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# The p-Arm Theory

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## **Abstract**

We introduce the p-Arm theory which give rise to a new mathematical object that we call the "p-exponential" which is invariant under p derivation. We calculate its derivate and we use this new function to solve differential equations. Next, we define its real and imaginary part which are the p-cosinus and the p-sinus respectively.

## Introduction

The Arm theory [1] gives a developpment on any p-th power function basis in changing of variable in the Arm formula. But for functions in  $\mathbb{C}[(u(z) - z_0)^p], p \in \mathbb{N}^*$  there is an other way (the p-Arm formula) to make this developpment : instead of changing the variable at the p-th powers, you can also derivate p times which will finally give the same result. This is the main idea behind the p-Arm theory.

The exponential function is the function which leaves invariant the operator in the Taylor formula i.e. :

$$\frac{\partial e^x}{\partial x} = e^x \quad (0.1)$$

So in constructing the p-Arm theory, we see that we need a "p-exponential"  $e_p^x$  function which leaves the operator of the p-Arm formula invariant :

$$\frac{\partial^p e_p^x}{\partial x^p} = e_p^x \quad ; \quad \frac{\partial^k e_p^x}{\partial x^k} \neq e_p^x \quad (0.2)$$

for  $1 \leq k < p$ . The answer to the question (0.2) is the definition of the p-exponential as follow :

$$e_p^x = \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \quad (0.3)$$

The p-Arm formula is not so much interesting itself because we already have the developpment by the Arm-theory, but this formula give rise to the p-exponential which is very interesting to study.

In studying the derivate of the p-exponential, we see that this operator acts like a shift operator on the p-exponential and we need a generalization of the "p-exponential" to also include its derivate. This generalized exponential function is :

$$e_{p,\mu}^x = \sum_{k=0}^{\infty} \frac{x^{pk+\mu}}{(pk+\mu)!} \quad (0.4)$$

for  $p, \mu \in \mathbb{N}^*$ . I know that there is already a generalized exponential function in the theory of the fractional calculus (see [2]) which is given by

$$E_\mu^y \equiv \sum_{k=0}^{\infty} \frac{t^{k-\mu}}{\Gamma(k+1-\mu)} \quad (0.5)$$

but which one I introduce here is more generalized because (0.4) has a multiplication and a shift whereas (0.5) has only a shift.

In the first section, we give the equivalent of the Arm formula for the p-Arm theory which we naturally call the p-Arm formula for function in  $\mathbb{C}[(u(z) - z_0)^p]$ .

In the second section, we give the equivalent shifted Arm formula for the p-Arm theory which we call the shifted p-Arm formula.

In the third section, we give the definition of the generalized exponential function. Next, we draw the six first real p-exponentials which is a beautiful graph. In effect, we explain why the p-th derivate of the p-exponential is itself. In this case, we calculate the derivate of the p-exponential. Thereby, we give the relation between the p-exponential and the traditional exponential. This is why we use this result to show that every function solving that its p-th derivate is itself can be expressed as a linear combination of p-exponential and we give the example of  $p = 2$ . Then defining the complex p-exponential, we give its real part called the p-cosinus and we draw the six first p-cosinus. Furthermore, we define the p-sinus which is the imaginary part of the complex p-exponential and we draw the six first of it. Finally, we define the p-tangent and we draw the six first p-tangent.

# 1 The p-Arm Formula

First we introduce the generalization to each basis  $u(z)$  of the well known Taylor formula which is written in the basis  $u(z) = z$  for each basis of the space  $\mathbb{C}[(u(z) - z_0)^p] = \text{span}\{1, (u(z) - z_0)^p, (u(z) - z_0)^{2p}, \dots\}$

**Theorem 1.**  $\forall u(z) \in \mathcal{C}(\mathbb{C})$  if  $\exists z \in \mathbb{C}$  such that  $u(z) = z_0 \in \mathbb{C}$  then  $\forall f(z) \in \mathbb{C}[(u(z) - z_0)^p]$

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[ \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} f(z) \right] (u(z) - z_0)^{pk} \quad (1.6)$$

**Proof :**

It's enough to show this formula on the basis  $\{(u(z) - z_0)^{pr}\}_{r \in \mathbb{N}}$ .

