



**HAL**  
open science

## The p-Arm Theory

Boris Arm

► **To cite this version:**

| Boris Arm. The p-Arm Theory. 2013. hal-00810535

**HAL Id: hal-00810535**

**<https://hal.science/hal-00810535>**

Preprint submitted on 10 Apr 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The p-Arm Theory

Arm Boris Nima  
*arm.boris@gmail.com*

## **Abstract**

We introduce the p-Arm theory which give rise to a new mathematical object that we call the "p-exponential" which is invariant under p derivation. We calculate its derivate and we use this new function to solve differential equations. Next, we define its real and imaginary part which are the p-cosinus and the p-sinus respectively.

## Introduction

The Arm theory [1] gives a developpment on any p-th power function basis in changing of variable in the Arm formula. But for functions in  $\mathbb{C}[(u(z) - z_0)^p]$ ,  $p \in \mathbb{N}^*$  there is an other way (the p-Arm formula) to make this developpment : instead of changing the variable at the p-th powers, you can also derivate p times which will finally give the same result. This is the main idea behind the p-Arm theory.

The exponential function is the function which leaves invariant the operator in the Taylor formula i.e. :

$$\frac{\partial e^x}{\partial x} = e^x \quad (0.1)$$

So in constructing the p-Arm theory, we see that we need a "p-exponential"  $e_p^x$  function which leaves the operator of the p-Arm formula invariant :

$$\frac{\partial^p e_p^x}{\partial x^p} = e_p^x \quad ; \quad \frac{\partial^k e_p^x}{\partial x^k} \neq e_p^x \quad (0.2)$$

for  $1 \leq k < p$ . The answer to the question (0.2) is the definition of the p-exponential as follow :

$$e_p^x = \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \quad (0.3)$$

The p-Arm formula is not so much interesting itself because we already have the developpment by the Arm-theory, but this formula give rise to the p-exponential which is very interesting to study.

In studying the derivate of the p-exponential, we see that this operator acts like a shift operator on the p-exponential and we need a generalization of the "p-exponential" to also include its derivate. This generalized exponential function is :

$$e_{p,\mu}^x = \sum_{k=0}^{\infty} \frac{x^{pk+\mu}}{(pk+\mu)!} \quad (0.4)$$

for  $p, \mu \in \mathbb{N}^*$ . I know that there is already a generalized exponential function in the theory of the fractional calculus (see [2]) which is given by

$$E_\mu^y \equiv \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \quad (0.5)$$

but which one I introduce here is more generalized because (0.4) has a multiplication and a shift whereas (0.5) has only a shift.

In the first section, we give the equivalent of the Arm formula for the  $p$ -Arm theory which we naturally call the  $p$ -Arm formula for function in  $\mathbb{C}[(u(z) - z_0)^p]$ .

In the second section, we give the equivalent shifted Arm formula for the  $p$ -Arm theory which we call the shifted  $p$ -Arm formula.

In the third section, we give the definition of the generalized exponential function. Next, we draw the six first real  $p$ -exponentials which is a beautiful graph. In effect, we explain why the  $p$ -th derivate of the  $p$ -exponential is itself. In this case, we calculate the derivate of the  $p$ -exponential. Thereby, we give the relation between the  $p$ -exponential and the traditional exponential. This is why we use this result to show that every function solving that its  $p$ -th derivate is itself can be expressed as a linear combination of  $p$ -exponential and we give the example of  $p = 2$ . Then defining the complex  $p$ -exponential, we give its real part called the  $p$ -cosinus and we draw the six first  $p$ -cosinus. Furthermore, we define the  $p$ -sinus which is the imaginary part of the complex  $p$ -exponential and we draw the six first of it. Finally, we define the  $p$ -tangent and we draw the six first  $p$ -tangent.

# 1 The p-Arm Formula

First we introduce the generalization to each basis  $u(z)$  of the well known Taylor formula which is written in the basis  $u(z) = z$  for each basis of the space  $\mathbb{C}[(u(z) - z_0)^p] = \text{span}\{1, (u(z) - z_0)^p, (u(z) - z_0)^{2p}, \dots\}$

**Theorem 1.**  $\forall u(z) \in \mathcal{C}(\mathbb{C})$  if  $\exists z \in \mathbb{C}$  such that  $u(z) = z_0 \in \mathbb{C}$  then  $\forall f(z) \in \mathbb{C}[(u(z) - z_0)^p]$

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[ \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} f(z) \right] (u(z) - z_0)^{pk} \quad (1.6)$$

**Proof :**

It's enough to show this formula on the basis  $\{(u(z) - z_0)^{pr}\}_{r \in \mathbb{N}}$ .

