

# The Arm Lie Group Theory Boris Arm

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# The Arm Lie Group Theory

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#### Abstract

We develop the Arm-Lie group theory which is a theory based on the exponential of a changing of matrix variable u(X). We define a corresponding u-adjoint action, the corresponding commutation relations in the Arm-Lie algebra and the u-Jacobi identity. Throught the exponentiation, Arm-Lie algebras become Arm-Lie groups. We give the example of  $\sqrt[p]{\mathfrak{so}(2)}$  and  $\sqrt[p]{\mathfrak{su}(2)}$ .

#### Introduction

The Arm theory [1] gives the generaized Taylor formula in any basis. It give rise to exponentials of any changing of variable u(x). While the Lie group theory has been built on the classical exponential, I wondered myself why not building a new Lie group theory based on exponential of changing of variable, i.e. on exponential :

$$e^{u(X)} \tag{0.1}$$

where u(X) is any changing of variable.

Besides, we can build an new adjoint action with this exponential in a new basis. This is the 'u-adjoint action' given in proposition 1 :

$$ad_u A.X = [A, u(X)] \tag{0.2}$$

The traditional Lie algebra satisfyied some commutation relations between their generators so I searched the corresponding commutation relations for this new structure. In fact the genrators of the Arm-Lie algebras  $\{X_1, ..., X_n, u(X_1), ..., u(X_n)\}$ , if n is the dimension, satisfy the following conditions

$$[X_i, u(X_j)] = \lambda_{ij}^k u(X_k)$$
  

$$[X_i, X_j] = \nu_{ij}^k u(X_k)$$
(0.3)

with  $\lambda_{ij}^{\ k}$  and  $\nu_{ij}^{\ k}$  the corresponding u-structure constant and structure constant respectively. In addition, there is a corresponding u-Jacobi identity in Arm-Lie algebras which is given by :

$$[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0$$
(0.4)

In the same way, Lie algebras become Lie groups throught exponentiation, we can take the exponential of each linear combinations which gives a group that we call a Arm-Lie group. This is why we check in the third section that the commutator of commutators of elements in the Arm-Lie algebras are still in the Arm-Lie algebra.

However, the big default of my construction is that I only find one example of the Arm-Lie algebra, but what a beautiful example :  $u^{-1}(\mathfrak{so}(2))$  and  $u^{-1}(\mathfrak{su}(2))$  with  $\forall p \in \mathbb{C}, u(x) = X^p$ . I think it is because the genrators of  $\mathfrak{su}(2)$  and  $\mathfrak{so}(2)$  are the exponential of something (i.e. their logarithm exist) but I am not sure yet. Nevertheless I hope there is other examples of Arm-Lie algebras. Moreover, the exponential of those two Arm-Lie algebras give two new groups which are Arm-Lie groups that we call rSO(2) and rSU(2) given by

$$rSO(2,\mathbb{C}) = \left\{ M \in M_l(\mathbb{C}) \mid {}^t M M = r \mathrm{Id}_2 \; ; \; r \in \mathbb{C} \right\}$$
(0.5)

and

$$rSU(2,\mathbb{C}) = \left\{ M \in M_l(\mathbb{C}) \mid MM^{\dagger} = r\mathrm{Id}_2 \ ; \ r \in \mathbb{C} \right\}$$
(0.6)

In the first section, we define the exponential in each basis of function u(X) of matrices. Next in the second section, we give the commutation relations between generators of the Arm-Lie algebra and the u-Jacobi identity. After in the third section, we show that commutators of commutators of elements of the Arm-Lie algebras are still in the Arm-Lie algebra. This is the condition which assure that the exponential of linear combinations will be groups which we call Arm-Lie groups. Furthermore in the fourth section we give the example of the p-th root of  $\mathfrak{so}(2)$  which is a trivial case because there is only two generators in the Arm-Lie algebra. Nonetheless it give rise to the new group rSO(2) (0.5) and we give elements of this groups. Finally in the fifth section, we give the example of the p-th root of  $\mathfrak{su}(2)$  which is not trivial because there is 6 generators in this Arm-Lie algebra. We give generating elements of its corresponding Arm-Lie group rSU(2) (0.6).

