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A note on solutions to controlled martingale problems and their conditioning

Julien CLAISSE* Denis TALAY† Xiaolu TAN‡

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Abstract

In this note, we rigorously justify a conditioning argument which is often (explicitly or implicitly) used to prove the dynamic programming principle in the stochastic control literature. To this end, we set up controlled martingale problems in an unusual way.

Key words. Stochastic control, martingale problem, dynamic programming principle.

MSC 2010. Primary 49L20; secondary 93E20.

1 Introduction

At the end of section 5 in [12], Y. Kabanov and C. Klüppelberg write: ‘The dynamic programming principle (DPP) has a clear intuitive meaning. (...) Our analysis in the literature reveals that it is difficult to find a paper with a self-contained and complete proof even if the model of interest is of the simplest class, for instance, a linear dynamics. Typically, some “formal” arguments are given indicating that the “rigorous” proofs can be found elsewhere, preferably in treatises on controlled Markov processes. Tracking further references, one can observe that they often deal with slightly different models, other definitions of optimality, “regular” controls and so on. For instance, in Fleming and Soner [6] and Yong and Zhou [22], the concept of control involves a choice of a stochastic basis.’ The authors provide a complete proof for their particular model by using a conditioning argument which, due to the specific value function under consideration, reduces to a disintegration of the Wiener measure.

For controlled diffusion processes problem, when the drift and diffusion coefficients depend continuously on the control, Krylov [13, Chap.3,Thm.6] proves the DPP by approximating the admissible controls by sequences of adapted processes with simple paths. His proof includes the same result as our Theorem 3.3 below, but restricted to these approximating admissible controls. Fleming and Soner [6,

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Chap.4, Sec.7] also proceed by approximating admissible controls; however they assume that the diffusion coefficient is smooth and strictly elliptic, which allows them to use smoothness properties of the solutions to Hamilton–Jacobi–Bellmann equations. El Karoui, Huu Nguyen and Jeanblanc [3] prove the DPP in the context of relaxed controls taking values in a compact and convex subset of the set of positive Radon measures equipped with the vague convergence topology. In Bouchard and Touzi [2] and Nutz [17], the authors invoke a flow property whose proof does not seem to be available in the literature and is not so obvious to us in general situations (see our discussion in Section 4).

In their context of stochastic games problems, Fleming and Souganidis [7] deduce sub-optimality and super-optimality dynamic principles from a conditioning property stated in their technical Lemma 1.11 whose proof is only sketched. As this lemma (reduced to the case of stochastic control problems) is actually crucial in several approaches to the DPP (see, e.g., its explicit or implicit use in Tang and Yong [21, Sec.4], Yong and Zhou [22, Lemma 3.2], Borkar [1, Proof of Theorem 1.1, Chap.3]), we find it useful to propose a precise formulation and a detailed justification, and to enlighten that properly defined controlled martingale problems are key ingredients in this context.

We here limit ourselves to examine stochastic control problems rather than stochastic game problems as in [7], which avoids additional technicalities and heavy notation.

2 Notation

Let $\mathbb{W} := C(\mathbb{R}^+, \mathbb{R}^d)$ be the canonical space of continuous functions from \mathbb{R}^+ to \mathbb{R}^d , equipped with Wiener measure \mathbb{W} . Denote the canonical filtration by $\mathbb{F} = (\mathcal{F}_s, s \geq 0)$ and the total σ -algebra by $\mathcal{F} := \bigvee_{s \geq 0} \mathcal{F}_s$. Denote by $\mathbf{0}$ the null element in \mathbb{W} .

For all t in \mathbb{R}^+ , w in \mathbb{W} , define the stopped path of w at time t by $w_{t \wedge \cdot} = (w_{t \wedge s}, s \geq 0)$. For all \mathbf{w} in \mathbb{W} , the concatenation path $\mathbf{w} \otimes_t w$ in \mathbb{W} is defined by

$$(\mathbf{w} \otimes_t w)_s := \begin{cases} \mathbf{w}_s, & \text{if } 0 \leq s \leq t, \\ w_s + \mathbf{w}_t - w_t, & \text{if } s \geq t. \end{cases}$$

Let now E be a Polish space. For all s in \mathbb{R}^+ and all E -valued \mathcal{F}_s -random variable f Doob's functional representation theorem (see, e.g., Lemma 1.13 in Kallenberg [10]) implies: $f(w) = f(w_{s \wedge \cdot})$ for all w in \mathbb{W} . Similarly, let g be an arbitrary E -valued \mathbb{F} -progressively measurable process; then $g(s, w) = g(s, w_{s \wedge \cdot})$ for all (s, w) in $\mathbb{R}^+ \times \mathbb{W}$.

