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Discussion on various model of multiple scattering in acoustics and elastic heterogeneous media

L. Le Marrec and G. Futhazar

IRMAR - Université Rennes 1, Campus Beaulieu, 35042 Rennes, France
loic.lemarrec@univ-rennes1.fr

A new formulation of the effective wave through a heterogeneous medium is the starting point of this paper. The coherent scattered field from a fixed inclusion is supposed to propagate with a complex effective wave number K. The objective is to find the expression of $K(\omega)$ according to the concentration of inclusion. This is performed in first order and traditional Foldy expression is obtained. Moreover the result at a second order differs from classical references as Linton-Martin formula. Each step of the method is detailed in order to discuss on the various consequences of proposed formulation. In particular, the solution for two coupled inclusions is detailed in order to highlight the consequences of hypotheses on the coupling regime.

1 Introduction

Multiple scattering methods [1] is applied to obtained some specifications of wave propagation through a random distribution of scatterer. This is performed by considering the coherent wave obtained by averaging the fields over all possible configurations of disorder [3, 4]. One of the main objective is to characterize an effective wave number. This later is generally defined as the wave number of the field exciting a representative scatterer. Several traditional steps are used to obtain such results some are statistical [2] some other are more based on the physical process. We propose here an original formulation that does not give new fundamental results but can be useful for specific problem typically when the source is no more a plane wave and when the physical contrast between matrix and inclusion is high.
The presentation is based on 2D-acoustical problem, where a random distribution of identical circular cylinders are confined in a specific domain.

2 Scattering formulation

We consider time-harmonic acoustical waves with angular frequency ω. Displacement $\vec{u}$ and pressure p in a medium of density ρ and celerity c are expressed thanks acoustical potential ψ: $\vec{u} = \nabla\psi$ and $p = -i\omega\rho\psi$. In homogeneous medium, the potential ψ satisfy the Helmholtz equation with wave number $k = \omega/c$ and is a linear combination of 2D harmonic eigen function $\phi_n(k\vec{r}) = J_n(kr)e^{in\theta}$ and $\hat{\phi}_n(k\vec{r}) = H_n(kr)e^{in\theta}$, where $H_n \equiv H_n^{(1)}$, r and θ are respectively the length and angle of vector position $\vec{r}$ according to a given frame.

2.1 Scattering by one inclusion

Consider a single inclusion with radius a. Material properties of inclusions are with the indices 1 whereas no indices are placed for the matrix case. Incoming, scattered and transmitted fields have to satisfy some restrictions. First, the incoming field ψ_i is supposed to be regular in the vicinity of the origin of the frame. Second the scattered field ψ_s must satisfy the Sommerfeld radiation condition. Third the transmitted field must be defined in the inclusion. According to these limitations, these fields can be expressed in a frame centered on the circular inclusion by $\psi_i(\vec{r}) = \sum_n c_n\phi_n(k\vec{r})$, $\psi_s(\vec{r}) = \sum_n a_n\hat{\phi}_n(k\vec{r})$ and $\psi_t(\vec{r}) = \sum_n b_n\phi_n(k_1\vec{r})$ where the coefficients c_n, a_n, b_n are the modal amplitude according to the specified frame. For perfect boundary condition continuity of displacement and pressure is imposed at the boundary, in term of potential: $\partial_r\psi_0 = \partial_r\psi_t$ and $\rho_0\psi_0 = \rho_1\psi_t$ at the boundary and $\psi_0 \equiv \psi_i + \psi_s$. Then we have to solve:

$$\begin{aligned} \sum_n \left(a_n\partial_r\hat{\phi}_n(k\vec{r}) + c_n\partial_r\phi_n(k\vec{r})\right) &= \sum_n b_n\partial_r\phi_n(k_1\vec{r}) \\ \rho\sum_n \left(a_n\hat{\phi}_n(k\vec{r}) + c_n\phi_n(k\vec{r})\right) &= \rho_1\sum_n b_n\phi_n(k_1\vec{r}) \end{aligned} \quad , \; r = a$$

We have $\partial_r\hat{\phi}_n(k\vec{r}) = k\, H_n'(ka)e^{in\theta}$ et $\partial_r\phi_n(k\vec{r}) = k\, J_n'(ka)e^{in\theta}$. Because $\int e^{in\theta}d\theta = 2\pi\delta_n$, the integration over θ leads to a linear system where solutions for each modes are uncoupled:

$$\begin{aligned} a_nH_n'(ka) - \frac{k_1}{k}b_nJ_n'(k_1a) &= -c_nJ_n'(ka) \\ a_nH_n(ka) - \frac{\rho_1}{\rho}b_nJ_n(k_1a) &= -c_nJ_n(k_1a) \end{aligned} \tag{1}$$