#### 1 The U-Exponential Map

Definition 1. We call the 'u-exponential' the function

$$e^{u(X)t} = \sum_{k=0}^{\infty} \frac{(u(X)t)^k}{k!}$$
 (1.7)

where  $u(X) \in C(M_l(\mathbb{C})), l \in \mathbb{N}$  is a function of matrices and X is a matrix.

Then we can compute the corresponding u-adjoint action

**Proposition 1.** The u-adjoint action corresponding to the u-exponential is given by

$$\mathrm{ad}_u A.X = [A, u(X)] \tag{1.8}$$

where [,] is the traditional Lie bracket.

Proof :

$$ad_{u}A.X = \lim_{t \to 0} \frac{d}{dt} \left[ e^{-u(X)t} A e^{u(X)t} \right]$$
$$= \lim_{t \to 0} \frac{d}{dt} \left[ (1 - u(X)t + ...) A (1 + u(X)t + ...) \right]$$
$$= \lim_{t \to 0} \frac{d}{dt} \left[ A + t(Au(X) - u(X)A) + ... \right]$$
$$= Au(X) - u(X)A$$
$$ad_{u}A.X = [A, u(X)]$$
(1.9)

We call  $\mathfrak{g}$  a Lie algebra.

#### 2 The Arm-Lie Algebra

**Definition 2.** A Arm-Lie algebra is a collection  $\{X_1, ..., X_n, u(X_1), ..., u(X_n)\}$  such that  $\{u(X_1), ..., u(X_n)\}$  is a basis of the Lie algebra  $\mathfrak{g}$ .

In the Arm-Lie algebra, we have the following relations

$$[X_i, u(X_j)] = \lambda_{ij}^k u(X_k)$$
  

$$[X_i, X_j] = \nu_{ij}^k u(X_k)$$
(2.10)

where  $\nu$  and  $\lambda$  are the structure constant and the u-structure constant respectively.

We can also define the corresponding u-Jacobi identity

$$[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0$$
(2.11)

#### 3 The Arm-Lie Group

First we recall the Campbell-Hausdorff formula

$$e^X e^Y = e^{Z(X,Y)}$$
 (3.12)

where

$$Z(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X-Y,[X,Y]]) - \frac{1}{24}[Y,[X,[X,Y]]] + \dots$$
(3.13)

The idea of the Arm-Lie group is that if we take generators of the Arm-Lie algebra  $X = X_i$  and  $Y = X_j$  then each terms of Z(X, Y) in (3.13) would be the commutators of elements of the Arm-Lie algebra :

$$[X_{k}, [..., [X_{f}, [X_{p}, X_{l}]]...]] = [X_{k}, [..., [X_{f}, \nu_{pl}^{t}u(X_{t})]...]]$$
  
$$= \nu_{pl}^{t} [X_{k}, [..., \lambda_{ft}^{r}u(X_{r})...]]$$
  
$$\vdots$$
  
$$[X_{k}, [..., [X_{f}, [X_{p}, X_{l}]]...]] = \nu_{pl}^{t} ... \lambda_{km}^{s}u(X_{s}) + ... + \lambda_{ab}^{c} ... \lambda_{de}^{f}u(X_{f})$$
(3.14)

and the first term of Z(X, Y) are given by  $X_i + X_j$ . Then we can conclude that the exponential of elements of an Arm-Lie algebra is a group which we naturally call Arm-Lie group.

In the moment I'm writing this article, the only Lie algebra which give Arm-Lie algebras are Lie algebra with generators which are the exponential of something (i.e. their logarithms exist). So we can start in studying the trivial one dimensional Arm-Lie algebra  $\sqrt[p]{\mathfrak{so}(2)}$  and its correspondinf Arm-Lie group.

