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# DECOMPOSABLE AND INDECOMPOSABLE ALGEBRAS OF DEGREE 8 AND EXPONENT 2

DEMBA BARRY

WITH AN APPENDIX BY ALEXANDER S. MERKURJEV

ABSTRACT. We study the decomposition of central simple algebras of exponent 2 into tensor products of quaternion algebras. We consider in particular decompositions in which one of the quaternion algebras contains a given quadratic extension. Let  $B$  be a biquaternion algebra over  $F(\sqrt{a})$  with trivial corestriction. A degree 3 cohomological invariant is defined and we show that it determines whether  $B$  has a descent to  $F$ . This invariant is used to give examples of indecomposable algebras of degree 8 and exponent 2 over a field of 2-cohomological dimension 3 and over a field  $\mathbb{M}(t)$  where the  $u$ -invariant of  $\mathbb{M}$  is 8 and  $t$  is an indeterminate. The construction of these indecomposable algebras uses Chow group computations provided by A. S. Merkurjev in Appendix.

## 1. INTRODUCTION

Let  $A$  be a central simple algebra over a field  $F$ . We say that  $A$  is *decomposable* if  $A \simeq A_1 \otimes_F A_2$  for two central simple  $F$ -algebras  $A_1$  and  $A_2$  both non isomorphic to  $F$ ; otherwise  $A$  is called *indecomposable*. Let  $K = F(\sqrt{a})$  be a quadratic separable extension of  $F$ . We say that  $A$  admits a *decomposition adapted* to  $K$  if  $K$  is in a quaternion subalgebra of  $A$ , that is,  $A \simeq (a, a') \otimes_F A'$  for some  $a' \in F$  and some subalgebra  $A' \subset A$ . If  $A$  is isomorphic to a tensor product of quaternion algebras, we will say that  $A$  is *totally decomposable*. Let  $B$  be a central simple algebra over  $K$ . The algebra  $B$  has a *descent* to  $F$  if there exists an  $F$ -algebra  $B'$  such that  $B \simeq B' \otimes_F K$ . It is clear that the algebra  $A$  admits a decomposition adapted to  $K$  if and only if the centralizer  $C_A K$  of  $K$  in  $A$  has a descent to  $F$ .

The first example of a non-trivial indecomposable central simple algebra of exponent 2 was given by Amitsur-Rowen-Tignol. More precisely, an explicit central division algebra of degree 8 and exponent 2 which has no quaternion subalgebra is constructed in [2]. Other examples of such algebras were given by Karpenko (see [11] and [12]) by computing torsion in Chow groups. In fact, 8 is the smallest possible degree for such an algebra by a well-known theorem of Albert which asserts that every algebra of exponent 2 and degree 4 is decomposable.

The decomposability question depends on the cohomological dimension of the ground field. Indeed, for obvious reasons there is no indecomposable algebra of exponent 2 over a field of cohomological dimension 0 or 1 (since the Brauer group is trivial in these cases). It follows from a result of Merkurjev (Theorem 3.1) that over a field of cohomological dimension 2 any central simple algebra of exponent

2 is isomorphic to a tensor product of quaternion algebras. On the other hand, the known examples of indecomposable algebras of exponent 2 are constructed over fields of cohomological dimension greater than or equal to 5.

The main goal of this article is to extend the existence of indecomposable algebras of exponent 2 to some fields of cohomological dimension smaller than or equal to 4. We also give an example over a field of rational functions in one variable over a field of  $u$ -invariant 8. The problem will be addressed through the study of the decomposability adapted to a quadratic extension of the ground field. More precisely, let  $K = F(\sqrt{a}) \subset A$  be a quadratic extension field. We first prove (Section 3) that if  $cd_2(F) \leq 2$  then  $K$  lies in a quaternion subalgebra of  $A$ . If  $\deg A = 8$ , we define a degree 3 cohomological invariant, depending only on the centralizer  $C_A K$  of  $K$  in  $A$ , which determines whether  $A$  admits a decomposition adapted to  $K$ , that is, whether  $C_A K$  has a descent to  $F$  (see Section 4). This invariant is used to give examples of indecomposable algebras of exponent 2 (Theorem 1.2 and Theorem 1.3). Although our invariant depends only on  $C_K A$ , it is nothing but a refinement of the invariant  $\Delta(A)$  defined by Garibaldi-Parimala-Tignol [8, §11] (see Remark 4.3). Through Remark 4.10 and the proof of Theorem 1.2, our invariant provides an example of indecomposable algebra  $A$  of degree 8 and exponent 2 such that  $\Delta(A)$  is nonzero, this is an answer to Garibaldi-Parimala-Tignol's question [8, Question 11.2].

For the proofs of Theorem 1.2 and Theorem 1.3 we need some results — injections (5.1) and (5.2) — on Chow groups of cycles of codimension 2. These results follow by Theorem A.9 and Theorem A.7 provided by A. S. Merkurjev in Appendix.

**Statement of main results.** Standard examples of fields of cohomological dimension 3 are  $k(t_1, t_2, t_3)$  or  $k((t_1))((t_2))((t_3))$  where  $k$  is an algebraically closed field and  $t_1, t_2, t_3$  are independent indeterminates over  $k$ . Recall also that the  $u$ -invariant of such fields is 8 by Tsen-Lang (see for instance [9, Theorem 1]). The following theorem shows that there is no indecomposable algebra of degree 8 and exponent 2 over these standard fields. We recall that the  $u$ -invariant of a field  $F$  is defined as

$$u(F) = \max\{\dim \varphi \mid \varphi \text{ anisotropic form over } F\}.$$

If no such maximum exists,  $u(F)$  is defined to be  $\infty$ .

**Theorem 1.1.** *There exists no indecomposable algebra of degree 8 and exponent 2 over a field of  $u$ -invariant smaller than or equal to 8.*

On the other hand, examples of indecomposable algebras do exist over a function field of one variable over a suitable field of  $u$ -invariant 8:

**Theorem 1.2.** *Let  $A$  be a central simple algebra of degree 8 and exponent 2 over a field  $F$  and let  $K = F(\sqrt{a})$  be a quadratic field extension of  $F$  contained in  $A$ . If  $K$  is not in a quaternion subalgebra of  $A$  then there exists an extension  $\mathbb{M}$  of  $F$  with  $u(\mathbb{M}) = 8$  such that the division algebra Brauer equivalent to*

$$A_{\mathbb{M}} \otimes_{\mathbb{M}} (a, t)_{\mathbb{M}(t)}$$

*is an indecomposable algebra of degree 8 and exponent 2 over  $\mathbb{M}(t)$ , where  $t$  is an indeterminate.*

To produce explicit examples, one may take for  $A$  any indecomposable algebra, as those constructed in [2] or [12]. Alternately, we give in [3] an example of a decomposable algebra satisfying the hypothesis of Theorem 1.2.

The following theorem shows that there exists an indecomposable algebra of degree 8 and exponent 2 over some field of 2-cohomological dimension 3. Let us recall that  $\text{CH}^2(\text{SB}(A))_{\text{tors}}$  is the torsion in the Chow group of cycles of codimension 2 over the Severi-Brauer variety  $\text{SB}(A)$  of  $A$  modulo rational equivalence. If  $A$  is of prime exponent  $p$  and index  $p^n$  (except the case  $p = 2 = n$ ) Karpenko showed in [12, Proposition 5.3] that if  $\text{CH}^2(\text{SB}(A))_{\text{tors}} \neq 0$  then  $A$  is indecomposable. Examples of such indecomposable algebras are given in [12, Corollary 5.4].

**Theorem 1.3.** *Let  $A$  be a central simple algebra of degree 8 and exponent 2 such that  $\text{CH}^2(\text{SB}(A))_{\text{tors}} \neq 0$ . Then there exists an extension  $\mathbb{M}$  of  $F$  with  $\text{cd}_2(\mathbb{M}) = 3$  such that  $A_{\mathbb{M}}$  is indecomposable.*

## 2. NOTATIONS AND PRELIMINARIES

Throughout this paper the characteristic of the base field  $F$  is assumed to be different from 2 and all algebras are associative and finite-dimensional over  $F$ . The main tools in this paper are central simple algebras, quadratic forms and Galois cohomology. Pierce's book [21] is a reference for the general theory of central simple algebras, references for quadratic form theory over fields are [5], [14] and [22].

Let  $F_s$  be a separable closure of  $F$ . For any integer  $n \geq 0$ ,  $H^n(F, \mu_2)$  denotes the Galois cohomology group

$$H^n(F, \mu_2) := H^n(\text{Gal}(F_s/F), \mu_2)$$

where  $\mu_2 = \{\pm 1\}$ . The group  $H^1(F, \mu_2)$  is identified with  $F^\times/F^{\times 2}$  by Kummer theory. For any  $a \in F^\times$ , we write  $(a)$  for the element of  $H^1(F, \mu_2)$  corresponding to  $aF^{\times 2}$ . The group  $H^2(F, \mu_2)$  is identified with the 2-torsion  $\text{Br}_2(F)$  in the Brauer group  $\text{Br}(F)$  of  $F$ , and we write  $[A]$  for the element of  $H^2(F, \mu_2)$  corresponding to a central simple algebra  $A$  of exponent 2. For more details on Galois cohomology the reader can consult Serre's book [24].

