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Some Poisson-Lie sigma models

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Abstract

We calculate the Poisson-Lie sigma model for every 4-dimensional Manin triples (function of its structure constant) and we give the 6-dimensional models for the Manin triples

$$(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*),$$

$$(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})),$$

$$(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C})) \text{ and}$$

$$(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))$$

1 Introduction

A Manin triples $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is a bialgebra $(\mathfrak{g}, \tilde{\mathfrak{g}}$ which don't intersect each others and a direct sum of this bialgebra $\mathfrak{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$). If the corresponding Lie groups have a Poisson structure, they are called Poisson-Lie groups. A Poisson-Lie sigma models is an action (3.13) calculated by a Poisson vector field matrix. [3] have deduced the extremal field which minimize the action of this models, which gives the motion equation (3.19). We calculate here the action and the equations of motion for some 6-dimensionals Manin triples and we give a general formula for each 4-dimensional Manin triples. The 6-dimensional Manin triples are $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*), (\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*), \mathfrak{sl}(2, \mathbb{C}), (\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))$ and $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))$.

2 Some Manin triples

The Drinfeld double D is defined as a Lie group such that its Lie algebra \mathfrak{D} equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ can be decomposed into a pair of maximally isotropic subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ such that \mathfrak{D} as a vector space is the direct sum of \mathfrak{g} and $\tilde{\mathfrak{g}}$. Any such decomposition written as an ordered set $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is called a Manin triples $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}}), (\mathfrak{D}, \tilde{\mathfrak{g}}, \mathfrak{g})$.

One can see that the dimensions of the subalgebras are equal and that bases $\{T_i\}, \{\tilde{T}^i\}$ in the subalgebras can be chosen so that

$$\langle T_i, T_j \rangle = 0, \quad \langle T_i, \tilde{T}^j \rangle = \langle \tilde{T}^j, T_i \rangle = \delta_i^j, \quad \langle \tilde{T}^i, \tilde{T}^j \rangle = 0 \quad (2.1)$$

This canonical form of the bracket is invariant with respect to the transformations

$$T'_i = T_k A_i^k, \quad \tilde{T}'^j = (A^{-1})^j_k \tilde{T}^k \quad (2.2)$$

Due to the ad-invariance of $\langle \cdot, \cdot \rangle$ the algebraic structure of \mathfrak{D} is

$$\begin{aligned} [T_i, T_j] &= c_{ij}^k T_k, & [\tilde{T}^i, \tilde{T}^j] &= f^{ij}_k \tilde{T}^k \\ [T_i, \tilde{T}^j] &= f^{jk}_i T_k - c_{ik}^j \tilde{T}^k \end{aligned}$$

There are just four types of nonisomorphic four-dimensional Manin triples.

Abelian Manin triples :

$$[T_i, T_j] = 0, \quad [\tilde{T}^i, \tilde{T}^j] = 0, \quad [T_i, \tilde{T}^j] = 0, \quad i, j = 1, 2 \quad (2.3)$$

Semi-Abelian Manin triples (only non trivial brackets are displayed) :

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [T_2, \tilde{T}^1] = T_2, \quad [T_2, \tilde{T}^2] = -T_1 \quad (2.4)$$

Type A non-Abelian Manin triples ($\beta \neq 0$) :

$$\begin{aligned} [T_1, T_2] &= T_2, & [\tilde{T}^1, \tilde{T}^2] &= \beta \tilde{T}^2 \\ [T_1, \tilde{T}^2] &= -\tilde{T}^2, & [T_2, \tilde{T}^1] &= \beta T_2, & [T_2, \tilde{T}^2] &= -\beta T_1 + \tilde{T}^1 \end{aligned}$$

Type B non-Abelian Manin triples :

$$\begin{aligned} [T_1, T_2] &= T_2, & [\tilde{T}^1, \tilde{T}^2] &= \tilde{T}^1 \\ [T_1, \tilde{T}^1] &= T_2, & [T_1, \tilde{T}^2] &= -T_1 - \tilde{T}^2, & [T_2, \tilde{T}^2] &= \tilde{T}^1 \end{aligned}$$

