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AN EXPLICIT FORMULA FOR THE CUBIC SZEGŐ EQUATION

PATRICK GÉRARD AND SANDRINE GRELLIER

Abstract. We derive an explicit formula for the general solution of the cubic Szegő equation and of the evolution equation of the corresponding hierarchy. As an application, we prove that all the solutions corresponding to finite rank Hankel operators are quasiperiodic.

1. Introduction

This paper is a continuation of the study of dynamical properties of an integrable system introduced by the authors in [2], [3]. As an evolution equation, the cubic Szegő equation is a simple model of non dispersive dynamics. More precisely, it can be identified as a first order Birkhoff normal form for a certain nonlinear wave equation, see [4]. As an Hamiltonian equation, it was proved in [2] to admit a Lax pair and finite dimensional invariant submanifolds corresponding to some finite rank conditions. In [3], action angle variables were introduced on generic subsets of the phase space, and on open dense subsets of the finite rank submanifolds. However, unlike the KdV equation or the one dimensional cubic nonlinear Schrödinger equation, this integrable system displays some degeneracy, since the collection of its conservation laws do not control the high regularity of the solution, as observed in [2]. An important consequence of this instability phenomenon is that the action angle variables cannot be extended to the whole phase space, even when restricted to one of the finite rank submanifolds. Our purpose in this paper is to prove a formula for the general solution of the initial value problem for this equation. In the case of generic data, this formula reduces to the one given by the action angle variables above. However, the formula enables to study the non generic case too, and allows in particular to establish the quasiperiodicity of all solutions lying in one of the above finite rank submanifolds, despite the already

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mentioned lack of a global system of action–angle variables. Finally, this formula is also very useful to revisit the instability phenomenon displayed in [2]. We now introduce the general setting of this equation.

1.1. **The setting.** Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, endowed with the Haar integral

$$\int_{\mathbb{T}} f := \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx \ .$$

On $L^2(\mathbb{T})$, we use the inner product

$$(f|g) := \int_{\mathbb{T}} f\overline{g} \ .$$

The family of functions $(\mathrm{e}^{ikx})_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$, on which the components of $f \in L^2(\mathbb{T})$ are the Fourier coefficients

$$\hat{f}(k) := (f|\mathrm{e}^{ikx}) \ .$$

We introduce the closed subspace

$$L^2_+(\mathbb{T}) := \{u \in L^2(\mathbb{T}) : \forall k < 0, \hat{u}(k) = 0\} \ .$$

Notice that elements $u \in L^2_+(\mathbb{T})$ identify to traces of holomorphic functions $\underline{u}$ on the unit disc D such that

$$\sup_{r<1} \int_0^{2\pi} |\underline{u}(r\mathrm{e}^{ix})|^2 dx < \infty \ ,$$

via the correspondence

$$\underline{u}(z) := \sum_{k=0}^{\infty} \hat{u}(k) z^k \ , \ z \in D \ , \ u(x) = \lim_{r\to 1} \underline{u}(r\mathrm{e}^{ix}) \ ,$$

which establishes a bijective isometry between $L^2_+(\mathbb{T})$ and the Hardy space of the disc.

We denote by Π the orthogonal projector from $L^2(\mathbb{T})$ onto $L^2_+(\mathbb{T})$, known as the Szegő projector,

$$\Pi\left(\sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{ikx}\right) = \sum_{k=0}^{\infty} \hat{f}(k)\mathrm{e}^{ikx} \ .$$

On $L^2_+(\mathbb{T})$, we introduce the symplectic form

$$\omega(h_1, h_2) = \mathrm{Im}(h_1|h_2) \ .$$

The densely defined energy functional

$$E(u) := \frac{1}{4} \int_{\mathbb{T}} |u|^4 \ ,$$

formally corresponds to the Hamiltonian evolution equation,

$$i\dot{u} = \Pi(|u|^2 u) \ , \tag{1}$$

which we called the cubic Szegő equation. In [2], we solved the initial value problem for this equation on the intersections of Sobolev spaces with $L^2_+(\mathbb{T})$. More precisely, define, for $s \geq 0$,

$$H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T}) = \{u \in L^2_+(\mathbb{T}) : \sum_{k=0}^{\infty} |\hat{u}(k)|^2 (1+k^2)^s < \infty\} \ .$$

Then equation (1) defines a smooth flow on $H^s_+(\mathbb{T})$ for $s > \frac{1}{2}$, and a continuous flow on $H^{\frac{1}{2}}_+(\mathbb{T})$. The main result of this paper provides an explicit formula for the solution of this initial value problem.

1.2. **Hankel operators and the explicit formula.** Let $u \in H^{\frac{1}{2}}_+(\mathbb{T})$. We denote by H_u the $\mathbb{C}$–antilinear operator defined on $L^2_+(\mathbb{T})$ as

$$H_u(h) = \Pi(u\overline{h}) \ , \ h \in L^2_+(\mathbb{T}) \ .$$

In terms of Fourier coefficients, this operator reads

$$\widehat{H_u(h)}(n) = \sum_{p=0}^{\infty} \hat{u}(n+p)\overline{\hat{h}(p)} \ .$$

In particular, its Hilbert–Schmidt norm is finite since $u \in H^{\frac{1}{2}}_+(\mathbb{T})$. We call H_u the Hankel operator of symbol u. Notice that this definition is different from the standard ones used in references [9], [11], where Hankel operators are rather defined as linear operators from L^2_+ into its orthogonal complement. The link between these two definitions can be easily established by means of the involution

$$f^\sharp(x) = \mathrm{e}^{-ix}\overline{f(x)} \ .$$

Notice that, with our definition, H_u satisfies the following self adjointness identity,

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1) \ , \ h_1, h_2 \in L^2_+(\mathbb{T}) \ . \tag{2}$$