Introducing the coefficient

$$\Delta_n = \frac{\rho c}{\rho_1 c_1}H_n(ka)J_n'(k_1a) - H_n'(ka)J_n(k_1a),$$

the scattering coefficients are $a_n = z_nc_n$ with $z_n = -\Re(\Delta_n)/\Delta_n$. For further development we introduce infinite vector $a := (\ldots, a_n, \ldots)$ and scattering matrix Z: $(Z)_{nm} := z_n\delta_{nm}$. According to these conventions we have $a = Z\,c$. In the special case of rigid inclusion $\partial_n\psi_0 = 0$ and for a void $\psi_0 = 0$; for such impenetrable scatterer the same methodology gives $z_n = -J_n'(ka)/H_n'(ka)$ and $z_n = -J_n(ka)/H_n(ka)$ respectively. Notice that these results have sense only for fields expressed in a frame centered on the cylinder.

2.2 Scattering by N inclusions

Let now consider N inclusions labeled s_j ($j = 1, \ldots, N$). For each inclusion s_j a centered frame is defined. The position of a point M is then given by the local position vector $\vec{r_j}$. The vector form s_i to s_j is $\vec{r_{ij}}$. The transmitted and scattered field of each inclusion expressed in its local frame is $\psi_t^j = \sum_n b_n^j(s_j)\phi_n(k_1\vec{r}_j)$ and $\psi_s^j = \sum_n a_n^j(s_j)\hat{\phi}_n(k_1\vec{r}_j)$. In order to solve the boundary condition for a given inclusion (say the first) all the field at its boundary must be given in the frame of s_1. According to the Graf's addition theorem [5] the field scattered by s_j at M is, in the frame of s_1:

$$\psi_s^j(M) = \sum_n a_n^1(s_j)\phi_n(k\vec{r_1}) \quad , \quad a^1(s_j) = \widehat{S}^{1j}a^j(s_j)$$

if $r_1 < r_{1j}$. This later restriction is of course fulfilled at the boundary of s_1 because inclusions do not overlap. The transformation matrix $\widehat{S}^{1j}$ from the frame j to the frame 1 is defined by $(\widehat{S}^{1j})_{nm} = \hat{\phi}_{n-m}(k\vec{r_{1j}})$. Using the same methodology than for a single scatterer one obtain a system of N equations:

$$a^i(s_i) = Z\left(c^i + \sum_{1\neq j}\widehat{S}^{ij}a^j(s_j)\right) \quad , \quad i = 1, \ldots, N \tag{2}$$

where c^i is the modal amplitude of the incoming wave in the s_i-frame. This system is well-posed if the position of each inclusion is known.

3 Multiple scattering problems

3.1 Coherent Potential Approximation

The coherent wave can be obtained by averaging the field over all possible configuration of scatterers. If the surface

concentration is φ, the mean field is then $\langle\psi\rangle = \varphi\langle\psi_1\rangle + (1-\varphi)\langle\psi_0\rangle$ where $\langle\psi_0\rangle$ and $\langle\psi_1\rangle$ are the mean of external and internal field. Coherent Potential Approximation (CPA) [1] supposes that the coherence is lost far from the sources and the boundary $\partial\mathcal{B}$ of the heterogeneous medium $\mathcal{B}$; in other word the CPA implies $\langle\psi_0\rangle = 0$ or $\langle\psi_1\rangle = 0$ equivalently. The incoming field is independent of the position of inclusion, therefore $\langle\psi_0\rangle = \psi_i + \langle\psi_s\rangle$ and the CPA implies:

$$\langle\psi_s\rangle = -\psi_i \tag{3}$$

If an inclusion is placed at M, $\langle\psi_s\rangle + \psi_i$ is nothing else than the exciting field used in literatures [6].

3.2 Ensemble average

A configuration is defined by the set of all positions of inclusion $\Lambda_N = \{\vec{r_{01}}, \vec{r_{02}}, \ldots, \vec{r_{0N}}\}$ according to a fixed frame. The configuration is governed by the probability density $p(\Lambda_N)$ and the ensemble average is defined by

$$\langle\psi_s\rangle = \int\!\!\ldots\!\!\int \psi_s(\Lambda_N)\, p(\Lambda_N)\, \mathrm{d}\vec{r_{0N}} \ldots \mathrm{d}\vec{r_{02}}\mathrm{d}\vec{r_{01}}$$

