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# The Arm Lie Group Theory

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## **Abstract**

We develop the Arm-Lie group theory which is a theory based on the exponential of a changing of matrix variable  $u(X)$ .

We define a corresponding  $u$ -adjoint action, the corresponding commutation relations in the Arm-Lie algebra and the  $u$ -Jacobi identity.

Through the exponentiation, Arm-Lie algebras become Arm-Lie groups.

We give the example of  $\mathfrak{p}/\mathfrak{so}(2)$  and  $\mathfrak{p}/\mathfrak{su}(2)$ .

## Introduction

The Arm theory [1] gives the generalized Taylor formula in any basis. It gives rise to exponentials of any changing of variable  $u(x)$ . While the Lie group theory has been built on the classical exponential, I wondered myself why not building a new Lie group theory based on exponential of changing of variable, i.e. on exponential :

$$e^{u(X)} \quad (0.1)$$

where  $u(X)$  is any changing of variable.

Besides, we can build a new adjoint action with this exponential in a new basis. This is the 'u-adjoint action' given in proposition 1 :

$$\text{ad}_u A.X = [A, u(X)] \quad (0.2)$$

The traditional Lie algebra satisfied some commutation relations between their generators so I searched the corresponding commutation relations for this new structure. In fact the generators of the Arm-Lie algebras  $\{X_1, \dots, X_n, u(X_1), \dots, u(X_n)\}$ , if  $n$  is the dimension, satisfy the following conditions

$$\begin{aligned} [X_i, u(X_j)] &= \lambda_{ij}^k u(X_k) \\ [X_i, X_j] &= \nu_{ij}^k u(X_k) \end{aligned} \quad (0.3)$$

with  $\lambda_{ij}^k$  and  $\nu_{ij}^k$  the corresponding u-structure constant and structure constant respectively. In addition, there is a corresponding u-Jacobi identity in Arm-Lie algebras which is given by :

$$[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0 \quad (0.4)$$

In the same way, Lie algebras become Lie groups through exponentiation, we can take the exponential of each linear combination which gives a group that we call an Arm-Lie group. This is why we check in the third section that the commutator of commutators of elements in the Arm-Lie algebras are still in the Arm-Lie algebra.

However, the big default of my construction is that I only find one example of the Arm-Lie algebra, but what a beautiful example :  $u^{-1}(\mathfrak{so}(2))$  and  $u^{-1}(\mathfrak{su}(2))$  with  $\forall p \in \mathbb{C}, u(x) = X^p$ . I think it is because the generators of  $\mathfrak{su}(2)$  and  $\mathfrak{so}(2)$  are the exponential of something (i.e. their logarithm exist) but I am not sure yet. Nevertheless I hope there are other examples of Arm-Lie algebras. Moreover, the exponential of those two Arm-Lie algebras give two new groups which are Arm-Lie groups that we call  $rSO(2)$  and  $rSU(2)$  given by

$$rSO(2, \mathbb{C}) = \left\{ M \in M_l(\mathbb{C}) \mid {}^t M M = r \text{Id}_2 ; r \in \mathbb{C} \right\} \quad (0.5)$$

and

$$rSU(2, \mathbb{C}) = \left\{ M \in M_l(\mathbb{C}) \mid M M^\dagger = r \text{Id}_2 ; r \in \mathbb{C} \right\} \quad (0.6)$$

In the first section, we define the exponential in each basis of function  $u(X)$  of matrices. Next in the second section, we give the commutation relations between generators of the Arm-Lie algebra and the u-Jacobi identity. After in the third section, we show that commutators of commutators of elements of the Arm-Lie algebras are still in the Arm-Lie algebra. This is the condition which assure that the exponential of linear combinations will be groups which we call Arm-Lie groups. Furthermore in the fourth section we give the example of the p-th root of  $\mathfrak{so}(2)$  which is a trivial case because there is only two generators in the Arm-Lie algebra. Nonetheless it give rise to the new group  $rSO(2)$  (0.5) and we give elements of this groups. Finally in the fifth section, we give the example of the p-th root of  $\mathfrak{su}(2)$  which is not trivial because there is 6 generators in this Arm-Lie algebra. We give generating elements of its corresponding Arm-Lie group  $rSU(2)$  (0.6).

## 1 The U-Exponential Map

**Definition 1.** We call the 'u-exponential' the function

$$e^{u(X)t} = \sum_{k=0}^{\infty} \frac{(u(X)t)^k}{k!} \quad (1.7)$$

where  $u(X) \in C(M_l(\mathbb{C}))$ ,  $l \in \mathbb{N}$  is a function of matrices and  $X$  is a matrix.