If  $k < r$  :

$$\begin{aligned} \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \partial_{u(z)}^{pk} (u(z) - z_0)^{pr} \\ &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \frac{(pr)!}{(p(r-k))!} (u(z) - z_0)^{p(r-k)} \\ \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= 0 \end{aligned} \quad (1.7)$$

If  $k > r$  :

$$\begin{aligned} \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial}{\partial u(z)} \right)^{pk} (u(z) - z_0)^{pr} \\ &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \partial_{u(z)}^{p(k-r)} (pr)! \\ \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= 0 \end{aligned} \quad (1.8)$$

If  $k = r$  :

$$\begin{aligned} \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= \lim_{z \rightarrow u^{-1}(z_0)} \frac{(pr)!}{(pk)!} \\ \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= 1 \end{aligned} \quad (1.9)$$

So we can see that :

$$\frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} = \delta_{k,r} \quad (1.10)$$

◆

## 2 The Shifted p-Arm Formula

If you have a function  $f \in \mathbb{C}[(u(z) - z_0)^{-p}] \oplus \mathbb{C}[(u(z) - z_0)^p]$ , you can know it if the coefficients on the negative basis are zeros before the infinity.

**Theorem 2.**  $\forall u(z) \in \mathcal{C}(\mathbb{C})$  if  $\exists z \in \mathbb{C}$  such that  $u(z) = z_0 \in \mathbb{C}$  then  $\forall f(z) \in \mathbb{C}[(u(z) - z_0)^p] \oplus \mathbb{C}[(u(z) - z_0)^{-p}]$

$$f(z) = \sum_{k=-m_p(u,f)}^{\infty} \frac{1}{(p(k + m_p(u,f)))!} \left[ \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{p(k+m_p(u,f))} (u(z)-z_0)^{pm_p(u,f)} f(z) \right] (u(z)-z_0)^{pk} \quad (2.11)$$

where the integer  $m_p(u, f) \in \mathbb{N}$  is given by :

$$m_p(u, f) = \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(f(z))}{p \ln(u(z) - z_0)} < \infty \quad (2.12)$$

**Proof :**

Let  $f(z)$  has the decomposition

$$f(z) = \sum_{k=-m_p(u,f)}^{\infty} \alpha_{pk} (u(z) - z_0)^{pk} = \sum_{k=0}^{\infty} \alpha_{p(k-m_p(u,f))} (u(z) - z_0)^{pk} (u(z) - z_0)^{-pm_p(u,f)} \quad (2.13)$$

where  $\alpha_{pk} = \langle f, (u(z) - z_0)^{pk} \rangle$ . Practically, we determine  $m_p$  in calculating

$$\begin{aligned} \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(f(z))}{p \ln(u(z) - z_0)} &= \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(\sum_{k=-m(u,f)}^{\infty} \alpha_k (u(z) - z_0)^k)}{p \ln(u(z) - z_0)} \\ &= \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(\alpha_{-m(u,f)} (u(z) - z_0)^{-m(u,f)})}{p \ln(u(z) - z_0)} \\ \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(f(z))}{p \ln(u(z) - z_0)} &= m(u, f) \end{aligned} \quad (2.14)$$

Inserting (2.13) in (1.6), we deduce

$$(u(z)-z_0)^{pm_p(u,f)} f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[ \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z)-z_0)^{pm_p(u,f)} f(z) \right] (u(z)-z_0)^{pk} \quad (2.15)$$

from which we deduce (2.11) in changing  $k' = k - m_p(u, f)$ .

◆

**Remark 1.** If you consider the shifted p-Arm formula (2.11) for  $p = 2, u(z) = e^{iz}$  and  $z_0 = 0$ , you will check that :

$$\cos^2(z) = \frac{e^{2iz} + 2 + e^{-2iz}}{4} \quad (2.16)$$

with  $m_2(e^{iz}, \cos^2) = 1$ .