A fundamental property of Hankel operators is their connection with the shift operator S, defined on $L^2_+(\mathbb{T})$ as

$$Su(x) = \mathrm{e}^{ix}u(x) \ .$$

This property reads

$$S^* H_u = H_u S = H_{S^* u} \ ,$$

where S^* denotes the adjoint of S. We denote by K_u this operator, and call it the shifted Hankel operator of symbol u. Notice that K_u

is Hilbert–Schmidt and self adjoint as well. As a consequence, operators H_u^2 and K_u^2 are $\mathbb{C}$–linear trace class positive operators on $L^2_+(\mathbb{T})$. Moreover, they are related by the following important identity,

$$K_u^2 = H_u^2 - (\cdot|u)u \ . \tag{3}$$

Theorem 1. *Let $u_0 \in H^{\frac{1}{2}}_+(\mathbb{T})$, and $u \in C(\mathbb{R}, H^{\frac{1}{2}}_+(\mathbb{T}))$ be the solution of equation (1) such that $u(0) = u_0$. Then*

$$\underline{u}(t,z) = ((I - z\mathrm{e}^{-itH^2_{u_0}}\mathrm{e}^{itK^2_{u_0}}S^*)^{-1}\mathrm{e}^{-itH^2_{u_0}}u_0 \,|\, 1) \ .$$

The proof of this theorem will be given in section 3. It is a non trivial consequence of the Lax pair structure recalled in section 2. Our second result concerns the special case of data u_0 such that H_{u_0} is of finite rank. In this case, operators $S^*, H^2_{u_0}, K^2_{u_0}$ act on a finite dimensional space containing u_0, and the implementation of the above formula reduces to diagonalization of matrices.

1.3. **Finite rank manifolds and quasiperiodicity.** Let d be a positive integer. We denote by $\mathcal{V}(d)$ the set of $u \in H^{\frac{1}{2}}_+(\mathbb{T})$ such that

$$\mathrm{rk}H_u = \left[\frac{d+1}{2}\right] \ , \ \mathrm{rk}K_u = \left[\frac{d}{2}\right] \ ,$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Using Kronecker's theorem [6], one can show that $\mathcal{V}(d)$ is a complex Kähler submanifold of $L^2_+(\mathbb{T})$ of dimension d — see the appendix of [2] —, consisting of rational functions of e^{ix}. More precisely, $\mathcal{V}(d)$ consists of functions of the form

$$u(x) = \frac{A(\mathrm{e}^{ix})}{B(\mathrm{e}^{ix})} \ ,$$

where A, B are polynomials with no common factors, B has no zero in the closed unit disc, $B(0) = 1$, and

- If $d = 2N$ is even, the degree of A is at most $N-1$ and the degree of B is exactly N.
- If $d = 2N+1$ is odd, the degree of A is exactly N and the degree of B is at most N.

Using the Lax pair structure recalled in section 2, $\mathcal{V}(d)$ is invariant through the flow of (1).

Theorem 2. *For every $u_0 \in \mathcal{V}(d)$, the map*

$$t \in \mathbb{R} \mapsto u(t) \in \mathcal{V}(d)$$

is quasiperiodic. More precisely, there exist a positive integer n, real numbers $\omega_1, \cdots, \omega_n$, and a smooth mapping

$$\Phi : \mathbb{T}^n \to \mathcal{V}(d)$$

such that, for every $t \in \mathbb{R}$,

$$u(t) = \Phi(\omega_1 t, \cdots, \omega_n t) \ .$$

In particular, for every $s > \frac{1}{2}$,

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} < +\infty \ . \tag{4}$$

Notice that property (4) was established in Theorem 7.1 of [2] under the additional generic assumption that u_0 belongs to $\mathcal{V}(d)_{\text{gen}}$, namely that the vectors $H_{u_0}^{2n}(1), n = 1, \dots, N = \left[\frac{d+1}{2}\right]$, are linearly independent. Our general formula allows us to extend property (4) to all data in $\mathcal{V}(d)$. However, it should be emphasized that, while it is clear from the arguments of Lemma 5 in [2] that estimate (4) is uniform if u_0 varies in a compact subset of $\mathcal{V}(d)_{\text{gen}}$, (4) does not follow from an a priori estimate on the whole of $\mathcal{V}(d)$, in the sense that one can find families of data (u_0^ε) in $\mathcal{V}(d)$, belonging to a compact subset of $\mathcal{V}(d)$, in particular bounded in all H^s, and such that

$$\sup_\varepsilon \sup_{t\in\mathbb{R}} \|u^\varepsilon(t)\|_{H^s} = \infty \ , \ s > \frac{1}{2} \ ,$$

see corollary 5 of [2]. We shall revisit this phenomenon in section 4 thanks to the explicit formula of Theorem 1.

Finally, let us mention that the generalization of property (4) to non finite rank solutions is an open problem.

1.4. **Organization of the paper.** Section 2 is devoted to recalling the crucial Lax pair structure attached to equation (1). As a fundamental consequence, $H_{u(t)}$ and $K_{u(t)}$ remain unitarily equivalent to their respective initial data. In section 3, we take advantage of this structure to derive Theorem 1. In section 4, we apply this theorem to the particular case of data u_0 belonging to $\mathcal{V}(3)$, which sheds a new light on the instability phenomenon. The next two sections are devoted to the proof of Theorem 2. As a preparation, we first generalize the explicit formula to Hamiltonian flows associated to energies

$$J^y(u) := ((I + yH_u^2)^{-1}(1)|1) \ ,$$

where y is a positive parameter. The quasi periodicity theorem then follows by observing, through an interpolation argument, that the map Φ in the statement of Theorem 2 can be defined as the value at time 1 of the Hamiltonian flow corresponding to a suitable linear combination of energies J^y.

2. The Lax pair structure

In this section, we recall the Lax pairs associated to the cubic Szegő equation, see [2], [3]. First we introduce the notion of a Toeplitz operator. Given $b \in L^\infty(\mathbb{T})$, we define $T_b : L^2_+ \to L^2_+$ as

$$T_b(h) = \Pi(bh) \ , \ h \in L^2_+ \ .$$

Notice that T_b is bounded and $T_b^* = T_{\bar{b}}$. The starting point is the following lemma.