Now we focus some six dimensional Manin triples. We recall that the commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of the Lie group $SL(2, \mathbb{C})$:

$$[T_1, T_2] = 2T_2, \quad [T_1, T_3] = -2T_3, \quad [T_2, T_3] = T_1 \quad (2.5)$$

The dual Lie algebra $\mathfrak{sl}(2, \mathbb{C})^*$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has the commutation relations :

$$[\tilde{T}^1, \tilde{T}^2] = \frac{1}{4} \tilde{T}^2, \quad [\tilde{T}^1, \tilde{T}^3] = \frac{1}{4} \tilde{T}^3, \quad [\tilde{T}^2, \tilde{T}^3] = 0 \quad (2.6)$$

There is a scalar product on $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*)$ such that (see [2]) :

$$(T_i, \tilde{T}^j) = \delta_i^j \quad (2.7)$$

Finally, we have that $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}))$ with this scalar product is also a Manin triples.

The Iwasawa decomposition allows us to decompose :

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2, \mathbb{C}) \oplus \mathfrak{sb}(2, \mathbb{C}) \quad (2.8)$$

where $\mathfrak{su}(2, \mathbb{C})$ is the Lie algebra of the Lie group $SU(2)$ with commutation relations :

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2 \quad (2.9)$$

$\mathfrak{sb}(2, \mathbb{C})$ is the Lie algebra of the Borel subgroup $SB(2, \mathbb{C})$ with commutation relations :

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [\tilde{T}^1, \tilde{T}^3] = \tilde{T}^3, \quad [\tilde{T}^2, \tilde{T}^3] = 0 \quad (2.10)$$

Here we can see in comparing (2.10) and (2.6) that $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})^*$.

The Iwasawa decomposition (2.8) allows us to identify $\mathfrak{sb}(2, \mathbb{C}) \simeq \mathfrak{su}(2, \mathbb{C})^*$. We define a scalar product on $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*)$ such that $(x, y) = \text{Im}(\text{Tr}(x|y))$. With this scalar product we have (see [2]) :

$$(T_i, \tilde{T}^j) = \delta_i^j \quad (2.11)$$

Finally we have that $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))$ with this scalar product is also a Manin triples.

3 Poisson-Lie sigma models

Given a Lie group M and a Poisson structure on it. We define the action of this model (see [1]) as :

$$S_1 = \int_{\Sigma} (\langle dgg^{-1}, A \rangle - \frac{1}{2} \langle A, (r - Ad_g r Ad_g) A \rangle) \quad (3.12)$$

where $g \in G$, $A = A_{\alpha}^i d\xi^{\alpha} X_i$ and $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical r matrix with \mathfrak{g} as the Lie algebra of G and $\{X_i\}$ as a basis of \mathfrak{g} . Note that the above action can be applied for simple or nonsemisimple Lie group G with ad-invariant symmetric bilinear nondegenerate form $\langle X_i, X_j \rangle = G_{ij}$ on the Lie algebra \mathfrak{g} . When the metric G_{ij} of Lie algebra is denegerate then the above action is not good. Here we use the following action instead of the above one :

$$S_2 = \int_{\Sigma} (dX_i \wedge A_i - \frac{1}{2} \mathcal{P}^{ij} A_i \wedge A_j) \quad (3.13)$$

where x are Lie group parameters with parametrization (e.g.)

$$\forall g \in G, g = e^{X_1 T_1} e^{X_2 T_2} \dots \quad (3.14)$$

where P^{ij} is the Poisson structure on the Lie group which for coboundary Poisson Lie groups it is obtained from

$$(\mathcal{P}(g))_\chi = b(g)a(g)^{-1} \quad (3.15)$$

We can obtain $a(g)^{-1}$ and $b(g)$ in computing :

$$(Ad_{g^{-1}})_\chi = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix} \quad (3.16)$$

$$(Ad_g)_\chi = \begin{pmatrix} a(g)^{-T} & -a(g)^{-T}b(g)^T d(g)^{-T} \\ 0 & d(g)^{-T} \end{pmatrix} \quad (3.17)$$

where T denotes the transpose matrix.