The shifted p-Arm formula gives rise to a new mathematical function which make one the limit in the formula (2.11).

### 3 The p-exponential

**Definition 1.** We define the generalised exponential function :

$$e_{p,\mu}^x = \sum_{k=0}^{\infty} \frac{x^{kp+\mu}}{(kp+\mu)!} \quad (3.17)$$

for  $p, \mu \in \mathbb{N}^*$ .

In the rest of this paper, we will call  $e_{p,0}^x = e_p^x$  the "p-exponential".

Now because we want see what are these new function, we draw the 6 first real p-exponentials :

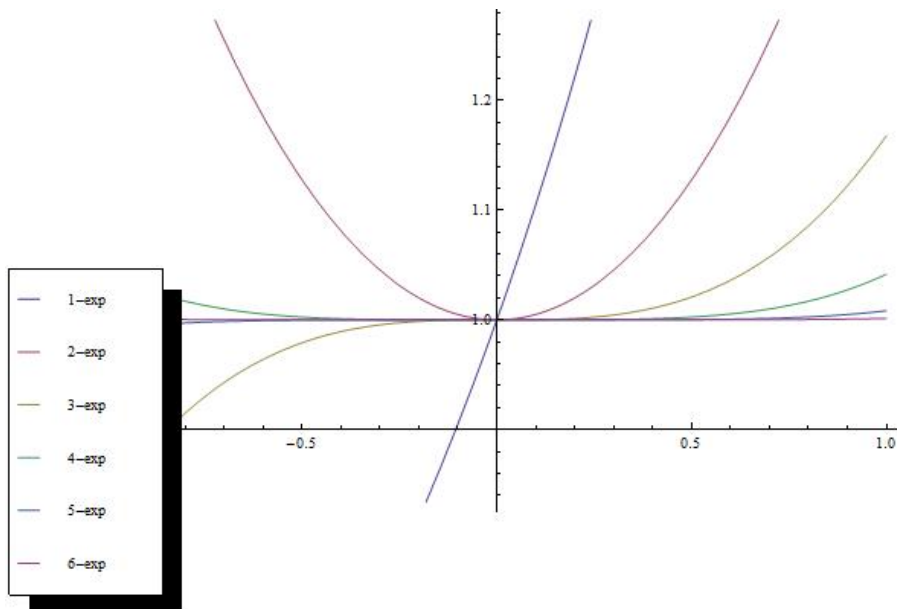


FIGURE 1 – The six first p-exponentials

We now explain why is this function interesting

**Proposition 1.** The p-exponential is a function such that

$$\frac{\partial^p e_p^x}{\partial x^p} = e_p^x \quad \text{and} \quad \frac{\partial^l e_p^x}{\partial x^l} \neq e_p^x \quad (3.18)$$

for each  $1 \leq l < p$ .

**Proof :**

$$\begin{aligned}
\frac{\partial^p e_p^x}{\partial x^p} &= \frac{\partial^p}{\partial x^p} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{(pk)!}{(pk-p)!} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{pk-p}}{(pk-p)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
\frac{\partial^p e_p^x}{\partial x^p} &= e_p^x
\end{aligned} \tag{3.19}$$

The second part of (3.18) is trivial.

◆

Now, we calculate the derivative of the p-exponential

**Proposition 2.** *The derivate of the p-exponential is given by :*

$$\frac{\partial e_p^x}{\partial x} = e_{p,p-1}^x \tag{3.20}$$

where  $p \in \mathbb{N}^*$ .

**Proof :**

$$\begin{aligned}
\frac{\partial e_p^x}{\partial x} &= \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} (pk) \frac{x^{pk-1}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} \\
\frac{\partial e_p^x}{\partial x} &= e_{p,p-1}^x
\end{aligned} \tag{3.21}$$

◆



**Remark 2.** Of course we have

$$\frac{\partial e_p^{u(x)}}{\partial x} = \frac{\partial u}{\partial x} e_{p,p-1}^{u(x)} \quad (3.22)$$

**Remark 3.** We see that because of (3.20), we have :

$$\frac{\partial^k e_p^x}{\partial x^k} = e_{p,p-k}^x \quad (3.23)$$

for  $1 \leq k \leq p$ . So the derivation acts like a shift operator on the  $p$ -exponential.

