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# ACOUSTICS 2012

## Wave decomposition method for identification of structural parameters

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This article deals with determination of the equation of motion of 1D and 2D structures from the measurement of the steady-state vibration response. The method is based on the modified IWC and McDaniel methods ([1]-[4]). The novelty is that we consider that the equation of motion of the structure is not known apriori. We can choose from some predefined set of models which *could* describe the structure. The choice of the best (optimal) model is achieved with the Bayesian information criterion. The method is applied to a thin aluminium beam under axial tension and to a back-board of an acoustic guitar. It is shown that in these cases the Bayesian information criterion provides useful means for choosing the most pertinent model for given data.

## 1 Introduction

This paper is dedicated to a model selection problem. The vibration problems are difficult to solve exactly, that is why there are many approximate models which are easier to solve but they neglect some physical phenomena. The obvious motivation for the model selection is to choose the simplest one which describes sufficiently well the physical problem. Let's consider for example a problem of calculation of the eigenfrequencies of a beam pinned at its ends. The most precise would probably be the 3D FEM analysis with a large number of degrees of freedom. If we restrain to analytical models in 1D, we can use the Timoshenko beam model or the simpler Euler-Bernoulli beam model (if the beam is sufficiently thin). We could also consider that the pinned ends contribute to the axial tension in the beam and add a tensile term in the models of Timoshenko and Euler-Bernoulli. If we knew all the necessary parameters describing those models we would choose the most precise model. But, generally, this is not the case. More often we want to determine those parameters by some inverse technique. This paper deals with a method called Bayesian information criterion (BIC) applied to the problem of determination an appropriate (optimal) model in vibration problems of beam and plates.

## 2 McDaniel inverse method for 1D problems

Before we can proceed to the model selection we must have some mathematical method which can fit the data with given model and provide the objective residuals. McDaniel [1] used for the first time a pseudo-local inverse method to determine the damping factor of the Euler beam structure. The basic idea of this method as follows. As an example we consider the equation of motion of an Euler beam in frequency domain

$$EI \frac{d^4 u}{dx^4} + \rho_L \omega^2 u = 0 \quad (1)$$

where  $EI$  is the beam stiffness,  $\omega$  is the angular frequency and  $\rho_L$  is the linear density. If there are no external forces in the observed region, the steady-state displacement field  $u$  must take the following form of the general solution

$$u_{gen}(EI, \rho_L) = \alpha_1 \sin(kx) + \alpha_2 \cos(kx) + \alpha_3 \sinh(kx) + \alpha_4 \cosh(kx) \quad (2)$$

where the wave vector  $k = \sqrt[4]{(\omega^2 \rho_L)/(EI)}$ . Whatever are the boundary conditions (which are often unknown) the real solution (dependent on the boundary conditions and excitation) is *included* in the general solution, in other words  $u \subset u_{gen}$ . We can define two functional operators:  $\mathbf{P}_{gen}$  - projector on the general solution space,  $\mathbf{Q}_{gen}$  - projector on

the space orthogonal to the general solution functional space. If the equation of motion describes perfectly the physical reality and we *know* its parameters  $EI/\rho_L$  then the  $u$  lies completely in the functional space  $u_{gen}$ , so  $\mathbf{P}_{gen}u = u$  and  $\mathbf{Q}_{gen}u = 0$ .

But in the realistic case, the Eq.(1) does not satisfy completely the real solution of the physical problem and we do not know exactly its parameter  $EI/\rho_L$ . So we can define the residual function

$$r = \mathbf{Q}_{gen}u \quad (3)$$

which shows the distance between the general solution and the measured displacement field. The optimal value of parameter  $EI/\rho_L$  is found by minimizing the norm of the residual  $r$

$$(EI/\rho_L)^* = \arg \min_{EI/\rho_L} \|r\| \quad (4)$$

In other words we optimize the general solution of the Eq.(1) in order to fit as closely as possible the measured data. The presented approach can be applied to a discrete problem in the following way. Let's consider the measured points  $x_1, \dots, x_N$ . Then the general solution in the discrete  $i$ -th point can be calculated using matrix  $\Psi$

