Reconsider the wave propagation problem set on the double cells period Ω_2 constituted of two masses, two springs. Denote by u_n and w_n the motions of two successive masses of the n^{th} period Ω_2 . By convention, w_n is associated to the "internal mass" which allows internal dynamic in the period. Express the double cells periodicity leads to consider u_n on the form:

$$u_n = U_0 e^{i(\omega t - \mathcal{K}_2 n(2l_0))} \quad (4)$$

First, the balance in the period leads to the following relation between w_n and the both motions u_n and u_{n+1}

$$w_n = \frac{u_n + u_{n+1}}{2 - \chi} \quad (5)$$

Then, the balance on the mass associated to u_n leads to

$$(u_{n+1} - 2u_n + u_{n-1}) - \chi(\chi - 4)u_n = 0 \quad (6)$$

and to the dispersion relation

$$4 \sin^2(\mathcal{K}_2(\omega) l_0) = \chi(4 - \chi) \quad (7)$$

Focus again situations of large scale evolution corresponding to $\mathcal{K}_2 2l_0 \approx 0 [2\pi]$. As stated previously, only the values in the first Brillouin zone are physical, thus $\mathcal{K}_2 2l_0 \ll 1$. Develop the left member of Eq. (7) near zero, leads to two frequency ranges :

- $\omega \ll \omega_0$, which is the low frequency case found in the wave study based on Ω_0 (in particular Eq. (7) indicates $w_n \approx (u_n + u_{n+1})/2$, then successive masses follow almost the same motion).

- $\omega \approx 2\omega_0$. Interestingly, $2\omega_0$ is the eigenfrequency of the simple eigenmode of the double cells period Ω_2 with periodic boundary conditions. Note that, for frequencies beyond $2\omega_0$, \mathcal{K}_2 becomes purely imaginary, hence phenomena are confined in a boundary layer without long distance propagation.

In the high frequency band $\omega \approx 2\omega_0$, Eq. (7) indicates $w_n \approx -(u_n + u_{n+1})/2$, meaning that the motions of successive masses are alternated. Therefore, the large scale phenomena driven by \mathcal{K}_2 , correspond to large modulations carried by the periodic eigenmode. This describes a typical situation of large modulation of high frequency waves.

2.1.3 Multi-cells period Ω_M

The same method leads to the following dispersion relation established for multiple cells period Ω_M made of M masses and M springs.

$$4 \sin^2\left(\frac{\mathcal{K}_M(\omega) M l_0}{2}\right) = -\left(\beta^{-\frac{M}{2}} - \beta^{\frac{M}{2}}\right)^2 \quad (8)$$

$$\beta = \frac{2 - \left(\frac{\omega}{\omega_0}\right)^2 + \frac{\omega}{\omega_0} \sqrt{\left(\frac{\omega}{\omega_0}\right)^2 - 4}}{2}$$

Usual long wave propagation are obtained for $\mathcal{K}_M M l_0 \approx 0$ and $\omega \ll \omega_0$. The modulation situations correspond to $\mathcal{K}_M M l_0 \approx 0$ for higher frequencies. Develop the left member of Eq. (8) near zero leads to several frequency bands, each of them being attached to one Ω_M periodic eigenmode frequency. All modes are double except the simple mode at $2\omega_0$ (when exists).

Those results evidence the possibility of large modulation of high frequency waves in frequency bands centered around the periodic eigenmode frequencies arising in multi-cells period. The existence of two different scales, related to the modes and to the modulation length, suggests the use of upscaling method. This aspect is undertaken in the next section.

2.2 Large modulation of high frequency waves in a periodic network of resonators

The scale separation between the multi-cells period Ω_M of size $l = M l_0$, and the modulation length L naturally introduces the scale ratio $\varepsilon_M = l/L = M l_0/L = M\varepsilon$. The macroscopic description is derived using the homogenization of discrete periodic media [6], [8] adapted to multi-cells period and high frequencies, briefly summarized herebelow.