Now we show an interesting relation which link the  $p$ -exponential with the traditional exponential.

**Proposition 3.** The link between the  $p$ -exponential and the usual exponential is given by :

$$\left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p^x = e^x \quad (3.24)$$

or equivalently :

$$\sum_{\mu=0}^{p-1} e_{p,\mu}^x = e^x \quad (3.25)$$

**Proof :**

$$\begin{aligned} \left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p^x &= e_p^x + \frac{\partial}{\partial x} e_p^x + \dots + \frac{\partial^{p-1}}{\partial x^{p-1}} e_p^x \\ &= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \dots + \frac{\partial^{p-1}}{\partial x^{p-1}} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} + \dots + \sum_{k=1}^{\infty} \frac{x^{pk-p+1}}{(pk-p+1)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} + \dots + \sum_{k=0}^{\infty} \frac{x^{pk+1}}{(pk+1)!} \\ &= e_p^x + e_{p,p-1}^x + \dots + e_{p,1}^x \\ \left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p^x &= e^x \end{aligned} \quad (3.26)$$

◆

Now we introduced the  $p$ -exponential, we can use it to solve some differential equations. In fact, this is why I created it, the exponential solve the limit of the first order differential equation in the traditional Taylor formula whereas the  $p$ -exponential solve the limit of the  $p$ th order differential equation in (1.6).

**Proposition 4.** *Let the differential equation*

$$\frac{\partial^p u(x)}{\partial x^p} = u(x) \quad (3.27)$$

$\exists \alpha_1, \dots, \alpha_p$  such that the solution of (3.27) can be expressed as :

$$u(x) = \sum_{k=1}^p \alpha_k e_p^{\omega_p^k x} \quad (3.28)$$

where  $\omega_p = e^{\frac{2i\pi}{p}}$  is the  $p$ -th root of unity.

**Proof :**

$$\begin{aligned} \frac{\partial^p u(x)}{\partial x^p} &= \sum_{k=1}^p \alpha_k \frac{\partial^p}{\partial x^p} e_p^{\omega_p^k x} \\ &= \sum_{k=1}^p \alpha_k \left( \frac{\partial(\omega_p^k x)}{\partial x} \frac{\partial}{\partial(\omega_p^k x)} \right)^p e_p^{\omega_p^k x} \\ &= \sum_{k=1}^p \alpha_k \omega_p^{pk} e_p^{\omega_p^k x} \\ \frac{\partial^p u(x)}{\partial x^p} &= u(x) \end{aligned} \quad (3.29)$$

◆

**Example :**

As an example of (3.27), we solve the well-know case :

$$\frac{\partial^2 u(x)}{\partial x^2} = u(x) \quad (3.30)$$

The formula (3.28) gives the solution :

$$\begin{aligned} u(x) &= \alpha_1 e_2^x + \alpha_2 e_2^{-x} \\ u(x) &= \alpha_1 \cosh(x) + \alpha_2 \cosh(-x) \end{aligned} \quad (3.31)$$

where  $\alpha_1, \alpha_2 \in \mathbb{C}$  depend on the initial conditions.

