



HAL
open science

Upper and lower bounds of low-frequency band gaps

Anton Krynkin

► **To cite this version:**

Anton Krynkin. Upper and lower bounds of low-frequency band gaps. Acoustics 2012, Apr 2012, Nantes, France. hal-00810790

HAL Id: hal-00810790

<https://hal.science/hal-00810790>

Submitted on 23 Apr 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



ACOUSTICS 2012

Upper and lower bounds of low-frequency band gaps

A. Krynkin

University of Bradford, School of Engineering, Design & Technology, Richmond Road,
Chesham Building, BD71DP Bradford, UK
a.krynkin@bradford.ac.uk

It is known that periodic structures (PS) exhibit frequency intervals where sound waves do not propagate (i.e. band gaps, BG). The existence of the band gaps is attributed to the periodicity (i.e. the Bragg BG) and the properties of a periodic element. The later substantially enhances the performance of the PS if it supports resonances that generate additional BG (resonant BG). By changing geometrical and physical properties of the periodic element the resonances can be observed below the first Bragg BG (associated with the half of sound wavelength) that makes PS effective treatment in the low frequency regime. The aim of this paper is to approximate the limiting frequencies (lower and upper bounds) of the generated low-frequency resonant BGs. The PS is represented by an array of thin elastic shells exhibiting multiple low-frequency resonances. In the vicinity of lower bound the approximation is found by means of the Rayleigh Identity which leads to the Foldy-type equation. The upper bound of the resonant BG is approximated with the help of matched asymptotic expansions. This gives an accurate approximation for the upper bound approaching the first Bragg BG where the contribution of effects related to the periodicity has to be taken into account.

1 Introduction

Structures supporting band gaps are of great interest for many years [1]. For the electromagnetic waves the theory on the periodic structures has been employed to describe their propagation through the crystals. Example of simple application of the periodic structures can be taken from the design of band-pass filters. In acoustics these structures can be used to effectively control sound at low-frequencies. Noise barriers are one of the potential applications of periodic structures. To optimize the performance of these structures one needs to know the frequencies of the band gap bounds.

The existence of band gaps attributed to the distance between the scatterers arranged in the periodic array have been studied experimentally, numerically and analytically. The homogenization technique derived for the periodic structures can be employed to find the band gap limiting frequencies [2, 3]. In particular the first band gap observed in the first Bruloin zone for the square arrangement of rigid scatterers (Neumann boundary conditions) is approximated by the following limiting frequencies [2]

$$\omega_1^2 = \omega_0^2(1 - 4\mathcal{F}) \quad \text{and} \quad \omega_2^2 = \omega_0^2(1 + 2\mathcal{F}). \quad (1)$$

The Bragg frequency $\omega_0 = \pi c/L$ corresponds to the half of wavelength that fits the distance between the scatterers (L). The results are based on the low filling fraction $\mathcal{F} = \pi a^2/L^2 \ll 1$. For the soft circular scatterers (Dirichlet boundary conditions) the first band gap starts from zero frequency [4, 5] $\omega_1 = 0$. Its upper limit can be approximated by the low-frequency approximations that give

$$\omega_2^2 = \frac{2}{\pi} \omega_0^2 [\log(L/a) - C]^{-1}, \quad (2)$$

within which constant C brings the periodic arrangement contribution. The second band gap related to the first Bragg frequency can be defined by the following limits [6]

$$\omega_1^2 = \omega_0^2 \quad \text{and} \quad \omega_2^2 = \omega_0^2(1 + 4\omega_0^2\eta). \quad (3)$$

This approximation involves parameter η that describes the influence of the periodic arrangement.

For the arrangement of the resonant scatterers the additional band gaps are observed [7, 8]. Their existence and position are attributed to the scatterer resonances (so-called local resonances). The homogenization technique can be again employed to find the approximate limiting frequencies. In this paper the resonant scatterer is represented by a cylindrical elastic shell that exhibits multiple resonances below the first Bragg frequency. Using the Rayleigh Identity [9] and

matched asymptotic expansions (MAE) it is possible to derive accurate approximations of the band gaps related to the local resonances.