If  $k < r$  :

$$\begin{aligned} \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \partial_{u(z)}^{pk} (u(z) - z_0)^{pr} \\ &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \frac{(pr)!}{(p(r-k))!} (u(z) - z_0)^{p(r-k)} \\ \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= 0 \end{aligned} \quad (1.7)$$

If  $k > r$  :

$$\begin{aligned} \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial}{\partial u(z)} \right)^{pk} (u(z) - z_0)^{pr} \\ &= \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \partial_{u(z)}^{p(k-r)} (pr)! \\ \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= 0 \end{aligned} \quad (1.8)$$

If  $k = r$  :

$$\begin{aligned} \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= \lim_{z \rightarrow u^{-1}(z_0)} \frac{(pr)!}{(pk)!} \\ \frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} &= 1 \end{aligned} \quad (1.9)$$

So we can see that :

$$\frac{1}{(pk)!} \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z) - z_0)^{pr} = \delta_{k,r} \quad (1.10)$$

◆

## 2 The Shifted p-Arm Formula

If you have a function  $f \in \mathbb{C}[(u(z) - z_0)^{-p}] \oplus \mathbb{C}[(u(z) - z_0)^p]$ , you can know it if the coefficients on the negative basis are zeros before the infinity.

**Theorem 2.**  $\forall u(z) \in \mathcal{C}(\mathbb{C})$  if  $\exists z \in \mathbb{C}$  such that  $u(z) = z_0 \in \mathbb{C}$  then  $\forall f(z) \in \mathbb{C}[(u(z) - z_0)^p] \oplus \mathbb{C}[(u(z) - z_0)^{-p}]$

$$f(z) = \sum_{k=-m_p(u,f)}^{\infty} \frac{1}{(p(k + m_p(u,f)))!} \left[ \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{p(k+m_p(u,f))} (u(z)-z_0)^{pm_p(u,f)} f(z) \right] (u(z)-z_0)^{pk} \quad (2.11)$$

where the integer  $m_p(u, f) \in \mathbb{N}$  is given by :

$$m_p(u, f) = \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(f(z))}{p \ln(u(z) - z_0)} < \infty \quad (2.12)$$

**Proof :**

Let  $f(z)$  has the decomposition

$$f(z) = \sum_{k=-m_p(u,f)}^{\infty} \alpha_{pk} (u(z) - z_0)^{pk} = \sum_{k=0}^{\infty} \alpha_{p(k-m_p(u,f))} (u(z) - z_0)^{pk} (u(z) - z_0)^{-pm_p(u,f)} \quad (2.13)$$

where  $\alpha_{pk} = \langle f, (u(z) - z_0)^{pk} \rangle$ . Practically, we determine  $m_p$  in calculating

$$\begin{aligned} \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(f(z))}{p \ln(u(z) - z_0)} &= \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(\sum_{k=-m(u,f)}^{\infty} \alpha_k (u(z) - z_0)^k)}{p \ln(u(z) - z_0)} \\ &= \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(\alpha_{-m(u,f)} (u(z) - z_0)^{-m(u,f)})}{p \ln(u(z) - z_0)} \\ \lim_{z \rightarrow u^{-1}(z_0)} -\frac{\ln(f(z))}{p \ln(u(z) - z_0)} &= m(u, f) \end{aligned} \quad (2.14)$$

Inserting (2.13) in (1.6), we deduce

$$(u(z)-z_0)^{pm_p(u,f)} f(z) = \sum_{k=0}^{\infty} \frac{1}{(pk)!} \left[ \lim_{z \rightarrow u^{-1}(z_0)} \left( \frac{\partial z}{\partial u} \frac{\partial}{\partial z} \right)^{pk} (u(z)-z_0)^{pm_p(u,f)} f(z) \right] (u(z)-z_0)^{pk} \quad (2.15)$$

from which we deduce (2.11) in changing  $k' = k - m_p(u, f)$ .

◆

**Remark 1.** If you consider the shifted p-Arm formula (2.11) for  $p = 2, u(z) = e^{iz}$  and  $z_0 = 0$ , you will check that :

$$\cos^2(z) = \frac{e^{2iz} + 2 + e^{-2iz}}{4} \quad (2.16)$$

with  $m_2(e^{iz}, \cos^2) = 1$ .

The shifted p-Arm formula gives rise to a new mathematical function which make one the limit in the formula (2.11).