where the integrations are performed in all the domain except in a circular disc C_M of radius a centered at the observation point M. Otherwise the field would be an internal field.
The total scattered field is $\psi_s = \sum_j \psi_s^j$. After ensemble average, the contributions of an inclusion is indistinguishable from the others. Therefore $\left\langle\psi_s^1\right\rangle = \left\langle\psi_s^{k\neq 1}\right\rangle$ and $\langle\psi_s\rangle = N\left\langle\psi_s^1\right\rangle$ if s_1 is chosen as a representative inclusion.
As presented by [7], $p(\Lambda_N) = p(\vec{r_{02}}\ldots\vec{r_{0N}}|\vec{r_{01}})p(\vec{r_{01}})$ where $p(\vec{r_{01}})$ is the probability to find s_1 ar $\vec{r}_{01}$ and $p(\vec{r_{02}}\ldots\vec{r_{0N}}|\vec{r_{01}})$ is the density probability to find the other inclusions s_j at r_{0j} ($j \neq 1$) if the position of s_1 is known. This later probability density can be decomposed as

$$p(\vec{r_{02}}\ldots\vec{r_{0N}}|\vec{r_{01}}) = p(\vec{r_{03}}\ldots\vec{r_{0N}}|\vec{r_{02}}, \vec{r_{01}})p(\vec{r_{02}}|\vec{r_{01}})$$

where $p(\vec{r_{02}}|\vec{r_{01}})$ is the conditional probability to find s_2 at $\vec{r}_{02}$ knowing the position of s_1. It is then useful to describe the following hierarchy

$$\begin{aligned}
\left\langle\psi_s^1\right\rangle &= \int \left\langle\psi_s^1\right\rangle_1 p(\vec{r_{01}})\, \mathrm{d}\vec{r_{01}} \\
\left\langle\psi_s^1\right\rangle_1 &= \int \left\langle\psi_s^1\right\rangle_{12} p(\vec{r_{02}}|\vec{r_{01}})\, \mathrm{d}\vec{r_{02}} \\
\left\langle\psi_s^1\right\rangle_{12} &= \int\!\!\ldots\!\!\int \psi_s^1(\Lambda_N)\, p(\vec{r}_{03}\ldots\vec{r}_{0N}|\vec{r}_{01}, \vec{r}_{02})\, \mathrm{d}\vec{r}_{03}\ldots\mathrm{d}\vec{r}_N
\end{aligned} \tag{4}$$

If the probability densities are supposed uniform:

$$p(\vec{r_{01}}) \approx \frac{1}{V}, \quad p(\vec{r_{02}}|\vec{r_{01}}) \approx \frac{1}{V}\mathcal{H}(r_{12} - 2a) = p(r_{12}) \tag{5}$$

where $V = N/n_0$ is the surface of the integration domain, $\mathcal{H}$ is the Heaviside step function and n_0 is the number of inclusion per unit area (or density of inclusion). In particular the CPA Eq.3 is at the first order of integration:

$$n_0 \int \left\langle\psi_s^1\right\rangle_1 \mathrm{d}\vec{r_{01}} = -\psi_i$$

4 Independent scattering

4.1 Discussion on its formulation

On the previous equation the objective is to give the more realistic sense to $\left\langle\psi_s^1\right\rangle_1$. In the traditional form of Independent Scattering Approximation (ISA) proposed by Foldy and other, one consider that the scattering pattern $f(\theta) = \sum_n z_n e^{in\theta}$ of the inclusion is not perturbed by ensemble average. Moreover the incoming field is replaced by an exciting field having an effective wave number K. Consider the case of incoming plane wave propagating along the $\vec{x}$ direction and probing s_1 placed at x_{01}, the radiation far from s_1 is

$$\left\langle\psi_s^1\right\rangle_1 = e^{iKx_{01}}\, f(\theta)\, H_0(kr_1) \tag{6}$$

The radial dependence of the scattering is governed by k. This is mainly motivated by the Sommerfeld radiation condition satisfied by ψ^1 before averaging over the position of all the inclusion except s_1. Moreover this formulation can be taken into question. First, Foldy approach leads to searching an exciting field solution of $(\Delta + K^2)\psi = 0$, where K is the effective wave number. Second, it looks reasonable to suppose that the scattering must be modified along the radial direction after this ensemble average: the global effect of other inclusions placed between s_1 and the observation point would induce additional dispersion and attenuation ($\mathcal{I}m(K) \leq 0$). In order to take into account this remarks, we propose to replace the previous equation by

$$\left\langle\psi_s^1\right\rangle_1 = e^{ikx_{01}}\, f(\theta)\, H_0(Kr_1) \tag{7}$$

where the effect of translation of s_1 induces simply a phase difference governed by k. According to this methods, the exciting field is considered to be the incoming wave in the first order of approximation. This proposed version of ISA in Eq.(7) is called ISA' in the following. If ISA' consists to suppose that s_1 is insonified only by the incoming wave $\left\langle\psi_s^1\right\rangle_{12}$ is consequently independent of the position of s_2 and by construction:

$$\left\langle\psi_s^1\right\rangle_1 = \int \left\langle\psi_s^1\right\rangle_{12}\, p(r_{12})\mathrm{d}\vec{r}_{02} = \left\langle\psi_s^1\right\rangle_{12}$$

the quasi-crystalline approximation (QCA) [8] is verified.
The contribution of other inclusions will be taken into account for methods using higher order where $\langle\psi_s\rangle_{12}$ depends on the position of s_2.