2.2.1 Discrete homogenization method

Following the previous analysis, among the M masses of n^{th} period Ω_M , the motion u_n of one of them is arbitrarily chosen as *leading variable* located at a *leading node* (the motions of other masses being related to the leading variables through the local balances in Ω_M). The leading variable carries the large scale modulations (denoted A). These latter are continuous functions of the macroscopic variable x coinciding with the leading nodes. Assuming the convergence when ε_M tends to zero, modulations are expanded in powers of ε_M

$$u_n = u(x = n l) = \sum_0^\infty \varepsilon_M^i A(n l) \quad (9)$$

The frequency is expanded *around the eigenfrequency* of one of the periodic eigenmodes of Ω_M , (say K^{th} mode)

$$\omega = \omega_K + \sum_1^\infty \varepsilon_M^i \omega \quad (10)$$

As the size of the period is small compared to the modulation length, the variations of the motions between neighboring leading nodes are expressed using Taylor's series, which introduces the macroscopic derivatives. As the motions of the non-leading masses are related to the leading variables, it only remains to express the balance equations at the leading

nodes. Then, introduction of the expansions and Taylor series of variables u_n leads to equations at different powers of ε_M . This study focuses on the leading order only.

2.2.2 Macroscopic descriptions

The above procedure has been applied to periods constituted by one, two and three cells.

For irreducible period $M = 1$ the classic homogenization at low frequency, described by Eq. (11) (" $''$ " stands for double spatial derivative, and A stands for 0A) is recovered. In this case the modulation equation coincides with the classic wave propagation equation.

For double cells period $M = 2$, the procedure applied around the eigenfrequency $\omega = 2\omega_0$ of the unique and *simple* periodic mode yields Eq. (12) that governs the modulation A of this mode at the leading order.

Modulations on triple cells period $M = 3$ around the eigenfrequency $\omega = \sqrt{3}\omega_0$ of the unique and *double* periodic mode are described by Eq. (13). Note that contrary to Eq. (12), the frequency term is always positive. Moreover, the Ω_3 modulation frequency is not an edge of the Brillouin zone, and could not be derived with antiperiodic conditions used in [7].

$$\Omega_0 : \quad k l_0^2 A'' + m^2 \omega^2 A = 0 \quad (11)$$

$$\Omega_2 : \quad k l_0^2 A'' + m \left(-\omega^2 + (2\omega_0)^2 \right) A = 0 \quad (12)$$

$$\Omega_3 : \quad k l_0^2 A'' + 4m \left(\omega - \sqrt{3}\omega_0 \right)^2 A = 0 \quad (13)$$

Replacing in Eq. (11, 12, 13) A'' by $-\mathcal{K}_M^2 A$, the expressions of \mathcal{K}_M^2 coincide with that given by Eq. (8) for $\mathcal{K}_M M l_0 \approx 0$.

Hence the exact solutions can be obtained independently via a multi-scale asymptotic method. In the next section, it is shown how the general principles of this approach developed for 1D discrete systems can be transposed to 3D porous media.

3 Large modulation of high frequency waves in periodic porous media

Let us now consider the case of a periodic 3D porous media (of period Ω , characteristic size l , porosity ϕ , pore domain Ω_f , and pores boundary Γ) and focus on the frequency range where the scale separation is lost. In a first approach, we consider that the thermal and viscous dissipation effects can be neglected, as previously. Then the local description within the pores is that of a perfect gas in adiabatic dynamic regime governed by the following equations (by linearity the harmonic time dependence $\exp(i\omega t)$ is skipped) :

$$\operatorname{div}(\gamma P^e \operatorname{grad}(u)) + \rho^e \omega^2 u = 0 \quad ; \quad \text{on } \Omega_f \quad (14)$$

$$u.n = 0 \quad ; \quad \text{on } \Gamma \quad ; \quad u \text{ } \Omega - \text{periodic} \quad (15)$$

This local problem presents exact solutions, i.e. the series of modes $u = \Phi_N$ for a discrete spectrum of frequency $\omega_N = 2\pi f_N$, $N \in \mathbb{N}$ (here and in the sequel capital indices refers to mode numbers). Before going further it is necessary to clarify the notion of period used in this study.

