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On the porous medium equations, fast diffusion equations and compressible Navier-Stokes equations, new results on the quasi-solutions and on the scaling of the equations

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Presented by

Abstract

In [3,4,5], we have developed a new tool called the *quasi solution* which approximate in some sense the compressible Navier-Stokes equation. In particular it allows to obtain global strong solution for the compressible Navier-Stokes equations with *large* initial data on the irrotational part of the velocity (*large* in the sense that the smallness assumption is subcritical in terms of scaling, it turns out that in this framework we are able to obtain large initial data in the energy space in dimension $N = 2$). In this paper we are interesting in studying in details this notion of *quasi solution* and in particular proving global weak solution, we also observe that for some choice of initial data (irrotationnal) we obtain some quasi solutions verifying the porous medium equation, the heat equation or the fast diffusion equation in function of the structure of the viscosity coefficients. Finally we show the convergence of the global weak solution of compressible Navier-Stokes equations to the quasi solutions when the pressure vanishing. We are also going to discuss the notion of scaling of the solution for compressible Navier-Stokes equations which justifies the notion of quasi solution.

Résumé

Sur les milieux poreux, les équations à diffusion rapides, les équations de Navier-Stokes compressibles, Nouveaux résultats sur les quasi-solutions ainsi que sur la notion de scaling

Dans [3,4,5], nous avons développé un nouvel outil appelé *quasi solution* qui approxime dans un certain sens les équations de Navier-Stokes compressible. En particulier cela permet d'obtenir des solutions fortes globales pour les équations de Navier-Stokes compressible avec des données initiales grandes sur la partie irrotationnelle de la vitesse (*grande* dans le sens d'une condition de petitesse sous critique, cela permet en particulier d'obtenir des solutions fortes globales en dimension $N = 2$ avec des données initiales grandes dans les espaces d'énergie). Dans ce papier nous sommes intéressé par étudier en détails ces *quasi solutions* et en particulier l'existence de solutions faibles globales, nous montrons que dans certains cas ces solutions vérifient les équations des milieux poreux, de l'équation de la chaleur et des équations à diffusion rapide en fonction de la structure des coefficients de viscosité. Nous montrons également la convergence des solutions faibles globales de Navier-Stokes coupressible vers ces quasi solutions lorsque la pression s'annule. Nous discutons enfin la notion de scaling pour les équations de Navier-Stokes compressible qui justifie cette notion de quasi solution.

Version française abrégée

Nous exhibons dans cette note des solutions aux quasi-solutions introduites dans [3,4,5] dans le contexte des équations de Navier-Stokes compressible. Rappelons d'abord le système de Navier-Stokes compressible modélisant un fluide compressible, les équations prennent la forme suivante :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \quad (1)$$

Ici $u = u(t, x) \in \mathbb{R}^N$ avec $N \geq 2$ correspond à la vitesse du liquide, $\rho = \rho(t, x) \in \mathbb{R}^+$ sa densité et on a $D(u) = \frac{1}{2}(\nabla u + {}^t \nabla u)$. La pression P s'écrit $P(\rho) = a\rho^\gamma$ avec $a > 0$ et $\gamma \geq 1$. $\mu(\rho) > 0$ et $N\lambda(\rho) + 2\mu(\rho) > 0$ sont les coefficients de viscosité. Dans la suite on suppose les coefficients de viscosités dégénérés et vérifiant en partie la relation algébrique suivante (voir Bresch, Desjardins dans [1]) :

$$\lambda(\rho) = 2\rho\mu'(\rho) - 2\mu(\rho). \quad (2)$$

On vérifie alors l'existence de quasi-solutions particulières définies comme dans [3,4,5], c'est à dire (ρ, u) est une quasi-solution du système (1) si (ρ, u) vérifie au sens des distributions :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \quad (3)$$

En particulier on montre que lorsque $\mu(\rho) = \mu\rho^\alpha$ avec $\alpha > 1 - \frac{1}{N}$ (cette condition est nécessaire afin d'assurer l'inégalité $N\lambda(\rho) + 2\mu(\rho) > 0$) alors $(\rho, -\frac{2\mu\alpha}{\alpha-1}\nabla\rho^{\alpha-1})$ est une quasi-solution si ρ vérifie au sens des solutions fortes (voir [7] chapitre 9) :

$$\partial_t \rho - 2\Delta\rho^\alpha = 0. \quad (4)$$

Lorsque $\alpha < 1$ on suppose que $u = 0$ lorsque $\rho = 0$.