For any quadratic form  $q$ , we denote by  $C(q)$  and  $d(q)$  the Clifford algebra and the discriminant of  $q$ . If  $q$  has dimension  $2m + 2$ , it is easy to see that  $C(q)$  is a tensor product of  $m + 1$  quaternion algebras. Conversely, any tensor product of quaternion algebras is a Clifford algebra. Let  $I^n F$  be the  $n$ -th power of the fundamental ideal  $IF$  of the Witt ring  $WF$ . Abusing notations, we write  $q \in I^n F$  if the Witt class of  $q$  lies in  $I^n F$ . We shall use frequently the following property: if  $q \in I^2 F$  (i.e,  $\dim q$  is even and its discriminant is trivial) the Clifford algebra of  $q$  has the form  $C(q) \simeq M_2(E(q))$  for some central simple algebra  $E(q)$  which is totally decomposable.

Now, let  $E/F$  be an extension of  $F$  and let  $q$  be a quadratic form defined over  $F$ . We say that  $E/F$  is *excellent* for  $q$  if the anisotropic part  $(q_E)_{\text{an}}$  of  $q_E$  is defined over  $F$ . If  $E/F$  is excellent for every quadratic form defined over  $F$ , the extension  $E/F$  is called *excellent*.

Generally, the biquadratic extensions are not excellent (see for instance [6, §5]) but we have the following result:

**Lemma 2.1.** *Assume that  $I^3F = 0$  and let  $L = F(\sqrt{a_1}, \sqrt{a_2})$  be a biquadratic extension of  $F$ . Then,  $L/F$  is excellent for the quadratic forms  $q \in I^2F$ . More precisely, for all  $q \in I^2F$  there exists  $q_0 \in I^2F$  such that  $(q_L)_{\text{an}} \simeq (q_0)_L$ .*

*Proof.* Let  $s : K = F(\sqrt{a_1}) \rightarrow F$  be the  $F$ -linear map such that  $s(1) = 0$  and  $s(\sqrt{a_1}) = 1$ . The corresponding Scharlau's transfer will be denoted by  $s_*$ . Notice that  $I^3K = 0$  because of  $I^3F = 0$  and the exactness of the sequence (see [5, Theorem 40.3])

$$\langle 1, -a_1 \rangle I^2F \longrightarrow I^3F \longrightarrow I^3K \xrightarrow{s_*} I^3F.$$

Let  $q \in I^2F$  be an anisotropic form such that  $q_L$  is isotropic. We first show that there exists a quadratic form  $q_0$  defined over  $F$  such that  $(q_0)_L \simeq (q_L)_{\text{an}}$ . If  $q_L$  is hyperbolic, there is nothing to show. We suppose  $q_L$  is not hyperbolic. It is well-known (see e.g. [14, Theorem 3.1, p.197]) that  $(q_K)_{\text{an}}$  is defined over  $F$ . If  $(q_K)_{\text{an}}$  remains anisotropic over  $L$ , we take  $q_0 = (q_K)_{\text{an}}$ . Otherwise, one has

$$(q_K)_{\text{an}} = \langle 1, -a_2 \rangle \otimes \langle \gamma_1, \dots, \gamma_r \rangle \perp q'$$

for some  $\gamma_1, \dots, \gamma_r \in K$  and some subform  $q'$  of  $(q_K)_{\text{an}}$  defined over  $K$  with  $q'_L$  anisotropic (see for instance [10, Proposition 3.2.1]). On the other hand, the form  $(q_K)_{\text{an}}$  being in  $I^2K$  and  $I^3K = 0$ , one has  $(q_K)_{\text{an}} \otimes \langle 1, -\gamma_1 \rangle = 0$ , that is,  $(q_K)_{\text{an}} \simeq \gamma_1 (q_K)_{\text{an}}$ . Thus, we may suppose  $\gamma_1 = 1$ .

We claim that the dimension of the form  $\langle 1, \gamma_2, \dots, \gamma_r \rangle$  is 1. Indeed, assume that  $r \geq 2$  and write  $(q_K)_{\text{an}} = \langle 1, -a_2 \rangle \otimes \langle 1, \gamma_2 \rangle \perp \varphi$  for some  $\varphi$  over  $K$ . Since  $\varphi_L$  is nonzero, it is clear that  $\varphi$  is nonzero. Let  $\alpha \in K^\times$  be represented by  $\varphi$ . The form  $\langle 1, -a_2 \rangle \otimes \langle 1, \gamma_2 \rangle \otimes \langle 1, \alpha \rangle$  is hyperbolic since  $I^3K = 0$ , hence  $\langle 1, -a_2 \rangle \otimes \langle 1, \gamma_2 \rangle$  represents  $-\alpha$ . It follows that  $(q_K)_{\text{an}}$  is isotropic, a contradiction. Hence  $r = 1$  and so  $(q_K)_{\text{an}} \simeq \langle 1, -a_2 \rangle \perp q'$ .

Now, we are going to show that  $q'_L \simeq (q_0)_L$  for some  $q_0$  defined over  $F$ . The Scharlau transfers  $s_*((q_K)_{\text{an}})$  and  $s_*(\langle 1, -a_2 \rangle)$  are both hyperbolic because  $(q_K)_{\text{an}}$  and  $\langle 1, -a_2 \rangle$  are defined over  $F$ . This implies that  $s_*(q')$  is hyperbolic, so  $q'$  is Witt equivalent to  $(q_0)_K$  for some  $q_0$  defined over  $F$ . We deduce from the excellence of  $K/L$  that  $q' \simeq (q_0)_K$ . Whence  $(q_L)_{\text{an}} \simeq (q_0)_L$ .

It remains to prove that  $q_0$  may be chosen to be in  $I^2F$ . Since  $(q_0)_L \in I^2L$ , the discriminant  $d(q_0) \in \{F^{\times 2}, a_1.F^{\times 2}, a_2.F^{\times 2}, a_1a_2.F^{\times 2}\}$ . If  $d(q_0) = 1$ , we are done. Otherwise, assume for example  $d(q_0) = a_1$  and write  $q_0 \simeq \langle c_1, \dots, c_m \rangle$ . The quadratic form  $q_1 \simeq \langle a_1c_1, c_2, \dots, c_m \rangle$  is such that  $(q_1)_L \simeq (q_0)_L$  and  $d(q_1) = 1$ ; that is  $q_1 \in I^2F$ . It suffices to replace  $q_0$  by  $q_1$ . This concludes the proof.  $\square$

### 3. ADAPTED DECOMPOSITION UNDER $cd_2(F) \leq 2$

In this section we assume that the 2-cohomological dimension  $cd_2(F) \leq 2$ . If the characteristic of  $F$  is different from 2, Merkurjev proved that division algebras of exponent 2 over  $F$  are totally decomposable. We use this observation to show that

any quadratic field extension of  $F$  in a central simple algebra over  $F$  of exponent 2 lies in a quaternion subalgebra. We also prove a related result for a biquadratic extension of  $F$ . First, let us recall the following result:

**Theorem 3.1** (Mercurjev). *Assume that  $cd_2(F) \leq 2$ . Then*

- (1) *Any central division algebra over  $F$  whose class is in  $\text{Br}_2(F)$  is totally decomposable.*
- (2) *A quadratic form  $q \in I^2F$  is anisotropic if and only if  $E(q)$  is a division algebra.*

*Proof.* See [9, Theorem 3]. □

Let  $A$  be a 2-power dimensional central simple algebra over  $F$  and let  $K \subset A$  be a quadratic extension of  $F$ . If  $A$  is not a division algebra,  $K$  is in any split quaternion algebra. We have the following general result in the division case:

**Proposition 3.2.** *Assume that  $cd_2(F) \leq 2$ . Let  $A$  be a central division algebra of exponent 2 over  $F$ . Every square-central element of  $A$  lies in a quaternion subalgebra.*