2 Rayleigh Identity

Consider acoustic problem where waves propagate through a doubly periodic square array of thin elastic shells with lattice constant L . The array of scatterers is surrounded by an acoustic environment with density ρ_o and speed of sound c_o . The elastic shell is characterised by its density ρ , the Young's modulus E and Poisson's ratio ν . In this paper we consider solution $p(\mathbf{r})$ of the Helmholtz equation defined in the polar coordinates $\mathbf{r} = (r, \theta)$. It is also assumed that the solution is time-harmonic $\exp(-i\omega t)$.

The scattering coefficient of the thin elastic shell can be approximated by

$$Z_n = -\frac{[J'_n(k_o a)]^2 [1 - (k_3 a)^2 + n^2]}{J'_n(k_o a) Y'_n(k_o a) [1 - (k_3 a)^2 + n^2] + [n^2 - (k_3 a)^2] Q}, \quad (4)$$

where $Z_{-n} = Z_n$, $k_o = \omega/c$, $Q = \rho_o(\rho a h)^{-1}$, a is the shell mid-surface radius, h is its half thickness and $k_3 = \omega/c_3$ with $c_3 = \sqrt{E/[\rho(1 - \nu^2)]}$. The coefficient (4) tends to the rigid scatterer limit as $Q \rightarrow 0$ so that

$$Z_n = -\frac{J'_n(k_o a)}{Y'_n(k_o a)}. \quad (5)$$

It is also noted that coefficients (4) have singularities that correspond to resonances of the elastic shell.

For the doubly periodic array, the solution $p(\mathbf{r})$ of the Helmholtz equation is also subject to the quasi-periodic conditions that are

$$p(\mathbf{r} + \mathbf{R}_j) = e^{i\boldsymbol{\beta} \cdot \mathbf{R}_j} p(\mathbf{r}), \quad \mathbf{R}_j = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2, \quad n_1, n_2 \in \mathbb{Z}, \quad (6)$$

within which vectors \mathbf{r} and \mathbf{R}_j are defined in Cartesian coordinates, \mathbf{R}_j defines the position of j -th lattice cell and $\boldsymbol{\beta}$ is wave vector.

Conditions (6) leads to the algebraic system of equations with respect to the unknown coefficients B_n , $-\infty < n < \infty$, that forms the Rayleigh Identity

$$B_n + \sum_{m=-\infty}^{+\infty} (-1)^{m-n} \sigma_{m-n}(k_o, \boldsymbol{\beta}) Z_m B_m = 0, \quad -\infty < n < \infty, \quad (7)$$

where $\sigma_n(k_o, \boldsymbol{\beta})$ represents the lattice sum [10]. This homogeneous algebraic system has to be truncated and then zeros of the determinant of coefficient matrix have to be found.

The latter gives dispersion relation between frequency ω and wave vector $\boldsymbol{\beta}$. In Figure 1 the dispersion relation is plotted in the first irreducible Brillouin zone defined by the following nodes $\Gamma = (0, 0)$, $M = (\pi, 0)$ and $K = (\pi, \pi)$. The first wide band gap is observed around $k_o L = 1.5$ that corresponds to the axisymmetric resonance of the thin elastic shell. The second wide band gap is positioned around the first Bragg frequency that corresponds to the case when half of sound wavelength fits the distance between the scatterers. The bound of the second band gap can be estimated by the equations (1) derived by McIver [2].

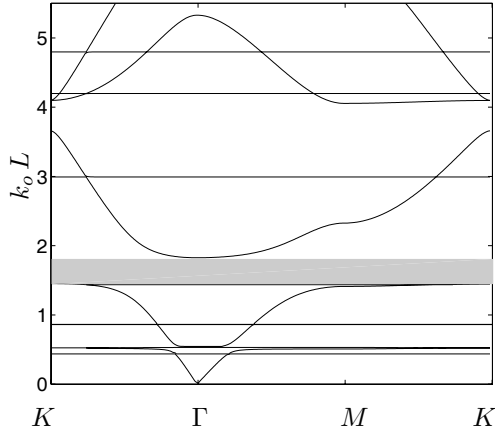


Figure 1: Eigenvalue solution obtained for the doubly periodic array of elastic shells [7].