The extremal fields (X, A) which minimize the action (3.13) have to satisfy the equation written locally (see [3]) as :

$$dX_i + \mathcal{P}^{ij}(X)A_j = 0 \quad (3.18)$$

$$dA_k + \frac{1}{2}\mathcal{P}^{ij},_k(X)A_i \wedge A_j = 0 \quad (3.19)$$

where $\mathcal{P}^{ij},_k = \partial_k \mathcal{P}^{ij}|_{X_k=0}$.

4 Poisson-Lie sigma model of any 4-dimensional Manin triple

We first calculate the matrix of the adjoint actions function of structure constant :

$$ad_{T_1} = \begin{pmatrix} 0 & c_{12}^1 & 0 & -f_{12}^1 \\ 0 & c_{12}^2 & f_{12}^1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{12}^1 & -c_{12}^2 \end{pmatrix}$$

$$ad_{T_2} = \begin{pmatrix} -c_{12}^1 & 0 & 0 & -f_{12}^2 \\ -c_{12}^2 & 0 & f_{12}^2 & 0 \\ 0 & 0 & c_{12}^1 & c_{12}^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To obtain the matrix \mathcal{P} , we calculate the adjoint action matrix of a general element $g = \prod_{i=1}^2 e^{\alpha_i T_i}$ by the formula :

$$(Ad_{\prod_{i=1}^2 e^{X_i T_i}})_\chi = \prod_{i=1}^2 e^{X_i (ad_{T_i})_\chi} \quad (4.20)$$

Similarly, we have :

$$(Ad_{(\prod_{i=1}^2 e^{X_i T_i})^{-1}})_\chi = \prod_{i=1}^2 e^{-X_{3-i} (ad_{T_{3-i}})_\chi} \quad (4.21)$$

We can deduce the matrix \mathcal{P}^{ij} :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -\mathcal{P}^{21} \\ \mathcal{P}^{21} & 0 \end{pmatrix}$$

where

$$\mathcal{P}^{21} = \frac{c_{12}^1(-1 + e^{c_{12}^2 X_1})f_1^{12} + c_{12}^2 e^{c_{12}^2 X_1 - c_{12}^1 X_2}(-1 + e^{c_{12}^1 X_2})f_2^{12}}{c_{12}^2 c_{12}^1} \quad (4.22)$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_{\Sigma} \sum_{i=1}^2 dX_i \wedge A_i - \mathcal{P}^{21} A_2 \wedge A_1 \quad (4.23)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - \mathcal{P}^{21} A_2 &= 0 \\ dX_2 + \mathcal{P}^{21} A_1 &= 0 \\ dA_1 - \frac{c_{12}^1 c_{12}^2 f_1^{12} + c_{12}^2 c_{12}^2 (-e^{-c_{12}^1 X_2} + 1) f_2^{12}}{c_{12}^2 c_{12}^1} A_1 \wedge A_2 &= 0 \\ dA_2 + \frac{c_{12}^2 e^{c_{12}^2 X_1} c_{12}^1 f_2^{12}}{c_{12}^2 c_{12}^1} A_2 \wedge A_1 &= 0 \end{aligned} \quad (4.24)$$

5 Poisson-Lie sigma model of $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})^*)$

We first calculate the matrix of the adjoint actions :

$$\begin{aligned} ad_{T_1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\ ad_{T_2} &= \frac{1}{4} \begin{pmatrix} 0 & 0 & 4 & 0 & -1 & 0 \\ -8 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \end{pmatrix} \\ ad_{T_3} &= \frac{1}{4} \begin{pmatrix} 0 & -4 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

To obtain the matrix \mathcal{P} , we calculate the adjoint action matrix of a general element $g = \prod_{i=1}^3 e^{\alpha_i T_i}$ by the formula :

$$(Ad_{\prod_{i=1}^3 e^{X_i T_i}})_\chi = \prod_{i=1}^3 e^{X_i (ad_{T_i})_\chi} \quad (5.25)$$

Similarly, we have :

$$(Ad_{(\prod_{i=1}^3 e^{X_i T_i})^{-1}})_\chi = \prod_{i=1}^3 e^{-X_i (ad_{T_i})_\chi} \quad (5.26)$$