Remark 1 On observe donc que la notion de quasi-solution est lié aux équations des milieux poreux ainsi que des équations à diffusion rapide.

Solutions autosimilaires

Nous allons également montrer que comme pour les milieux poreux les équations de Navier-Stokes compressible admettent une invariance par échelle. En effet si (ρ, u) est une solution de (16) alors pour tout $\lambda > 0$:

$$\rho_\lambda(t, x) = \lambda^\alpha \rho(\lambda t, \lambda^\beta x) \quad \text{et} \quad u_\lambda(t, x) = \lambda^{\alpha_1} u(\lambda t, \lambda^\beta x),$$

est une solution de (16) avec $\alpha_1 + \beta = 1$, $(\lambda - 1)\alpha + 2\beta = 1$, $\alpha(\gamma - 1) + \beta = \alpha_1 + 1$. Cela implique que :

$$\alpha = \frac{-1}{\lambda - \gamma}, \quad \alpha_1 = \frac{1 - \gamma}{2(\lambda - \gamma)} \quad \text{et} \quad \beta = \frac{2\lambda - \gamma - 1}{2(\lambda - \gamma)}.$$

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1. Introduction

This paper is devoted to prove the existence of quasi solutions for compressible Navier-Stokes equations with degenerate viscosity coefficients. Let recall the definition of quasi solutions introduced in [3,4,5]. Let $\rho(t, x)$ and $u(t, x)$ denote the density and the velocity of a barotropic compressible viscous fluid (as usual, ρ is a non-negative function and u is a vector field in \mathbb{R}^N with $N \geq 2$).

Définition 1.1 We say that (ρ, u) is a quasi solution if (ρ, u) verifies in distribution sense:

$$\begin{cases} \frac{\partial}{\partial t} \rho + \operatorname{div}(\rho u) = 0, \\ \frac{\partial}{\partial t} (\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) \operatorname{Du}) - \nabla(\lambda(\rho) \operatorname{div} u) = 0, \\ (\rho, u)_{t=0} = (\rho_0, u_0) \end{cases} \quad (5)$$

For a more precise definition we refer to the definition of global weak solution in [6] p6 where we assume here that $P(\rho) = 0$.

Remark 1 Here λ and μ verifies the condition (2) and the conditions (8)-(12) of [6]. In particular we have classical energy estimates by multiplying the momentum equation by u except that we have no information on the density of the type $\rho^\gamma \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^N))$ as for compressible Navier-Stokes equation when ($P(\rho) = \rho^\gamma$ with $\gamma \geq 1$) because here $P(\rho) = 0$. However using the entropy discovered in [1] we can prove that $\sqrt{\rho} \nabla \varphi(\rho)$ belongs in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^N))$ with $\varphi'(\rho) = \frac{\mu'(\rho)}{\rho}$ and that ρ belongs in $L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^N))$. It will be sufficient to prove the stability of global weak solution.

Remark 2 Let us remark that by using this notion of quasi solution in [3] we obtain global strong solution with initial data small in subcritical space for the scaling of the equations. In this sense quasi solutions are good approximate in order to obtain global strong solution with large initial data in terms of scaling (in particular in dimension $N = 2$ we can choose large initial data in energy space).

We now are going to investigate the existence of such quasi solution for the viscosity coefficients verifying (2). More precisely as in [3] we are going to search in a first time irrotational solution under the form $u(t, x) = \nabla c(t, x)$. Let us assume now that:

$$\mu(\rho) = \mu \rho^\alpha \text{ with } \alpha > 0 \text{ and } \lambda(\rho) = 2(\alpha - 1)\mu \rho^\alpha, \quad (6)$$

with $\alpha > 1 - \frac{1}{N}$ in order to insure the relation $2\mu(\rho) + N\lambda(\rho) > 0$. Then we observe that $\mu(\rho)$ and $\lambda(\rho)$ verify the relation (2). We obtain the following result.