Then we can compute the corresponding u-adjoint action

**Proposition 1.** The u-adjoint action corresponding to the u-exponential is given by

$$\text{ad}_u A.X = [A, u(X)] \quad (1.8)$$

where  $[ , ]$  is the traditional Lie bracket.

**Proof :**

$$\begin{aligned} \text{ad}_u A.X &= \lim_{t \rightarrow 0} \frac{d}{dt} \left[ e^{-u(X)t} A e^{u(X)t} \right] \\ &= \lim_{t \rightarrow 0} \frac{d}{dt} \left[ (1 - u(X)t + \dots) A (1 + u(X)t + \dots) \right] \\ &= \lim_{t \rightarrow 0} \frac{d}{dt} \left[ A + t(Au(X) - u(X)A) + \dots \right] \\ &= Au(X) - u(X)A \\ \text{ad}_u A.X &= [A, u(X)] \end{aligned} \quad (1.9)$$

◆

We call  $\mathfrak{g}$  a Lie algebra.

## 2 The Arm-Lie Algebra

**Definition 2.** A Arm-Lie algebra is a collection  $\{X_1, \dots, X_n, u(X_1), \dots, u(X_n)\}$  such that  $\{u(X_1), \dots, u(X_n)\}$  is a basis of the Lie algebra  $\mathfrak{g}$ .

In the Arm-Lie algebra, we have the following relations

$$\begin{aligned} [X_i, u(X_j)] &= \lambda_{ij}^k u(X_k) \\ [X_i, X_j] &= \nu_{ij}^k u(X_k) \end{aligned} \quad (2.10)$$

where  $\nu$  and  $\lambda$  are the structure constant and the u-structure constant respectively.

We can also define the corresponding u-Jacobi identity

$$[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0 \quad (2.11)$$

### 3 The Arm-Lie Group

First we recall the Campbell-Hausdorff formula

$$e^X e^Y = e^{Z(X,Y)} \quad (3.12)$$

where

$$Z(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X - Y, [X, Y]]) - \frac{1}{24}[Y, [X, [X, Y]]] + \dots \quad (3.13)$$

The idea of the Arm-Lie group is that if we take generators of the Arm-Lie algebra  $X = X_i$  and  $Y = X_j$  then each terms of  $Z(X, Y)$  in (3.13) would be the commutators of elements of the Arm-Lie algebra :

$$\begin{aligned} [X_k, [\dots, [X_f, [X_p, X_l]]\dots]] &= [X_k, [\dots, [X_f, \nu_{pl}^t u(X_t)]\dots]] \\ &= \nu_{pl}^t [X_k, [\dots, \lambda_{ft}^r u(X_r)\dots]] \\ &\vdots \\ [X_k, [\dots, [X_f, [X_p, X_l]]\dots]] &= \nu_{pl}^t \dots \lambda_{km}^s u(X_s) + \dots + \lambda_{ab}^c \dots \lambda_{de}^f u(X_f) \end{aligned} \quad (3.14)$$

and the first term of  $Z(X, Y)$  are given by  $X_i + X_j$ . Then we can conclude that the exponential of elements of an Arm-Lie algebra is a group which we naturally call Arm-Lie group.

In the moment I'm writing this article, the only Lie algebra which give Arm-Lie algebras are Lie algebra with generators which are the exponential of something (i.e. their logarithms exist). So we can start in studying the trivial one dimensional Arm-Lie algebra  $\sqrt[p]{\mathfrak{so}(2)}$  and its correspondinf Arm-Lie group.

### 4 $\sqrt[p]{\mathfrak{so}(2)}$

We consider the changing of variable :

$$u(X) = X^p \quad (4.15)$$

for  $p \in \mathbb{C}^*$ . It's well known that the generator basis of  $\mathfrak{so}(2)$  is given by the first Pauli matrix multiply by  $\sqrt{-1}$  :

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p \quad (4.16)$$

Because  $i\sigma_1$  is a basis of  $\mathfrak{so}(2)$ , if we want to know the basis of  $u^{-1}(\mathfrak{so}(2)) = \sqrt[p]{\mathfrak{so}(2)}$ , we have to calculate the p-th root of  $i\sigma_1$ . Then you can check the validity of the relation :

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \quad (4.17)$$

So with (4.17), it is very easy to calculate the p-th root of  $i\sigma_1$  :

$$\begin{aligned} \sqrt[p]{i\sigma_1} &= \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\ &= \cos\left(\frac{\pi}{2p}\right) \text{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ X_1 = \sqrt[p]{i\sigma_1} &= \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix} \end{aligned}$$

where  $\text{Id}_2$  is of course the identity 2-dimensional matrix. This Arm-Lie algebra is trivial because there is only 2 generators  $i\sigma_1 = (X_1)^p$  and  $\sqrt[p]{i\sigma_1} = X_1$  satisfying the relations :

$$\begin{aligned} [X_1, X_1^p] &= 0 \\ [X_1, X_1] &= 0 \end{aligned} \tag{4.18}$$