4.2 Effective wavenumber

We fixe the reference frame at the point M where the CPA has to be verified. In this frame, the incoming field is simply c_0^0 because $\phi_n(k\vec{0}) = \delta_n$. Suppose that s_1 is placed at $\vec{r}_{01}$ in the reference frame. Its scattering measured at M and obtained after averaging all other inclusions is:

$$\left\langle\psi_s^1\right\rangle_1 = \sum_n \left\langle a_n^1\right\rangle_1 \hat{\phi}_n(K\vec{r_{10}})$$

where a_n^j denotes $a_n^j(s_j)$ in the following.
If ISA' is used, the exciting field is the incoming field: $\left\langle a^1\right\rangle_1 \equiv Zc^1 = ZS^{10}c^0$ where the transformation S^{10} is defined by $(S^{10})_{nm} = \phi_{n-m}(k\vec{r}_{10}) = \phi_{m-n}(k\vec{r}_{01})$ according to Graf's theorem [5]. Then, the CPA supposes that K satisfies:

$$n_0 \sum_n \int (ZS^{10}c^0)_n \hat{\phi}_n(K\vec{r_{10}})\, \mathrm{d}\vec{r}_{01} = -c_0^0 \tag{8}$$

We have $\hat{\phi}_n(K\vec{r_{10}}) = (-1)^n\hat{\phi}_n(K\vec{r_{01}})$ and the integral is:

$$(-1)^n z_n \sum_m c_m^0 \int \phi_{m-n}(k\vec{r_{01}})\hat{\phi}_n(K\vec{r_{01}})\, \mathrm{d}\vec{r}_{01} \tag{9}$$

Using Green Theorem, the surface integral can be replaced by a contour integration:

$$\int_{\mathcal{B}} \phi_{m-n}(k\vec{r})\hat{\phi}_n(K\vec{r})\,\mathrm{d}\vec{r} = \int_{\partial\mathcal{B}} \frac{\hat{\phi}_n(K\vec{r})\partial_{\vec{n}}\phi_{m-n}(k\vec{r}) - \phi_{m-n}(k\vec{r})\partial_{\vec{n}}\hat{\phi}_n(K\vec{r})}{K^2-k^2}\,\mathrm{d}\vec{r} \tag{10}$$

what generalize the case of ([7]-Eq.(67)) for an incoming plane wave.
Here $\mathcal{B}$ is defined by $r > a$ as mentioned in section 3.2. The contribution from infinity is null due to the radial attenuation imposed by $\hat{\phi}_n(K\vec{r})$. Therefor $\partial\mathcal{B}$ is mainly $\partial\mathcal{C}_M$ the contour of circle $\mathcal{C}_M$ with radius a. In the reference frame the external normal is $\vec{n} = -\vec{e}_r$. Using $\partial_{\vec{n}}\phi_n(k\vec{r}) = -kJ_n'(kr)e^{in\theta}$ at $r = a$ and the orthogonality relation $\int e^{im\theta}\mathrm{d}\theta = 2\pi\delta_m$ the previous equation gives after simplification

$$\int_{\mathcal{B}} \phi_{m-n}(kr\vec{_{01}})\hat{\phi}_n(K\vec{r}_{01})\,\mathrm{d}\vec{r}_{01} = 2\pi a^2(-1)^n\mathcal{N}_n(k,K)\delta_m$$

with:

$$\mathcal{N}_n(k,K) = \frac{kaJ_n'(ka)H_n(Ka) - KaJ_n(ka)H_n'(Ka)}{(ka)^2-(Ka)^2}$$

Remembering $\varphi = n_0\pi a^2$ and simplify by c_0^0, the CPA is:

$$2\varphi\sum_n z_n\mathcal{N}_n(k,K) = -1 \tag{11}$$

The effective wavenumber must satisfy this equation. An approximate form can be found if Ka is close enough from ka: $Ka = ka + \epsilon$ with $|\epsilon| \ll 1$. Taylor expansion to the first order of the function $H_n(Ka)$ and $H_n'(Ka)$ in the numerator of $\mathcal{N}$ gives:

$$\mathcal{N}_n(k,K) \approx -\frac{2i}{\pi}\frac{1}{(ka)^2-(Ka)^2}$$

and then the traditional form of the ISA is obtained:

$$K^2 = k^2 - 4in_0\sum_n z_n$$

Finally the new formulation of ISA' does not introduce particular effect and is efficient with the CPA if the hole correction formulated in section 3.2 is respected. The method has been developed without care on the form and directivity of the incoming wave, the general expression of the effective wave number is independent of source.