For all (t, \mathbf{w}) in $\mathbb{R}^+ \times \mathbb{W}$ and all E -valued \mathcal{F} -random variable f , define the shifted \mathcal{F} -random variable $f^{t, \mathbf{w}}$ by

$$\forall w \in \mathbb{W}, \quad f^{t, \mathbf{w}}(w) := f(\mathbf{w} \otimes_t w).$$

Notice that, the path \mathbf{w} being fixed, $f^{t, \mathbf{w}}$ is independent of \mathcal{F}_t . Similarly, let g be any E -valued \mathbb{F} -progressively measurable process, define the shifted \mathbb{F} -progressively measurable process by $g^{t, \mathbf{w}} := (g_s^{t, \mathbf{w}}, s \geq 0)$. Again it is clear that $g^{t, \mathbf{w}}$ is independent of \mathcal{F}_t .

Finally, denote \mathbb{W}^2 by $\bar{\mathbb{W}}$. In all the sequel, we identify $\bar{\mathbb{W}}$ with the canonical space of continuous functions from \mathbb{R}^+ to \mathbb{R}^{2d} . In particular, we naturally define

the filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_s, s \geq 0)$, the σ -algebra $\bar{\mathcal{F}}$, the concatenation of paths and the shifted processes on $\bar{\mathbb{W}}$, and the canonical element in $\bar{\mathbb{W}}$ is usually denoted by $\bar{w} = (w, w')$.

3 Conditioning strong solutions to controlled SDEs

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space embedded with a standard Brownian motion $(B_s, s \geq 0)$. Denote by $\mathbb{B} = (\mathcal{B}_s, s \geq 0)$ the natural filtration generated by the Brownian motion. Let in addition U be a Polish space.

Hypothesis 3.1. *Let b be a map from $\mathbb{R}^+ \times \mathbb{W} \times U$ to \mathbb{R}^d and σ a map from $\mathbb{R}^+ \times \mathbb{W} \times U$ to the space S_d of square matrices of order d . Assume that:*

- (i) b and σ are Borel measurable,
- (ii) for all u in U , $b(\cdot, \cdot, u)$ and $\sigma(\cdot, \cdot, u)$ are \mathbb{F} -progressively measurable,
- (iii) b and σ are bounded.

Denote by \mathcal{U} the collection of all U -valued \mathbb{B} -progressively measurable processes and, for all t in \mathbb{R}^+ , denote by \mathcal{U}^t the collection of all $\nu \in \mathcal{U}$ independent of \mathcal{B}_t . Given a control ν in \mathcal{U} , consider the following system of controlled stochastic differential equations (SDEs):

$$dX_s = b(s, X, \nu_s)ds + \sigma(s, X, \nu_s)dB_s. \quad (1)$$

A strong solution to the equation (1) with control ν in \mathcal{U} and initial condition (t, \mathbf{x}) in $\mathbb{R}^+ \times \mathbb{W}$ is a \mathbb{B} -progressively measurable process $(X_\theta^{t, \mathbf{x}, \nu}, \theta \geq 0)$ such that, for all θ in $[t, +\infty)$,

$$X_\theta^{t, \mathbf{x}, \nu} = \mathbf{x}_t + \int_t^\theta b(s, X^{t, \mathbf{x}, \nu}, \nu_s)ds + \int_t^\theta \sigma(s, X^{t, \mathbf{x}, \nu}, \nu_s)dB_s, \quad \mathbb{P} - a.s. \quad (2)$$

and $X_\theta^{t, \mathbf{x}, \nu} = \mathbf{x}_\theta$ for all θ in $[0, t]$.