3.1 Irreducible period and selected period

A periodic media is generally defined from its irreducible period Ω_0 . However, any integer combination (n, m, p) in the three directions of space of Ω_0 define an other period $\Omega = \bigcup_{n,m,p} \Omega_0$ of the media. In the usual homogenization approach, the periodic problems (and solutions) at the local scale are shown to be independent on the definition of the period. This results from the quasi-static regime at the local scale.

Now, when considering dynamics at the local scale, the modes necessarily depend on the definition of the volume (the period Ω) on which the periodic boundary conditions apply (on $\partial\Omega$). Indeed, the "family" of mode increases as the number of irreducible periods included in the period Ω grows. For instance, the series of modes of double period, $\Omega = \bigcup_{2,1,1} \Omega_0$, includes the Ω_0 modes ("duplicated") and new modes specific to the double period. These latter modes respect the periodicity condition on $\partial\Omega$ but not on the internal boundary defined by $\partial\Omega_0$.

Therefore, we have to specify the period Ω on which the periodic mode is defined : contrary to low frequency homogenization, the selected period Ω becomes a parameter of the description. This is implicitly included in the sequel, where the analysis is performed for any type of $\Omega = \bigcup_{n,m,p} \Omega_0$. This procedure guarantees to reach the whole family of large modulation phenomena, associated to modes existing in the collection of periods.

Notice that a subgroup of Ω -modes can be determined by using "phase shifted" periodic boundary condition on the irreducible period Ω_0 . For instance as [7] an antiperiodic condition Ω_0 enables to identify a subgroup of modes of double period, $\Omega = \bigcup_{2,1,1} \Omega_0$ (or a $\exp(i\pi/n)$ -periodic condition for $\Omega = \bigcup_{n,1,1} \Omega_0$, etc...). However, this convenient procedure does not give access to the whole series of modes.

3.2 Modulation and asymptotic method

Choice a period Ω and select for example the K^{th} mode i.e. the couple (ω_K, Φ_K) . By construction, mode Φ_K repeated Ω -periodically, gives rise to a high frequency wave of constant modal amplitude, or equivalently a high frequency wave of infinite modulation length. Contrary to infinite length, a large modulation length implies, (i) a non-constant amplitude (of mode Φ_K) varying at a scale L much larger than the period l , and (ii) a frequency ω distinct but nevertheless close to the eigenfrequency ω_K , ω_K since when $\varepsilon = l/L \rightarrow 0$, then $\omega \rightarrow \omega_K$.

To describe the situation of modulated K^{th} mode when the modulation scale ratio is small, i.e. $\varepsilon \ll 1$, we apply the multi-scale asymptotic method well established in the framework of homogenisation [9] of periodic media. In this purpose, two space variables, y and $x = \varepsilon y$ - associated respectively to the variations at the cell and modulation scales - are introduced, the usual derivative are changed into $\partial/\partial y + \varepsilon \partial/\partial x$, the variable u is expanded in power of ε , each term (specified by ante exponents) being Ω periodic, and the frequency is also expanded in ε -power around ω_K :

$$u(x, y) = \sum_0^\infty \varepsilon^i u^i(x, y) \quad \omega = \omega_K + \sum_1^\infty \varepsilon^i \omega^i \quad (16)$$

The process, similar to [7], is achieved as follows : the expansions are introduced in Eq. (14-15) rewritten with the two-

scale derivatives, then the terms of same power in ε are identified, and the problems obtained in series are solved up to obtain the equation governing the large scale phenomena at the leading order.

3.3 Modulation of simple mode

We focus here on the case where the considered mode Φ_K is simple (double modes are addressed in the next section).