### 3 The p-exponential

**Definition 1.** We define the generalised exponential function :

$$e_{p,\mu}^x = \sum_{k=0}^{\infty} \frac{x^{kp+\mu}}{(kp+\mu)!} \quad (3.17)$$

for  $p, \mu \in \mathbb{N}^*$ .

In the rest of this paper, we will call  $e_{p,0}^x = e_p^x$  the "p-exponential".

Now because we want see what are these new function, we draw the 6 first real p-exponentials :

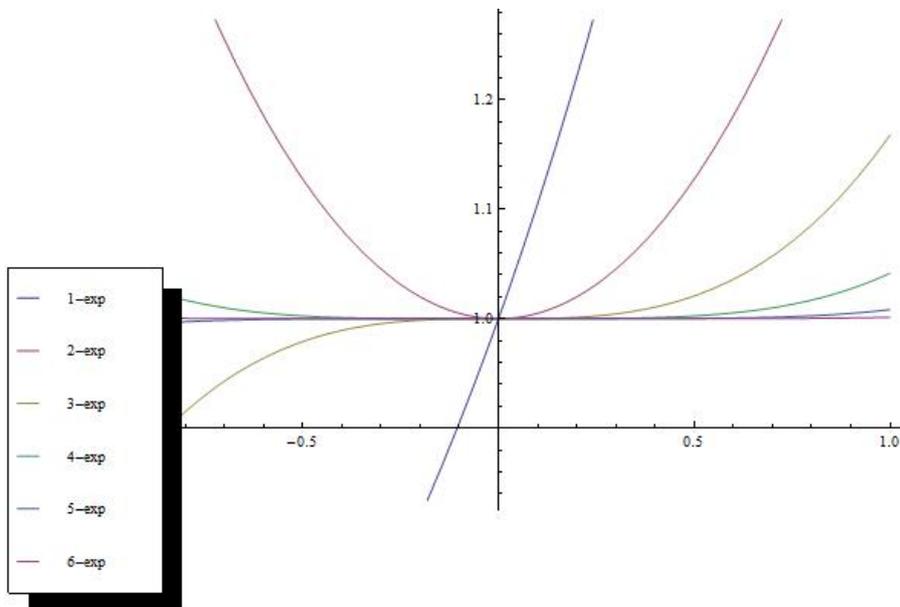


FIGURE 1 – The six first p-exponentials

We now explain why is this function interesting

**Proposition 1.** The p-exponential is a function such that

$$\frac{\partial^p e_p^x}{\partial x^p} = e_p^x \quad \text{and} \quad \frac{\partial^l e_p^x}{\partial x^l} \neq e_p^x \quad (3.18)$$

for each  $1 \leq l < p$ .

**Proof :**

$$\begin{aligned}
\frac{\partial^p e_p^x}{\partial x^p} &= \frac{\partial^p}{\partial x^p} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{(pk)!}{(pk-p)!} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{pk-p}}{(pk-p)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
\frac{\partial^p e_p^x}{\partial x^p} &= e_p^x
\end{aligned} \tag{3.19}$$

The second part of (3.18) is trivial.

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Now, we calculate the derivative of the p-exponential

**Proposition 2.** *The derivate of the p-exponential is given by :*

$$\frac{\partial e_p^x}{\partial x} = e_{p,p-1}^x \tag{3.20}$$

where  $p \in \mathbb{N}^*$ .

**Proof :**

$$\begin{aligned}
\frac{\partial e_p^x}{\partial x} &= \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\
&= \sum_{k=1}^{\infty} (pk) \frac{x^{pk-1}}{(pk)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} \\
\frac{\partial e_p^x}{\partial x} &= e_{p,p-1}^x
\end{aligned} \tag{3.21}$$

◆

**Remark 2.** Of course we have

$$\frac{\partial e_p^{u(x)}}{\partial x} = \frac{\partial u}{\partial x} e_{p,p-1}^{u(x)} \quad (3.22)$$

**Remark 3.** We see that because of (3.20), we have :

$$\frac{\partial^k e_p^x}{\partial x^k} = e_{p,p-k}^x \quad (3.23)$$

for  $1 \leq k \leq p$ . So the derivation acts like a shift operator on the  $p$ -exponential.

Now we show an interesting relation which link the  $p$ -exponential with the traditional exponential.