Theorem 1.1 We have particular global weak solution solution of the system (1) of the form $(\rho, u = -\frac{2\mu\alpha}{\alpha-1} \nabla \rho^{\alpha-1})$ when $\alpha \neq 1$ with ρ solving the following system in the sense of strong solution introduced by Vázquez in [7] chapter 8 p185 :

$$\begin{cases} \partial_t \rho - 2\mu \Delta \rho^\alpha = 0, \\ \rho(0, \cdot) = \rho_0. \end{cases} \quad (7)$$

Remark 3 Let us point out that when $\rho = 0$ the velocity is not defined when $0 < \alpha < 1$ that is why we assume that $u = 0$ on the vacuum set. In other case we could give sense to pu as in [6].

Remark 4 We can observe as in [2] that if we consider the compressible Navier-Stokes equation with a friction term apu and a pressure of the form $2\mu\rho^\alpha$ then the same solution than theorem 1.2 verify a such system.

Remark 5 We recognize here the so called equation of the porous medium when $\alpha > 1$ and of the fast diffusion when $0 < \alpha < 1$. We refer for more details on the theory to the books of J-L Vázquez (see [7,8]).

Proof: Let us assume in a first time that the solution (ρ, u) of system (5) are classical, we are going to search solution under the form: $(\rho, -\frac{2\mu\alpha}{\alpha-1} \nabla \rho^{\alpha-1})$. The mass equation give us:

$$\partial_t \rho - 2\Delta \rho^\alpha = 0 \quad (8)$$

Let us check that the second equation is compatible with the first and keep an irrotational structure. First we have:

$$\begin{aligned}\partial_t(\rho u) &= -\frac{2\alpha}{\alpha-1}\partial_t(\rho\nabla\rho^{\alpha-1}) = -2\partial_t\nabla\rho^\alpha. \\ \operatorname{div}(\rho u \otimes u) &= \frac{4\alpha^2}{(\alpha-\frac{1}{2})^2}(\Delta\rho^{\alpha-\frac{1}{2}}\nabla\rho^{\alpha-\frac{1}{2}} + \frac{1}{2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2).\end{aligned}\tag{9}$$

Indeed we have:

$$\operatorname{div}(\rho u \otimes u)_j = \frac{4\alpha^2}{(\alpha-1)^2} \sum_i \partial_i(\rho\partial_i\rho^{\alpha-1}\partial_j\rho^{\alpha-1}) = \frac{4\alpha^2}{(\alpha-\frac{1}{2})^2}(\Delta\rho^{\alpha-\frac{1}{2}}\partial_j\rho^{\alpha-\frac{1}{2}} + \frac{1}{2}\partial_j|\nabla\rho^{\alpha-\frac{1}{2}}|^2)$$

Next we have:

$$\begin{aligned}-\operatorname{div}(2\rho^\alpha Du) &= \frac{4\alpha}{\alpha-1}\operatorname{div}(\rho^\alpha\nabla\nabla\rho^{\alpha-1}) = \frac{4\alpha}{\alpha-1}(\rho^\alpha\nabla\Delta\rho^{\alpha-1} + \nabla\rho^\alpha \cdot \nabla\nabla\rho^{\alpha-1}). \\ -\nabla(\lambda(\rho)\operatorname{div}u) &= 2(\alpha-1)\nabla(\rho^\alpha\operatorname{div}(\frac{2\alpha}{\alpha-1}\nabla\rho^{\alpha-1})) = 4\alpha\nabla(\rho^\alpha\Delta\rho^{\alpha-1}).\end{aligned}\tag{10}$$