## 4 $\sqrt[p]{\mathfrak{so}(2)}$

We consider the changing of variable :

$$u(X) = X^p \tag{4.15}$$

for  $p \in \mathbb{C}^*$ . It's well known that the generator basis of  $\mathfrak{so}(2)$  is given by the first Pauli matrix multiplify by  $\sqrt{-1}$ :

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p$$
 (4.16)

Because  $i\sigma_1$  is a basis of  $\mathfrak{so}(2)$ , if we want to know the basis of  $u^{-1}(\mathfrak{so}(2)) = \sqrt[p]{\mathfrak{so}(2)}$ , we have to calculate the p-th root of  $i\sigma_1$ . Then you can check the validity of the relation :

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$
(4.17)

So with (4.17), it is very easy to calculate the p-th root of  $i\sigma_1$ :

$$\sqrt[p]{i\sigma_1} = \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$
$$= \cos\left(\frac{\pi}{2p}\right) \operatorname{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$X_1 = \sqrt[p]{i\sigma_1} = \left(\frac{\cos\left(\frac{\pi}{2p}\right) - \sin\left(\frac{\pi}{2p}\right)}{\sin\left(\frac{\pi}{2p}\right) - \cos\left(\frac{\pi}{2p}\right)}\right)$$

where Id<sub>2</sub> is of course the identity 2-dimensional matrix. This Arm-Lie algebra is trivial because there is only 2 generators  $i\sigma_1 = (X_1)^p$  and  $\sqrt[p]{i\sigma_1} = X_1$  satisfying the relations :

$$\begin{bmatrix} X_1, X_1^{p} \end{bmatrix} = 0 \begin{bmatrix} X_1, X_1 \end{bmatrix} = 0$$
 (4.18)

Of course (4.18) implies that the generator of the Arm-Lie algebra satisfy the u-Jacobi identity (2.11). Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie group :

$$\exp(\sqrt[p]{\mathfrak{so}(2)}) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid e^{z_1 \sqrt[p]{i\sigma_1} + z_2 i\sigma_1} \right\}$$

$$(4.19)$$

To have an idea of the elements of the group (4.19), we can explicit :

$$\exp(z_1 X_1) = \exp\left(z_1 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix}$$
(4.20)

and the well-known element of SO(2):

$$\exp\left(z_2 X_1^{p}\right) = \begin{pmatrix} \cos\left(z_2\right) & -\sin\left(z_2\right) \\ \sin\left(z_2\right) & \cos\left(z_2\right) \end{pmatrix}$$

$$(4.21)$$

Hence we can identify this Arm-Lie algebra to

$$\exp(\sqrt[p]{\mathfrak{so}(2)}) \equiv rSO(2,\mathbb{C}) \equiv \left\{ M \in M_n(\mathbb{C}) \mid {}^t M M = r \operatorname{Id}_2 \; ; \; r \in \mathbb{C} \right\}$$
(4.22)

for  $\frac{1}{p} \neq 1[2]$  and just SO(2) if  $\frac{1}{p} = 1[2]$ .

Remark 1. Of course the cosinus and the sinus of a complex arument is well defined

$$\cos(z) = \cos(\operatorname{Re}(z))\cosh(\operatorname{Im}(z)) - i\sin(\operatorname{Re}z)\sinh(\operatorname{Im}(z))$$
  

$$\sin(z) = \sin(\operatorname{Re}(z))\cosh(\operatorname{Im}(z)) + i\cos(\operatorname{Re}(z))\sinh(\operatorname{Im}(z))$$
(4.23)

where Re and Im denote the real and imaginary parts respectively.