**Proposition 3.** The link between the  $p$ -exponential and the usual exponential is given by :

$$\left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p^x = e^x \quad (3.24)$$

or equivalently :

$$\sum_{\mu=0}^{p-1} e_{p,\mu}^x = e^x \quad (3.25)$$

**Proof :**

$$\begin{aligned} \left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p^x &= e_p^x + \frac{\partial}{\partial x} e_p^x + \dots + \frac{\partial^{p-1}}{\partial x^{p-1}} e_p^x \\ &= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \frac{\partial}{\partial x} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \dots + \frac{\partial^{p-1}}{\partial x^{p-1}} \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \sum_{k=1}^{\infty} \frac{x^{pk-1}}{(pk-1)!} + \dots + \sum_{k=1}^{\infty} \frac{x^{pk-p+1}}{(pk-p+1)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{pk}}{(pk)!} + \sum_{k=0}^{\infty} \frac{x^{pk+p-1}}{(pk+p-1)!} + \dots + \sum_{k=0}^{\infty} \frac{x^{pk+1}}{(pk+1)!} \\ &= e_p^x + e_{p,p-1}^x + \dots + e_{p,1}^x \\ \left( \sum_{\mu=0}^{p-1} \frac{\partial^\mu}{\partial x^\mu} \right) e_p^x &= e^x \end{aligned} \quad (3.26)$$

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Now we introduced the  $p$ -exponential, we can use it to solve some differential equations. In fact, this is why I created it, the exponential solve the limit of the first order differential equation in the traditional Taylor formula whereas the  $p$ -exponential solve the limit of the  $p$ th order differential equation in (1.6).

**Proposition 4.** *Let the differential equation*

$$\frac{\partial^p u(x)}{\partial x^p} = u(x) \quad (3.27)$$

$\exists \alpha_1, \dots, \alpha_p$  such that the solution of (3.27) can be expressed as :

$$u(x) = \sum_{k=1}^p \alpha_k e_p^{\omega_p^k x} \quad (3.28)$$

where  $\omega_p = e^{\frac{2i\pi}{p}}$  is the  $p$ -th root of unity.

**Proof :**

$$\begin{aligned} \frac{\partial^p u(x)}{\partial x^p} &= \sum_{k=1}^p \alpha_k \frac{\partial^p}{\partial x^p} e_p^{\omega_p^k x} \\ &= \sum_{k=1}^p \alpha_k \left( \frac{\partial(\omega_p^k x)}{\partial x} \frac{\partial}{\partial(\omega_p^k x)} \right)^p e_p^{\omega_p^k x} \\ &= \sum_{k=1}^p \alpha_k \omega_p^{pk} e_p^{\omega_p^k x} \\ \frac{\partial^p u(x)}{\partial x^p} &= u(x) \end{aligned} \quad (3.29)$$

◆

**Example :**

As an example of (3.27), we solve the well-know case :

$$\frac{\partial^2 u(x)}{\partial x^2} = u(x) \quad (3.30)$$

The formula (3.28) gives the solution :

$$\begin{aligned} u(x) &= \alpha_1 e_2^x + \alpha_2 e_2^{-x} \\ u(x) &= \alpha_1 \cosh(x) + \alpha_2 \cosh(-x) \end{aligned} \quad (3.31)$$

where  $\alpha_1, \alpha_2 \in \mathbb{C}$  depend on the initial conditions.