5 Second order

5.1 Scattering by two inclusions

First consider the scattering problem Eq.(2) for two inclusions. It reduces to solve the system:

$$\begin{aligned} a^1 &= Z\left(c^1 + \widehat{S}^{12}a^2\right) \\ a^2 &= Z\left(c^2 + \widehat{S}^{21}a^1\right) \end{aligned} \tag{12}$$

Hence a^1 is solution of:

$$(I - Z\widehat{S}^{12}Z\widehat{S}^{21})a^1 = Z(I+\widehat{S}^{12}ZS^{21})c^1 = Zc^1 + Z\widehat{S}^{12}Zc^2 \tag{13}$$

The contribution of each terms can be highlighted by considering the far field interaction where $kr_{12} \gg 1$. In this case, the asymptotic form of Bessel functions for large argument can be used:

$$H_n(z) \approx H_0(z)e^{-in\frac{\pi}{2}}, \quad J_n(z) \approx H_0(z)\frac{e^{-in\frac{\pi}{2}} + e^{in\frac{\pi}{2}}}{2}, \quad z \gg 1$$

Introducing these hypotheses gives the following forms for the two main matrices:

$$\begin{aligned} (Z\widehat{S}^{12}Z\widehat{S}^{21})_{nm} &= H_0^2(kr_{12})z_n e^{in(-\frac{\pi}{2}-\theta_{12})}e^{im(-\frac{\pi}{2}+\theta_{12})}f(\pi) \\ (Z\widehat{S}^{12}Z)_{nm} &= H_0(kr_{12})z_n e^{-in(\frac{\pi}{2}+\theta_{12})}z_m e^{im(\frac{\pi}{2}+\theta_{12})} \end{aligned}$$

where $f(\pi) = \sum_p(-1)^p z_p$ is the backscattering contribution of the far field pattern.
Consider an incoming plane wave with direction θ_i. The incoming field at any point M can be decomposed:

$$\psi_i(M) = \psi_i^1 e^{ikr_1\cos(\theta_1-\theta_i))} = \psi_i^1\sum_n i^n J_n(kr)e^{in(\theta_1-\theta_i)}$$

where ψ_i^1 is the incoming field at the center of s_1. In other word we have $c_n^1 = \psi_i^1 e^{in(\frac{\pi}{2}-\theta_i)}$ and $c_m^2 = \psi_i^2 e^{im(\frac{\pi}{2}-\theta_i)}$. It is then possible to express each vectors:

$$\begin{aligned} \left(Z\widehat{S}^{12}Z\widehat{S}^{21}a^1\right)_n &= H_0^2(kr_{12})f(\pi)z_n e^{in(-\frac{\pi}{2}-\theta_{12})}\sum_m a_m^1 e^{im(-\frac{\pi}{2}+\theta_{12})} \\ \left(Zc^1\right)_n &= \psi_i^1 z_n e^{in(\frac{\pi}{2}-\theta_i)} \\ \left(Z\widehat{S}^{12}Zc^2\right)_n &= H_0(kr_{12})z_n e^{-in(\frac{\pi}{2}+\theta_{12})}\psi_i^2\sum_m z_m e^{im(\pi+\theta_{12}-\theta_i)} \end{aligned}$$

The far field contribution of the inclusion s_1 is

$$\psi_s^1 \sim H_0(kr_1)\mathcal{F}(\theta_1) \qquad \text{with} \qquad \mathcal{F}(\theta_1) = \sum_n a_n e^{in(\theta_1-\frac{\pi}{2})}$$

Strictly speaking, $\mathcal{F}$ is not the scattering pattern f as it depends on the probing wave. Moreover it gives the same informations.
Multiply each terms of Eq.(13) by $e^{in(-\frac{\pi}{2}+\theta_1)}$ and sum over n give:

$$\begin{aligned} \sum_n a_n^1 e^{in(\theta_1-\frac{\pi}{2})} &= \mathcal{F}(\theta_1) \\ \sum_n\left(Z\widehat{S}^{12}Z\widehat{S}^{21}a^1\right)_n e^{in(\theta_1-\frac{\pi}{2})} &= H_0^2(kr_{12})f(\pi)f(\theta_1-\theta_{21})\mathcal{F}(\theta_{12}) \\ \sum_n\left(Z^1c^1\right)_n e^{in(\theta_1-\frac{\pi}{2})} &= f(\theta_1-\theta_i)\psi_i^1 \\ \sum_n\left(Z\widehat{S}^{12}Zc^2\right)_n e^{in(\theta_1-\frac{\pi}{2})} &= H_0(kr_{12})f(\theta_1-\theta_{21})f(\theta_{21}-\theta_i)\psi_i^2 \end{aligned}$$