Finally we have from (9):

$$\operatorname{div}(\rho u \otimes u) = \frac{4\alpha}{(\alpha-1)}\Delta\rho^{\alpha-1}\nabla\rho^\alpha + \frac{2\alpha^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}} + \frac{2\alpha^2}{(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2.\tag{11}$$

by using the fact that:

$$\begin{aligned}\Delta\rho^{\alpha-\frac{1}{2}} &= \sum_i \partial_i(\frac{\alpha-\frac{1}{2}}{\alpha-1}\partial_i\rho^{\alpha-1}\rho^{\frac{1}{2}}) = \frac{\alpha-\frac{1}{2}}{\alpha-1}\rho^{\frac{1}{2}}\Delta\rho^{\alpha-1} + \frac{1}{2}(\alpha-\frac{1}{2})\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2. \\ \frac{4\alpha^2}{(\alpha-\frac{1}{2})^2}\Delta\rho^{\alpha-\frac{1}{2}}\nabla\rho^{\alpha-\frac{1}{2}} &= \frac{4\alpha}{(\alpha-1)}\Delta\rho^{\alpha-1}\nabla\rho^\alpha + \frac{2\alpha^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}\end{aligned}$$

Finally by combining (10) and (11) we obtain:

$$\begin{aligned}\operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\rho^\alpha Du) &= \frac{4\alpha}{(\alpha-1)}\nabla(\rho^\alpha\Delta\rho^{\alpha-1}) + \frac{4\alpha}{\alpha-1}\nabla\rho^\alpha \cdot \nabla\nabla\rho^{\alpha-1} \\ &\quad + \frac{2\alpha^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}} + \frac{2\alpha^2}{(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2\end{aligned}\tag{12}$$

Now since we have:

$$\nabla\rho^\alpha \cdot \nabla\nabla\rho^{\alpha-1} = \frac{\alpha(\alpha-1)}{2(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2 - \frac{\alpha(\alpha-1)}{2(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}$$

we finally reduce (12) to the following equation:

$$\operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\rho^\alpha Du) = \frac{4\alpha}{(\alpha-1)}\nabla(\rho^\alpha\Delta\rho^{\alpha-1}) + \frac{4\alpha^2}{(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2\tag{13}$$

Finally using (9), (10) and (13) we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\rho^\alpha Du) - \nabla(\lambda(\rho)\operatorname{div}u) &= \\ = -2\nabla(\partial_t\rho^\alpha - \frac{2\alpha}{(\alpha-1)}\rho^\alpha\Delta\rho^{\alpha-1} - \frac{2\alpha^2}{(\alpha-\frac{1}{2})^2}|\nabla\rho^{\alpha-\frac{1}{2}}|^2 - 2\alpha\rho^\alpha\Delta\rho^{\alpha-1}) &= -2\alpha\nabla(\rho^{\alpha-1}(\partial_t\rho - 2\Delta\rho^\alpha)).\end{aligned}$$

This concludes the proof inasmuch as via the above equation the momentum equation is compatible to the mass equation and verify the same equation. \square

Remark 6 More generally we have solution of the form $(\rho, -2\nabla\varphi(\rho))$ with ρ verifying:

$$\partial_t \rho - 2\operatorname{div}(\rho \nabla \varphi(\rho)) = 0.$$

Remark 7 We have recognized the so called porous medium equation and the fast diffusion equation. Unlike the heat equation $\alpha = 1$, when $\alpha > 1$ this equation exhibits finite speed of propagation in the sense that solutions associated to compactly supported initial data remains compactly supported in space variable at all times. The situation is different in the case of fast diffusion equation $0 < \alpha < 1$, indeed in this case infinite propagation occurs. Let us mention that when α is in the interval $(0, m_c)$ with $m_c = \max(0, \frac{N-2}{N})$ then it can appears a phenomena of extinction of the solution in finite time (it is not the case in our framework because $\alpha > m_c$). Let us recall that if the initial data is strictly positive then the unique global weak solution is classical (see the proposition 7.21 p 177 in [7]). We refer to [7,8] for more details on such equations..

Finally we obtain the following theorems.