Lemma 1. *Let $a, b, c \in H^s_+$, $s > \frac{1}{2}$. Then*

$$H_{\Pi(a\bar{b}c)} = T_{a\bar{b}} H_c + H_a T_{b\bar{c}} - H_a H_b H_c \ .$$

Proof. Given $h \in L^2_+$, we have

$$\begin{aligned} H_{\Pi(a\bar{b}c)}(h) &= \Pi(a\bar{b}c\bar{h}) = \Pi(a\bar{b}\Pi(c\bar{h})) + \Pi(a\bar{b}(I-\Pi)(c\bar{h})) \\ &= T_{a\bar{b}} H_c(h) + H_a(g) \ , \ g := b\overline{(I-\Pi)(c\bar{h})} \ . \end{aligned}$$

Since $g \in L^2_+$,

$$g = \Pi(g) = \Pi(b\bar{c}h) - \Pi(b\overline{\Pi(c\bar{h})}) = T_{b\bar{c}}(h) - H_b H_c(h) \ .$$

This completes the proof. □

Using Lemma 1 with $a = b = c = u$, we get

$$H_{\Pi(|u|^2 u)} = T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3 \ . \tag{5}$$

Theorem 3. *Let $u \in C^\infty(\mathbb{R}, H^s_+), s > \frac{1}{2}$, be a solution of (1). Then*

$$\begin{aligned} \frac{dH_u}{dt} &= [B_u, H_u] \ , \ B_u := \frac{i}{2} H_u^2 - i T_{|u|^2} \ , \\ \frac{dK_u}{dt} &= [C_u, K_u] \ , \ C_u := \frac{i}{2} K_u^2 - i T_{|u|^2} \ . \end{aligned}$$

Proof. Using equation (1) and identity (5),

$$\frac{dH_u}{dt} = H_{-i\Pi(|u|^2 u)} = -i H_{\Pi(|u|^2 u)} = -i(T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3) \ .$$

Using the antilinearity of H_u, this leads to the first identity. For the second one, we observe that

$$K_{\Pi(|u|^2 u)} = H_{\Pi(|u|^2 u)} S = T_{|u|^2} H_u S + H_u T_{|u|^2} S - H_u^3 S \ . \tag{6}$$

Moreover, notice that

$$T_b(Sh) = S T_b(h) + (bSh|1) \ .$$

In the case $b = |u|^2$, this gives

$$T_{|u|^2} Sh = S T_{|u|^2} h + (|u|^2 Sh|1) \ .$$

Moreover,

$$(|u|^2 Sh|1) = (u|u\overline{Sh}) = (u|K_u(h)) \ .$$

Consequently,

$$H_u T_{|u|^2} Sh = K_u T_{|u|^2} h + (K_u(h)|u)u \ .$$

Coming back to (6), we obtain

$$K_{\Pi(|u|^2 u)} = T_{|u|^2} K_u + K_u T_{|u|^2} - (H_u^2 - (\cdot|u)u) K_u \ .$$

Using identity (3), this leads to

$$K_{\Pi(|u|^2 u)} = T_{|u|^2} K_u + K_u T_{|u|^2} - K_u^3 \ . \tag{7}$$

The second identity is therefore a consequence of antilinearity and of

$$\frac{dK_u}{dt} = -iK_{\Pi(|u|^2u)} \ .$$

□

Observing that B_u, C_u are linear and antiselfadjoint, we obtain, following a classical argument due to Lax [7],

Corollary 1. *Under the conditions of Theorem 3, define $U = U(t)$, $V = V(t)$ the solutions of the following linear ODEs on $\mathcal{L}(L^2_+)$,*

$$\frac{dU}{dt} = B_u U \ , \ \frac{dV}{dt} = C_u V \ , \ U(0) = V(0) = I \ .$$

Then $U(t), V(t)$ are unitary operators and

$$H_{u(t)} = U(t)H_{u(0)}U(t)^* \ , \ K_{u(t)} = V(t)K_{u(0)}V(t)^* \ .$$

3. Proof of the formula

In this section, we prove Theorem 1. Our starting point is the following identity, valid for every $v \in L^2_+$,

$$\underline{v}(z) = ((I - zS^*)^{-1}v|1) \ , \ z \in D \ . \tag{8}$$

Indeed, the Taylor coefficient of order n of the right hand side at $z = 0$ is

$$((S^*)^n v|1) = (v|S^n 1) = \hat{v}(n) \ ,$$

which coincides with the Taylor coefficient of order n of the left hand side. Let $u \in C^\infty(\mathbb{R}, H^s_+)$ be a solution of (1), $s > \frac{1}{2}$. Applying (8) to $v = u(t)$ and using the unitarity of $U(t)$, we get

$$\underline{u}(t,z) = ((I - zS^*)^{-1}u(t)|1) = (U(t)^*(I - zS^*)^{-1}u(t)|U(t)^*1) \ ,$$

which yields

$$\underline{u}(t,z) = ((I - zU(t)^*S^*U(t))^{-1}U(t)^*u(t)|U(t)^*1) \ . \tag{9}$$

We shall identify successively $U(t)^*1, U(t)^*u(t)$, and the restriction of $U(t)^*S^*U(t)$ on the range of H_{u_0}. We begin with $U(t)^*1$,

$$\frac{d}{dt}U(t)^*1 = -U(t)^*B_u(1) \ ,$$

and

$$B_u(1) = \frac{i}{2}H^2_u(1) - iT_{|u|^2}(1) = -\frac{i}{2}H^2_u(1) \ .$$

Hence

$$\frac{d}{dt}U(t)^*1 = \frac{i}{2}U(t)^*H^2_u(1) = \frac{i}{2}H^2_{u_0}U(t)^*1 \ ,$$

where we have used corollary 1. This yields

$$U(t)^*1 = \mathrm{e}^{i\frac{t}{2}H^2_{u_0}}(1) \ . \tag{10}$$

Consequently,

$$U(t)^*u(t) = U(t)^*H_{u(t)}(1) = H_{u_0}U(t)^*(1) = H_{u_0}\mathrm{e}^{i\frac{t}{2}H_{u_0}^2}(1) \ ,$$

and therefore

$$U(t)^*u(t) = \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}(u_0) \ . \tag{11}$$

Finally,

$$U(t)^*S^*U(t)H_{u_0} = U(t)^*S^*H_{u(t)}U(t) = U(t)^*K_{u(t)}U(t) \ ,$$

and therefore

$$U(t)^*S^*U(t)H_{u_0} = U(t)^*V(t)K_{u_0}V(t)^*U(t) \ . \tag{12}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}U(t)^*V(t) &= -U(t)^*B_{u(t)}V(t) + U(t)^*C_{u(t)}V(t) = U(t)^*(C_{u(t)} - B_{u(t)})V(t) \\ &= \frac{i}{2}U(t)^*(K_{u(t)}^2 - H_{u(t)}^2)V(t) = \frac{i}{2}(U(t)^*V(t)K_{u_0}^2 - H_{u_0}^2U(t)^*V(t)) \ .\end{aligned}$$