## 5 $\sqrt[p]{\mathfrak{su}(2)}$

We still consider the changing of variable :

$$u(X) = X^p \tag{5.24}$$

for  $p \in \mathbb{C}^*$ . It's well known that the generator basis of  $\mathfrak{su}(2)$  is given by the Pauli matrices multiplify by  $\sqrt{-1}$ :

$$i\sigma_{1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_{1}) = (X_{1})^{p}$$

$$i\sigma_{2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = u(X_{2}) = (X_{2})^{p}$$

$$i\sigma_{3} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = u(X_{3}) = (X_{3})^{p}$$
(5.25)

Because  $\{i\sigma_1, i\sigma_2, i\sigma_3\}$  is a basis of  $\mathfrak{su}(2)$ , if we want to know the basis of  $u^{-1}(\mathfrak{so}(2)) = \sqrt[p]{\mathfrak{su}(2)}$ , we have to calculate the p-th root of  $i\sigma_1, i\sigma_2$  and  $i\sigma_3$ . Then you can check the validity of the relation :

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$
$$i\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right)$$
$$i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)$$

So with (5.26), it is very easy to calculate the p-th root of  $i\sigma_1, i\sigma_2, i\sigma_3$ :

$$\sqrt[p]{i\sigma_1} = \exp\left(\frac{\pi}{2p}\begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}\right) \\
= \cos\left(\frac{\pi}{2p}\right) \operatorname{Id}_2 + \sin\left(\frac{\pi}{2p}\right)\begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix} \\
X_1 = \sqrt[p]{i\sigma_1} = \left(\frac{\cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right)}{\sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right)}\right) \\
\sqrt[p]{i\sigma_2} = \exp\left(\frac{\pi}{2p}\begin{pmatrix} 0 & i\\ i & 0 \end{pmatrix}\right) \\
= \cos\left(\frac{\pi}{2p}\right) \operatorname{Id}_2 + \sin\left(\frac{\pi}{2p}\right)\begin{pmatrix} 0 & i\\ i & 0 \end{pmatrix} \\
X_2 = \sqrt[p]{i\sigma_2} = \left(\frac{\cos\left(\frac{\pi}{2p}\right) & i\sin\left(\frac{\pi}{2p}\right)}{i\sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right)}\right) \\
\sqrt[p]{i\sigma_3} = \exp\left(\frac{\pi}{2p}\begin{pmatrix} i & 0\\ 0 & -i \end{pmatrix}\right) \\
X_3 = \sqrt[p]{i\sigma_3} = \left(\frac{\exp\left(\frac{\pi}{2p}\right) & 0\\ 0 & \exp\left(-\frac{\pi}{2p}\right) & 0\\ 0 & \exp\left(-\frac{\pi}{2p}\right) & 0 \\
\end{array}\right)$$

where Id<sub>2</sub> is of course the identity 2-dimensional matrix. This Arm-Lie algebra has 6 generators  $i\sigma_1 = (X_1)^p, i\sigma_2 = (X_2)^p, i\sigma_3 = (X_3)^p$  and  $\sqrt[p]{i\sigma_1} = X_1, \sqrt[p]{i\sigma_2} = X_2, \sqrt[p]{i\sigma_3} = X_3$  satisfying the relations :

$$[X_{1}, (X_{2})^{p}] = -2\sin\left(\frac{\pi}{2p}\right) (X_{3})^{p}$$
  

$$[X_{2}, (X_{3})^{p}] = -2\sin\left(\frac{\pi}{2p}\right) (X_{1})^{p}$$
  

$$_{3}, (X_{1})^{p}] = -2\sin\left(\frac{\pi}{2p}\right) (X_{2})^{p}$$
(5.26)

and

$$[X_{1}, X_{2}] = -2\sin^{2}\left(\frac{\pi}{2p}\right) (X_{3})^{p}$$
  

$$[X_{2}, X_{3}] = -2\sin^{2}\left(\frac{\pi}{2p}\right) (X_{1})^{p}$$
  

$$[X_{3}, X_{1}] = -2\sin^{2}\left(\frac{\pi}{2p}\right) (X_{2})^{p}$$
(5.27)