We can deduce the matrix \mathcal{P}^{ij} :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -\frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} & -\frac{X_3}{4}e^{-2X_1} \\ \frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} & 0 & \frac{X_2 X_3}{2} \\ \frac{X_3}{4}e^{-2X_1} & -\frac{X_2 X_3}{2} & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_\Sigma \sum_{i=1}^3 dX_i \wedge A_i + \left(\frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} \right) A_1 \wedge A_2 + \frac{X_3}{4}e^{-2X_1} A_1 \wedge A_3 - \frac{X_2 X_3}{2} A_2 \wedge A_3 \quad (5.27)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - \left(\frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} \right) A_2 - \frac{X_3}{4}e^{-2X_1} A_3 &= 0 \\ dX_2 + \left(\frac{X_2}{4}(1 + X_2 X_3)e^{2X_1} \right) A_1 + \frac{X_2 X_3}{2} A_3 &= 0 \\ dX_3 + \frac{X_3}{4}e^{-2X_1} A_1 - \frac{X_2 X_3}{2} A_2 &= 0 \\ dA_1 - \frac{X_2}{2}(1 + X_2 X_3) A_1 \wedge A_2 + \frac{X_3}{2} A_1 \wedge A_3 &= 0 \\ dA_2 - \frac{e^{2X_1}}{4} A_1 \wedge A_2 + \frac{X_3}{2} A_2 \wedge A_3 &= 0 \\ dA_3 - \frac{X_2^2}{4} e^{2X_1} A_1 \wedge A_2 - \frac{e^{-2X_1}}{4} A_1 \wedge A_3 + X_2 A_2 \wedge A_3 &= 0 \end{aligned}$$

6 Poisson-Lie sigma model of $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C})^*, \mathfrak{sl}(2, \mathbb{C}))$

Now to obtain this Poisson Lie sigma model, we have to change $T_i \rightarrow \tilde{T}^i$ and $\tilde{T}^i \rightarrow T_i$ of the previous model. And we can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \mathcal{P}^{ij} for this model :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -2e^{\frac{X_1}{4}} X_2 & 2e^{\frac{X_1}{4}} X_3 \\ 2e^{\frac{X_1}{4}} X_2 & 0 & 2 - \frac{1}{2}e^{\frac{X_1}{2}}(4 + X_2 X_3) \\ -2e^{\frac{X_1}{4}} X_3 & -2 + \frac{1}{2}e^{\frac{X_1}{2}}(4 + X_2 X_3) & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_{\Sigma} \sum_{i=1}^3 dX_i \wedge A_i + 2e^{\frac{X_1}{4}} X_2 A_1 \wedge A_2 - 2e^{\frac{X_1}{4}} X_3 A_1 \wedge A_3 + \left(-2 + \frac{1}{2}e^{\frac{X_1}{2}}(4 + X_2 X_3)\right) A_2 \wedge A_3 \quad (6.28)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - 2e^{\frac{X_1}{4}} X_2 A_2 + 2e^{\frac{X_1}{4}} X_3 A_3 &= 0 \\ dX_2 + 2e^{\frac{X_1}{4}} X_2 A_1 + \left(2 - \frac{1}{2}e^{\frac{X_1}{2}}(4 + X_2 X_3)\right) A_3 &= 0 \\ dX_3 - 2e^{\frac{X_1}{4}} X_3 A_1 - \left(2 - \frac{1}{2}e^{\frac{X_1}{2}}(4 + X_2 X_3)\right) A_2 &= 0 \\ dA_1 - \frac{X_2}{2} A_1 \wedge A_2 + \frac{X_3}{4} A_1 \wedge A_3 - \frac{1}{4}(4 + X_2 X_3) A_2 \wedge A_3 &= 0 \\ dA_2 - 2e^{\frac{X_1}{4}} A_1 \wedge A_2 - \frac{1}{2}e^{\frac{X_1}{4}} A_2 \wedge A_3 &= 0 \\ dA_3 + 2e^{\frac{X_1}{4}} A_1 \wedge A_3 - \frac{e^{\frac{X_1}{2}} X_2}{2} A_2 \wedge A_3 &= 0 \end{aligned}$$