Theorem 1.2 Let $1 < \gamma < p$ with $p = +\infty$ if $N = 2$ and $p = 3$ if $N = 3$. Assume that we have a sequence (ρ_n, u_n) of weak solutions of system (5) satisfying the entropies of [6] with initial data ρ_0^n and u_0^n such that:

$$\rho_0^n \geq 0, \quad \rho_0^n \rightarrow \rho_0 \text{ in } L^1(\mathbb{R}^N), \quad \rho_0^n u_0^n \rightarrow \rho_0 u_0 \text{ in } L^1(\mathbb{R}^N), \quad (14)$$

and satisfy the following bounds (with C constant independent on n):

$$\int_{\mathbb{R}^N} \rho_0^n \frac{|u_0^n|^2}{2} < C, \quad \int_{\mathbb{R}^N} \sqrt{\rho_0^n} |\nabla \varphi(\rho_0^n)|^2 dx < C \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_0^n \frac{1 + |u_0^n|^2}{2} \ln(1 + |u_0^n|^2) dx < C. \quad (15)$$

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n} u_n)$ converges strongly to a weak solution $(\rho, \sqrt{\rho} u)$ of (5) satisfying entropy inequalities of [6].

Theorem 1.3 Assume that there exists global weak solution $(\rho_\epsilon, u_\epsilon)$ verifying the definition of [6] with the conditions (8) – (12) of [6] (see also theorem 2.1 in [6]) of the system:

$$\begin{cases} \partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon u_\epsilon) = 0, \\ \partial_t (\rho_\epsilon u_\epsilon) + \operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \operatorname{div}(2\mu(\rho_\epsilon) D(u_\epsilon)) - \nabla(\lambda(\rho_\epsilon) \operatorname{div} u_\epsilon) + \epsilon \nabla \rho_\epsilon^\gamma = 0, \\ (\rho_\epsilon, u_\epsilon)_{/t=0} = (\rho_0, u_0). \end{cases} \quad (16)$$

then $(\rho_\epsilon, u_\epsilon)$ converges in distribution sense to a quasi-solution (ρ, u) when ϵ goes to 0 with initial data (ρ_0, u_0) . (Here (ρ_0, u_0) verifies the entropies of [6]).

Proof: Concerning the stability of global weak solution, assume the existence of a sequel $(\rho_n, u_n)_{n \in \mathbb{N}}$ of global weak solution in the sense of [6] then it suffices to observe that we have as in [1,6] the following uniform bounds in n where we have multiplied the momentum equation by u_n and by $\nabla \varphi(\rho_n)$:

$$\begin{aligned} & \int_{\mathbb{R}^N} \rho_n |u_n(t, x)|^2(t, x) dx + \int_0^t \int_{\mathbb{R}^N} \mu(\rho_n) |Du_n|^2 dx dt + \int_0^t \int_{\mathbb{R}^N} \lambda(\rho_n) |\operatorname{div} u_n|^2 dx dt \leq \int_{\mathbb{R}^N} \rho_0^n |u_0^n|^2(x) dx. \\ & \int_{\mathbb{R}^N} [\rho_n |u_n(t, x)|^2 + \rho_n |\nabla \varphi(\rho_n)|^2(t, x)] dx + \int_0^t \int_{\mathbb{R}^N} \mu(\rho_n) |\nabla u_n|^2 dx dt + \int_0^t \int_{\mathbb{R}^N} \lambda(\rho_n) |\operatorname{div} u_n|^2 dx dt \\ & \leq C \left(\int_{\mathbb{R}^N} (\rho_0^n |u_0^n|^2(x) + \rho_0^n |\nabla \varphi(\rho_0^n)|^2(x)) dx \right). \end{aligned} \quad (17)$$

Finally as in [6] it remains to obtain a gain of integrability on the velocity u_n , it is obvious by multiplying the momentum equation by $(1 + \ln(1 + |u_n|^2))u_n$ and by bootstrap argument assuming the condition (11)

as in [6]. In particular we show that $\rho_n \frac{1+|u_n|^2}{2} \ln(1+|u_n|^2)$ is uniformly bounded in $L^\infty((0, T), L^1(\mathbb{R}^N))$ for any $T > 0$. By using these different entropies as in [6] we can easily show the convergence in distribution sense of (ρ_n, u_n) to (ρ, u) via compactness argument.