3.3.1 Leading order

The problem encountered at the leading order reads :

$$\gamma P^e \operatorname{div}_y(\operatorname{grad}_y({}^0u)) + \rho^e \omega_K^2 {}^0u = 0 \quad \text{on } \Omega_f \quad (17)$$

$${}^0u.n = 0, \text{ on } \Gamma; \quad {}^0u \text{ } \Omega - \text{periodic} \quad (18)$$

Mode Φ_K being simple, the solution is on the form

$${}^0u(x, y) = A(x)\Phi_K(y)$$

where $A(x)$ is the slow varying amplitude of mode Φ_K .

3.3.2 First order

The following problem set at the local scale is similar to the previous problem, except the presence of ${}^1S({}^0u)$:

$$\gamma P^e [\operatorname{div}_y(\operatorname{grad}_y({}^1u)) + {}^1S({}^0u)] + \rho^e \omega_K^2 {}^1u = 0 \quad \text{on } \Omega_f \quad (19)$$

$${}^1S({}^0u) = \quad (20)$$

$$\operatorname{div}_y(\operatorname{grad}_x({}^0u)) + \operatorname{div}_x(\operatorname{grad}_y({}^0u)) + 2\omega_K(\rho^e/\gamma P^e) {}^1\omega {}^0u$$

$${}^1u.n = 0, \text{ on } \Gamma; \quad {}^1u \text{ } \Omega - \text{periodic}$$

To handle this problem let first establish that, following the Fredholm alternative, we have :

$$\langle {}^1S({}^0u).{}^0u \rangle = A(x) \langle {}^1S(A(x)\Phi_K).\Phi_K \rangle = 0 \quad (21)$$

where $\langle F \rangle$ stands for $\frac{1}{\Omega_f} \int_{\Omega_f} F dv$. In this aim, notice that, from the divergence theorem, the impenetrability condition (on Γ) and the periodic condition (on $\partial\Omega_f - \Gamma$) :

$$\begin{aligned} \langle \operatorname{div}_y(\operatorname{grad}_y({}^1u)).{}^0u \rangle &= \\ -\langle \operatorname{grad}_y({}^1u).\operatorname{grad}_y({}^0u) \rangle &= \langle \operatorname{div}_y(\operatorname{grad}_y({}^0u)).{}^1u \rangle \end{aligned}$$

Then taking the scalar product of Eq. (19) by 0u and of Eq. (17) by 1u and subtracting provides equality Eq. (21). This latter can further be simplified since (the non capital indices stand for 3D-space direction) :

$$\begin{aligned} &\langle \operatorname{div}_y(\operatorname{grad}_x({}^0u)) + \operatorname{div}_x(\operatorname{grad}_y({}^0u)).\Phi_K \rangle \\ &= A_{,x_j} \langle \Phi_{Kj} \operatorname{div}_y(\Phi_K) + \Phi_{Ik} \Phi_{Kj,y_k} \rangle \\ &= \operatorname{grad}_x(A) \cdot \frac{1}{\Omega_f} \int_{\partial\Omega_f} (\Phi_K \otimes \Phi_K).n ds = 0 \end{aligned}$$

because of the impenetrability and periodic conditions. Consequently Eq. (21) simply reduces to (with the notation $\Phi.\Phi = |\Phi|^2$) :

$$2\omega_K(\rho^e/\gamma P^e) {}^1\omega [A(x)]^2 \langle |\Phi_K|^2 \rangle = 0$$

therefore the first corrector of frequency necessarily vanishes:

$${}^1\omega = 0$$

Now, recalling that the modes $\{\Phi_N, N \in \mathbb{N}\}$ form an orthonormal basis, the solution 1u is looked in the form of linear combination (depending on the macrovariable x) of the Φ_N :

$${}^1u = B(x)\Phi_K + \sum_{N \neq K} a_N(x)\Phi_N$$

The coefficients a_N are deduced by taking (i) the scalar product of Eq. (19) by Φ_N , and (ii) the scalar product of the modal equation of Φ_N (equivalent of Eq. (17) where ω_K is replaced by ω_N) by 1u , then subtracting. This yields :

$$\begin{aligned} \langle {}^1S({}^0u).\Phi_N \rangle &= -(\rho^e/\gamma P^e)(\omega_K^2 - \omega_N^2) \langle {}^1u.\Phi_N \rangle \\ &= -(\rho^e/\gamma P^e)(\omega_K^2 - \omega_N^2) a_N \langle |\Phi_N|^2 \rangle \end{aligned}$$