*Proof.* Assume that  $\deg A = 2^m$ . By Theorem 3.1, the algebra  $A$  decomposes into a tensor product of  $m$  quaternion algebras. We may associate with  $M_2(A)$  a  $(2m+2)$ -dimensional quadratic form  $q \in I^2F$  in such a way that  $C(q) \simeq M_2(A)$ . This form  $q$  is anisotropic by Theorem 3.1 (2). The form  $q$  being in  $I^2F$ , the center of the Clifford algebra  $C_0(q)$  is  $F \times F$  and  $C_0(q) \simeq C_+(q) \times C_-(q)$  with  $A \simeq C_+(q) \simeq C_-(q)$ . Let  $x \in A - F$  be such that  $x^2 = a \in F^\times$ . The element  $z = x^{-1}\sqrt{a}$ , which is different from 1 and  $-1$ , is such that  $z^2 = 1$ ; hence  $A_{F(\sqrt{a})}$  is not a division algebra. Then Theorem 3.1 and [14, Theorem 3.1, p.197] indicate that the form  $q$  has a subform  $\alpha\langle 1, -a \rangle$  for some  $\alpha \in F^\times$ . Write  $q \simeq \alpha\langle 1, -a \rangle \perp q_1$  and let  $V_q$  be the underlying space of  $q$ . Let  $e_1, \dots, e_{2m+2}$  be an orthogonal basis of  $V_q$  which corresponds to this diagonalization. One has:

$$\begin{cases} e_1e_2, e_1e_3 \in C_0(q) \\ (e_1e_2)^2 = \alpha^2a, (e_1e_3)^2 = -\alpha^2q(e_3) \\ (e_1e_2)(e_1e_3) = -(e_1e_3)(e_1e_2). \end{cases}$$

Let  $z = e_1 \dots e_{2m+2}$ . Since  $q \in I^2F$ , we have  $z^2 \in F^{\times 2}$ . Let  $z^2 = \lambda^2$  with  $\lambda \in F^\times$  and set  $z_+ = \frac{1}{2}(1 + \lambda^{-1}z)$  and  $z_- = \frac{1}{2}(1 - \lambda^{-1}z)$ . So,  $z_+, z_-$  are the primitive central idempotents in  $C_0(q)$ . The elements  $(e_1e_2)z_+$  and  $(e_1e_3)z_+$  generate a quaternion subalgebra isomorphic to  $(a, a')$  in  $C_+(q) \simeq A$  where  $a' = (e_1e_3)^2$ . Hence,  $A$  contains some element  $y$  such that  $y^2 = a$  and  $y$  lies in a quaternion subalgebra of  $A$ . By the Skolem-Noether Theorem  $x$  and  $y$  are conjugated. Therefore,  $x$  is in a quaternion subalgebra. □

Recall that if  $cd_2(F) \leq 2$  then  $I^3F = 0$  ([15, Theorem A5']). This fact allows to use Lemma 2.1 in the proof of the following results.

**Theorem 3.3.** *Assume that  $cd_2(F) \leq 2$ . Let  $A$  be a central division algebra of degree  $2^n$  and exponent 2 over  $F$ . Let  $L = F(\sqrt{a_1}, \sqrt{a_2})$  be a biquadratic extension of  $F$*

contained in  $A$ . There exist quaternion algebras  $Q_1, Q_2$  and a subalgebra  $A' \subset A$  over  $F$  such that  $F(\sqrt{a_i}) \subset Q_i$  ( $i = 1, 2$ ) and  $A \simeq Q_1 \otimes Q_2 \otimes A'$ .

*Proof.* Let  $q \in I^2F$  be an anisotropic quadratic form such that  $C(q) \simeq M_2(A)$  (Theorem 3.1). Notice that the index of  $A_L$  is  $2^{n-2}$ . Since  $cd_2(L) = 2$ , it follows by [25, Lemma 8] that

$$\dim(q_L)_{\text{an}} = 2 \log_2 \text{ind}(A_L) + 2 = 2n - 2.$$

By Lemma 2.1, there is  $q' \in I^2F$  such that  $(q_L)_{\text{an}} \simeq q'_L$ . Put  $\psi = q - q'$ . The form  $\psi_L$  being hyperbolic it follows by [14, Theorem 4.3, p.444] that there exist quadratic forms  $\varphi_1, \varphi_2$  such that

$$\psi = \langle 1, -a_1 \rangle \otimes \varphi_1 \perp \langle 1, -a_2 \rangle \otimes \varphi_2 \text{ in } WF.$$

Since  $\psi \in I^2F$ , we must have  $\varphi_1, \varphi_2 \in IF$ . Taking the Witt-Clifford invariant of each side of the identity above, we obtain

$$A \otimes A' \sim (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2))$$

where  $A'$  is such that  $M_2(A') = C(q')$ . This yields that

$$A \simeq (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \otimes A'.$$

□

If  $\deg A = 8$ , the above theorem holds without the division condition on  $A$  as shows the following:

**Proposition 3.4.** *Assume that  $cd_2(F) \leq 2$ . Let  $A$  be a central simple algebra of degree 8 and exponent 2 over  $F$ . Let  $L = F(\sqrt{a_1}, \sqrt{a_2})$  be a biquadratic extension of  $F$  contained in  $A$ . There exist quaternion algebras  $Q_1, Q_2$  and  $Q$  over  $F$  such that  $F(\sqrt{a_i}) \subset Q_i$  ( $i = 1, 2$ ) and  $A \simeq Q_1 \otimes Q_2 \otimes Q$ .*

*Proof.* We are going to argue on the index  $\text{ind}(A)$  of  $A$ : if  $\text{ind}(A) \leq 2$ , the statement is clear since  $L$  lies in any split algebra of degree 4. If  $\text{ind}(A) = 8$ , the result follows from Theorem 3.3.

Suppose  $\text{ind}(A) = 4$ . Let  $q \in I^2F$  be an anisotropic quadratic form such that  $C(q) \sim M_2(A)$ . Such a form exists by Merkurjev's Theorem. Since  $A_L$  is split or of index 2, the form  $q_L$  is either hyperbolic or equivalent to a multiple of the anisotropic norm form,  $n_Q$ , of some quaternion algebra  $Q$  defined over  $L$ . Notice that every multiple of  $n_Q$  is isometric to  $n_Q$ . We may consider  $n_Q$  as being defined over  $F$  because of Lemma 2.1; so  $Q$  is defined over  $F$ . Moreover,  $(q - n_Q)_L$  is hyperbolic. Put  $\psi = q$  if  $\text{ind}(A_L) = 1$  and  $\psi = q - n_Q$  if  $\text{ind}(A_L) = 2$ ; the form  $\psi_L$  is hyperbolic. As in the proof of Theorem 3.3 there exist quadratic forms  $\varphi_1, \varphi_2 \in IF$  such that

$$A \otimes Q \sim (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \text{ if } \text{ind}(A_L) = 2,$$

and

$$A \sim (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \text{ if } \text{ind}(A_L) = 1.$$



This yields that

$$A \simeq (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \otimes M_2(F) \quad \text{or} \quad A \simeq (a_1, d(\varphi_1)) \otimes (a_2, d(\varphi_2)) \otimes Q.$$

□

**Remark 3.5.** For central simple algebras of degree 8 and exponent 2 over  $F$  with  $cd_2(F) = 2$ , Proposition 4.9 cannot be generalized to the triquadratic extensions of  $F$ . For example, let  $F = \mathbb{C}(x, y)$  where  $\mathbb{C}$  is the field of complex numbers and  $x, y$  are independent indeterminates over  $\mathbb{C}$ . Notice that  $cd_2(F) = 2$ . Since all 5-dimensional quadratic forms over  $F$  are isotropic by Tsen-Lang (see for instance [14, p.376]), Theorem 3.1 yields that index equals 2 for every division algebra of exponent 2 over  $F$ . Let  $a_1 = x^2(1+y)^2 - 4xy$ ,  $a_2 = 1-x$ ,  $a_3 = y$  and  $M = F(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ . It is shown in [6, (5.8)] that there exists an algebra  $A$  of exponent 2 and degree 8 over  $F$  containing  $M$  such that  $A$  admits no decomposition of the form  $(a_1, a'_1) \otimes (a_2, a'_2) \otimes (a_3, a'_3)$  with  $a'_1, a'_2, a'_3 \in F$ .

#### 4. DEGREE 3 INVARIANT

In this section, we assume no condition on the 2-cohomological dimension of  $F$ . Let  $K = F(\sqrt{a})$  be a quadratic field extension of  $F$ . Throughout this section  $B$  is a biquaternion algebra over  $K$  such that  $\text{cor}_{K/F}(B)$  is trivial. In fact, this condition on  $\text{cor}_{K/F}(B)$  means that  $B$  has a descent to  $F$  up to Brauer equivalence because of the exactness of the sequence (see for instance [1, (4.6)])

$$\text{Br}_2(F) \xrightarrow{r_{K/F}} \text{Br}_2(K) \xrightarrow{\text{cor}_{K/F}} \text{Br}_2(F).$$

We are going to attach to  $B$  a cohomological invariant which determines whether  $B$  has a descent to  $F$ . This descent problem is another way of studying the question of adapted decomposition. Indeed, let  $A$  be a central simple algebra over  $F$  of degree greater than or equal to 8 whose restriction is  $B$ . As explained in the introduction,  $B$  has a descent to  $F$  if and only if  $A$  admits a decomposition adapted to  $K$ .