A simple form of the scattering pattern is then solution of:

$$\begin{aligned} &\mathcal{F}(\theta_1) - H_0^2(kr_{12})f(\pi)f(\theta_1-\theta_{21})\mathcal{F}(\theta_{12}) = \\ &f(\theta_1-\theta_i)\psi_i^1 + H_0(kr_{12})f(\theta_1-\theta_{21})f(\theta_{21}-\theta_i)\psi_i^2 \end{aligned}$$

For observation along s_2 direction $\theta_1 = \theta_{12} = \theta_{21} + \pi$:

$$\mathcal{F}(\theta_{12}) = \frac{f(\theta_{12}-\theta_i)\psi_i^1 + H_0(kr_{12})f(\pi)f(\theta_{21}-\theta_i)\psi_i^2}{1 - H_0^2(kr_{12})f(\pi)^2}$$

And after rearrangement:

$$\begin{aligned} &\mathcal{F}(\theta_1) = f(\theta_1-\theta_i)\psi_i^1 + \ldots \\ &H_0(kr_{12})f(\theta_1-\theta_{21})\frac{f(\theta_{21}-\theta_i)\psi_i^2 + H_0(kr_{12})f(\pi)f(\theta_{12}-\theta_i)\psi_i^1}{1 - H_0^2(kr_{12})f^2(\pi)} \end{aligned}$$

Of course if the $O(H_0^2(kr_{12}))$ terms are neglected we observe a more intuitive from of the far field contribution:

$$\mathcal{F}(\theta_1) = f(\theta_1-\theta_i)\psi_i^1 + H_0(kr_{12})f(\theta_1-\theta_{21})f(\theta_{21}-\theta_i)\psi_i^2$$

The far field pattern is obtained by a weak coupling with the second inclusion: f and H_0 acts as an angular and radial transfer function. Neglecting $O(H_0^2(kr_{12}))$ is equivalent to neglect $Z\widehat{S}^{12}Z\widehat{S}^{21}$ in Eq.(13). In other words $Z\widehat{S}^{12}Z\widehat{S}^{21}$ is mainly a near field contribution. As shown in figure 1 the contribution of $Z\widehat{S}^{12}Z\widehat{S}^{21}$ looks far from negligible for the near field contribution.

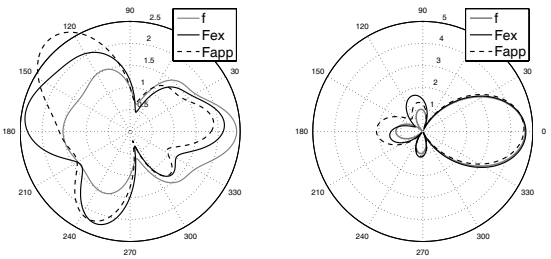

Figure 1: $\mathcal{F}$ evaluated by solving Eq.(13). Exact form (Fex) and obtained by neglecting $Z\widehat{S}^{12}Z\widehat{S}^{21}$ (Fapp), exact form for a single inclusion (f). Left: rigid inclusion; right: small contrast $(\rho_1, c_1) = \frac{3}{4}(\rho, c)$. Incidence along $\vec{x}$ and $\theta_{12} = \pi/3$. Case with $ka = 3$ and $kr_{12} = 2ka$

5.2 Ensemble average

For N inclusions the scattering amplitude is given in Eq.(2). Ensemble average on all inclusions except s_1 gives:

$$\left\langle a^1 \right\rangle_1 = Z\left(c^1 + \sum_{k\neq 1}\int \widehat{\mathbf{S}}^{1k}\left\langle a^k\right\rangle_{1k}\ p(r_{1k})\,\mathrm{d}\vec{r}_{1k}\right)$$

Here $\widehat{\mathbf{S}}^{ij}$ is the same as $\widehat{S}^{ij}$ but evaluated with K in place of k because Graf's Theorem is applied on:

$$\left\langle \psi_s^1\right\rangle_1 = \sum_n \left\langle a_n^1\right\rangle_1\ \hat{\phi}_n(K\vec{r}_1)$$