As U is Polish, one again can apply Doob's functional representation theorem: for all ν in \mathcal{U} , there exists a \mathbb{F} -progressively measurable function g_ν from $\mathbb{R}^+ \times \mathbb{W}$ to U such that for all s in \mathbb{R}^+ and ω in Ω ,

$$\nu_s(\omega) = g_\nu(s, B(\omega)) = g_\nu(s, B_{s \wedge \cdot}(\omega)).$$

Let t in \mathbb{R}^+ and $\hat{\omega} \in \Omega$ be fixed. Define the shifted control process $(\nu_s^{t, \hat{\omega}}, s \geq 0)$ as follows: for all ω in Ω , s in \mathbb{R}^+

$$\nu_s^{t, \hat{\omega}}(\omega) := g_\nu^{t, B(\hat{\omega})}(s, B(\omega)) = g_\nu(s, B(\hat{\omega}) \otimes_t B(\omega)).$$

Notice that, for all fixed $\hat{\omega}$, $\nu^{t, \hat{\omega}}$ belongs to \mathcal{U}^t .

Stochastic control problems involve the choice of a class of admissible controls. We now formulate our hypothesis on this class.

Hypothesis 3.2. *Let $\mathcal{A} \subset \mathcal{U}$ be the collection of admissible controls and \mathcal{A}^t be the subset of those which are independent of \mathcal{B}_t . We assume that, for all admissible control ν , t in \mathbb{R}^+ and \mathbb{P} -almost all $\hat{\omega}$ in Ω , the shifted control $\nu^{t, \hat{\omega}}$ is also admissible.*

Theorem 3.3. *Under Hypotheses 3.1 and 3.2, assume that there exists a unique strong solution to (1) for all admissible control and initial condition. Let (t, \mathbf{x}, ν) be in $\mathbb{R}^+ \times \mathbb{W} \times \mathcal{A}$ and τ be a finite \mathbb{B} -stopping time. Then for all measurable function $f : \mathbb{W} \rightarrow \mathbb{R}^+$,*

$$\mathbb{E} \left[f(X^{t, \mathbf{x}, \nu}) \middle| \mathcal{B}_\tau \right] (\omega) = F \left(\tau(\omega), X_{\tau \wedge \cdot}^{t, \mathbf{x}, \nu}(\omega), \nu^{\tau(\omega), \omega} \right), \quad \mathbb{P}(d\omega) - a.s., \quad (3)$$

where, for all s in \mathbb{R}^+ , \mathbf{y} in \mathbb{W} , μ in \mathcal{A}^s ,

$$F(s, \mathbf{y}, \mu) := \mathbb{E} [f(X^{s, \mathbf{y}, \mu})].$$

Remark 3.4. (i) *To ensure the existence and uniqueness of a strong solution to (1) for all control and initial condition, a sufficient condition is that the functions b and σ are uniformly Lipschitz in \mathbf{x} , i.e., there exists $L > 0$ such that for all s in \mathbb{R}^+ , (\mathbf{x}, \mathbf{y}) in \bar{W} and u in U ,*

$$|b(t, \mathbf{x}, u) - b(t, \mathbf{y}, u)| + \|\sigma(t, \mathbf{x}, u) - \sigma(t, \mathbf{y}, u)\| \leq L \sup_{0 \leq s \leq t} |\mathbf{x}_s - \mathbf{y}_s|.$$

For a proof, see, e.g., Rogers and Williams [15]. For more general conditions, see, e.g., Protter [19] or Jacod and Memin [9].