From the usual integral transformations and using the impenetrability and periodic conditions of the modes, the left hand side term reads :

$$\langle {}^1S(A(x)\Phi_K).\Phi_N \rangle = \operatorname{grad}_x(A) \cdot \langle \Phi_N \operatorname{div}_y(\Phi_K) - \Phi_K \operatorname{div}_y(\Phi_N) \rangle$$

The two latter equalities define the coefficients $a_N(x)$ that depend linearly on $\operatorname{grad}_x(A)$. To sum up :

$${}^1u = B(x)\Phi_K - \sum_{N \neq K} \frac{\gamma P^e}{\rho^e(\omega_K^2 - \omega_N^2)} \frac{\operatorname{grad}_x(A) \cdot \widehat{\Phi_N \Phi_K}}{\langle |\Phi_N|^2 \rangle} \Phi_N$$

where, to lighten the writing, we introduced the following notation for the combined vector build from two vectors Φ and Ψ :

$$\widehat{\Phi \Psi} = \langle \Phi \operatorname{div}_y(\Psi) - \Psi \operatorname{div}_y(\Phi) \rangle = -\widehat{\Psi \Phi}$$

The expression of 1u explicitly means that the modulation of the mode Φ_K generates a redistribution of field of motion on the other modes.

3.3.3 Second order

Accounting for the fact that ${}^1\omega = 0$, the problem at the second order reads:

$$\gamma P^e [\operatorname{div}_y(\operatorname{grad}_y({}^2u)) + {}^1S({}^1u) + {}^2S({}^0u)] + \rho^e \omega_K^2 {}^2u = 0 \quad (22)$$

$$\begin{aligned} {}^2S({}^0u) &= \operatorname{div}_x(\operatorname{grad}_x({}^0u)) + 2\omega_K(\rho^e/\gamma P^e) {}^2\omega {}^0u \\ {}^2u.n &= 0, \text{ on } \Gamma; \quad {}^2u \text{ } \Omega - \text{periodic} \end{aligned} \quad (23)$$

Following the same reasoning as for the first order, it is straight forward to establish that :

$$\langle [{}^2S({}^0u) + {}^1S({}^1u)].{}^0u \rangle = 0 \quad (24)$$

and introducing the expressions of 0u and 1u :

$$\begin{aligned} &\langle {}^2S(A(x)\Phi_K).\Phi_K \rangle + \langle {}^1S(B(x)\Phi_K).\Phi_K \rangle \\ &+ \sum_{N \neq K} \langle {}^1S(a_N(x)\Phi_N).\Phi_K \rangle = 0 \end{aligned}$$

Each term of this equality can be explicitated :

$$\begin{aligned} \langle {}^2S(A(x)\Phi_K).\Phi_K \rangle &= \\ \langle \operatorname{div}_x(\operatorname{grad}_x(A(x)\Phi_K)).\Phi_K \rangle &+ A(x)2\omega_K(\rho^e/\gamma P^e) {}^2\omega \langle |\Phi_K|^2 \rangle \\ &= A_{,x_p x_q} \langle \Phi_{Kp} \Phi_{Kq} \rangle + A(x)2\omega_K(\rho^e/\gamma P^e) {}^2\omega \langle |\Phi_K|^2 \rangle \end{aligned}$$

Then, as in Eq. (21), $\langle {}^1S(B(x)\Phi_K). \Phi_K \rangle = 0$.