The contribution of $s_{k\neq 1}$ are indistinguishable and are associated to $N-1 \approx N$ representative inclusion s_2. Hence:

$$\left\langle a^1\right\rangle_1 = Z\left(c^1 + N\int \widehat{\mathbf{S}}^{12}\left\langle a^2\right\rangle_{12}\ p(r_{12})\,\mathrm{d}\vec{r}_{12}\right) \qquad (14)$$

or equivalently:

$$\left\langle a^1\right\rangle_{12} = Z\left(c^1 + N\,\widehat{\mathbf{S}}^{12}\left\langle a^2\right\rangle_{12}\right) \qquad (15)$$

5.3 Linton-Martin closure assumption

In order to solve the previous equation it is necessary to have a relation between $\left\langle a^1\right\rangle_1$ and $\left\langle a^2\right\rangle_{12}$ in Eq.(14) or $\left\langle a^1\right\rangle_{12}$ and $\left\langle a^2\right\rangle_{12}$ in Eq.(15).
The Linton-Martin approximation [7] is based on the QCA where we suppose

$$\left\langle a^2\right\rangle_{12} \approx S^{21}\left\langle a^1\right\rangle_1$$

according to the ISA. The objective is then to solve:

$$\left(I - N\int \widehat{\mathbf{S}}^{12}ZS^{21}\ p(r_{12})\,\mathrm{d}\vec{r}_{12}\right)\left\langle a^1\right\rangle_1 = Zc^1 \qquad (16)$$

In fact the dispersion solution is found supposing no probing wave: $c^1 \equiv 0$ looking for non trivial solution of the previous equation. This can be made by defining $K(\omega)$ solution of

$$\det\left(I - N\int \widehat{\mathbf{S}}^{12}ZS^{21}\ p(r_{12})\,\mathrm{d}\vec{r}_{12}\right) = 0 \qquad (17)$$

5.4 Two inclusions - closure assumptions

Moreover, according to Eq.(2) it is possible to express the contribution of $\left\langle a^2\right\rangle_{12}$:

$$Z\left(c^2 + \widehat{\mathbf{S}}^{21}\left\langle a^1\right\rangle_{12} + \sum_{k\neq(1,2)}\int \widehat{\mathbf{S}}^{2k}\left\langle a^k\right\rangle_{12k}\ p(\vec{r}_{0k}|\vec{r}_{01},\vec{r}_{02})\mathrm{d}\vec{r}_{0k}\right)$$

again contribution of $s_{k\neq(1,2)}$ are indistinguishable and can be attributed to $N-2 \approx N$ representative inclusion s_3 to obtain

$$\left\langle a^2\right\rangle_{12} = Z\left(c^2 + \widehat{\mathbf{S}}^{21}\left\langle a^1\right\rangle_{12} + N\int \widehat{\mathbf{S}}^{23}\left\langle a^3\right\rangle_{123}\ p(\vec{r}_{03}|\vec{r}_{01},\vec{r}_{02})\mathrm{d}\vec{r}_{03}\right)$$

If the contribution of $\left\langle a^3\right\rangle_{123}$ is supposed null, the coupling between three inclusions is neglected, but the pair coupling between s_2 and s_1 is fully represented by:

$$\left\langle a^2\right\rangle_{12} = Z\left(c^2 + \widehat{\mathbf{S}}^{21}\left\langle a^1\right\rangle_{12}\right)$$

Introducing this relation in Eq.(14) gives a more complete form of the second order ensemble average, where the contribution of $\left\langle a^2\right\rangle_{12}$ has not been lowered by QCA:

$$(I - N\,Z\widehat{\mathbf{S}}^{12}Z\widehat{\mathbf{S}}^{21})\left\langle a^1\right\rangle_{12} = Z(I + N\,\widehat{\mathbf{S}}^{12}ZS^{21})c^1$$

If one consider $c^1 \equiv 0$, the dispersion relation is found to be solution of

$$\det\left(I - N\int \widehat{\mathbf{S}}^{12}Z\widehat{\mathbf{S}}^{21}\ p(r_{12})\,\mathrm{d}\vec{r}_{12}\right) = 0 \qquad (18)$$

An alternative is to compute $\left\langle a^1\right\rangle_1$ from the previous equation. Numerically, this can be done by left-multiply the right hand side by $(I - N\,Z\widehat{\mathbf{S}}^{12}Z\widehat{\mathbf{S}}^{21})^{-1}$ then integrating the obtained matrix. But this matrix is ill conditioned and integration is time consuming.
An alternative is to suppose that $\left\langle a^1\right\rangle_1$ is solution of:

$$(I - N\,Z\langle B\rangle_1)\left\langle a^1\right\rangle_1 = Z\,(I + N\,\langle D\rangle_1)\,c^1 \qquad (19)$$

with
$$\langle B\rangle_1 = \int \widehat{\mathbf{S}}^{12}Z\widehat{\mathbf{S}}^{21}p(r_{12})\mathrm{d}\vec{r}_2$$
$$\langle D\rangle_1 = \int \widehat{\mathbf{S}}^{12}ZS^{21}p(r_{12})\mathrm{d}\vec{r}_2$$