(ii) *We here suppose that b and σ are bounded to apply, in our Section 6, the theorem 6.1.3 in [20] as it is stated. However, classical localization arguments allow one to deal, e.g., with functions satisfying: there exists $C > 0$ such that, for all (t, \mathbf{x}) in $\mathbb{R}^+ \times \mathbb{W}$,*

$$\sup_{u \in U} (|b(t, \mathbf{x}, u)| + \|\sigma(t, \mathbf{x}, u)\|) \leq C \left(1 + \sup_{0 \leq s \leq t} |\mathbf{x}_s| \right).$$

(iii) *Instead of considering positive functions f one may consider functions with suitable growth conditions at infinity.*

(iv) *It is not clear how to define the measurability of the function F . However Equality (3) shows that the r.h.s. is a measurable function of ω except on a \mathcal{B}_τ -measurable null event (or, equivalently, is a r.v. defined on the \mathbb{P} -completion of the σ -field \mathcal{B}_τ).*

The proof of Theorem 3.3 is postponed to Section 6. We conclude this section by showing how it is used to solve stochastic control problems. Let $\Phi : \mathbb{W} \rightarrow \mathbb{R}^+$ be a positive reward function. Define the value function of the control problem by

$$V(t, \mathbf{x}) := \sup_{\nu \in \mathcal{A}} \mathbb{E} [\Phi(X^{t, \mathbf{x}, \nu})]. \quad (4)$$

Proposition 3.5. *Suppose that the conditions in Theorem 3.3 hold true. For all (t, \mathbf{x}) in $\mathbb{R}^+ \times \mathbb{W}$, it holds*

$$V(t, \mathbf{x}) = \sup_{\mu \in \mathcal{A}^t} \mathbb{E} [\Phi(X^{t, \mathbf{x}, \mu})]. \quad (5)$$

Suppose in addition that the value function V is measurable. Then, for all (t, \mathbf{x}) in $\mathbb{R}^+ \times \mathbb{W}$ and all \mathbb{B} -stopping times τ taking values in $[t, \infty)$, one has

$$V(t, \mathbf{x}) \leq \sup_{\nu \in \mathcal{A}} \mathbb{E} [V(\tau, X_{\tau \wedge \cdot}^{t, \mathbf{x}, \nu})]. \quad (6)$$

Proof. Equality (5) follows from Theorem 3.3 in the particular case $\tau \equiv t$. Then Theorem 3.3 readily implies (6). \square

Remark 3.6. *Inequality (6) is the ‘easy’ part of the dynamic programming principle (DPP). Equality (5), combined with the continuity of the value function, is a key step to classical proofs of the difficult part of the DPP.*

4 Discussions on Theorem 3.3

The intuitive meaning of Theorem 3.3 is as follows.

Given the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, suppose that $(\mathbb{P}_{\hat{\omega}}, \hat{\omega} \in \Omega)$ is a regular conditional probability distribution (r.c.p.d.) of \mathbb{P} given \mathcal{B}_τ (for a definition, see, e.g., Stroock and Varadhan [20]). For \mathbb{P} -a.e. $\hat{\omega}$, one clearly has $\mathbb{P}_{\hat{\omega}}(\nu_s = \nu_s^{\tau(\hat{\omega}), \hat{\omega}}, s \geq \tau(\hat{\omega})) = 1$, $\mathbb{P}_{\hat{\omega}}(X_s^{t, \mathbf{x}, \nu} = X_s^{t, \mathbf{x}, \nu}(\hat{\omega}), 0 \leq s \leq \tau(\hat{\omega})) = 1$, and $(B_s)_{s \geq \tau(\hat{\omega})}$ is still a Brownian motion under $\mathbb{P}_{\hat{\omega}}$. Equality (3) would be obvious, when $X^{t, \mathbf{x}, \nu}$ solves the following equation under \mathbb{P} and $\mathbb{P}_{\hat{\omega}}$ for \mathbb{P} -almost all $\hat{\omega}$:

$$X_{\theta \vee \tau}^{t, \mathbf{x}, \nu} = X_\tau^{t, \mathbf{x}, \nu} + \int_\tau^{\theta \vee \tau} b(s, X_\tau^{t, \mathbf{x}, \nu}, \nu_s) ds + \int_\tau^{\theta \vee \tau} \sigma(s, X_\tau^{t, \mathbf{x}, \nu}, \nu_s) dB_s. \quad (7)$$

However this may not be true because the stochastic integral involved in (7) (or in (2)) depends on the reference probability measure.

One possible solution to this issue is to use a pathwise construction of stochastic integrals under different probability measures, that is, to construct a universal process such that (2) or (7) holds true under the probability measure \mathbb{P} as well as the conditional probabilities $\mathbb{P}_{\hat{\omega}}$. For such a construction, see Nutz [18] and references therein, noticing that the construction in [18] uses the median limit which assumes the axiom of choice and the Continuum Hypothesis.