We infer

$$U(t)^*V(t) = \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{i\frac{t}{2}K_{u_0}^2} \ .$$

Plugging this identity into (12), we obtain

$$\begin{aligned}U(t)^*S^*U(t)H_{u_0} &= \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{i\frac{t}{2}K_{u_0}^2}K_{u_0}\mathrm{e}^{-i\frac{t}{2}K_{u_0}^2}\mathrm{e}^{i\frac{t}{2}H_{u_0}^2} \\ &= \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}K_{u_0}\mathrm{e}^{i\frac{t}{2}H_{u_0}^2} \\ &= \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*H_{u_0}\mathrm{e}^{i\frac{t}{2}H_{u_0}^2} \\ &= \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*\mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}H_{u_0} \ .\end{aligned}$$

We conclude that, on the range of H_{u_0},

$$U(t)^*S^*U(t) = \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*\mathrm{e}^{-i\frac{t}{2}H_{u_0}^2} \ . \tag{13}$$

It remains to plug identities (10), (11), (13) into (9). We finally obtain

$$\begin{aligned}\underline{u}(t,z) &= ((I - z\mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*\mathrm{e}^{-i\frac{t}{2}H_{u_0}^2})^{-1}\mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}(u_0)|\mathrm{e}^{i\frac{t}{2}H_{u_0}^2}(1)) \\ &= ((I - z\mathrm{e}^{-itH_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*)^{-1}\mathrm{e}^{-itH_{u_0}^2}(u_0)|1) \ ,\end{aligned}$$

which is the claimed formula in the case of data $u_0 \in H_+^s, s > \frac{1}{2}$. The case $u_0 \in H_+^{\frac{1}{2}}$ follows by a simple approximation argument. Indeed, we know from [2], Theorem 2.1, that, for every $t \in \mathbb{R}$, the mapping $u_0 \mapsto u(t)$ is continuous on $H_+^{\frac{1}{2}}$. On the other hand, the maps $u_0 \mapsto H_{u_0}, K_{u_0}$ are continuous from $H_+^{\frac{1}{2}}$ into $\mathcal{L}(L_+^2)$. Since $H_{u_0}^2, K_{u_0}^2$ are selfadjoint, the operator

$$\mathrm{e}^{-itH_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*$$

has norm at most 1. Hence, for $z \in D$, the right hand side of the formula is continuous from $H_+^{\frac{1}{2}}$ into $\mathbb{C}$.

4. An example

This section is devoted to revisiting sections 6.1, 6.2 of [2] by means of the explicit formula. Given $\varepsilon \in \mathbb{R}$, we define

$$u_0^\varepsilon(x) = \mathrm{e}^{ix} + \varepsilon \ .$$

It is easy to check that $u_0^\varepsilon \in \mathcal{V}(3)$, hence the corresponding solution u^ε of (1) is valued in $\mathcal{V}(3)$, and consequently reads

$$u^\varepsilon(t,x) = \frac{a^\varepsilon(t)\mathrm{e}^{ix} + b^\varepsilon(t)}{1 - p^\varepsilon(t)\mathrm{e}^{ix}} \ ,$$

with $a^\varepsilon(t) \in \mathbb{C}^*, b^\varepsilon(t) \in \mathbb{C}, p^\varepsilon(t) \in D, a^\varepsilon(t) + b^\varepsilon(t)p^\varepsilon(t) \neq 0$. We are going to calculate these functions explicitly. We start with the special case $\varepsilon = 0$. In this case, $|u_0^0| = 1$, hence

$$u^0(t,x) = \mathrm{e}^{-it}u_0^0(x)$$

so

$$a^0(t) = \mathrm{e}^{-it} \ , \ b^0(t) = 0 \ , \ p^0(t) = 0 \ .$$

We come to $\varepsilon \neq 0$. The operators $H_{u_0}^2, K_{u_0}^2, S^*$ act on the range of $H_{u_0^\varepsilon}$, which is the two dimensional vector space spanned by $1, \mathrm{e}^{ix}$. In this basis, the matrices of these three operators are respectively

$$\mathcal{M}(H_{u_0}^2) = \begin{pmatrix} 1+\varepsilon^2 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \ , \ \mathcal{M}(K_{u_0}^2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ \mathcal{M}(S^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ .$$

The eigenvalues of $H_{u_0}^2$ are

$$\rho_\pm^2 = 1 + \frac{\varepsilon^2}{2} \pm \varepsilon\sqrt{1 + \frac{\varepsilon^2}{4}} \ ,$$

hence the matrix of the exponential is given by

$$\begin{aligned} \mathcal{M}\left(\mathrm{e}^{-itH_{u_0}^2}\right) &= \frac{\mathrm{e}^{-it\rho_+^2} - \mathrm{e}^{-it\rho_-^2}}{\rho_+^2 - \rho_-^2}\mathcal{M}(H_{u_0}^2) + \frac{\rho_-^2\mathrm{e}^{-it\rho_+^2} - \rho_+^2\mathrm{e}^{-it\rho_-^2}}{\rho_-^2 - \rho_+^2} I \\ &= \frac{\mathrm{e}^{-i\Omega t}}{2\omega}\left(-2i\sin(\omega t)\,\mathcal{M}(H_{u_0}^2) + (2\omega\cos(\omega t) + 2i\Omega\sin(\omega t))I\right) \\ \text{where } \ \omega &:= \varepsilon\sqrt{1+\frac{\varepsilon^2}{4}} \ , \ \Omega := 1 + \frac{\varepsilon^2}{2}. \end{aligned}$$

We obtain

$$\mathrm{e}^{-itH_{u_0}^2}(u_0) = \frac{\mathrm{e}^{-i\Omega t}}{2\omega}\left(-2i\varepsilon\Omega\sin(\omega t) + 2\varepsilon\omega\cos(\omega t) + (2\omega\cos(\omega t) - i\varepsilon^2\sin(\omega t))\mathrm{e}^{ix}\right) \ ,$$

$$\mathcal{M}\left(\mathrm{e}^{-itH_{u_0}^2}\mathrm{e}^{itK_{u_0}^2}S^*\right) = \frac{\mathrm{e}^{-it\frac{\varepsilon^2}{2}}}{2\omega}\begin{pmatrix} 0 & 2\omega\cos(\omega t) - i\varepsilon^2\sin(\omega t) \\ 0 & -2i\varepsilon\sin(\omega t) \end{pmatrix} \ ,$$