Let  $\varphi$  be an Albert form of  $B$ , that is,  $\varphi$  is a 6-dimensional form in  $I^2K$  such that  $C_0(\varphi) \simeq C_+(\varphi) \times C_-(\varphi)$  with  $C_+(\varphi) \simeq C_-(\varphi) \simeq B$ . Recall that an Albert form is unique up to a scalar. We have the following lemma where  $e_3$  denotes the Arason invariant  $I^3F \rightarrow H^3(F, \mu_2)$ :

**Lemma 4.1.** *We keep the above notations. One has the following statements:*

- (1) *Scharlau's transfer  $s_*(\varphi)$  lies in  $I^3F$ .*
- (2) *For any  $\lambda \in K^\times$ , we have  $e_3(s_*(\varphi)) = e_3(s_*(\langle \lambda \rangle \otimes \varphi)) + \text{cor}_{K/F}((\lambda) \cdot [B])$ .*
- (3) *For  $\lambda \in K^\times$ , the following are equivalent:*
  - (i)  $s_*(\langle \lambda \rangle \otimes \varphi) = 0$ .
  - (ii)  $e_3(s_*(\varphi)) = \text{cor}_{K/F}((\lambda) \cdot [B])$ .



*Proof.* (1) Since the diagram

$$\begin{array}{ccc} I^2 K & \xrightarrow{s_*} & I^2 F \\ e_2 \downarrow & & \downarrow e_2 \\ \mathrm{Br}_2(K) & \xrightarrow{\mathrm{cor}_{K/F}} & \mathrm{Br}_2(F) \end{array}$$

commutes (see [1, (4.18)]) and  $\mathrm{cor}_{K/F}(B) = 0$ , we have  $e_2(s_*(\varphi)) = 0$ . Hence,

$$s_*(\varphi) \in \mathrm{Ker}(e_2 : I^2 F \rightarrow \mathrm{Br}_2(F)) = I^3 F.$$

(2) From the identity

$$s_*(\varphi) = s_*(\langle \lambda \rangle \otimes \varphi + \langle 1, -\lambda \rangle \otimes \varphi) = s_*(\langle \lambda \rangle \otimes \varphi) + s_*(\langle 1, -\lambda \rangle \otimes \varphi)$$

we have

$$e_3(s_*(\varphi)) = e_3(s_*(\langle \lambda \rangle \otimes \varphi)) + e_3(s_*(\langle 1, -\lambda \rangle \otimes \varphi)). \quad (4.1)$$

On the other hand, since the diagram

$$\begin{array}{ccc} I^3 K & \xrightarrow{s_*} & I^3 F \\ e_3 \downarrow & & \downarrow e_3 \\ H^3(K, \mu_2) & \xrightarrow{\mathrm{cor}_{K/F}} & H^3(F, \mu_2) \end{array}$$

commutes ([1, (5.7)]), one has

$$\begin{aligned} e_3(s_*(\langle 1, -\lambda \rangle \otimes \varphi)) &= \mathrm{cor}_{K/F}(e_3(\langle 1, -\lambda \rangle \otimes \varphi)) \\ &= \mathrm{cor}_{K/F}((\lambda) \cdot e_2(\varphi)) \\ &= \mathrm{cor}_{K/F}((\lambda) \cdot [C_+(\varphi)]) \\ &= \mathrm{cor}_{K/F}((\lambda) \cdot [B]). \end{aligned}$$

Therefore the relation (4.1) becomes

$$e_3(s_*(\varphi)) = e_3(s_*(\langle \lambda \rangle \otimes \varphi)) + \mathrm{cor}_{K/F}((\lambda) \cdot [B])$$

as desired.

(3) This point follows immediately from (2) because  $s_*(\langle \lambda \rangle \otimes \varphi) = 0$  if and only if  $e_3(s_*(\langle \lambda \rangle \otimes \varphi)) = 0$  by the Arason-Pfister Hauptsatz (see for instance [5, Chap. 4])  $\square$

It follows from Lemma 4.1 (2) that  $e_3(s_*(\varphi))$  modulo  $\mathrm{cor}_{K/F}((K^\times) \cdot [B])$  does not depend on the choice of the Albert form  $\varphi$ . Therefore we may define an invariant:

**Definition 4.2.** Let  $B$  be a biquaternion algebra over  $K$  such that  $\mathrm{cor}_{K/F}(B) = 0$ . The invariant

$$\delta_{K/F}(B) \in \frac{H^3(F, \mu_2)}{\mathrm{cor}_{K/F}((K^\times) \cdot [B])}$$

is the class of  $e_3(s_*(\varphi))$ .

**Remark 4.3.** If  $A$  is a central simple  $F$ -algebra of degree 8 containing  $K$ , and if  $B = C_A K$ , then  $\delta_{K/F}(B)$  is the image in  $H^3(F, \mu_2) / \text{cor}_{K/F}((K^\times) \cdot [B])$  of the discriminant  $\Delta(A, \sigma) \in H^3(F, \mu_2)$  of any symplectic involution  $\sigma$  on  $A$  which leaves  $K$  elementwise fixed. This follows by comparing the definition of  $\delta_{K/F}(B)$  with Garibaldi-Parimala-Tignol [8, Proposition 8.1]. Therefore, the image of  $\delta_{K/F}(B)$  in  $H^3(F, \mu_2) / (F^\times) \cdot [A]$  is the invariant  $\Delta(A)$  defined in [8, §11] (Note that  $[B] = \text{res}_{K/F}[A]$ , so by the projection formula  $\text{cor}_{K/F}((K^\times) \cdot [B]) = (N_{K/F}(K^\times)) \cdot [A] \subset (F^\times) \cdot [A]$ ).

The main property of the invariant is the following result:

**Proposition 4.4.** *The algebra  $B$  has a descent to  $F$  if and only if  $\delta_{K/F}(B) = 0$ .*

*Proof.* It follows from the definition that  $\delta_{K/F}(B) = 0$  if and only if there exists  $\lambda \in K^\times$  such that the equivalent conditions (i) and (ii) of Lemma 4.1 hold. We are going to prove that  $B$  has a descent to  $F$  if and only if (i) holds. We know that  $B$  has a descent to  $F$  if and only if there is  $\varphi_0 \in I^2 F$  with  $\dim \varphi_0 = 6$  such that

$$B \simeq C_+(\varphi) \simeq C_+(\varphi_0) \otimes K \simeq C_+(\langle \varphi_0 \rangle_K).$$

In other words,  $B$  has a descent to  $F$  if and only if there exists  $\lambda \in K^\times$  such that  $\varphi \simeq \langle \lambda \rangle \otimes (\varphi_0)_K$  because of the uniqueness of the Albert form up to similarity. Therefore to get the statement, it suffices to show for given  $\lambda \in K^\times$ , there exists  $\varphi_0 \in I^2 F$  with  $\dim \varphi_0 = 6$  and  $\varphi \simeq \langle \lambda \rangle \otimes (\varphi_0)_K$  if and only if  $s_*(\langle \lambda \rangle \otimes \varphi) = 0$ .

Suppose  $B$  has a descent to  $F$ , that is,  $\varphi \simeq \langle \lambda \rangle \otimes (\varphi_0)_K$ . We have automatically  $s_*(\langle \lambda \rangle \otimes \varphi) = 0$ . Conversely, assume that  $s_*(\langle \lambda \rangle \otimes \varphi) = 0$ . Then, as in the proof of Lemma 2.1 there exists a quadratic form  $\varphi_0 \in I^2 F$  with  $\langle \lambda \rangle \otimes \varphi \simeq (\varphi_0)_K$ . That concludes the proof.  $\square$

Now, let us denote by  $X$  the Weil transfer of  $\text{SB}(B)$  over  $F$ . Such a transfer exists (see for instance [4, (2.8)] or [23, Chapter 4]) and is projective ([13, Corollary 2.4]) since  $\text{SB}(B)$  is a projective variety. Moreover, we have

$$X_K \simeq \text{SB}(B) \times \text{SB}(B)$$

(see [4, (2.8)]). We denote by  $F(X)$  the function field of  $X$ . Notice that  $F(X)$  splits  $B$ ; that also means  $\varphi$  is hyperbolic over  $F(X)$ .

For any integer  $d \geq 1$ , let  $\mathbb{Q}/\mathbb{Z}(d-1) = \varinjlim \mu_n^{\otimes(d-1)}$ , where  $\mu_n$  is the group of  $n$ -th roots of unity in  $F_s$ . We let (see [7, Appendix A])  $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$  be the Galois cohomology group with coefficients in  $\mathbb{Q}/\mathbb{Z}(d-1)$ . By definition, one has the canonical map  $H^3(F, \mu_{2n}^{\otimes 2}) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ . On the other hand, using the surjectivity of the map  $H^3(F, \mu_{2n}^{\otimes 2}) \rightarrow H^3(F, \mu_n^{\otimes 2})$  (see [18]), the infinite long exact sequence in cohomology induced by the natural exact sequence of Galois modules

$$1 \rightarrow \mu_2 \rightarrow \mu_{2n}^{\otimes 2} \rightarrow \mu_n^{\otimes 2} \rightarrow 1$$

shows that the canonical map  $H^3(F, \mu_2) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2))$  is injective.