Of course (4.18) implies that the generator of the Arm-Lie algebra satisfy the u-Jacobi identity (2.11). Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie group :

$$\exp(\sqrt[p]{\mathfrak{so}(2)}) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid e^{z_1 \sqrt[p]{i\sigma_1} + z_2 i\sigma_1} \right\} \tag{4.19}$$

To have an idea of the elements of the group (4.19), we can explicit :

$$\exp(z_1 X_1) = \exp\left(z_1 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \tag{4.20}$$

and the well-known element of  $SO(2)$  :

$$\exp(z_2 X_1^p) = \begin{pmatrix} \cos(z_2) & -\sin(z_2) \\ \sin(z_2) & \cos(z_2) \end{pmatrix} \tag{4.21}$$

Hence we can identify this Arm-Lie algebra to

$$\exp(\sqrt[p]{\mathfrak{so}(2)}) \equiv rSO(2, \mathbb{C}) \equiv \left\{ M \in M_n(\mathbb{C}) \mid {}^t M M = r \text{Id}_2 ; r \in \mathbb{C} \right\} \tag{4.22}$$

for  $\frac{1}{p} \neq 1[2]$  and just  $SO(2)$  if  $\frac{1}{p} = 1[2]$ .

**Remark 1.** *Of course the cosinus and the sinus of a complex arument is well defined*

$$\begin{aligned} \cos(z) &= \cos(\text{Re}(z)) \cosh(\text{Im}(z)) - i \sin(\text{Re}(z)) \sinh(\text{Im}(z)) \\ \sin(z) &= \sin(\text{Re}(z)) \cosh(\text{Im}(z)) + i \cos(\text{Re}(z)) \sinh(\text{Im}(z)) \end{aligned} \tag{4.23}$$

where  $\text{Re}$  and  $\text{Im}$  denote the real and imaginary parts respectively.

## 5 $\sqrt[p]{\mathfrak{su}(2)}$

We still consider the changing of variable :

$$u(X) = X^p \quad (5.24)$$

for  $p \in \mathbb{C}^*$ . It's well known that the generator basis of  $\mathfrak{su}(2)$  is given by the Pauli matrices multiplied by  $\sqrt{-1}$  :

$$\begin{aligned} i\sigma_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p \\ i\sigma_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = u(X_2) = (X_2)^p \\ i\sigma_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = u(X_3) = (X_3)^p \end{aligned} \quad (5.25)$$

Because  $\{i\sigma_1, i\sigma_2, i\sigma_3\}$  is a basis of  $\mathfrak{su}(2)$ , if we want to know the basis of  $u^{-1}(\mathfrak{su}(2)) = \sqrt[p]{\mathfrak{su}(2)}$ , we have to calculate the p-th root of  $i\sigma_1, i\sigma_2$  and  $i\sigma_3$ . Then you can check the validity of the relation :

$$\begin{aligned} i\sigma_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\ i\sigma_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \\ i\sigma_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\left(\frac{\pi}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \end{aligned}$$



So with (5.26), it is very easy to calculate the p-th root of  $i\sigma_1, i\sigma_2, i\sigma_3$  :

$$\begin{aligned}
\sqrt[p]{i\sigma_1} &= \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\
&= \cos\left(\frac{\pi}{2p}\right) \text{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
X_1 = \sqrt[p]{i\sigma_1} &= \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix} \\
\sqrt[p]{i\sigma_2} &= \exp\left(\frac{\pi}{2p} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \\
&= \cos\left(\frac{\pi}{2p}\right) \text{Id}_2 + \sin\left(\frac{\pi}{2p}\right) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
X_2 = \sqrt[p]{i\sigma_2} &= \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & i \sin\left(\frac{\pi}{2p}\right) \\ i \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix} \\
\sqrt[p]{i\sigma_3} &= \exp\left(\frac{\pi}{2p} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \\
X_3 = \sqrt[p]{i\sigma_3} &= \begin{pmatrix} \exp\left(\frac{\pi}{2p}\right) & 0 \\ 0 & \exp\left(-\frac{\pi}{2p}\right) \end{pmatrix}
\end{aligned}$$

where  $\text{Id}_2$  is of course the identity 2-dimensional matrix. This Arm-Lie algebra has 6 generators  $i\sigma_1 = (X_1)^p, i\sigma_2 = (X_2)^p, i\sigma_3 = (X_3)^p$  and  $\sqrt[p]{i\sigma_1} = X_1, \sqrt[p]{i\sigma_2} = X_2, \sqrt[p]{i\sigma_3} = X_3$  satisfying the relations :