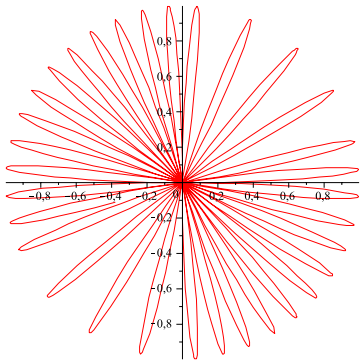

FIGURE 4.1. The trajectory of p^ε for small ε.

and finally

$$\begin{aligned} a^\varepsilon(t) &= \mathrm{e}^{-it(1+\varepsilon^2)} \ , \ b^\varepsilon(t) = \mathrm{e}^{-it(1+\varepsilon^2/2)} \left(\varepsilon \cos(\omega t) - i \frac{2+\varepsilon^2}{\sqrt{4+\varepsilon^2}} \sin(\omega t) \right) \\ p^\varepsilon(t) &= -\frac{2i}{\sqrt{4+\varepsilon^2}} \sin(\omega t)\, \mathrm{e}^{-it\varepsilon^2/2} \ , \ \omega := \frac{\varepsilon}{2}\sqrt{4+\varepsilon^2} \ . \end{aligned}$$

The important feature of such dynamics concerns the regime $\varepsilon \to 0$. Though $p^0(t) \equiv 0$, $p^\varepsilon(t)$ may visit small neighborhoods of the unit circle at large times. Specifically, at time $t^\varepsilon = \pi/(2\omega) \sim \pi/(2\varepsilon)$, we have $|p^\varepsilon(t)| \sim 1 - \varepsilon^2$. A consequence is that the momentum density,

$$\begin{aligned} \mu_n(t^\varepsilon) &:= n|\hat{u}^\varepsilon(t^\varepsilon, n)|^2 = n|a^\varepsilon(t^\varepsilon) + b^\varepsilon(t^\varepsilon)p^\varepsilon(t^\varepsilon)|^2 |p^\varepsilon(t^\varepsilon)|^{2(n-1)} \\ &= n\frac{\varepsilon^4}{(4+\varepsilon^2)^2}\left(1 - \frac{\varepsilon^2}{4+\varepsilon^2}\right)^{n-1} , \end{aligned}$$

which satisfies

$$\sum_{n=1}^{\infty} \mu_n(t^\varepsilon) = Tr(K^2_{u^\varepsilon(t^\varepsilon)}) = Tr(K^2_{u_0^\varepsilon}) = 1 \ ,$$

becomes concentrated at high frequencies

$$n \simeq \frac{1}{\varepsilon^2} \ .$$

This induces the following instability of H^s norms

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \simeq \frac{1}{\varepsilon^{2s-1}} \ , \ s > \frac{1}{2} \ ,$$

a phenomenon of the same nature as the one displayed by Colliander, Keel, Staffilani, Takaoka and Tao in [1]. This proves in particular that

conservation laws do not control H^s regularity for $s > \frac{1}{2}$. Notice that, as already mentioned at the end of subsection 1.3 of the introduction, the family (u_0^ε) approaches u_0^0, which is a non generic element of $\mathcal{V}(3)$, since $H^2_{u_0}$ admits 1 as a double eigenvalue.

This example naturally leads to the question of large time behavior of the H^s norm of individual solutions for $s > \frac{1}{2}$. We are going to answer this question in the special case of finite rank solutions by proving the quasi periodicity theorem in the next two sections.

5. Generalization to the Szegő hierarchy

The Szegő hierarchy was introduced in [2] and used in [3]. For the convenience of the reader, and because our notation is slightly different, we shall recall the main facts here. For $y > 0$ and $u \in H^{\frac{1}{2}}_+$, we set

$$J^y(u) = ((I + yH_u^2)^{-1}(1)|1) \ .$$

Notice that the connection with the Szegő equation is made by

$$E(u) = \frac{1}{4}(\partial_y^2 J^y_{|y=0} - (\partial_y J^y_{|y=0})^2) \ .$$

For every $s > \frac{1}{2}$, J^y is a smooth real valued function on H^s_+, and its Hamiltonian vector field is given by

$$X_{J^y}(u) = 2iyw^y H_u w^y \ , \ w^y := (I + yH_u^2)^{-1}(1) \ ,$$

which is a Lipschitz vector field on bounded subsets of H^s_+. This fact is a consequence of the following lemma, where we collect basic estimates. We recall that the Wiener algebra W is the space of $f \in L^2_+$ such that

$$\|f\|_W := \sum_{k=0}^{\infty} |\hat{f}(k)| < \infty \ .$$

Lemma 2. *Let $f, u, v \in L^2_+$.*

$$\begin{aligned}
\|H_u f\|_W &\leq \|u\|_W \|f\|_W \ , \\
\|H_u f\|_{H^{s-\frac{1}{2}}} &\leq \|u\|_{H^s} \|f\|_{L^2} \ , \ s \geq \frac{1}{2} \ , \\
\|H_u f\|_{H^s} &\leq \|u\|_{H^s} \|f\|_W \ , \ s \geq 0 \ , \\
\|w^y\|_{H^s} &\leq (1 + y\|u\|^2_{H^s}) \ , \ s > 1 \ , \\
\|fg\|_{H^s} &\leq C_s(\|f\|_W \|g\|_{H^s} + \|g\|_W \|f\|_{H^s}) \ , \\
\|X_{J^y}(u) - X_{J^y}(v)\|_{H^s} &\leq C_s(R, y)\|u - v\|_{H^s} \ , \ s > 1 \ , \|u\|_{H^s} + \|v\|_{H^s} \leq R \ .
\end{aligned}$$

Proof. The first three estimates are straightforward consequences of the formula

$$\widehat{H_u f}(k) = \sum_{\ell=0}^{\infty} \hat{u}(k + \ell)\overline{\hat{f}(\ell)} \ .$$

The fourth estimate comes from these estimates and the fact that

$$w^y = 1 - yH_u^2 w^y \ , \ \|w^y\|_{L^2} \leq 1 \ .$$

The fifth estimate is obtained by decomposing

$$\widehat{fg}(k) = \sum_{\ell=0}^{\infty} \hat{f}(k-\ell)\hat{g}(\ell) = \sum_{|k-\ell|\leq \ell} \hat{f}(k-\ell)\hat{g}(\ell) + \sum_{|k-\ell|>\ell} \hat{f}(k-\ell)\hat{g}(\ell) \ .$$