Another possible way is to extend to controlled SDEs the flow property enjoyed by strong solutions in the sense of Ikeda and Watanabe [8]. However, as mentioned in the introduction, this property seems questionable to us for controlled SDEs (1). The issue is that, for each control process, Equality (2) holds true except on a null set which depends on this control.

We thus follow another strategy. We notice that (3) concerns the probability law of the controlled process. This leads us to introduce a controlled martingale problem formulation which allows us to justify (3) with weak conditions under which strong solutions may even not exist.

5 Conditioning solutions to controlled martingale problems

The notion of controlled martingale problems appeared a long time ago: see, e.g., Fleming [5]. They are usually posed on the state space of the controlled process. Here we introduce a different formulation.

We start with defining new maps. Let b and σ satisfy the conditions in Hypothesis 3.1. Denote again \mathcal{U} the collection of all U -valued \mathbb{F} -progressively measurable processes and \mathcal{U}^t the subset of controls independent of \mathcal{F}_t . For all (t, ν) in $\mathbb{R}^+ \times \mathcal{U}$,

define the maps $\bar{b}^{t,\nu} : [t, \infty) \times \bar{W} \rightarrow \mathbb{R}^d$ and $\bar{\sigma}^{t,\nu} : [t, \infty) \times \bar{W} \rightarrow S_{2d}$ by

$$\begin{aligned}\bar{b}^{t,\nu}(s, \bar{w}) &:= \begin{pmatrix} b(s, w, \nu_s(w')) \\ 0 \end{pmatrix}, \quad s \geq t, \\ \bar{\sigma}^{t,\nu}(s, \bar{w}) &:= \begin{pmatrix} \sigma(s, w, \nu_s(w')) \\ \text{Id}_d \end{pmatrix}, \quad s \geq t,\end{aligned}$$

where $\bar{w} = (w, w')$ is in \bar{W} . These maps result from the following observation. Suppose that the equation (1) has a strong solution $X^{t,\mathbf{x},\nu}$ as defined in Section 3. Then the process $\bar{X}^{t,\mathbf{x},\nu} := (X^{t,\mathbf{x},\nu}, B)$ solves

$$\bar{X}_\theta^{t,\mathbf{x},\nu} = \begin{pmatrix} \mathbf{x}_t \\ B_t \end{pmatrix} + \int_t^\theta \bar{b}^{t,\nu}(s, \bar{X}^{t,\mathbf{x},\nu}) ds + \int_t^\theta \bar{\sigma}^{t,\nu}(s, \bar{X}^{t,\mathbf{x},\nu}) dB_s, \quad \theta \geq t.$$

To these maps we associate the following differential operator acting on functions φ in $C^2(\mathbb{R}^{2d})$:

$$\bar{L}_s^{t,\nu} \varphi(\bar{w}) := \bar{b}^{t,\nu}(s, \bar{w}) \cdot D\varphi(\bar{w}_s) + \frac{1}{2} \text{Tr}(\bar{a}^{t,\nu}(s, \bar{w}) D^2 \varphi(\bar{w}_s)),$$

where Tr stands for the trace operator and

$$\bar{a}^{t,\nu}(s, \bar{w}) := \bar{\sigma}^{t,\nu}(s, \bar{w}) \bar{\sigma}^{t,\nu}(s, \bar{w})^*.$$

Now, for all $t \geq 0$, ν be in \mathcal{U} , and φ in $C^2(\mathbb{R}^{2d})$, define the process $(\bar{M}_\theta^{t,\nu,\varphi}, \theta \geq t)$ on the enlarged space \bar{W} by

$$\bar{M}_\theta^{t,\nu,\varphi}(\bar{w}) := \varphi(\bar{w}_\theta) - \int_t^\theta \bar{L}_s^{t,\nu} \varphi(\bar{w}) ds, \quad \theta \geq t. \quad (8)$$

We now introduce our two notions of controlled martingale problems.