Finally, each $\langle {}^1S(a_N(x)\Phi_N). \Phi_K \rangle$ is determined similarly as the coefficient $\langle {}^1S(A(x)\Phi_K). \Phi_N \rangle$ expressed previously:

$$\begin{aligned} \langle {}^1S(a_N(x)\Phi_N). \Phi_K \rangle &= \text{grad}_x(a_N(x)). \widehat{\Phi_N \Phi_K} \\ &= -A_{x,p,q} \frac{\gamma P^e}{\rho^e(\omega_K^2 - \omega_N^2)} \widehat{(\Phi_N \Phi_K)_p} \widehat{(\Phi_N \Phi_K)_q} \end{aligned}$$

3.3.4 Features of simple mode modulation

The above results can be summarised as follows. The leading order governing equation for the modulation at the frequency ω of the simple mode (ω_K, Φ_K) is of the form :

$$\gamma P^e \text{div}_x(T \text{grad}_x(A(x))) + \rho^e(\omega^2 - \omega_K^2)A(x) = 0$$

where we use the $o(\varepsilon^2)$ -approximation : $\omega^2 - \omega_K^2 = 2\omega(\varepsilon^2 2\omega_K)$ and where the tensor T associated to mode Φ_K is given by :

$$T = \frac{\langle \Phi_K \otimes \Phi_K \rangle}{\langle |\Phi_K|^2 \rangle} - \frac{\gamma P^e}{\rho^e} \sum_{N \neq K} \frac{\widehat{\Phi_N \Phi_K} \otimes \widehat{\Phi_N \Phi_K}}{\langle |\Phi_N|^2 \rangle \langle |\Phi_K|^2 \rangle (\omega_K^2 - \omega_N^2)}$$

Note that the modulation is driven by a strictly macroscopic equation. Parameters of this latter are fully determined from the dynamic properties of the period (modes) and the frequency, independently of the boundary conditions. These results are consistent with the assumption of a scale separation between the long scale modulation carried by the local mode. They constitute a 3D generalisation of the result established in section 2 on the 1-D network of resonators for the modulation of the simple mode.

From its expression, the tensor T is $O(1)$, symmetric by construction, and therefore diagonalizable. Thus, any 3-D modulation of the considered mode can be decomposed into specific modulations along the three principal directions of T . The principal values, denoted by T_α , $\alpha = 1, 2, 3$ may be positive or negative, and are expected to be different. This evidences the anisotropy of the modulation phenomena.

Amplitude of the modulation in the α -principal direction is governed by (' stands for spatial derivative) :

$$\gamma P^e T_\alpha A'' + \rho^e(\omega^2 - \omega_K^2)A = 0$$

and the amplitude variation takes the classic form (x stands for the macro variable in principal direction α) :

$$A(x) = A_+ \exp(+ikx) + A_- \exp(-ikx)$$

where

$$\kappa(\omega) = \sqrt{\frac{\rho^e(\omega^2 - \omega_K^2)}{T_\alpha \gamma P^e}} = \frac{\omega_K}{c_{\text{sound}}} \sqrt{\frac{2}{T_\alpha} \left(\frac{\omega}{\omega_K} - 1 \right)}$$

Thus, provided that $(\omega - \omega_K)/T_\alpha > 0$, the modulation oscillates and propagates with a "wavelength" $\Lambda_\alpha \gg \lambda_K$ (λ_K is the wavelength in air at ω_K) highly dependent on the frequency :

$$\Lambda_\alpha(\omega) = \lambda_K \sqrt{\frac{T_\alpha}{2} \frac{1}{\frac{\omega}{\omega_K} - 1}} = O\left(\frac{c_{\text{sound}}}{\sqrt{\omega - \omega_K} \sqrt{\omega_K}}\right) \gg \lambda_K$$

Conversely, when $(\omega - \omega_K)/T_\alpha < 0$, the modulation behaves as "diffusive wave". Oscillations present an exponential decay, and the penetration depth, of the order of $|\Lambda_\alpha|$, is highly

frequency dependent. Note the asymmetry of modulation behaviour on both side of the modal frequency ω_K . Further, because of anisotropy, the asymmetry can be inverted between different principal directions (if two principal values are of opposite sign). Obviously, only the propagative phenomena gives rise to long correlation lengths of high frequency waves.