$$u_{gen}^{(i)} = \Psi_{ij} \alpha_j \quad (5)$$

$$\Psi_{i,*} = [\sin(kx_i), \cos(kx_i), \sinh(kx_i), \cosh(kx_i)] \quad (6)$$

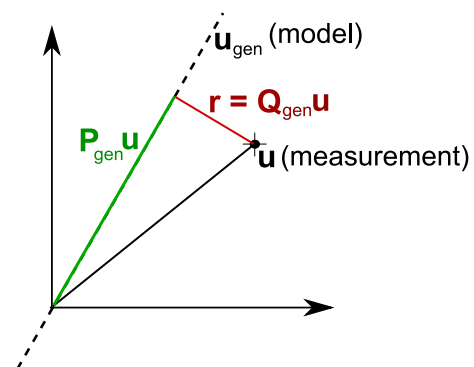


Figure 1: Schematic representation of the relations between the measured data  $\mathbf{u}$ , general solution (model)  $\mathbf{u}_{gen}$ , projected data on the space of the general solution and the residual.

It can be shown that in the discrete case, the projector to the general solution space is a square matrix  $\mathbf{P}_{gen} = \Psi\Psi^{-1}$  and the projector to the orthogonal (residual) space is  $\mathbf{Q}_{gen} = \mathbf{I} - \Psi\Psi^{-1}$ . The inverse of the  $\Psi$  matrix is meant in the pseudo-inverse sense, because this matrix is strongly rectangular.

The final residual vector of the inverse method can be defined as

$$\mathbf{r}^* = \mathbf{Q}_{gen}^* \mathbf{u} \quad (7)$$

where the  $\mathbf{Q}_{gen}^*$  is the resulting residual operator optimized for the solution given by Eq.(7) and  $\mathbf{u}$  the displacement function  $u$  measured in discrete measurement points. The simplified schematic representation of the problem in two dimensions (two measurement points) is shown at the Figure 1. The *measurement* point  $\mathbf{u}$  remains fixed, while it is the general space (represented by line  $\mathbf{u}_{gen}$ ) which is optimized to diminish the norm of the residual vector  $\mathbf{r}$ .

All the above considerations can be applied to any mathematical model of a vibration of 1D structures which can be described by the displacement function  $u$  and one ordinary differential equation (this equation should be self-adjoint). In this way we can deal with the string, thin and thick beams, thin sandwich composite beams and hybrid equations between beam and string (typically thick bass guitar string or beam under heavy tension).

### 3 Modified IWC method for 2D problems

The McDaniel method cannot be used directly for equations of motion describing vibration in 2D because the general solution does not exist in the closed form. It can only be approximated by some set of functions which are all particular solutions of the homogenous equation. Let's consider the most simple Kirchhoff plate equation of motion defined by

$$D\Delta u + \rho_s \omega^2 u = 0 \quad (8)$$

where  $D$  is the plate stiffness and  $\rho_s$  is the surface density of the plate. Berthaut [2] was the first who studied the correlation between the solution of the Eq.(8) and plane waves travelling in different directions. He showed that his method called Inverse Wave Correlation (IWC) can be used for estimation of the dispersion curves of the equation of motion. Chardon [5] used the same principle but in a different sense. He decomposed the measured wavefield into a *sum* of plane waves all satisfying the equation on motion in question. This sum of the plane waves serves as a "pseudo-general" solution of the McDaniel method for 1D structures:

$$u_{gen}(x_j, y_j) = \sum_{n=1}^N \sum_{m=1}^{N_{roots}} \alpha_{nm} \exp(ik_m(x_j \cos \theta_n + y_j \sin \theta_n)) \quad (9)$$

where  $N$  is the number of plane waves directions forming the basis and  $N_{roots}$  is the number of roots of the dispersion equation associated to the Eq.(8). Chardon considered only propagating waves ( $k_m > 0$ ) and placed his investigation far from the boundaries to avoid the presence of the evanescent waves. Indeed, it was proven by Colton [6] that the sinusoidal plane waves form dense space of the general solution of the Helmholtz equation of the 2nd order. A similar proof for 4th order equation like the Kirchhoff plate was not found to the author's knowledge. None the less, we use the modified method of Chardon and we use all the roots  $\{k_m\}$  to be closer to the general solution.

The inverse technique is the same as in the preceding McDaniel case with projectors  $\mathbf{P}_{gen}$  and  $\mathbf{Q}_{gen}$ . The matrix  $\Psi$  is defined as follows

$$\Psi_{j,[m,n]} = [\exp(ik_m(x_j \cos \theta_n + y_j \sin \theta_n))]_{m,n=1}^4 \quad (10)$$

The complexity of using this method in 2D is the choice of the number of plane waves directions. The number should not be too low, so that the general solution is not too restrained and it should not be too high either in order not to overfit the measured data. In our case we made use of the BIC criterion described below to do the optimal choice of the plane waves number. The advantage of using the BIC criterion is omitting of the human factor in the choice of the size of the (pseudo-) general solution. We would naturally include as many plane wave as possible to get closer to the general solution. But after applying the BIC criterion we can see that there is a optimal number of waves which describes sufficiently well the vibration field without adding to much free parameters.