Definition 5.1. *Given (t, \mathbf{x}) in $\mathbb{R}^+ \times \mathbb{W}$ and ν in \mathcal{U} , a probability measure $\bar{\mathbb{P}}^{t,\mathbf{x},\nu}$ on $(\bar{W}, \bar{\mathcal{F}})$ is a solution to the controlled martingale problem associated to (1) with control ν and initial condition (t, \mathbf{x}) if, for all function φ in $C_c^2(\mathbb{R}^{2d})$, the process $(\bar{M}_\theta^{t,\nu,\varphi}, \theta \geq t)$ is a $\bar{\mathbb{F}}$ -martingale under $\bar{\mathbb{P}}^{t,\mathbf{x},\nu}$, $\bar{\mathbb{P}}^{t,\mathbf{x},\nu}(w_s = \mathbf{x}_s, \forall 0 \leq s \leq t) = 1$, and $\bar{\mathbb{P}}^{t,\mathbf{x},\nu}(w' \in A) = \mathbb{W}(A)$ for every A in \mathcal{F}_t , where \mathbb{W} stands for the Wiener measure.*

Definition 5.2. *Given $(t, \mathbf{x}, \mathbf{y})$ in $\mathbb{R}^+ \times \bar{W}$ and μ in \mathcal{U}^t , a probability measure $\bar{\mathbb{P}}^{t,\mathbf{x},\mathbf{y},\mu}$ on $(\bar{W}, \bar{\mathcal{F}})$ is a solution to the shifted controlled martingale problem associated to (1) with control μ and initial condition $(t, \mathbf{x}, \mathbf{y})$ if, for all function φ in $C_c^2(\mathbb{R}^{2d})$, the process $(\bar{M}_\theta^{t,\mu,\varphi}, \theta \geq t)$ is a $\bar{\mathbb{F}}$ -martingale under $\bar{\mathbb{P}}^{t,\mathbf{x},\mathbf{y},\mu}$ and $\bar{\mathbb{P}}^{t,\mathbf{x},\mathbf{y},\mu}(w_s = \mathbf{x}_s, w'_s = \mathbf{y}_s, \forall 0 \leq s \leq t) = 1$.*

Before stating our main result we reformulate Hypothesis 3.2 on the set of admissible controls in the context of the canonical space \bar{W} .

Hypothesis 5.3. *Let $\mathcal{A} \subset \mathcal{U}$ be the collection of admissible controls and \mathcal{A}^t be the subset of those which are independent of \mathcal{F}_t . We assume that, for all admissible control ν , for all t in \mathbb{R}^+ and \mathbb{W} -almost all w' in \mathbb{W} , the shifted control $\nu^{t,w'}$ is also admissible.*

We now are in a position to rigorously state and prove the conditioning property which sustains the DPP.

Theorem 5.4. *Under Hypotheses 3.1 and 5.3, assume that there exists a unique solution to both martingale problems in Definitions 5.1 and 5.2 associated to (1) for each admissible control and initial condition. Let $(t, \mathbf{x}, \nu) \in \mathbb{R}^+ \times \mathbb{W} \times \mathcal{A}$, τ be a $\bar{\mathbb{F}}$ -stopping time taking value in $[t, \infty)$. Let $(\bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu}, \bar{\mathbf{w}} = (\mathbf{w}, \mathbf{w}') \in \bar{\mathbb{W}})$ be a regular conditional probability of $\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}$ given $\bar{\mathcal{F}}_\tau$. Then*

$$\bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu} = \bar{\mathbb{P}}^{\tau(\bar{\mathbf{w}}), \bar{\mathbf{w}}, \nu^{\tau(\bar{\mathbf{w}}), \mathbf{w}'}} , \quad \bar{\mathbb{P}}^{t, \mathbf{x}, \nu}(d\bar{\mathbf{w}}) - a.s. \quad (9)$$

Equality (9) shows that the r.h.s. is a measurable function of $\bar{\mathbf{w}}$ except on a $\bar{\mathcal{F}}_\tau$ -measurable null event (see Remark 3.4 (iv)).