As for the last estimate, we set

$$w^y[u] := (I + yH_u^2)^{-1}(1) \ .$$

We write

$$\|w^y[u]-w^y[v]\|_{L^2} = y\|(I+yH_u^2)^{-1}(H_v^2-H_u^2)(I+yH_v^2)^{-1}(1)\|_{L^2} \leq yR\|u-v\|_{H^s} \ .$$

Then, by using again the first two inequalities,

$$w^y[u] - w^y[v] = y(H_v^2(w^y[v]) - H_u^2(w^y[u]))$$

leads to

$$\|w^y[u] - w^y[v]\|_{H^s} \leq C(R,y)\|u-v\|_{H^s} \ .$$

Using moreover the fact that H^s is an algebra, this yields the desired estimate. □

By the Cauchy–Lipschitz theorem, the evolution equation

$$\dot{u} = X_{J^y}(u) \tag{14}$$

admits local in time solutions for every initial data in H^s_+ for $s > 1$, and the lifetime is bounded from below if the data are bounded in H^s_+. We shall see that this evolution equation admits a Lax pair structure similar to the one in section 2.

Theorem 4. *For every $u \in H^s_+$, we have*

$$\begin{aligned} H_{iX_{J^y}(u)} &= H_u F_u^y + F_u^y H_u \ , \\ K_{iX_{J^y}(u)} &= K_u G_u^y + G_u^y K_u \ , \\ G_u^y(h) &:= -yw^y\,\Pi(\overline{w^y}\,h) + y^2 H_u w^y\,\Pi(\overline{H_u w^y}\,h) \ , \\ F_u^y(h) &:= G_u^y(h) - y^2(h|H_u w^y)H_u w^y \ . \end{aligned}$$

If $u \in C^\infty(\mathcal{I}, H^s_+)$ is a solution of equation (14) on a time interval $\mathcal{I}$, then

$$\begin{aligned} \frac{dH_u}{dt} &= [B_u^y, H_u] \ , \ \frac{dK_u}{dt} = [C_u^y, K_u] \ , \\ B_u^y &= -iF_u^y \ , \ C_u^y = -iG_u^y \ . \end{aligned}$$

Proof.

Lemma 3. *We have the following identity,*

$$H_{aH_u(a)}(h) = H_u(a)H_a(h) + H_u(a\Pi(\overline{a}h) - (h|a)a) \ .$$

Proof.

$$H_{aH_u(a)}(h) = \Pi(aH_u(a)\overline{h}) = H_u(a)H_a(h) + \Pi(H_u(a)(I-\Pi)(a\overline{h}))\ .$$

On the other hand,

$$(1-\Pi)(a\overline{h}) = \overline{\Pi(\overline{a}h)} - (a|h)\ .$$

The lemma follows by plugging the latter formula into the former one. □

Let us complete the proof. Using the identity

$$w^y = 1 - yH_u^2 w^y,$$

and Lemma 3 with $a = H_u(w^y)$, we get

$$\begin{aligned} & H_{w^y\, H_u(w^y)}(h) = H_{H_u(w^y)}(h) - yH_{H_u(w^y)H_u^2(w^y)}(h) \\ = \ & H_{H_u(w^y)}(h) - yH_u^2(w^y)H_{H_u(w^y)}(h) - yH_u\left(H_u(w^y)\Pi(\overline{H_u(w^y)}h) - (h|H_u(w^y))H_u(w^y)\right) \\ = \ & w^y\, H_{H_u(w^y)}(h) - yH_u\left(H_u(w^y)\Pi(\overline{H_u(w^y)}h) - (h|H_u(w^y))H_u(w^y)\right) \\ = \ & w^y\, \Pi(\overline{w^y}\, H_u h) - yH_u\left(H_u(w^y)\Pi(\overline{H_u(w^y)}h) - (h|H_u(w^y))H_u(w^y)\right)\ . \end{aligned}$$

We therefore have obtained

$$H_{w^y\, H_u(w^y)} = L_u^y H_u + H_u R_u^y$$

where L_u^y and R_u^y are the following self adjoint operators,

$$L_u^y(h) = w^y\, \Pi(\overline{w^y}\, h)\ ,\ R_u^y(h) = -y\left(H_u(w^y)\Pi(\overline{H_u(w^y)}h) - (h|H_u(w^y))H_u(w^y)\right)\ .$$

Consequently, since $H_{w^y\, H_u(w^y)}$ is self adjoint,

$$H_{w^y\, H_u(w^y)} = \frac{1}{2}(L_u^y + R_u^y)H_u + H_u\frac{1}{2}(L_u^y + R_u^y)\ .$$

Multiplying by $-2y$, we obtain the desired formula, since

$$F_u^y = -y(L_u^y + R_u^y)\ .$$

We now come to the second identity. From the first one, we get

$$K_{iX_{Jy}(u)} = H_{iX_{Jy}(u)}S = H_u F_u^y S + F_u^y K_u\ . \tag{15}$$

For every $h, v \in L^2_+$, we use

$$\Pi(\overline{v}Sh) = S\Pi(\overline{v}h) + (Sh|v)$$

and infer

$$\begin{aligned} F_u^y Sh &= -yw^y\, \Pi(\overline{w^y}\, Sh) + y^2 H_u w^y\, \Pi(\overline{H_u w^y}\, Sh) - y^2(Sh|H_u w^y)H_u w^y \\ &= SG_u^y h - y(Sh|w^y)w^y = SG_u^y h + y^2(Sh|H_u^2 w^y)w^y \\ &= SG_u^y h + y^2(H_u(w^y)|K_u(h))w^y\ , \end{aligned}$$

where we have used $w^y = 1 - yH_u^2 w^y$ again. Plugging this identity into (15), we obtain the claim.