Since  $\varphi$  is hyperbolic over  $F(X)$ , it is clear that

$$e_3(s_*(\varphi)) \in \text{Ker} \left( H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(X), \mathbb{Q}/\mathbb{Z}(2)) \right).$$

On the other hand, since  $F(X)$  splits  $B$ , we have

$$\text{cor}_{K/F}((K^\times) \cdot [B]) \subset \text{Ker} \left( H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(X), \mathbb{Q}/\mathbb{Z}(2)) \right).$$

We have just proved the following consequence:

**Corollary 4.5.** *The invariant  $\delta_{K/F}(B)$  is in the quotient group*

$$\frac{\text{Ker} \left( H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(X), \mathbb{Q}/\mathbb{Z}(2)) \right)}{\text{cor}_{K/F}((K^\times) \cdot [B])}. \quad (4.2)$$

**Remark 4.6.** The quotient (4.2) is canonically identified with the torsion  $\text{CH}^2(X)_{\text{tors}}$  (see [20] or [19]). Therefore, we may view  $\delta_{K/F}(B)$  as belonging to this group.

We now study the behavior of  $\delta_{K/F}(B)$  under an odd degree extension. Let  $\mathbb{F}$  be an odd degree extension of  $F$ . We denote by  $K\mathbb{F}$  the quadratic extension  $\mathbb{F}(\sqrt{a})$ . The following result shows that if  $B$  has no descent to  $F$ , then the same holds for  $B_{\mathbb{F}}$ .

**Proposition 4.7.** *The scalar extension map*

$$\frac{\text{Ker} \left( H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(X), \mathbb{Q}/\mathbb{Z}(2)) \right)}{\text{cor}_{K/F}((K^\times) \cdot [B])} \longrightarrow \frac{\text{Ker} \left( H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(\mathbb{F}(X_{\mathbb{F}}), \mathbb{Q}/\mathbb{Z}(2)) \right)}{\text{cor}_{K\mathbb{F}/\mathbb{F} }(((K\mathbb{F})^\times) \cdot [B_{K\mathbb{F}}])}$$

*is an injection.*

*Proof.* Let  $M \supset K \supset F$  be a separable biquadratic extension of  $K$  in  $B$  which splits  $B$ ; so  $[M : F] = 8$ . Since  $M$  splits  $B$ , we have

$$X_M \simeq \text{SB}(B)_M \times \text{SB}(B)_M \simeq \mathbf{P}_M^3 \times \mathbf{P}_M^3$$

where  $\mathbf{P}_M^3$  denotes the projective 3-space considered as variety. The function field  $M(\mathbf{P}_M^3)$  of  $\mathbf{P}_M^3$  being a purely transcendental extension of  $M$ , the function field  $M(X_M)$  is a purely transcendental extension of  $M$ . So, we have

$$H^3(M, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow H^3(M(X_M), \mathbb{Q}/\mathbb{Z}(2)).$$

Let  $\xi \in \text{Ker} \left( H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(X), \mathbb{Q}/\mathbb{Z}(2)) \right)$  and denote by  $\xi_{\mathbb{F}}$  its image in  $H^3(\mathbb{F}, \mathbb{Q}/\mathbb{Z}(2))$  under the scalar extension. Suppose  $\xi_{\mathbb{F}} = \text{cor}_{K\mathbb{F}/\mathbb{F}}((\lambda) \cdot [B_{K\mathbb{F}}])$  for some  $\lambda \in (K\mathbb{F})^\times$ . Let  $A$  be a central simple  $F$ -algebra such that  $r_{K/F}(A) = B$ . Note that such an algebra  $A$  exists because the sequence

$$\text{Br}_2(F) \xrightarrow{r_{K/F}} \text{Br}_2(K) \xrightarrow{\text{cor}_{K/F}} \text{Br}_2(F)$$

is exact. One has

$$\begin{aligned}
 [\mathbb{F} : F]\xi = \text{cor}_{\mathbb{F}/F}(\xi_{\mathbb{F}}) &= \text{cor}_{\mathbb{F}/F}\left(\text{cor}_{K\mathbb{F}/\mathbb{F}}((\lambda) \cdot [B_{K\mathbb{F}}])\right) \\
 &= \text{cor}_{\mathbb{F}/F}\left(N_{K\mathbb{F}/\mathbb{F}}(\lambda) \cdot [A_{\mathbb{F}}]\right) \\
 &= N_{\mathbb{F}/F}\left(N_{K\mathbb{F}/\mathbb{F}}(\lambda)\right) \cdot [A] \\
 &= N_{K/F}\left(N_{K\mathbb{F}/K}(\lambda)\right) \cdot [A] \\
 &= \text{cor}_{K/F}\left(N_{K\mathbb{F}/K}(\lambda) \cdot [A_K]\right) \in \text{cor}_{K/F}((K^\times) \cdot [B]).
 \end{aligned}$$

This implies that the order of  $\xi$  is odd in

$$\frac{\text{Ker}\left(H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))\right)}{\text{cor}_{K/F}((K^\times) \cdot [B])}$$

since  $[\mathbb{F} : F]$  is odd. On the other hand, consider the following commutative diagram where the vertical maps are given by scalar extension and the horizontal maps are restriction and corestriction maps

$$\begin{array}{ccccc}
 H^3(F, \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & H^3(M, \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \\
 \downarrow & & \downarrow & & \downarrow \\
 H^3(F(X), \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & H^3(M(X_M), \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))
 \end{array}$$

The image of  $\xi$  by the top row is  $[M : F]\xi = 8\xi$ . Moreover, a diagram chase shows that  $8\xi$  is trivial in  $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$  because  $\xi$  is in the kernel of the left vertical map, and the central vertical map is injective. So, the order of  $\xi$  is then prime to  $[\mathbb{F} : F]$ . It follows that  $\xi \in \text{cor}_{K/F}((K^\times) \cdot [B])$ . The proof is complete.  $\square$

Now, assume that  $A$  a central simple algebra of degree 8 and exponent 2 over  $F$ . Let  $K = F(\sqrt{a})$  be a quadratic field extension of  $F$  contained in  $A$  and let  $\mathbb{F}$  be an odd degree extension of  $F$ . Denote by  $B = C_A K$  the centralizer of  $K$  in  $A$ .

**Remark 4.8.** (1) The algebra  $A$  admits a decomposition adapted to  $K$  if and only if  $B$  has a descent to  $F$ . In other words,  $A$  admits a decomposition adapted to  $K$  if and only if  $\delta_{K/F}(B) = 0$ . So,  $\delta_{K/F}(B) \neq 0$  if  $A$  is indecomposable.

(2) One deduces from Proposition 4.7 that if  $K$  is not in a quaternion subalgebra of  $A$ , then the same holds over  $\mathbb{F}$ .

(3) Let  $\text{Ker}(\text{res})$  be the kernel of the restriction map

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow H^3(F(\text{SB}(A)), \mathbb{Q}/\mathbb{Z}(2)).$$

The quotient group

$$\text{Ker}(\text{res})/[A] \cdot H^1(F, \mathbb{Q}/\mathbb{Z}(2))$$

is identified with  $\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}}$  in [19]. Arguing as in the proof of Proposition 4.7, we may see that the scalar extension map

$$\mathrm{CH}^2(\mathrm{SB}(A))_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(\mathrm{SB}(A)_{\mathbb{F}})_{\mathrm{tors}}$$

is injective. This injection may be also deduced from [12, Corollary 1.2 and Proposition 1.3].

Let  $A'$  be the division algebra Brauer equivalent to

$$A' \sim A \otimes_F (a, t)_{F(t)}$$

where  $t$  is an indeterminate over  $F$ . Note that

$$A'_K \sim A_K \otimes K(t) \sim B_{K(t)}.$$

The following proposition shows that if  $\delta_{K/F}(B)$  is nonzero then the invariant  $\Delta(A')$ , defined in [8, §11], is nonzero.

**Proposition 4.9.** *The scalar extension map*

$$\frac{H^3(F, \mu_2)}{\mathrm{cor}_{K/F}((K^\times) \cdot [B])} \longrightarrow \frac{H^3(F(t), \mu_2)}{(F(t)^\times) \cdot [A']} \quad (4.3)$$

is an injection.