The first wide band gap is observed below the Bragg frequency and its bounds are of main interest in this paper. It is immediately noted that the lower bound ω_1 of this band gap tends to the value of the local resonance [11]. This could be explained by the standing wave generated by the axisymmetric resonance. That gives

$$\omega_1^2 = \omega_0^2 + \pi c_o^2 Q, \quad (8)$$

where term Q defines the interaction of the acoustic environment with the elastic shell and $\omega_0 = c_3/a$ corresponds to the resonance of shell in vacuum. Note that the lower limit of this band gap is independent of the geometry of the periodic array.

To derive the upper bound one can use Foldy's type dispersion relation given by

$$\beta^2 = k_o^2 - \frac{4}{A} F, \quad (9)$$

where $F = \sum_{n=-\infty}^{+\infty} Z_n$ represents the far field pattern. This equation is true for the wavelength much bigger than the radius of scatterer and the lattice constant that is $k_o a \ll 1$ and $k_o L \ll 1$. For the local resonances observed far below the first Bragg frequency equation (9) gives accurate approximations for the upper bounds of the corresponding band gaps. In the first irreducible Brillouin zone the upper bound is found in the vicinity of $\beta L = 0$. In particular for the axisymmetric resonance with the dominant scattering coefficient Z_0 the dispersion relation (9) is expanded in Taylor series with respect to $k_o a \ll 1$ that gives

$$\omega_2^2 = \omega_1^2 + \mathcal{F}(2\omega_1^2 - \omega_0^2), \quad \text{as } \beta L \rightarrow 0, \mathcal{F} \rightarrow 0, \quad (10)$$

Figure 2 illustrates the accuracy of Foldy's approximations. It is observed that by approaching the first Bragg frequency the accuracy of Foldy's approximation is deteriorating. In particular this is true for the upper bound of the band gap generated by the axisymmetric resonance. For the upper limit the accuracy range is decreased to 5.5% compared to 3.5% observed around the upper bound ($k_o L = 0.55$) of the band gap generated by the shell resonance with index $n = 1$.

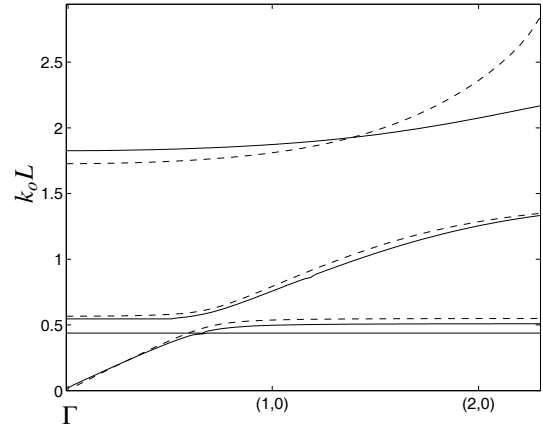


Figure 2: Foldy approximation (9) (dashed line) compared with the semi-analytical solution of Rayleigh Identity (7) (solid line).

To create more accurate approximation one needs to include the influence of the quasi-periodic conditions defined in (6). This requires to relax the condition $k_o L \ll 1$ imposed earlier to derive the Foldy's approximation (9). For this reason the technique based on the matched asymptotic expansions is employed in the next section.

3 Matched asymptotic expansions

The asymptotic expansions are found in outer and inner regions. Near the scatterer the solution $p(\mathbf{r})$ is subject to the boundary conditions imposed on its surface. The characteristic length of this region is the radius a of scatterer that defines the inner scaling of radius vector \mathbf{r} as

$$\xi = r/a. \quad (11)$$

In the outer region the solution $p(\mathbf{r})$ is less affected by the geometry and physical properties of the scatterer that results in the following coordinate scaling

$$\zeta = k_o r. \quad (12)$$

It is only required that wavelength is much bigger than radius of the scatterer ($k_o a \ll 1$). This allows $k_o L = O(1)$.