The last formulae are straightforward consequences of the antilinearity of H_u and K_u. □

Using Theorem 4 in a similar way to section 2, we derive

Corollary 2. *Under the conditions of Theorem 4, assuming moreover $0 \in \mathcal{I}$, define $U^y = U^y(t)$, $V^y = V^y(t)$ the solutions of the following linear ODEs on $\mathcal{L}(L^2_+)$,*

$$\frac{dU^y}{dt} = B^y_u U^y \ , \ \frac{dV^y}{dt} = C^y_u V^y \ , \ U^y(0) = V^y(0) = I \ .$$

Then $U^y(t), V^y(t)$ are unitary operators and

$$H_{u(t)} = U^y(t) H_{u(0)} U^y(t)^* \ , \ K_{u(t)} = V^y(t) K_{u(0)} V^y(t)^* \ .$$

At this stage, we are going to generalize slightly the setting, for the needs of the next section. Let $y_1, \dots, y_n$ be positive numbers, and $a_1, \dots, a_n$ be real numbers. We consider the functional

$$\hat{J}(u) = \sum_{k=1}^n a_k J^{y_k}(u) = (f(H_u^2)1|1) \ , \ f(s) := \sum_{k=1}^n \frac{a_k}{1 + y_k s} \ ,$$

and the evolution equation

$$\dot{u} = X_{\hat{J}}(u) \ . \tag{16}$$

By linearity from Theorem 4, it is clear that the solution of (16) satisfies

$$\frac{dH_u}{dt} = [\hat{B}_u, H_u] \ , \ \frac{dK_u}{dt} = [\hat{C}_u, K_u] \ , \tag{17}$$

with

$$\hat{B}_u = \sum_{k=1}^n a_k B^{y_k}_u \ , \ \hat{C}_u = \sum_{k=1}^n a_k C^{y_k}_u \ . \tag{18}$$

Corollary 3. *Let u be a solution of equation (16) on some time interval $\mathcal{I}$ containing 0, define $\hat{U} = \hat{U}(t)$, $\hat{V} = \hat{V}(t)$ the solutions of the following linear ODEs on $\mathcal{L}(L^2_+)$,*

$$\frac{d\hat{U}}{dt} = \hat{B}_u \hat{U} \ , \ \frac{d\hat{V}}{dt} = \hat{C}_u \hat{V} \ , \ \hat{U}(0) = \hat{V}(0) = I \ .$$

Then $\hat{U}(t), \hat{V}(t)$ are unitary operators and

$$H_{u(t)} = \hat{U}(t) H_{u(0)} \hat{U}(t)^* \ , \ K_{u(t)} = \hat{V}(t) K_{u(0)} \hat{V}(t)^* \ .$$

As a consequence of this corollary, if we start from an initial datum $u(0)$ such that $H_{u(0)}$ is a trace class operator, then $H_{u(t)}$ is trace class for every t, with the same trace norm. By Peller's theorem [11], Chap. 6, Theorem 1.1, the trace norm of H_u is equivalent to the norm of u in the Besov space $B^1_{1,1}$, which is contained into W and contains H^s_+ for every $s > 1$. Consequently, if $u(0) \in H^s_+$ for some $s > 1$, then $u(t)$ stays bounded in W. We claim that, if $u(0)$ is in $\mathcal{V}(d)$, the evolution

can be continued for all time. Moreover, since the ranks of $H_{u(t)}$ and $K_{u(t)}$ are conserved in view of Corollary 3, this evolution takes place in $\mathcal{V}(d)$ if $u(0) \in \mathcal{V}(d)$.

Corollary 4. *The equation (16) defines a smooth flow on H^s_+ for every $s > 1$ and on $\mathcal{V}(d)$ for every d.*

In view of the Gronwall lemma, the statement is an easy consequence of the following estimate.

Lemma 4. *Let $R, y \geq 0, s > 1$ be given. There exists $C(d, R, y, s) > 0$ such that, for every $u \in \mathcal{V}(d)$ with $\|u\|_W \leq R$,*

$$\|X_{J^y}(u)\|_{H^s} \leq C(d, R, y, s)(1 + \|u\|_{H^s}) \ .$$

Proof. By using Lemma 2, we are reduced to prove

$$\|w^y\|_W \leq B(d, R, y) \ .$$

We set $N = \left[\frac{d+1}{2}\right]$. The above estimate is an easy consequence of

$$(I + H_u^2)^{-1} = \sum_{k=0}^{N} a_k H_u^{2k} \ ,$$

with $|a_k| \leq 1$ for $k = 0, \ldots, N$. In fact, the Cayley–Hamilton theorem yields

$$(H_u^2)^{N+1} = \sum_{k=1}^{N} (-1)^{k-1} S_k (H_u^2)^{N-k+1} \ , \ S_k := \sum_{\ell_1 < \cdots < \ell_k} \rho_{\ell_1}^2 \ldots \rho_{\ell_k}^2 \ ,$$

and one can easily check that

$$a_k = (-1)^k \ \frac{1 + \sum_{j=1}^{N-k} S_j}{1 + \sum_{j=1}^{N} S_j} \ , \ k = 0, \ldots, N \ .$$

where $\rho_1^2 \geq \cdots \geq \rho_N^2$ are the positive eigenvalues of H_u^2, listed with their multiplicities. □

Remark 1. *For general data $u(0) \in H^s_+$, one can prove similarly that the solution can be continued for all time if $y\|u(0)\|_{H^s}$ is small enough, or just if $y \operatorname{Tr}|H_{u(0)}|$ is small enough.*

Our next step is to derive an explicit formula for the solution of (16) along the same lines as in section 3. The starting points are the formulae

$$\begin{aligned} B_u^y(1) &= iyJ^y(u)w^y \\ C_u^y - B_u^y &= -iy^2(\cdot|H_u w^y)H_u w^y \\ &= iyJ^y(u)((I + yH_u^2)^{-1} - (I + yK_u^2)^{-1}) \ , \end{aligned}$$

where we have used the identity $K_u^2 = H_u^2 - (\cdot|u)u$. This leads to

$$\begin{aligned} \hat{B}_u(1) &= ig(H_u^2)(1) \ , \ g(s) := \sum_{k=1}^{n} \frac{a_k y_k J^{y_k}(u)}{1+y_k s} \ , \\ \hat{C}_u - \hat{B}_u &= i(g(H_u^2) - g(K_u^2)) \ . \end{aligned}$$

Arguing exactly as in section 3, we obtain the following formula.