3.4 Double mode modulation

We address now the case of multiple modes. For simplicity, we consider the situation of double mode (say Φ_K, Ψ_K with $\langle \Phi_K, \Psi_K \rangle = 0$) at the frequency ω_K .

3.4.1 Derivation of the modulation equation

In presence of double modes, the solution of the leading order problem Eq. (17, 18) becomes :

$${}^0u(x, y) = A(x)\Phi_K(y) + B(x)\Psi_K(y)$$

The next problem is identical to Eq. (19, 20), and introduces the same term ${}^1S({}^0u)$. However, the orthogonality condition derived from the Fredholm alternative applies now to both modes Φ_K, Ψ_K and we have :

$$\langle {}^1S({}^0u). \Phi_K \rangle = 0 \quad ; \quad \langle {}^1S({}^0u). \Psi_K \rangle = 0$$

As seen section 3.3.2 :

$$\langle {}^1S(A(x)\Phi_K). \Phi_K \rangle = \langle {}^1S(B(x)\Psi_K). \Psi_K \rangle = 0$$

thus the orthogonality conditions simplify into :

$$\langle {}^1S(B(x)\Psi_K). \Phi_K \rangle = 0 \quad ; \quad \langle {}^1S(A(x)\Phi_K). \Psi_K \rangle = 0$$

that gives the two following coupled equations :

$$\begin{aligned} \text{grad}_x(B(x)). \widehat{\Phi_K \Psi_K} &= -2\omega_K(\rho^e/\gamma P^e) {}^1\omega A(x) \langle |\Phi_K|^2 \rangle \\ \text{grad}_x(A(x)). \widehat{\Psi_K \Phi_K} &= -2\omega_K(\rho^e/\gamma P^e) {}^1\omega B(x) \langle |\Psi_K|^2 \rangle \end{aligned}$$

3.4.2 Features of double mode modulation

Eliminating B (for example) in the above set, and, using the $o(\varepsilon)$ -approximation : ${}^1\omega\varepsilon = \omega - \omega_K$, we derive the following leading order set of equations for modulation at frequency ω of the double mode $(\omega_K, \Phi_K, \Psi_K)$:

$$\gamma P^e \text{div}_x(R \text{grad}_x(A(x))) + \rho^e 4(\omega - \omega_K)^2 A(x) = 0$$

$$B(x) = \text{grad}_x(A(x)). \frac{\widehat{\Phi_K \Psi_K}}{\langle |\Psi_K|^2 \rangle} \frac{\gamma P^e}{\rho^e} \frac{1}{2\omega_K(\omega - \omega_K)}$$

where the tensor R is defined by :

$$R = \frac{\gamma P^e}{\rho^e} \frac{\widehat{\Phi_K \Psi_K} \otimes \widehat{\Phi_K \Psi_K}}{\omega_K^2 \langle |\Psi_K|^2 \rangle \langle |\Phi_K|^2 \rangle}$$

and has the property (recall that λ_K is the wavelength in air at ω_K) :

$$R = O\left(\frac{c_{\text{sound}}^2}{\omega_K^2} \frac{(\langle |\Psi_K| \rangle \langle |\Phi_K| \rangle)^2}{(\lambda_K)^2} \frac{1}{\langle |\Psi_K|^2 \rangle \langle |\Phi_K|^2 \rangle}\right) = O(1)$$