We are now going to prove that if we have some global weak solution $(\rho_\epsilon, u_\epsilon)$ for the system (16) in the sense of the definition in [6], then these global weak solution converge in distribution sense to a quasi-solution with initial data (ρ_0, u_0) . It suffices then to obtain the same uniform entropies in ϵ than in [6] for the sequel $(\rho_\epsilon, u_\epsilon)$. Similarly we have:

$$\begin{aligned} \int_{\mathbb{R}^N} [\rho_\epsilon |u_\epsilon(t, x)|^2(t, x) + \frac{\epsilon}{\gamma-1} \rho_\epsilon^\gamma] dx + \int_0^t \int_{\mathbb{R}^N} \mu(\rho_\epsilon) |Du_\epsilon|^2 dx dt + \int_0^t \int_{\mathbb{R}^N} \lambda(\rho_\epsilon) |\operatorname{div} u_\epsilon|^2 dx dt \\ \leq \int_{\mathbb{R}^N} [\rho_0 |u_0|^2(x) + \frac{\epsilon}{\gamma-1} \rho_0^\gamma] dx. \end{aligned} \quad (18)$$

$$\begin{aligned} \int_{\mathbb{R}^N} [\rho_\epsilon |u_\epsilon(t, x)|^2 + \rho_\epsilon |\nabla \varphi(\rho_\epsilon)|^2(t, x)] dx + \int_0^t \int_{\mathbb{R}^N} \mu(\rho_\epsilon) |\nabla u_\epsilon|^2 dx dt + \int_0^t \int_{\mathbb{R}^N} \lambda(\rho_\epsilon) |\operatorname{div} u_\epsilon|^2 dx dt \\ + \epsilon \int_0^t \int_{\mathbb{R}^N} \nabla \varphi(\rho_\epsilon) \cdot \nabla \rho_\epsilon^\gamma dx dt \leq C \left(\int_{\mathbb{R}^N} (\rho_0 |u_0|^2(x) + \rho_0 |\nabla \varphi(\rho_0)|^2(x) + \frac{\epsilon}{\gamma-1} \rho_0^\gamma(x)) dx \right). \end{aligned}$$

By using the lemma 3.2 of [6] with $\forall \delta \in (0, 2)$, we have that:

$$\begin{aligned} \int_{\mathbb{R}^N} \rho_\epsilon \frac{1+|u_\epsilon|^2}{2} \ln(1+|u_\epsilon|^2)(t, x) dx + \nu \int_0^t \int_{\mathbb{R}^N} \mu(\rho_\epsilon) (1 + \ln(1+|u_\epsilon|^2)) |Du_\epsilon|^2(t, x) dx dt \\ \leq C \int_0^t \int_{\mathbb{R}^N} \mu(\rho_\epsilon) |\nabla u_\epsilon|^2(t, x) dx dt + C_\delta \epsilon \int_0^t \left(\int_{\mathbb{R}^N} \frac{\rho_\epsilon^{2\gamma-\frac{\delta}{2}}}{\mu(\rho_\epsilon)^{\frac{2}{2-\delta}}} dx \right) dt. \end{aligned} \quad (19)$$

The key point consists in observing that the right hand side of (19) is uniformly bounded in ϵ . The last step corresponds to use the same compactness argument than in [6] to show that $(\rho_\epsilon, u_\epsilon)$ converges in distribution sense to a quasi-solution (ρ, u) when ϵ goes to 0 with initial data (ρ_0, u_0) . Let us point out that $\epsilon \rho_\epsilon^\gamma$ goes to 0 in distribution sense. Indeed it suffice to observe that ρ_ϵ and $\sqrt{\rho_\epsilon} \nabla \varphi(\rho_\epsilon)$ are uniformly bounded in ϵ respectively in $L^\infty((0, T), L^1(\mathbb{R}^N))$ and $L^\infty((0, T), L^2(\mathbb{R}^N))$ for any $T > 0$. In particular by Sobolev embedding and interpolation we obtain that $\epsilon^{1-\alpha} \rho_\epsilon^\gamma$ with $\alpha > 0$ is uniformly bounded in L^1_{loc} what means that $\epsilon \rho_\epsilon^\gamma$ goes to 0. \square

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