The presented method can be used to any equation of motion describing the plate with the only one equation like the Kirchhoff plate, Mindlin plate, Kirchhoff orthotropic plate, membrane (Helmholtz equation).

### 4 BIC criterion

Bayesian information criterion (BIC) is a statistical tool which predicts the best choice of models from a given finite set of models. It can be used for any problem of optimization where there exists an objective likelihood function. The criterion searches for the maximum of the likelihood function over the considered models but penalizes the models with more free parameters. This is important because in vibration problems we can always have more complicated models which give better fit to the measured data but the price to pay is the complexity of the model and high possibility of unstable determination of these parameters. If we consider that our inverse problem (either McDaniel or IWC) has a residual vector which is normally distributed then we can write the BIC function for a model  $m$  as follows (for example [7])

$$\text{BIC}_m = n \ln \left( \frac{\sum r_i^2}{n} \right) + K_m \ln n \quad (11)$$

where  $n$  is the number of measured independent data,  $\mathbf{r}$  is the residual defined by Eq.(7),  $K_m$  is the number of model parameters. The absolute value of BIC does not have importance, what counts are the differences of BIC for different models in question. If we define the difference

$$\Delta_m = \text{BIC}_m - \min(\text{BIC}) \quad (12)$$

then we can determine the posterior probabilities that a given model  $m$  is the closest to given measured data ([7])

$$p_m = \frac{\exp(-\Delta_m/2)}{\sum_j \exp(-\Delta_j/2)} \quad (13)$$

### 5 Experimental identification of optimal 1D model

The method of model selection was first applied to the determination of suitable model for a thin aluminium beam.

The beam was submitted to four levels of static axial forces (0, 500, 1000 and 2000N). Beam length was 400mm, its width 20mm and its thickness 2mm. It was transversely driven by a shaker with pseudo-random excitation bandwidth 0-10kHz. The steady-state response was measured at 43 points by scanning laser vibrometer. The prestress axial force could be measured independently by static tensile meter which used the mechanical deformation of the rigid support of the beam.

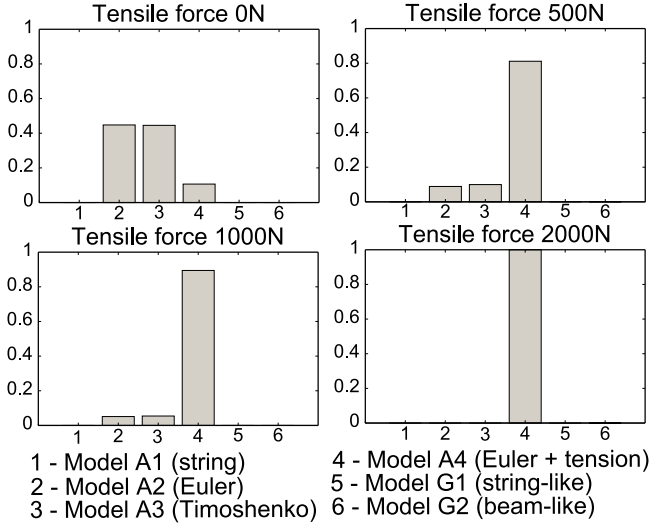


Figure 2: BIC posterior probabilities of different models considered in McDaniel inverse problem for a aluminium beam under axial tension. Some probabilities are negligible, so we cannot see anything.

All the models considered in this analysis shown in Figure 2 can be summarized in linear differential equation of the form:

$$A_0 u(x) + A_2 \frac{d^2 u(x)}{dx^2} + A_4 \frac{d^4 u(x)}{dx^4} = 0 \quad (14)$$

The coefficients  $A_i$  depend of chosen model. Their overview is in the Table 1. From the analysis of the posterior probabilities shown in Figure 2 we can see that the Euler-Bernoulli model with axial force is preferred once the axial force reaches significant level of 500N. Interesting phenomenon is the near equality of the Euler-Bernoulli and Timoshenko models. This is not surprising because the beam is sufficiently thin. Results showing the determination of the Euler-Bernoulli model with axial force are shown at Figure 3. We can see a good agreement between the results obtained by McDaniel inverse method (dynamic) and the static methods (3-point bending for example).