3.1 Outer expansion

In the outer region the scatterers are replaced by the point sources O_j in each cell and the solution is subject to the quasi-periodic conditions (6). This gives

$$\Psi(r, \theta) = \sum_{n=-\infty}^{+\infty} A_n \sum_{\mathbf{R}_j \in \Lambda \setminus \{0\}} e^{i\mathbf{B}\mathbf{R}_j} H_n^{(1)}(k r_j) e^{in\theta_j}, \quad (13)$$

where (r_j, θ_j) are the local coordinates with origin at O_j .

The leading order of coefficients A_n , $n \in \mathbb{Z}$ is estimated by the small parameter [13, Section 6.8] that is

$$\eta = \frac{1}{K - \log \epsilon}, \quad (14)$$

with unknown constant K . Thus

$$A_n = \eta \hat{A}_n. \quad (15)$$

The use of addition theorem [12] and transformation to the reciprocal lattice Λ^* in equation (13) converts it to the form that includes lattice sum [6]

$$\sigma_n(k_o, \boldsymbol{\beta}) = -\delta_{0,n} + i\sigma_n^Y(k_o, \boldsymbol{\beta}), \quad -\infty < n < \infty. \quad (16)$$

It is assumed here that for $n = 0$ imaginary part of lattice sum (16) contributes to the outer solution as

$$\sigma_0^Y(k_o, \boldsymbol{\beta}) = \frac{\delta}{\eta}, \quad \text{with } \delta = O(1). \quad (17)$$

Then expanding outer solution in terms of the inner coordinate ξ up to order η leads to

$$\Psi^{(\eta,n)}(\xi) = i\hat{A}_0 \left[\delta - \frac{2}{\pi} + \eta \frac{2}{\pi} (K + \gamma - \log 2 + \log \xi) \right]. \quad (18)$$

3.2 Inner expansion

The inner solution has to satisfy the boundary conditions imposed on the surface of single thin elastic shell. This gives

$$\psi(r, \theta) = \sum_{n=-\infty}^{+\infty} B_n^J [J_n(k_o r) + Z_n Y_n(k_o r)] e^{in\theta} \quad (19)$$

where B_n^J is unknown constant and Z_n is defined by equation (4). In the vicinity of axisymmetric resonance scattering coefficient Z_0 is $O(\eta)$ that gives

$$Z_0 = \delta_Z \eta, \quad \text{with } \delta_Z = O(1). \quad (20)$$

Expanding the inner solution (19) up to the order η we arrive at

$$\psi^{(\eta,n)}(\xi) = B_0^J \left[1 - \frac{2}{\pi} \delta_Z + \eta \frac{2}{\pi} \delta_Z \log \xi \right]. \quad (21)$$

3.3 Matching

Matching the outer solution (18) and the inner solution (21) we compare firstly factors of $\log \xi$ in order η that is

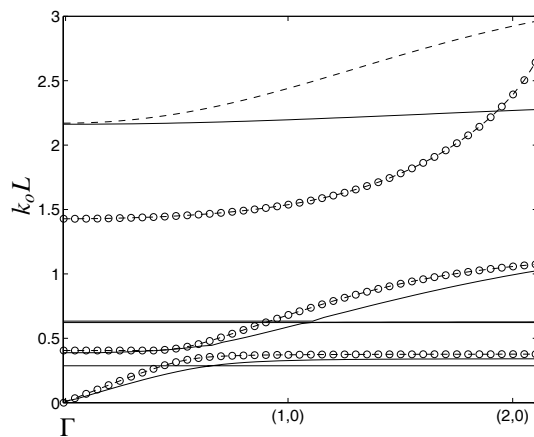
$$\hat{A}_0 = -i\delta_Z B_0^J. \quad (22)$$

The consistency of the leading orders in (18) and (21) requires that

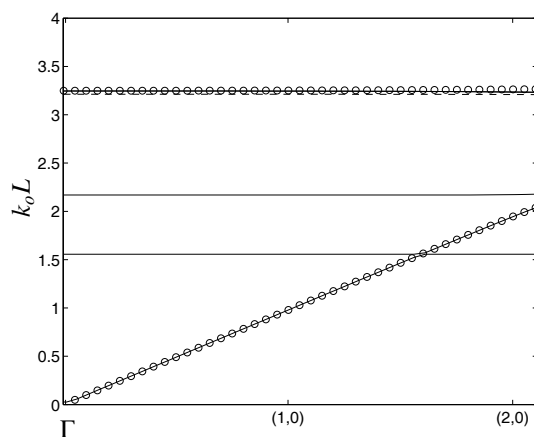
$$Z_0 \sigma_0^Y - 1 = 0. \quad (23)$$

The derivation of dispersion relation (23) is based on equations (17), (20) and (22).