Theorem 5. *The solution u of equation (16) with initial data $u(0) = u_0 \in H_+^s, s > 1$, is given by*

$$\underline{u}(t,z) = ((I - z\mathrm{e}^{2itg(H_{u_0}^2)}\mathrm{e}^{-2itg(K_{u_0}^2)}S^*)^{-1}\mathrm{e}^{2itg(H_{u_0}^2)}u_0 \,|\, 1) \ , \ z \in D \ , \tag{19}$$

where

$$g(s) := \sum_{k=1}^{n} \frac{a_k y_k J^{y_k}(u)}{1+y_k s} \ .$$

6. Proof of the quasiperiodicity theorem

In this section, we prove Theorem 2. Let $u_0 \in \mathcal{V}(d)$ be given. Firstly we show that $t \mapsto u(t)$ is a quasi periodic function valued into $\mathcal{V}(d)$. Denote by Σ the union of the spectra of $H_{u_0}^2$ and $K_{u_0}^2$. We claim that it is enough to prove that, for any function $\omega : \Sigma \to \mathbb{T}$, the formula

$$\underline{\Phi(\omega)}(z) = ((I - z\mathrm{e}^{-i\omega}(H_{u_0}^2)\mathrm{e}^{i\omega}(K_{u_0}^2)S^*)^{-1}\mathrm{e}^{-i\omega}(H_{u_0}^2)u_0 \,|\, 1) \ , \ z \in D \ ,$$

defines an element $\Phi(\omega) \in \mathcal{V}(d)$. Indeed, if this is established, Theorem 1 exactly claims that $u(t) = \Phi(t\omega)$, where, for every $s \in \Sigma$, $\omega(s) = s \bmod 2\pi$. Moreover, it is clear from the above formula that $\Phi(\omega)$ is a rational function with coefficients smoothly dependent on $\omega \in \mathbb{T}^\Sigma$, so that Φ is smooth as a map from $\mathbb{T}^\Sigma$ to $\mathcal{V}(d)$.

Let $\omega \in \mathbb{T}^\Sigma$. For each $s \in \Sigma$, we represent $\omega(s)$ by some element of $[0, 2\pi)$, still denoted by $\omega(s)$. Fix $n = |\Sigma|$ and let $y_1, \ldots, y_n$ be n positive numbers pairwise distinct. Then the matrix

$$\left(\frac{1}{1+y_k s}\right)_{k=1,\ldots,n, s\in\Sigma}$$

is invertible, hence the linear system

$$\omega(s) = -2\sum_{k=1}^{n} \frac{a_k y_k J^{y_k}(u_0)}{1+y_k s} \ , \ s \in \Sigma$$

has a unique solution $a_1, \ldots, a_n$. Using Theorem 5, $\Phi(\omega)$ is the value at time $t = 1$ of the solution u of equation (16) with parameters $a_1, \ldots, a_n, y_1, \ldots, y_n$. By Corollary 4, it belongs to $\mathcal{V}(d)$. This proves quasi periodicity.

Since Φ is a continuous mapping, $\Phi(\mathbb{T}^\Sigma)$ is a compact subset of $\mathcal{V}(d)$. On the other hand, for every s, the H^s norm is continuous on $\mathcal{V}(d)$. It is

therefore bounded on this compact subset, which contains the integral curve issued from u_0. This completes the proof of Theorem 2.

Remark 2. *It is tempting to adapt the above proof of quasi periodicity to non finite rank solutions. However, even assuming that one can define a flow on H^s_+ for all y with convenient estimates for large y, this strategy meets a serious difficulty. Indeed, on the one hand, the construction of a Hamiltonian flow on H^s_+ for*

$$\hat{J}(u) = (f(H_u^2)1|1)$$

requires a minimal regularity for f, say C^1, which, if f is represented as

$$f(s) = \int_0^\infty \frac{a(y)}{1+ys}\, d\mu(y)$$

for some positive measure μ and some function a on $\mathbb{R}_+$, imposes a decay condition as

$$\int_0^\infty y|a(y)|\, d\mu(y)\ .$$

On the other hand, Σ is made of a sequence of positive numbers converging to 0 and of its limit, and the interpolation problem

$$\omega(s) = -2\int_0^\infty \frac{ya(y)J^y(u_0)}{1+ys}\, d\mu(y)$$

would have a solution only if $\omega : \Sigma \to \mathbb{T}$ is continuous on Σ. Unfortunately, the space $C(\Sigma, \mathbb{T})$ is not compact, neither for the simple convergence, nor for the uniform convergence. Therefore the question of large time dynamics of non finite rank solutions of the cubic Szegő equation remains widely open.

References

[1] Colliander J., Keel M., Staffilani G., Takaoka H., Tao, T., *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrodinger equation*, Inventiones Math.181 (2010), 39–113.

[2] Gérard, P., Grellier, S., *The cubic Szegö equation* , Ann. Scient. Éc. Norm. Sup. 43 (2010), 761-810.

[3] Gérard, P., Grellier, S., *Invariant Tori for the cubic Szegö equation*, Invent. math. 187 (2012), 707–754.

[4] Gérard, P., Grellier, S., *Effective integrable dynamics for a certain nonlinear wave equation*, Analysis and PDEs, 5 (2012), 1139–1155.

[5] Kappeler, T., Pöschel, J. : *KdV & KAM*, A Series of Modern Surveys in Mathematics, vol. 45, Springer-Verlag, 2003.

[6] Kronecker, L. : *Zur Theorie der Elimination einer Variabeln aus zwei algebraischen Gleichungen* Monatsber. Königl. Preuss. Akad. Wiss. (Berlin), 535-600 (1881). Reprinted in *Leopold Kronecker's Werke*, vol. 2, 113–192, Chelsea, 1968.

[7] Lax, P. : *Integrals of Nonlinear equations of Evolution and Solitary Waves*, Comm. Pure and Applied Math. 21, 467-490 (1968).

[8] Lax, P. : *Periodic solutions of the the KdV equation.* Comm. Pure Appl. Math. 28 , 141–188 (1975).

[9] Nikolskii, N. K. : *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz.* Translated from the French by Andreas Hartmann. Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[10] Novikov, S. P. : *The periodic problem for the Korteweg-de Vries equation.* Funkt. Anal. i Prilozhen. 8 (1974), 54-66.

[11] Peller, V. V.: *Hankel operators and their applications.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[12] Zakharov, V. E., Shabat, A. B.: *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media.* Soviet Physics JETP 34 (1972), no. 1, 62–69.

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