6 Proofs

Proof of Theorem 5.4 The result is a direct consequence of the stability of the martingale property under conditioning (see, e.g., Theorem 1.2.10 in Stroock and Varadhan [20]). Let $(\bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu}, \bar{\mathbf{w}} = (\mathbf{w}, \mathbf{w}') \in \bar{\mathbb{W}})$ be a r.c.p.d. of $\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}$ given $\bar{\mathcal{F}}_\tau$. From Theorem 6.1.3 in [20], there exists a null set N in $\bar{\mathcal{F}}_\tau$ such that, for all function φ in $C_c^2(\mathbb{R}^{2d})$, the process $(\bar{M}_\theta^{t, \nu, \varphi}, \theta \geq \tau(\bar{\mathbf{w}}))$ defined by (8) is a $\bar{\mathbb{F}}$ -martingale under $\bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu}$ for all $\bar{\mathbf{w}}$ in $\bar{\mathbb{W}} \setminus N$.

Let $\bar{\mathbf{w}}$ in $\bar{\mathbb{W}} \setminus N$ be fixed. Observe that

$$\begin{aligned} \bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu} \left(\bar{b}^{t, \nu}(s, \bar{w}) = \bar{b}^{\tau(\bar{\mathbf{w}}), \nu^{\tau(\bar{\mathbf{w}}), \mathbf{w}'}}(s, \bar{w}), \forall s \geq \tau(\bar{\mathbf{w}}) \right) &= 1, \\ \bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu} \left(\bar{\sigma}^{t, \nu}(s, \bar{w}) = \bar{\sigma}^{\tau(\bar{\mathbf{w}}), \nu^{\tau(\bar{\mathbf{w}}), \mathbf{w}'}}(s, \bar{w}), \forall s \geq \tau(\bar{\mathbf{w}}) \right) &= 1. \end{aligned}$$

Hence, for all function φ in $C_c^2(\mathbb{R}^{2d})$, the process $(\bar{M}^{\tau(\bar{\mathbf{w}}), \nu^{\tau(\bar{\mathbf{w}}), \mathbf{w}'}, \varphi}, \theta \geq \tau(\bar{\mathbf{w}}))$ is a $\bar{\mathbb{F}}$ -martingale under $\bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu}$.

By uniqueness of the solution to the shifted controlled martingale problem associated to (1) with control $\nu^{\tau(\bar{\mathbf{w}}), \mathbf{w}'}$ and initial condition $(\tau(\bar{\mathbf{w}}), \bar{\mathbf{w}})$, we deduce that

$$\bar{\mathbb{P}}_{\bar{\mathbf{w}}}^{t, \mathbf{x}, \nu} = \bar{\mathbb{P}}^{\tau(\bar{\mathbf{w}}), \bar{\mathbf{w}}, \nu^{\tau(\bar{\mathbf{w}}), \mathbf{w}'}}$$

for all $\bar{\mathbf{w}} \in \bar{\mathbb{W}} \setminus N$. □

Proof of Theorem 3.3 Under the hypotheses of Theorem 3.3, the hypotheses of Theorem 5.4 are satisfied. Indeed, for all (t, \mathbf{x}) in $\mathbb{R}^+ \times \mathbb{W}$ and ν in \mathcal{A} , the law of $(X^{t, \mathbf{x}, \nu}, B)$ on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ provides a solution to the controlled martingale problem associated to (1) with control ν and initial condition (t, \mathbf{x}) . The uniqueness follows directly from a corollary to Theorem 4.1.1 in Ikeda-Watanabe [8] (or from Corollary 5.4.9 in Karatzas and Shreve [11]). Hence, for all (t, \mathbf{x}, ν) in $\mathbb{R}^+ \times \mathbb{W} \times \mathcal{A}$,

$$\mathcal{L}^{\mathbb{P}}(X^{t, \mathbf{x}, \nu}, B) = \bar{\mathbb{P}}^{t, \mathbf{x}, \nu}. \quad (10)$$

Similarly, for all μ be in \mathcal{A}^t , the law $(X^{t, \mathbf{x}, \mu}, \mathbf{y} \otimes_t B)$ is the unique solution to the shifted controlled martingale problem associated to (1) with control μ and initial condition $(t, \mathbf{x}, \mathbf{y})$. Hence

$$\mathcal{L}^{\mathbb{P}}(X^{t, \mathbf{x}, \nu}, \mathbf{y} \otimes_t B) = \bar{\mathbb{P}}^{t, \mathbf{x}, \mathbf{y}, \nu}. \quad (11)$$