Now we define the p-cosinus and p-sinus functions

**Definition 2.** *The p-cosinus is the real part of the complex exponential given by*

$$\cos_p(x) = \frac{e_p^{ix} + e_p^{-ix}}{2} \tag{3.32}$$

We draw the 6 first p-cosinus

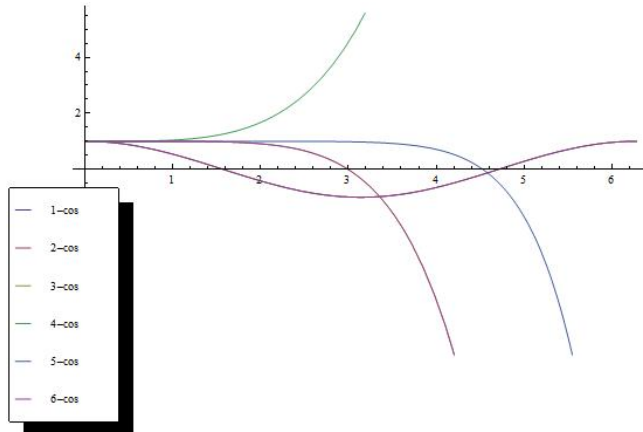


FIGURE 2 – The six first p-cosinus

**Definition 3.** *The p-sinus is the imaginary part of the complex exponential given by*

$$\sin_p(x) = \frac{e_p^{ix} - e_p^{-ix}}{2i} \tag{3.33}$$

We draw the 6 first p-sinus

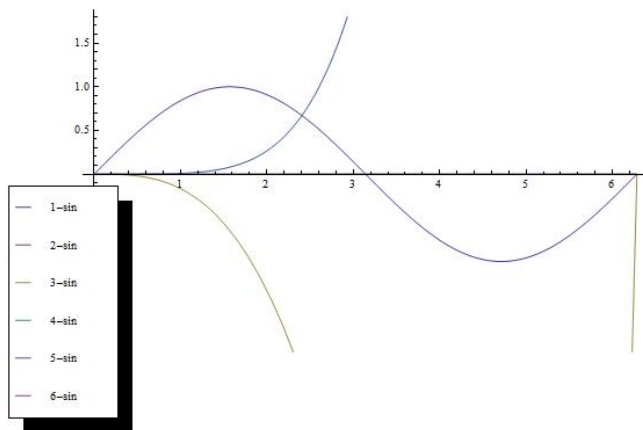


FIGURE 3 – The six first p-sinus

**Definition 4.** The  $p$ -tangent is given by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)} \tag{3.34}$$

We draw the 6 first  $p$ -tangent

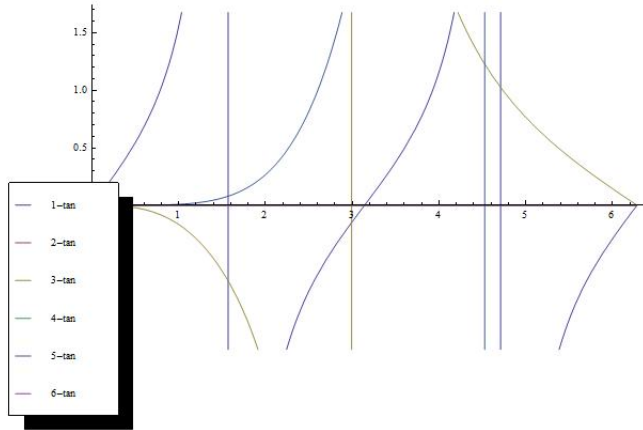


FIGURE 4 – The six first  $p$ -tangent

## Discussion

Even if the limit of the sum of two elements seems to be

$$\lim_{x+y \rightarrow \infty} e_p^{x+y} = \frac{1}{p} e^x e^y \quad (3.35)$$

on the graph for  $p \geq 2$ , I didn't find a simple relation between the sum of arguments and the product of exponentials. In a same way, we don't have an equivalent of the Moivre formula which links the  $n$ -th power of the exponential with the multiplication with  $n$  of the argument. However this relation seems to exist on the graph if we consider it in the infinity limit :

$$\lim_{x \rightarrow \infty} e_p^{nx} = \frac{1}{p} (e^x)^n \quad (3.36)$$

for  $p \geq 2$

In addition I also search for the value of the module of the p-exponential but it seems to not have a fixed valued on the graph. So on the graph, it seems to be :

$$\lim_{x \rightarrow \infty} \cos_p^2(x) + \sin_p^2(x) = \infty \quad (3.37)$$

for  $p \geq 3$ . There is an exception for  $p = 2$  because  $e_2 = \cosh$  and we have that :

$$|e_2^{ix}| = \cos(x) \quad (3.38)$$

For now, I didn't find yet the inverse function of the p-exponential or of the generalized exponential function. I tried finding an expression for the derivate of the "p-logarithm" :

$$\left. \frac{\partial \ln_p(x)}{\partial x} \right|_{x=e_p^x} = \frac{1}{e_{p,p-1}^x} \quad (3.39)$$

but we need a relation between the p-exponential  $e_p^x$  and its derivate  $e_{p,p-1}^x$  other than the derivation relation itself.

## Références

- [1] Arm B. N., The Arm Theory
- [2] Bologna M., Short Introduction to Fractional Calculus