$$\begin{aligned}
[X_1, (X_2)^p] &= -2 \sin\left(\frac{\pi}{2p}\right) (X_3)^p \\
[X_2, (X_3)^p] &= -2 \sin\left(\frac{\pi}{2p}\right) (X_1)^p \\
[(X_3)^p, (X_1)^p] &= -2 \sin\left(\frac{\pi}{2p}\right) (X_2)^p
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
[X_1, X_2] &= -2 \sin^2\left(\frac{\pi}{2p}\right) (X_3)^p \\
[X_2, X_3] &= -2 \sin^2\left(\frac{\pi}{2p}\right) (X_1)^p \\
[X_3, X_1] &= -2 \sin^2\left(\frac{\pi}{2p}\right) (X_2)^p
\end{aligned} \tag{5.27}$$

Of course (5.27) and (5.26) imply that the generators of the Arm-Lie algebra  $\sqrt[p]{su(2)}$  satisfy the u-Jacobi identity. Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie

group :

$$\exp(\sqrt[p]{\mathfrak{su}(2)}) = \left\{ (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 \mid e^{z_1 \sqrt[p]{i\sigma_1} + z_2 i\sigma_1 + z_3 \sqrt[p]{i\sigma_2} + z_4 i\sigma_2 + z_5 \sqrt[p]{i\sigma_3} + z_6 i\sigma_3} \right\} \quad (5.28)$$

To have an idea of the elements of the group (4.19), we can explicit :

$$\begin{aligned} \exp(z_1 X_1) &= \exp\left(z_1 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \\ \exp(z_3 X_3) &= \exp\left(z_3 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) & i \sin\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) \\ i \sin\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \\ \exp(z_5 X_5) &= \exp\left(z_5 \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \exp\left(iz_5 \sin\left(\frac{\pi}{2p}\right)\right) & 0 \\ 0 & \exp\left(z_5 \sin\left(-\frac{\pi}{2p}\right)\right) \end{pmatrix} \end{aligned} \quad (5.29)$$

and the well-known elements of  $SU(2)$  :

$$\begin{aligned} \exp(z_2 X_1^p) &= \begin{pmatrix} \cos(z_2) & -\sin(z_2) \\ \sin(z_2) & \cos(z_2) \end{pmatrix} \\ \exp(z_4 X_2^p) &= \begin{pmatrix} \cos(z_4) & i \sin(z_4) \\ i \sin(z_4) & \cos(z_4) \end{pmatrix} \\ \exp(z_6 X_3^p) &= \begin{pmatrix} \exp(iz_6) & 0 \\ 0 & \cos(-iz_6) \end{pmatrix} \end{aligned} \quad (5.30)$$

Hence we can identify this Arm-Lie algebra to

$$\exp(\sqrt[p]{\mathfrak{su}(2)}) \equiv rSU(2, \mathbb{C}) \equiv \left\{ M \in M_n(\mathbb{C}) \mid M^\dagger M = r \text{Id}_2 ; r \in \mathbb{C} \right\} \quad (5.31)$$

for  $\frac{1}{p} \neq 1[2]$  and just  $SU(2)$  if  $\frac{1}{p} = 1[2]$ .

## Discussion

Unfortunately, I searched other Arm-Lie algebras among classic Lie algebras but I didn't find. I think it's because many other classic groups are not the exponential of something (i.e. their logarithms do not exist) but I'm not sure. I find this structure only for  $\mathfrak{su}(2)$  and its subalgebra  $\mathfrak{so}(2)$ . I tried with  $\mathfrak{so}(3)$  and  $\mathfrak{sl}(2)$  but it didn't work. I develop this theory in order to classify what I found but I hope there is other Lie algebras which are Arm-Lie algebras. There is not a lot of example of Arm-Lie algebra but  $\sqrt[2]{\mathfrak{su}(2)}$  is a beautiful and very fundamental example.

I also tried this theory for other  $u(X)$  for example I took  $u(X) = \exp(\exp(X))$  but it is just a shifted case of the traditional case. I was also limited by the choice of changing of variable that I was able to do because I had to find the inverse function of matricial variable which is sometimes hard to do. I also tried function as sin or cos but it gave the identity or zero respectively. The only good changing of variable which I found was  $u(X) = X^p$  which is a lot of changing for  $p \in \mathbb{C}$  but it finally give the same result for all  $p$ .

Finally we can imagine an other Lie group theory based on the 'p-exponential' which I introduced in [2] but now I think that I will give the same result as the usual Lie group theory but with the p-exponential instead of the traditional exponential. Maybe I will explore this way in an other work.

## Références

- [1] Arm B. N., The Arm Theory
- [2] Arm B. N., The p-Arm Theory