The characteristics of the modulation of double and simple modes differs significantly. The expression of the symmetric

tensor R implies that there is an *unique* non-zero principal value R_K associated to the principal vector $\widehat{\Phi_K \Psi_K}$. This latter defines the unique possible direction of modulation. In this direction the amplitude of the modulation is driven by :

$$\gamma P^e R_K A'' + \rho^e 4(\omega - \omega_K)^2 A = 0$$

where

$$R_K = \frac{\gamma P^e}{\rho^e} \frac{|\Phi_K \Psi_K|^2}{\omega_K^2 \langle |\Psi_K|^2 \rangle \langle |\Phi_K|^2 \rangle} > 0$$

The amplitude variation takes also the classic form :

$$A(x) = A_+ \exp(+i\tilde{k}x) + A_- \exp(-i\tilde{k}x)$$

with the real valued "wave number" :

$$\tilde{k}(\omega) = \sqrt{\frac{\rho^e 4(\omega - \omega_K)^2}{R_K \gamma P^e}} = \frac{\omega_K}{c_{sound}} \sqrt{\frac{4}{R_K} \left(\frac{\omega}{\omega_K} - 1 \right)^2}$$

Thus, the modulation is always a propagating phenomena. Its "wavelength" $\tilde{\Lambda}_K$ is linked to the frequency by the relation :

$$\tilde{\Lambda}_K(\omega) = \lambda_K \sqrt{\frac{R_K}{4} \frac{1}{\left| \frac{\omega}{\omega_K} - 1 \right|}} = O\left(\frac{c_{sound}}{|\omega - \omega_K|}\right) \gg \lambda_K$$

Note that, contrary to simple modes, the modulation behaviour is symmetric on both side of the modal frequency ω_K . The modulation wavelength is inversely proportional to the frequency shift $\omega - \omega_K$, for double mode, and to the square root of the frequency shift $\sqrt{\omega - \omega_K}$ for simple mode. Consequently, the spectrum band for modulation of double modes is wider than for simple mode and longer correlation lengths can be expected.

4 Conclusion

The phenomena of large modulation of high frequency acoustic waves in periodic porous media has been investigated through three different approaches. Periodic networks of Helmholtz resonators, reduced to 1D-spring-mass system, are first studied analytically, then studied by multi-scale method applied to periodic discrete media. 3D-porous media are addressed by multi-scale asymptotic method in continuum mechanics. These several analysis lead to similar conclusions.

The originality of this study is to depart from the classical framework of homogenization of periodic media based upon the principle of scale separation. This latter is commonly understood in the sense that a physical variable, relevant for the phenomena (pressure in poro-acoustics) varies at the macro-scale. It follows as essential consequences, that :

- the local regime in the period is quasi-static at the leading order (or at least in subdomain of the period),
- the relevant physical variables are preserved identical though the up-scaling,
- a unique "equivalent media" description applies on the whole low frequency range,
- the homogenized description is independent of the selected period (irreducible or not).

Considering higher frequencies implies that the physical variables vary locally according to the modes. However, when the mode amplitude presents large scale modulation, another type of scale separation on which the asymptotic multi-scale procedure applies. The derived description and

usual homogenization are fundamentally different in nature meanwhile both lead to macro scale formulation :

- through the up-scaling process, the type of variable are changed (from pressure to mode amplitude), in addition to the change of differential operator,
- contrary to an unique "equivalent continuum media", a *family* of descriptions is derived for modulation phenomena,
- each description is attached to a specific eigenmode carrying the high frequency modulated wave,
- consequently, a given description depends of the selected period (irreducible or not) and is valid only in the vicinity of the considered eigenfrequency.

Modulation phenomena involve "full" dynamics at the local scale. Consequently, it differs from Rayleigh scattering (frequency lower than eigenmodes) [5], and from inner resonance in highly contrasted materials [4]. "Full" dynamics in periodic media are usually described by the Floquet-Bloch theory [10]. The specificity of the present approach is to extract, from the comprehensive Floquet-Bloch modal space, the particular frequency bands enabling large modulations, therefore large correlation, of high frequency acoustic perturbations. It also provides the governing modulation equation.

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