Of course (5.27) and (5.26) imply that the generators of the Arm-Lie algebra  $\sqrt[p]{su(2)}$  satisfy the u-Jacobi identity. Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie

group :

$$\exp(\sqrt[p]{\mathfrak{su}(2)}) = \left\{ (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 \mid e^{z_1 \sqrt[p]{i\sigma_1} + z_2 i\sigma_1 + z_3 \sqrt[p]{i\sigma_2} + z_4 i\sigma_2 + z_5 \sqrt[p]{i\sigma_3} + z_6 i\sigma_3} \right\}$$
(5.28)

To have an idea of the elements of the group (4.19), we can explicit :

$$\exp(z_1 X_1) = \exp\left(z_1 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \\ \exp(z_3 X_3) = \exp\left(z_3 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) & i \sin\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) \\ i \sin\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \\ \exp(z_5 X_5) = \exp\left(z_5 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \exp\left(iz_5 \sin\left(\frac{\pi}{2p}\right)\right) & 0 \\ 0 & \exp\left(z_5 \sin\left(-\frac{\pi}{2p}\right)\right) \end{pmatrix} \right)$$
(5.29)

and the well-known elements of SU(2) :

$$\exp\left(z_2 X_1^{p}\right) = \begin{pmatrix} \cos\left(z_2\right) & -\sin\left(z_2\right) \\ \sin\left(z_2\right) & \cos\left(z_2\right) \end{pmatrix}$$
$$\exp\left(z_4 X_2^{p}\right) = \begin{pmatrix} \cos\left(z_4\right) & i\sin\left(z_4\right) \\ i\sin\left(z_4\right) & \cos\left(z_4\right) \end{pmatrix}$$
$$\exp\left(z_6 X_3^{p}\right) = \begin{pmatrix} \exp\left(iz_6\right) & 0 \\ 0 & \cos\left(-iz_6\right) \end{pmatrix}$$
(5.30)

Hence we can identify this Arm-Lie algebra to

$$\exp(\sqrt[p]{\mathfrak{su}(2)}) \equiv rSU(2,\mathbb{C}) \equiv \left\{ M \in M_n(\mathbb{C}) \mid M^{\dagger}M = r \operatorname{Id}_2 \ ; \ r \in \mathbb{C} \right\}$$
(5.31)

for  $\frac{1}{p} \neq 1[2]$  and just SU(2) if  $\frac{1}{p} = 1[2]$ .

#### Discussion

Unfortunately, I searched other Arm-Lie algebras among classics Lie algebras but I didn't find. I think it's because many other classic groups are not the exponential of something (i.e. their logarithms do not exist) but I'm not sure. I find this structure only for  $\mathfrak{su}(2)$  and its subalgebra  $\mathfrak{so}(2)$ . I tried with  $\mathfrak{so}(3)$  and  $\mathfrak{sl}(2)$  but it didn't work. I develop this theory in order to classify what I found but I hope there is other Lie algebras which are Arm-Lie algebras. There is not a lot of example of Arm-Lie algebra but  $\sqrt[p]{\mathfrak{su}(2)}$  is a beautiful and very fondamental example.

I also tried this theory for other u(X) for example I took  $u(X) = \exp(\exp(X))$  but it is just a shifted case of the traditional case. I was also limited by the choice of changing of variable that I was able to do because I had to find the inverse function of matricial variable which is sometimes hard to do. I also tried function as sin or cos but it gave the identity or zero respectively. The only good changing of variable which I found was  $u(X) = X^p$  which is a lot of changing for  $p \in \mathbb{C}$  but it finally give the same result for all p.

Finally we can imagine an other Lie group theory based on the 'p-exponential' which I introduced in [2] but now I think that I will give the same result as the usual Lie group theory but with the p-exponential instead of the traditional exponential. Maybe I will explore this way in an other work.

## Références

- [1] Arm B. N., The Arm Theory
- [2] Arm B. N., The p-Arm Theory