*Proof.* The map is clearly well-defined because

$$\mathrm{cor}_{K/F}(K^\times \cdot [B]) \subset \mathrm{cor}_{K(t)/F(t)}(K(t)^\times \cdot [B_{K(t)}]) \subset (F(t)^\times) \cdot [A'].$$

Let  $\xi \in H^3(F, \mu_2)$ . Suppose  $\xi = f[A'] = f([A] + [(a, t)])$  for some  $f \in F(t)^\times$ . Consider the residue map

$$\partial_t : H^3(F(t), \mu_2) \longrightarrow H^2(F, \mu_2)$$

where  $F(t)$  is equipped with its  $t$ -adic valuation  $v_t$  (see for instance [7, §7]). We have

$$\partial_t(\xi) = \partial_t(f[A] + f[(a, t)]) = 0.$$

Since  $f$  is taken modulo  $F(t)^{\times 2}$ , we may assume that  $v_t(f)$  is either 0 or 1.

If  $v_t(f) = 1$ , that is,  $f = tf_0$  for some  $t$ -adic unit  $f_0$ , then

$$\begin{aligned} \partial_t(\xi) = \partial_t(f[A] + f[(a, t)]) &= [A] + \partial_t((tf_0, a, t)) \\ &= [A] + \partial_t((-f_0, a, t)) \\ &= [A] + [(-f_0(0), a)] = 0. \end{aligned}$$

Therefore  $A$  is Brauer equivalent to  $(-f_0(0), a)$ . It follows that the two quotients of the map (4.3) are trivial. In this case, there is nothing to show.

Suppose  $v_t(f) = 0$ . One has

$$\partial_t(f[A] + f[(a, t)]) = [(f(0), a)] = 0.$$

On the other hand,  $\xi = f(0)[A]$  by the specialization map  $\ker \partial_t \rightarrow H^3(F, \mu_2)$  associated with  $t$  at 0. Since  $(f(0), a) = 0$ , we deduce that  $\xi \in (N_{K/F}(K^\times)) \cdot [A] = \mathrm{cor}_{K/F}((K^\times) \cdot [B])$ . The proof is complete.  $\square$

**Remark 4.10.** According to Remark 4.3, the image of  $\delta_{K/F}(B)$  by the scalar extension map (4.3) is  $\Delta(A')$ . The proof of Theorem 1.2 provides an example of indecomposable algebra  $A'$  of degree 8 and exponent 2 such that  $\Delta(A')$  is nonzero.

## 5. PROOFS OF THE MAIN STATEMENTS

**5.1. Proof of Theorem 1.1.** We start out by the following lemma (a part of [9, Theorem 2]):

**Lemma 5.1.** *Every central simple  $F$ -algebra of exponent 2 is Brauer equivalent to a tensor product of at most  $\frac{1}{2}u(F) - 1$  quaternion algebras.*

*Proof.* Let  $A$  be a central simple  $F$ -algebra of exponent 2. Let  $\varphi \in I^2F$  be an anisotropic quadratic form such that  $C(\varphi) \sim A$  (by Merkurjev's Theorem [16]). The form  $\varphi$  being anisotropic,  $\dim \varphi \leq u(F)$ . Let  $\varphi'$  be a subform of  $\varphi$  of codimension 1. The algebra  $C(\varphi)$  is Brauer equivalent to  $C_0(\varphi')$  which is a tensor product of  $\frac{1}{2}(\dim(\varphi') - 1) = \frac{1}{2}(\dim \varphi) - 1$  quaternion algebras.  $\square$

Now, let  $A$  be a degree 8 and exponent 2 algebra over a field of  $u$ -invariant smaller than or equal to 8. By Lemma 5.1,  $A$  is Brauer equivalent to a tensor product of three quaternion algebras. Therefore, this equivalence is an isomorphism by dimension count; this proves that  $A$  is decomposable. This concludes the proof of Theorem 1.1.

**5.2. A consequence.** Let  $U$  and  $V$  be smooth complete geometrically irreducible varieties over  $F$ . In the appendix, Merkurjev gives conditions under which the scalar extension map  $\text{CH}(V) \rightarrow \text{CH}(V_{F(U)})$  is injective. For instance, let  $A$  be central simple  $F$ -algebra of degree 8 and exponent 2 and let  $K$  be a quadratic separable extension of  $F$  contained in  $A$ . Denote by  $X$  the Weil transfer of  $\text{SB}(C_A K)$  over  $F$ . Let  $q$  be a quadratic form over  $F$  with  $\dim q \geq 9$ . It follows from Theorem A.9 that the scalar extension map

$$\text{CH}^2(X)_{\text{tors}} \rightarrow \text{CH}^2(X_{F(q)})_{\text{tors}} \quad (5.1)$$

is injective, where  $F(q)$  is the function field of the projective quadric defined by  $q = 0$ . This property does not hold anymore if one replaces  $X$  by  $\text{SB}(A)$ . Indeed, Theorem 1.1 implies the following:

**Corollary 5.2.** *Let  $A$  be a central simple algebra of degree 8 and exponent 2 over  $F$ . Assume that  $\text{CH}^2(\text{SB}(A))_{\text{tors}} \neq 0$ . Then there exist an extension  $F'$  of  $F$  and a 9-dimensional quadratic form  $q$  defined over  $F_\ell$  such that the scalar extension map*

$$\text{CH}^2(\text{SB}(A)_{F'})_{\text{tors}} \rightarrow \text{CH}^2(\text{SB}(A)_{F'(q)})_{\text{tors}}$$

*is not injective.*

For the proof we need the following particular class of fields of  $u$ -invariant at most 8: starting with any field  $F$  over which there is an anisotropic quadratic form

of dimension 8, one defines a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_\infty = \bigcup_i F_i =: M_8^u(F)$$

inductively as follows: if  $F_{i-1}$  is already given, the field  $F_i$  is the composite of all function fields  $F_{i-1}(\varphi)$ , where  $\varphi$  ranges over (the isometry classes of) all 9-dimensional forms over  $F_{i-1}$ . Clearly, any 9-dimensional form over  $\mathbb{M}$  is isotropic. Therefore  $u(M_8^u(F)) \leq 8$ . Such a construction is due to Merkurjev (see [17]).

*Proof of Corollary 5.2.* Let  $\mathbb{M} = M_8^u(F)$  be an extension as above; recall that  $u(\mathbb{M}) \leq 8$ . The algebra  $A_{\mathbb{M}}$  being decomposable by Theorem 1.1, we have  $\text{CH}^2(\text{SB}(A)_{\mathbb{M}})_{\text{tors}} = 0$ . Since  $\mathbb{M} = \bigcup_i F_i$ , there exists  $\ell$  such that  $\text{CH}^2(\text{SB}(A)_{F_\ell})_{\text{tors}} \neq 0$  and  $\text{CH}^2(\text{SB}(A)_{F_{\ell+1}})_{\text{tors}} = 0$ . By definition,  $F_{\ell+1}$  is the composite of all function fields  $F_\ell(q)$ , where  $q$  ranges over all 9-dimensional forms over  $F_\ell$ . Hence, there exists an extension  $F'$  of  $F_\ell$  and a 9-dimensional form  $q$  over  $F_\ell$  such that  $\text{CH}^2(\text{SB}(A)_{F'})_{\text{tors}} \neq 0$  and  $\text{CH}^2(\text{SB}(A)_{F'(q)})_{\text{tors}} = 0$  as was to be shown.  $\square$

**5.3. Proof of Theorem 1.2.** Put  $\mathbb{M} = M_8^u(F)$  and  $B = C_A K$ . As in the previous section we denote by  $A'$  be the division algebra Brauer equivalent to  $A \otimes_F (a, t)_{F(t)}$ .

*First proof.* Recall that  $\delta_{K/F}(C_A K)$  is in  $\text{CH}^2(X)_{\text{tors}}$  by Remark 4.6, where  $X$  is the Weil transfer of  $\text{SB}(C_A K)$ . Since  $\delta_{K/F}(C_A K) \neq 0$  (see Remark 4.8), it follows by injection (5.1) that the extension  $\mathbb{M}(\sqrt{a})$  is not in a quaternion subalgebra of  $A_{\mathbb{M}}$ . By [26, Proposition 2.10] (or [3]) the algebra  $A'_{\mathbb{M}}$  is an indecomposable algebra of degree 8 and exponent 2. This concludes the proof.  $\square$

*Second proof.* Consider the following injections

$$\frac{H^3(F, \mu_2)}{\text{cor}_{K/F}((K^\times) \cdot [B])} \hookrightarrow \frac{H^3(\mathbb{M}, \mu_2)}{\text{cor}_{\mathbb{M}K/\mathbb{M}}((\mathbb{M}K)^\times) \cdot [B_{\mathbb{M}K}]} \hookrightarrow \frac{H^3(\mathbb{M}(t), \mu_2)}{(\mathbb{M}(t)^\times) \cdot [A'_{\mathbb{M}}]}$$

where the first is due to Merkurjev (injection (5.1)) and the second by Proposition 4.9. Since  $\delta_{K/F}(B) \neq 0$  and its image by the composite of these above injections is  $\Delta(A'_{\mathbb{M}})$ , we have  $\Delta(A'_{\mathbb{M}}) \neq 0$ . This also means  $A'_{\mathbb{M}}$  is indecomposable by Garibaldi-Parimala-Tignol [8, §11].  $\square$

**5.4. Proof of Theorem 1.3.** Here, we also need a particular class of fields of cohomological dimension at most 3: starting with any field  $F$  over which there is an anisotropic 3-fold Pfister form, one defines a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_\infty = \bigcup_i F_i =: M_3^{cd}(F)$$

inductively as follows: the field  $F_{2i+1}$  is the maximal odd degree extension of  $F_{2i}$ ; the field  $F_{2i+2}$  is the composite of all the function fields  $F_{2i+1}(\pi)$ , where  $\pi$  ranges over all 4-fold Pfister forms over  $F_{2i+1}$ . The arguments used by Merkurjev in [17], show that  $cd_2(M_3^{cd}(F)) \leq 3$ . Merkurjev used such a technique for constructing fields of cohomological dimension 2 over which there exist anisotropic quadratic forms



of dimension  $2d$  for an arbitrary integer  $d$ , i.e, counterexamples to Kaplansky's conjecture in the theory of quadratic forms.