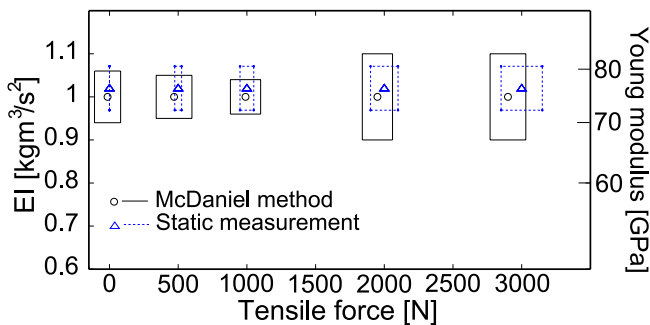


Figure 3: Results showing the optimized values for the Euler-Bernoulli beam with tensile force.

Model	Name	Parameters
A1	String	$A_0 = \rho_L \omega^2$ $A_2 = T$
A2	Euler-Bernoulli	$A_0 = \rho_L \omega^2$ $A_4 = EI$
A3	Timoshenko	$A_0 = -\rho_L \omega^2 + \frac{\rho_L \omega^4 I}{k A^2 G}$ $A_2 = \omega^2 \rho_L \frac{I}{A} (1 + \frac{E}{k G})$ $A_4 = EI$
A4	Euler + Force	$A_0 = -\rho_L \omega^2$ $A_2 = -T$ $A_4 = EI$
G1	String-like	$A_0 = \rho_L \omega^2$ $A_2 = A_2(\omega)$
G2	Beam-like	$A_0 = -\rho_L \omega^2$ $A_4 = A_4(\omega)$

Table 1: Different string/beam models. Models designed A are analytical with parameters constant upon frequency. Models designed G are pseudo-general, their parameters can change with frequency. Parameters:  $T$ -tensile force in the string,  $E$ -Young's modulus,  $G$ -Shear modulus,  $I$ -section moment,  $\rho_L$ - linear density,  $\omega$  - angular frequency,  $A$  section of the beam,  $k=5/6$  for rectangular section.

## 6 Experimental identification of optimal 2D model

As an example of a rather complicated 2D structure we studied the backboard of the acoustic guitar model Stag 536. It is made of 4mm thick spruce wood and it is stiffened by three parallel wooden stiffeners on the inner side of the guitar (Figure 4). So the plate is essentially anisotropic and not homogenous in space. Two regions were studied: the zone A which represents the entire backboard was studied at low frequencies around 500Hz and the zone B which is situated between the two adjacent stiffeners represents homogenous wood was studied at higher frequencies around 1500Hz. Modified IWC method was used to identify the coefficients of the isotropic and orthotropic Kirchhoff plate equations from the vibration fields measured in the zone A and B.

The orthotropic Kirchhoff plate equation has the form

$$-\rho_s \omega^2 u + D_1 \frac{\partial^4 u}{\partial x^4} + D_3 \frac{\partial^4 u}{\partial y^4} + (D_2 + D_4) \frac{\partial^4 u}{\partial x^2 \partial y^2} = 0 \quad (15)$$

where  $\rho_s$  is the surface density and coefficients  $D_i$  describe the bending stiffness of the plate. If the plate is isotropic then  $D_1 = D_3 = D$  and  $D_2 + D_4 = 2D$ .

We identified the two models (isotropic and orthotropic) from the vibration fields using the modified IWC method. On Figure 5 we can see the posterior probabilities determined for two considered models Kirchhoff isotropic and orthotropic plate. The analysis run on the data at lower frequencies (500 - 1000 Hz) used whole zone A (stiffeners were neglected). On the other hand at higher frequencies (1-2kHz) the same analysis was performed on restrained zone B (homogenous anisotropic wood without stiffeners). From the Figure 5 we can say that at the lower frequencies isotropic Kirchhoff model is better while at the higher frequencies it is the orthotropic model which is more adapted.