Let (t, \mathbf{x}, ν) in $\mathbb{R}^+ \times \mathbb{W} \times \mathcal{A}$ be fixed. Let $f : \mathbb{W} \rightarrow \mathbb{R}^+$ be a positive \mathcal{F} -measurable function. Let Y be an arbitrary positive \mathcal{B}_τ -random variable. From

Doob's functional representation theorem, there exist a \mathbb{F} -stopping time $\tilde{\tau}$ and an \mathcal{F} -measurable positive function g_Y defined on \mathbb{W} such that

$$\tau(\omega) = \tilde{\tau}(B(\omega)) \quad \text{and} \quad Y(\omega) = g_Y(B_{\tau \wedge \cdot}(\omega)) = g_Y(B(\omega)).$$

The random time $\bar{\tau}(w, w') := \tilde{\tau}(w')$ on $\bar{\mathbb{W}}$ clearly is an $\bar{\mathbb{F}}$ -stopping time. Successively using (10), (11) and (9), we obtain

$$\begin{aligned} \mathbb{E} [f(X^{t, \mathbf{x}, \nu}(\omega)) Y(\omega)] &= \mathbb{E}^{\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}} [f(w) g_Y(w')] \\ &= \mathbb{E}^{\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}} \left[\mathbb{E}^{\bar{\mathbb{P}}_{\bar{w}}^{t, \mathbf{x}, \nu}} [f(\mathbf{w})] g_Y(w') \right] \\ &= \mathbb{E}^{\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}} \left[\mathbb{E}^{\bar{\mathbb{P}}^{\bar{\tau}(\bar{w}), \bar{w}, \nu^{\bar{\tau}(\bar{w}), w'}}} [f(\mathbf{w})] g_Y(w') \right] \\ &= \mathbb{E} \left[F \left(\tau(\omega), X^{t, \mathbf{x}, \nu}(\omega), \nu^{\tau(\omega), \omega} \right) Y(\omega) \right] \\ &= \mathbb{E} \left[F \left(\tau(\omega), X_{\tau \wedge \cdot}^{t, \mathbf{x}, \nu}(\omega), \nu^{\tau(\omega), \omega} \right) Y(\omega) \right], \end{aligned}$$

where $(\bar{\mathbb{P}}_{\bar{w}}^{t, \mathbf{x}, \nu}, \bar{w} \in \bar{\mathbb{W}})$ is a r.c.p.d. of $\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}$ given $\bar{\mathcal{F}}_{\bar{\tau}}$. This completes the proof. \square

7 Conclusion and perspectives

We have rigorously justified a classical key argument in the proof of the DPP under weak hypotheses. To go further and prove the DPP, one usually needs that the value function satisfies some semi-continuity or continuity property (see, e.g., [1, 2, 3, 6, 7, 13, 21, 22] among many other references). Recent advances allow one to obtain the DPP without such regularity properties: see, e.g., Neufeld and Nutz [16], El Karoui and Tan [4].

In principle, Theorems 3.3 and 5.4 can be extended to controlled stochastic differential equations driven by Poisson random measures. However, even for uncontrolled systems, the uniqueness of solutions to classical martingale problems under weak hypotheses on the coefficients is a difficult issue: see, e.g., Lepeltier and Marchal [14].

Finally, it is natural to define an optimal control problem in terms of our controlled martingale problems by setting the value function as

$$\bar{V}(t, \mathbf{x}) := \sup_{\nu \in \mathcal{A}} \mathbb{E}^{\bar{\mathbb{P}}^{t, \mathbf{x}, \nu}} [f(w)].$$

In view of Theorem 5.4, the conclusions in Proposition 3.5 hold true for \bar{V} . Thus an interesting issue is to seek fairly general conditions on b and σ under which \bar{V} satisfies the DPP and is a viscosity solution to a Hamilton–Jacobi–Bellmann equation.

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