7 Poisson-Lie sigma model of $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}))$

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \mathcal{P}^{ij} for this model :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3 & -\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3 \\ \cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3 & 0 & -1 + \cos X_2 \cos X_3 \\ \cos X_3 \sin X_1 \sin X_3 + \cos X_1 \sin X_3 & 1 - \cos X_2 \cos X_3 & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$\begin{aligned} S_2 &= \int_{\Sigma} \sum_{i=1}^3 dX_i \wedge A_i - (-\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3) A_1 \wedge A_2 \\ &\quad - (-\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3) A_1 \wedge A_3 - (-1 + \cos X_2 \cos X_3) A_2 \wedge A_3 \end{aligned} \quad (7.29)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 + (-\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3) A_2 + (-\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3) A_3 &= 0 \\ dX_2 + (\cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3) A_1 + (-1 + \cos X_2 \cos X_3) A_3 &= 0 \\ dX_3 + (\cos X_3 \sin X_1 \sin X_3 + \cos X_1 \sin X_3) A_1 (1 - \cos X_2 \cos X_3) A_2 &= 0 \\ dA_1 + \sin X_3 A_1 \wedge A_2 - \cos X_3 \sin X_2 A_1 \wedge A_3 &= 0 \\ dA_2 - \cos X_1 \cos X_3 A_1 \wedge A_2 - \cos X_3 \sin X_1 A_1 \wedge A_3 &= 0 \\ dA_3 + \sin X_1 A_1 \wedge A_2 - \cos X_1 A_1 \wedge A_3 &= 0 \end{aligned}$$

8 Poisson-Lie sigma model of $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sb}(2, \mathbb{C}), \mathfrak{su}(2, \mathbb{C}))$

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix \mathcal{P}^{ij} for this model :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -e^{X_1} X_3 & -e^{X_1} X_2 \\ e^{X_1} X_3 & 0 & \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2)) \\ e^{X_1} X_2 & \frac{1}{2}(-1 + e^{2X_1}(1 + X_2^2 + X_3^2)) & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_{\Sigma} \sum_{i=1}^3 dX_i \wedge A_i + e^{X_1} X_3 A_1 \wedge A_2 + e^{X_1} X_2 A_1 \wedge A_3 - \frac{1}{2}(1 - e^{2X_1}(1 + 2X_2^2 + 2X_3^2)) A_2 \wedge A_3 \quad (8.30)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - e^{X_1} X_3 A_2 - e^{X_1} X_2 A_3 &= 0 \\ dX_2 + e^{X_1} X_3 A_1 + \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2)) A_3 &= 0 \\ dX_3 + e^{X_1} X_2 A_1 - \frac{1}{2}(1 - e^{2X_1}(1 + X_2^2 + X_3^2)) A_2 &= 0 \\ dA_1 - X_3 A_1 \wedge A_2 - X_2 A_1 \wedge A_3 - (1 + X_2^2 + X_3^2) A_2 \wedge A_3 &= 0 \\ dA_2 - e^{X_1} A_1 \wedge A_3 &= 0 \\ dA_3 - e^{X_1} A_1 \wedge A_2 &= 0 \end{aligned}$$

9 Discussion

We gives here the Poisson-Lie sigma models of some Manin triples. Concerning the general formula (9.32), we have to say that this is no problem when c_{12}^2 and c_{12}^1 is zero because

$$\mathcal{P}^{21} = \frac{(-1 + e^{c_{12}^2 X_1}) f_1^{12}}{c_{12}^2} + \frac{e^{c_{12}^2 X_1 - c_{12}^1 X_2} (-1 + e^{c_{12}^1 X_2}) f_2^{12}}{c_{12}^2} \quad (9.31)$$

which can be approximate by

$$\mathcal{P}^{21} = (X_1 + \frac{c_{12}^2}{2} X_1^2 + \dots) f_1^{12} + e^{c_{12}^2 X_1 - c_{12}^1 X_2} (X_2 + \frac{c_{12}^1}{2} X_2^2 + \dots) f_2^{12} \quad (9.32)$$

We tried to obtain the equivalent formula for $n = 3$ but the calculus was too hard.

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