Now we define the p-cosinus and p-sinus functions

**Definition 2.** *The p-cosinus is the real part of the complex exponential given by*

$$\cos_p(x) = \frac{e_p^{ix} + e_p^{-ix}}{2} \tag{3.32}$$

We draw the 6 first p-cosinus

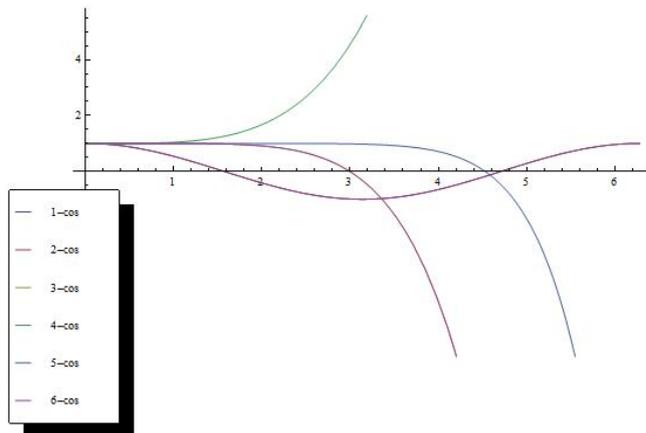


FIGURE 2 – The six first p-cosinus

**Definition 3.** *The p-sinus is the imaginary part of the complex exponential given by*

$$\sin_p(x) = \frac{e_p^{ix} - e_p^{-ix}}{2i} \tag{3.33}$$

We draw the 6 first p-sinus

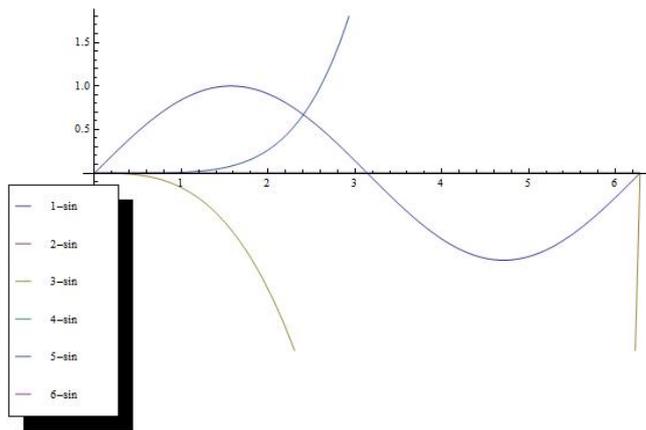


FIGURE 3 – The six first p-sinus

**Definition 4.** The  $p$ -tangent is given by

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)} \tag{3.34}$$

We draw the 6 first  $p$ -tangent

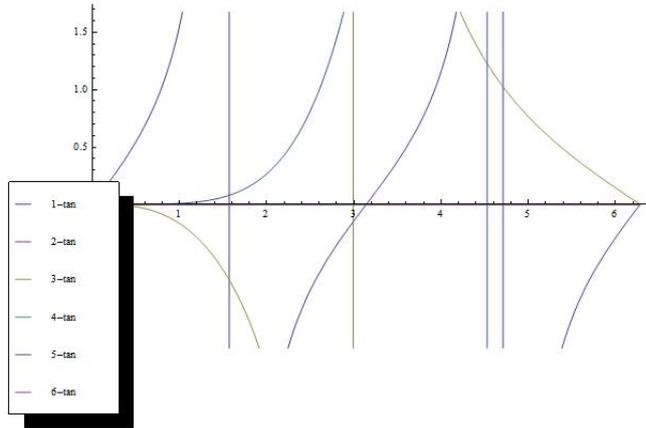


FIGURE 4 – The six first  $p$ -tangent

## Discussion

Even if the limit of the sum of two elements seems to be

$$\lim_{x+y \rightarrow \infty} e_p^{x+y} = \frac{1}{p} e^x e^y \quad (3.35)$$

on the graph for  $p \geq 2$ , I didn't find a simple relation between the sum of arguments and the product of exponentials. In a same way, we don't have an equivalent of the Moivre formula which links the  $n$ -th power of the exponential with the multiplication with  $n$  of the argument. However this relation seems to exist on the graph if we consider it in the infinity limit :

$$\lim_{x \rightarrow \infty} e_p^{nx} = \frac{1}{p} (e^x)^n \quad (3.36)$$

for  $p \geq 2$

In addition I also search for the value of the module of the p-exponential but it seems to not have a fixed valued on the graph. So on the graph, it seems to be :

$$\lim_{x \rightarrow \infty} \cos_p^2(x) + \sin_p^2(x) = \infty \quad (3.37)$$

for  $p \geq 3$ . There is an exception for  $p = 2$  because  $e_2 = \cosh$  and we have that :

$$|e_2^{ix}| = \cos(x) \quad (3.38)$$

For now, I didn't find yet the inverse function of the p-exponential or of the generalized exponential function. I tried finding an expression for the derivate of the "p-logarithm" :

$$\left. \frac{\partial \ln_p(x)}{\partial x} \right|_{x=e_p^x} = \frac{1}{e_{p,p-1}^x} \quad (3.39)$$

but we need a relation between the p-exponential  $e_p^x$  and its derivate  $e_{p,p-1}^x$  other than the derivation relation itself.

## Références

- [1] Arm B. N., The Arm Theory
- [2] Bologna M., Short Introduction to Fractional Calculus