This method supposes that $\left\langle a^1\right\rangle_{12}$ is not affected by integration. And QCA is replaced by the following closure assumption:

$$\left\langle a^1\right\rangle_{12} \approx \left\langle a^1\right\rangle_1$$

without imposing a particular form of $\left\langle a^1\right\rangle_1$ or $\left\langle a^2\right\rangle_1$.
The section 5.1 gives an other point of view for comparing the LM-formulation Eq.(16) and the proposed one Eq.(19): with Eq.(19) the near field coupling looks more accurate.

5.5 Analytical integration

In order to give an explicit form of $\left\langle a^1\right\rangle_1$, $\langle D\rangle_1$ and $\langle B\rangle_1$ have to been evaluated first.

5.5.1 Calculation of $\langle D\rangle_1$

$$(\langle D\rangle_1)_{nm} = \sum_p z_p(-1)^{p-m}\int_{\mathcal{B}\backslash C_1}\hat{\phi}_{n-p}(Kr_{12})\phi_{p-m}(kr_{12})\mathrm{d}\vec{r}_{12}$$

According to Eq.(10) the integration is performed on ∂C_1 of radius $2a$ and gives:

$$8\pi a^2(-1)^{p-n}\mathcal{N}_{n-p}(2k,2K)\delta_{nm}$$

The matrix $\langle D\rangle_1$ is diagonal with a general term

$$(\langle D\rangle_1)_{nn} = 8\pi a^2\sum_p z_p\mathcal{N}_{n-p}(2k,2K)$$

5.5.2 Calculation of $\langle B\rangle_1$

$$(\langle B\rangle_1)_{nm} = \sum_p z_p(-1)^{p-n}\int_{\mathcal{B}\backslash C_1}\hat{\phi}_{n-p}(Kr_{12})\hat{\phi}_{p-m}(Kr_{12})\mathrm{d}\vec{r}_{12}$$

Green Therorem can't be used because the argument of the Bessel functions are the same. Integration along θ_{12} gives $n = m$ and the matrix is diagonal with n^{th} term:

$$(\langle B\rangle_1)_{nn} = 2\pi\sum_p z_p\int_{2a}^{\infty}H_{n-p}(Kr_{12})H_{n-p}(Kr_{12})\mathrm{d}r_{12}$$

Using [9]-eq.10.22.5 the integral is $2a^2\mathcal{P}_{n-p}(2K)$ with

$$\mathcal{P}_n(K) = H_n^2(Ka) - H_{n-1}(Ka)H_{n+1}(Ka)$$

Hence

$$(\langle B\rangle_1)_{nn} = 4\pi a^2\sum_p z_p\mathcal{P}_{n-p}(2K)$$

5.5.3 Calculation of $\left\langle a^1\right\rangle_1$ and $\left\langle a^1\right\rangle$

Because matrix $\langle B\rangle_1$ and $\langle D\rangle_1$ are diagonal $\left\langle a^1\right\rangle_1$ is simply:

$$\left\langle a^1\right\rangle_1 = GZc^1$$

where G is the diagonal matrix:

$$(G)_{nn} = \frac{1 + 8\varphi\sum_p z_p\mathcal{N}_{n-p}(2k,2K)}{1 - 4\varphi\sum_p z_p\mathcal{P}_{n-p}(2K)}$$

As G is independent of the position of s_1, the integration over all the positions of s_1 in $\mathcal{B}\backslash C_M$ gives similar result than in section 4.2. We obtain:

$$2\varphi\sum_n z_n\frac{1 + 8\varphi\sum_p z_p\mathcal{N}_{n-p}(2k,2K)}{1 - 4\varphi\sum_p z_p\mathcal{P}_{n-p}(2K)}\mathcal{N}_n(k,K) = -1 \qquad (20)$$

It is a proposition of implicit solution of K for a second order model.

6 Conclusion

The proposed formulation of the multiple scattering problem is consistent with the Foldy approximation. At higher order this formulation induces a reformulation of the closure assumption. Among the various solutions the comparison with the 2-inclusions problem supposes that Linton-Martin formulation is based on weak coupling between two representative inclusions. Current numerical investigations are performed in order to give quantitative comparison.

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