Figure 3 demonstrates the improvement in approximation of the band gap upper limiting frequency near the axisymmetric resonance. For the bigger radius of elastic shell ($a = 0.0375$) m the upper bound of the band gap is affected by the effects associated with the periodicity. As a result the approximation of the upper bound based on Foldy's equation (9) is not valid anymore (the results are accurate within 30%). However the results based on MAE technique are accurate within 1%.



(a)



(b)

Figure 3: Foldy approximation (9) (dashed line with circles) and MAE results (23) (dashed line) compared with the semi-analytical solution of Rayleigh Identity (7) (solid line). (a) $a = 0.0375$ m. (b) $a = 0.007$ m.

4 Conclusion

The Foldy's type dispersion relation can be used to estimate the limiting frequencies of the band gaps generated by the local resonances. This however cannot be used to accurately estimate the limiting frequencies close to the first Bragg band gap. The improved dispersion relation based on matched asymptotic expansions includes the effect associated with the periodicity. The approach described in this paper can also be employed to find the bounds of band gaps related to the local resonances of higher order.

Acknowledgments

Author is grateful for the support from The University of Bradford and Prof. K. Horoshenkov.

References

- [1] L. Brillouin, *Wave propagation in periodic structures*, McGraw-Hill, New York (1946).

- [2] P. McIver, “Approximations to wave propagation through doubly-periodic arrays of scatterers”, *Waves Random Complex Media*, **17** 439–453 (2006).
- [3] R. V. Craster, J.D. Kaplunov and A.V. Pichugin, “High-frequency homogenization for periodic media”, *Proc. R. Soc. A*, **466** 2341–2362 (2010).
- [4] N.A. Nicorovici, R.C. McPhedran and L.C. Botten, “Photonic band gaps for arrays of perfectly conducting cylinders”, *Phys. Rev. E*, **52**, 1135–1145 (1995).
- [5] C.G. Poulton, R.C. McPhedran, N.A. Nicorovici, L.C. Botten and A.B. Movchan, “Asymptotics of photonic band structures for doubly-periodic arrays”, *Proceedings of the IUTAM Symposium on Mechanical and Electromagnetic Waves in Structured Media*, Kluwer Academic Publishers, New York (2001) 227–238.
- [6] A. Krynkin and P. McIver, “Approximations to wave propagation through a lattice of Dirichlet scatterers”, *Waves Random Complex Media*, **19** 347–365 (2009).
- [7] A. Krynkin, O. Umnova, A.Y.B Chong, S. Taherzadeh and K. Attenborough, “Predictions and measurements of sound transmission through a periodic array of elastic shells in air”, *J. Acoust. Soc. Am.*, **128**, 3496–3506 (2010).
- [8] A. Krynkin, O. Umnova, A.Y.B Chong, S. Taherzadeh and K. Attenborough, “Scattering by coupled resonating elements in air”, *J. Phys. D: Appl. Phys.*, **44**, 125501 (2011).
- [9] Movchan, A.B., Movchan, N.V. and Poulton, C.G., *Asymptotic Models of Fields in Dilute and Densely Packed Composites*, Imperial College Press, London (2002).
- [10] C.M. Linton, “Lattice sums for the Helmholtz equation”, *SIAM Review*, **52**, 630–674 (2010).
- [11] A.B. Movchan and S. Guenneau, “Split-ring resonators and localized modes”, *Phys. Rev. B*, **70**, 125116 (2004).
- [12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1972).
- [13] D.G. Crighton, A.P. Dowling, J.E. Ffowcs Williams, M. Heckl and F.G. Leppington, *Modern Methods in Analytical Acoustics*, Springer-Verlag, London (1992).