Now, let  $A$  be a central simple algebra of degree 8 and exponent 2 such that  $\text{CH}^2(\text{SB}(A))_{\text{tors}} \neq 0$ . Put  $\mathbb{M} = M_3^{cd}(F)$  and let  $\mathbb{F}$  be an odd degree extension of  $F$ . The scalar extension map

$$\text{CH}^2(\text{SB}(A))_{\text{tors}} \longrightarrow \text{CH}^2(\text{SB}(A)_{\mathbb{F}})_{\text{tors}}$$

is injective by Remark 4.8. On the other hand, let  $\pi$  be a 4-fold Pfister form over  $\mathbb{F}$ . It follows from Theorem A.7 that the scalar extension map

$$\text{CH}^2(\text{SB}(A)_{\mathbb{F}})_{\text{tors}} \longrightarrow \text{CH}^2(\text{SB}(A)_{\mathbb{F}(\pi)})_{\text{tors}} \quad (5.2)$$

is injective. We deduce from these two latter injections that  $\text{CH}^2(\text{SB}(A)_{\mathbb{M}})_{\text{tors}} \neq 0$ ; and so  $A_{\mathbb{M}}$  is indecomposable. The algebra  $A_{\mathbb{M}}$  being indecomposable, we must have  $cd_2(\mathbb{M}) > 2$  by Theorem 3.1. Hence,  $cd_2(\mathbb{M}) = 3$ . This completes the proof.

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## APPENDIX A.

## ON THE CHOW GROUP OF CYCLES OF CODIMENSION 2

BY ALEXANDER S. MERKURJEV

Let  $X$  be an algebraic variety over  $F$ . We write  $A^i(X, K_n)$  for the homology group of the complex

$$\coprod_{x \in X^{(i-1)}} K_{n-i+1}(F(x)) \xrightarrow{\partial} \coprod_{x \in X^{(i)}} K_{n-i}(F(x)) \xrightarrow{\partial} \coprod_{x \in X^{(i+1)}} K_{n-i-1}(F(x)),$$

where  $K_j$  are the Milnor  $K$ -groups and  $X^{(i)}$  is the set of points in  $X$  of codimension  $i$  (see [7, §5]). In particular,  $A^i(X, K_i) = \text{CH}^i(X)$  is the Chow group of classes of codimension  $i$  algebraic cycles on  $X$ .

Let  $X$  and  $Y$  be smooth complete geometrically irreducible varieties over  $F$ .

**Proposition A.1.** *Suppose that for every field extension  $K/F$  we have:*

- (1) *The natural map  $\text{CH}^1(X) \rightarrow \text{CH}^1(X_K)$  is an isomorphism of torsion free groups,*
- (2) *The product map  $\text{CH}^1(X_K) \otimes K^\times \rightarrow A^1(X_K, K_2)$  is an isomorphism.*

Then the natural sequence

$$0 \longrightarrow (\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)) \oplus \mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(X_{F(Y)})$$

is exact.

*Proof.* Consider the spectral sequence

$$E_1^{p,q} = \coprod_{y \in Y^{(p)}} A^q(X_{F(y)}, K_{2-p}) \implies A^{p+q}(X \times Y, K_2)$$

for the projection  $X \times Y \longrightarrow Y$  (see [7, Cor. 8.2]). The nonzero terms of the first page are the following:

$$\mathrm{CH}^2(X_{F(Y)})$$

$$A^1(X_{F(Y)}, K_2) \longrightarrow \coprod_{y \in Y^{(1)}} \mathrm{CH}^1(X_{F(y)})$$

$$A^0(X_{F(Y)}, K_2) \longrightarrow \coprod_{y \in Y^{(1)}} A^0(X_{F(y)}, K_1) \longrightarrow \coprod_{y \in Y^{(2)}} \mathrm{CH}^0(X_{F(y)}).$$

Then  $E_1^{2,0} = \coprod_{y \in Y^{(2)}} \mathbb{Z}$  is the group of cycles on  $X$  of codimension 2 and  $E_1^{1,0} = \coprod_{y \in Y^{(1)}} F(y)^\times$  as  $X$  is complete. It follows that  $E_2^{2,0} = \mathrm{CH}^2(Y)$ .

By assumption, the differential  $E_1^{0,1} \longrightarrow E_1^{1,1}$  is identified with the map

$$\mathrm{CH}^1(X) \otimes (F(Y)^\times \longrightarrow \coprod_{y \in Y^{(1)}} \mathbb{Z}).$$

Since  $Y$  is complete and  $\mathrm{CH}^1(X)$  is torsion free, we have  $E_2^{0,1} = \mathrm{CH}^1(X) \otimes F^\times$  and  $E_\infty^{1,1} = E_2^{1,1} = \mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)$ .

The edge map

$$A^1(X \times Y, K_2) \longrightarrow E_2^{0,1} = \mathrm{CH}^1(X) \otimes F^\times$$

is split by the product map

$$\mathrm{CH}^1(X) \otimes F^\times = A^1(X, K_1) \otimes A^0(Y, K_1) \longrightarrow A^1(X \times Y, K_2),$$

hence the edge map is surjective. Therefore, the differential  $E_2^{0,1} \longrightarrow E_2^{2,0}$  is trivial and hence  $E_\infty^{2,0} = E_2^{2,0} = \mathrm{CH}^2(Y)$ . Thus, the natural homomorphism

$$\mathrm{CH}^2(Y) \longrightarrow \mathrm{Ker}(\mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(X_{F(Y)}))$$

is injective and its cokernel is isomorphic to  $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)$ . The statement follows.  $\square$

**Example A.2.** Let  $X$  be a projective homogeneous variety of a semisimple algebraic group over  $F$ . There exist an étale  $F$ -algebra  $E$  and an Azumaya  $E$ -algebra  $A$  such that for  $i = 0$  and  $1$ , we have an exact sequence

$$0 \longrightarrow A^1(X, K_{i+1}) \longrightarrow K_i(E) \xrightarrow{\rho} H^{i+2}(F, \mathbb{Q}/\mathbb{Z}(i+1)),$$

where  $\rho(x) = N_{E/F}((x) \cup [A])$  (see [4] and [5]). If the algebras  $E$  and  $A$  are split, then  $\rho$  is trivial and for every field extension  $K/F$ ,

$$\mathrm{CH}^1(X) \simeq K_0(E) \simeq K_0(E \otimes K) \simeq \mathrm{CH}^1(X_K),$$

$$A^1(X_K, K_2) \simeq K_1(E \otimes K) \simeq K_0(E) \otimes K^\times \simeq \mathrm{CH}^1(X_K) \otimes K^\times.$$

Therefore, the condition (1) and (2) in Proposition A.1 hold. For example, if  $X$  is a smooth projective quadric of dimension at least 3, then  $E = F$  and  $A$  is split.

Now consider the natural complex

$$\mathrm{CH}^2(X) \oplus (\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)) \longrightarrow \mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)}). \quad (\text{A.1})$$

**Proposition A.3.** *Suppose that*

- (1) *The Grothendieck group  $K_0(Y)$  is torsion-free,*
- (2) *The product map  $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times Y)$  is an isomorphism.*

*Then the sequence (A.1) is exact.*

*Proof.* It follows from the assumptions that the map  $K_0(Y) \longrightarrow K_0(Y_{F(X)})$  is injective and the kernel of the natural homomorphism  $K_0(X \times Y) \longrightarrow K_0(Y_{F(X)})$  coincides with

$$I_0(X) \otimes K_0(Y),$$

where  $I_0(X)$  is the kernel of the rank homomorphism  $K_0(X) \longrightarrow \mathbb{Z}$ .