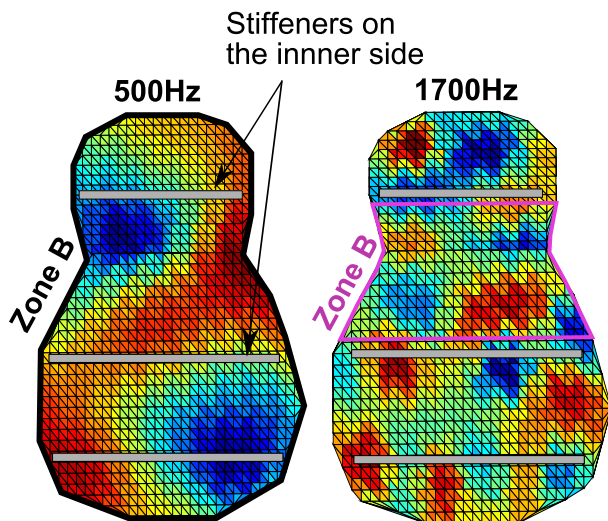


Figure 4: Vibration shapes at 500 and 1700Hz of the backboard of the acoustic guitar. Zone A contains the entire backboard while the zone B lies between the two adjacent stiffeners.

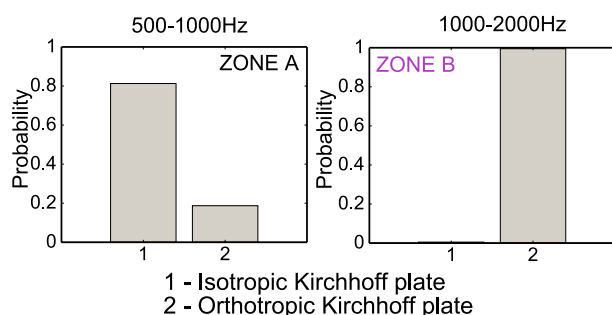


Figure 5: Posterior probabilities determined by the BIC of the two considered models describing the guitar backboard.

As the mass of the plate is unknown we cannot determine coefficients  $D_i$  directly but we can determine the velocities of the propagating plane waves in dependence of angle of propagation. Let's consider a plane wave propagating in the direction  $\mathbf{n}=(n_x, n_y)$  and wave vector  $\mathbf{k}=\mathbf{k}\mathbf{n}$

$$u = \exp(i(\omega t - \mathbf{k}\mathbf{x})) \quad (16)$$

Then inserting Eq.(16) into Eq.(15) we get the dispersion equation for  $\mathbf{k}$  and determine the phase velocity  $c(n_x, n_y, \omega)$ . It is useful to consider the phase velocity divided by square root of  $\omega$  because this variable is independent of frequency

$$\frac{c}{\sqrt{\omega}} = \frac{\sqrt{\omega}}{k} = \sqrt[4]{\frac{D_1 n_x^4 + (D_2 + D_4) n_x^2 n_y^2 + D_3 n_y^4}{\rho_s}} \quad (17)$$

On Figure 6 we can clearly see the anisotropic nature of the wood measured at the zone B. This is not surprising because zone B is uniquely composed of wood without stiffeners. The surprising result of this analysis is that the guitar backplate is behaving as isotropic at low frequencies. In other words we can say that the anisotropic effect of wood is compensated by added stiffeners.

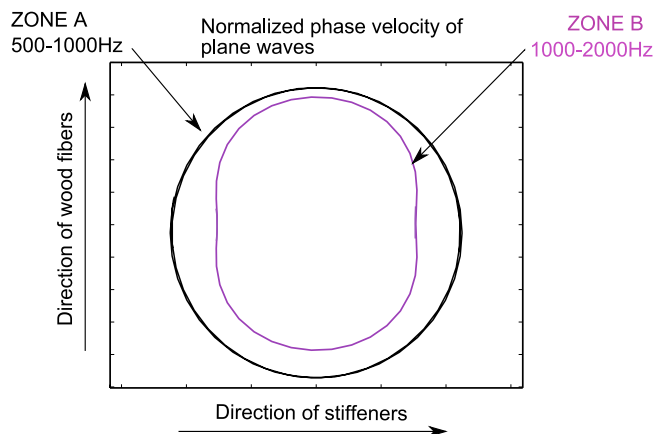


Figure 6: Normalized phase velocities as results of the inverse problem defined by Eq.(17).

## 7 Conclusion

In this article we showed the applicability of the BIC criterion used within the McDaniel and IWC inverse methods. This criterion enables us to choose which model is the most appropriate to describe the vibration problem. The method was applied to the determination of the equation of motion of the aluminium beam under axial tension and to the determination of equation of motion of an acoustic guitar backboard. Interestingly, we observed that this backboard stiffened by three stiffeners behaves almost like an isotropic plate at low frequencies, while at high frequencies we see clearly the anisotropic nature of wooden plate.

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