The kernel of the second homomorphism in the sequence (A.1) is generated by the classes of closed integral subschemes  $Z \subset X \times Y$  that are not dominant over  $X$ . By Riemann-Roch (see [2]), we have  $[Z] = -c_2([O_Z])$  in  $\mathrm{CH}^2(X \times Y)$ , where  $c_i : K_0(X \times Y) \longrightarrow \mathrm{CH}^i(X \times Y)$  is the  $i$ -th Chern class map. As

$$[O_Z] \in \mathrm{Ker}(K_0(X \times Y) \longrightarrow K_0(Y_{F(X)})) = I_0(X) \otimes K_0(Y),$$

it suffices to show that  $c_2(I_0(X) \otimes K_0(Y))$  is contained in the image  $M$  of the first map in the sequence (A.1).

The formula  $c_2(x + y) = c_2(x) + c_1(x)c_1(y) + c_2(y)$  shows that it suffices to prove that for all  $a, a' \in I_0(X)$  and  $b, b' \in K_0(Y)$ , the elements  $c_1(ab) \cdot c_1(a'b')$  and  $c_2(ab)$  are contained in  $M$ . This follows from the formulas (see [1, Remark 3.2.3 and Example 14.5.2]):  $c_1(ab) = mc_1(a) + nc_1(b)$  and

$$c_2(ab) = \frac{m^2 - m}{2}c_1(a)^2 + mc_2(a) + (nm - 1)c_1(a)c_1(b) + \frac{n^2 - n}{2}c_1(b)^2 + nc_2(b),$$

where  $n = \mathrm{rank}(a)$  and  $m = \mathrm{rank}(b)$ .  $\square$

**Example A.4.** If  $Y$  is a projective homogeneous variety, then the condition (1) holds by [6]. If  $X$  is a projective homogeneous variety of a semisimple algebraic group  $G$  over  $F$  and the Tits algebras of  $G$  are split, then it follows from [6] that the condition (2) also holds for any  $Y$ . For example, if the even Clifford algebra of a nondegenerate quadratic form is split, then the corresponding projective quadric  $X$  satisfies (2) for any  $Y$ .

For any field extension  $K/F$ , let  $K^s$  denote the subfield of elements that are algebraic and separable over  $F$ .

**Proposition A.5.** *Suppose that for every field extension  $K/F$  we have:*

- (1) *The natural map  $\mathrm{CH}^1(X) \rightarrow \mathrm{CH}^1(X_K)$  is an isomorphism,*
- (2) *The natural map  $\mathrm{CH}^1(Y_{K^s}) \rightarrow \mathrm{CH}^1(Y_K)$  is an isomorphism.*

*Then the sequence (A.1) is exact.*

*Proof.* Consider the spectral sequence

$$E_1^{p,q}(F) = \coprod_{x \in X^{(p)}} A^q(Y_{F(x)}, K_{2-p}) \implies A^{p+q}(X \times Y, K_2) \quad (\text{A.2})$$

for the projection  $X \times Y \rightarrow X$ . The nonzero terms of the first page are the following:

$$\mathrm{CH}^2(Y_{F(X)})$$

$$A^1(Y_{F(X)}, K_2) \longrightarrow \coprod_{x \in X^{(1)}} \mathrm{CH}^1(Y_{F(x)})$$

$$A^0(Y_{F(X)}, K_2) \longrightarrow \coprod_{x \in X^{(1)}} A^0(Y_{F(x)}, K_1) \longrightarrow \coprod_{x \in X^{(2)}} \mathrm{CH}^0(Y_{F(x)}).$$

As in the proof of Proposition A.1, we have  $E_2^{2,0}(F) = \mathrm{CH}^2(X)$ . For a field extension  $K/F$ , write  $C(K)$  for the factor group

$$\mathrm{Ker}(\mathrm{CH}^2(X_K \times Y_K) \rightarrow \mathrm{CH}^2(Y_{K(X)})) / \mathrm{Im}(\mathrm{CH}^2(X_K) \rightarrow \mathrm{CH}^2(X_K \times Y_K)).$$

The spectral sequence (A.2) for the varieties  $X_K$  and  $Y_K$  over  $K$  yields an isomorphism  $C(K) \simeq E_2^{1,1}(K)$ . We have a natural composition

$$\mathrm{CH}^1(X_K) \otimes \mathrm{CH}^1(Y_K) \rightarrow E_1^{1,1}(K) \rightarrow E_2^{1,1}(K) \simeq C(K).$$

We claim that the group  $C(F)$  is generated by images of the compositions

$$\mathrm{CH}^1(X_K) \otimes \mathrm{CH}^1(Y_K) \rightarrow C(K) \xrightarrow{N_{K/F}} C(F)$$

over all finite separable field extensions  $K/F$  (here  $N_{K/F}$  is the norm map for the extension  $K/F$ ).

The group  $C(F)$  is generated by images of the maps

$$\varphi_x : \mathrm{CH}^1(Y_{F(x)}) \longrightarrow E_2^{1,1}(F) \simeq C(F)$$

over all points  $x \in X^{(1)}$ . Pick such a point  $x$  and let  $K := F(x)^s$  be the subfield of elements that are separable over  $F$ . Then  $K/F$  is a finite separable field extension. Let  $x' \in X_K^{(1)}$  be a point over  $x$  such that  $K(x') \simeq F(x)$ . Then  $\varphi_x$  coincides with the composition

$$\mathrm{CH}^1(Y_{K(x')}) \longrightarrow C(K) \xrightarrow{N_{K/F}} C(F).$$

By assumption, the map  $\mathrm{CH}^1(Y_K) \longrightarrow \mathrm{CH}^1(Y_{K(x')})$  is an isomorphism, hence the image of  $\varphi_x$  coincides with the image of

$$[x'] \otimes \mathrm{CH}^1(Y_K) \longrightarrow C(K) \xrightarrow{N_{K/F}} C(F),$$

whence the claim.

As  $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$  is an isomorphism for every field extension  $K/F$ , the projection formula shows that the map  $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y) \longrightarrow C(F)$  is surjective. The statement follows.  $\square$

**Example A.6.** Let  $Y$  be a projective homogeneous variety with the  $F$ -algebras  $E$  and  $A$  as in Example A.2. If  $A$  is split, then  $\mathrm{CH}^1(Y_K) = K_0(E \otimes K)$  for every field extension  $K/F$ . As  $K^s$  is separably closed in  $K$ , the natural map  $K_0(E \otimes K^s) \longrightarrow K_0(E \otimes K)$  is an isomorphism, therefore, the condition (2) holds.

Write  $\widetilde{\mathrm{CH}}^2(X \times Y)$  for the cokernel of the product map  $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y) \longrightarrow \mathrm{CH}^2(X \times Y)$ . We have the following commutative diagram:

$$\begin{array}{ccccccc} & & & & \mathrm{CH}^2(X) & & \\ & & & & \downarrow & \searrow & \\ & & & & & & \mathrm{CH}^2(X_{F(Y)}) \\ 0 & \longrightarrow & \mathrm{CH}^2(Y) & \longrightarrow & \widetilde{\mathrm{CH}}^2(X \times Y) & \longrightarrow & \\ & & & \searrow & \downarrow & & \\ & & & & \mathrm{CH}^2(Y_{F(X)}) & & \end{array}$$

Proposition A.1 gives conditions for the exactness of the row in the diagram and Propositions A.3 and A.5 - for the exactness of the column in the diagram.

A diagram chase yields together with Propositions A.1, A.3 and A.5 yields the following statements.

**Theorem A.7.** *Let  $X$  and  $Y$  be smooth complete geometrically irreducible varieties such that for every field extension  $K/F$ :*

- (1) *The natural map  $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$  is an isomorphism of torsion free groups,*
- (2) *The natural map  $\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(X_K)$  is injective,*

- (3) The product map  $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$  is an isomorphism,
- (4) The Grothendieck group  $K_0(Y)$  is torsion-free,
- (5) The product map  $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times Y)$  is an isomorphism.

Then the natural map  $\mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)})$  is injective.

**Remark A.8.** The conditions (1) – (3) hold for a smooth projective quadric  $X$  of dimension at least 7 by [3, Theorem 6.1] and Example A.2. By Example A.4, the conditions (4) and (5) hold if the even Clifford algebra of  $X$  is split and  $Y$  is a projective homogeneous variety.

**Theorem A.9.** Let  $X$  and  $Y$  be smooth complete geometrically irreducible varieties such that for every field extension  $K/F$ :

- (1) The natural map  $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$  is an isomorphism of torsion free groups,
- (2) The natural map  $\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(X_K)$  is injective,
- (3) The product map  $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$  is an isomorphism,
- (4) The natural homomorphism  $\mathrm{CH}^1(Y_{K^s}) \rightarrow \mathrm{CH}^1(Y_K)$  is an isomorphism.

Then the natural map  $\mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)})$  is injective.

**Remark A.10.** The conditions (1) – (3) hold for a smooth projective quadric  $X$  of dimension at least 7 and a projective homogeneous variety  $Y$  with the split Azumaya algebra by Remark A.8 and Example A.6.

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