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On the infinity Laplacian Equation on Graph with Applications to image and Manifolds Processing

Elmoataz Abderrahim¹, Desquesnes Xavier¹, Lakhdari Zakaria¹
and Lezoray Olivier¹

¹ GREYC CNRS UMR 6072
6. Boulevard Marchal juin 14050 Caen
{abderrahim.elomataz-billah,lezoray,desquesnes}@unicaen.fr

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Abstract. *In this paper, an adaptation of the infinity laplacian equation to the case of weighted graph is proposed. Our approach is based on a family of discrete morphological local and nonlocal gradient on graphs expressed by partial difference equations. Our formulation generalize local and nonlocal configurations in the context of image processing and extend this equation to the processing any data that can be represented by a graph in any dimension. An example of applications will be given to illustrate the use of infinity-harmonic functions for local and nonlocal inpainting.*

1 Introduction

The ∞ -Laplacian equation is currently at the interface of different mathematical fields and applications. It was studied by Aronson [1] and has been the subject of renewed interest in several recent works. It has many applications as differential game, image processing [2], optimal transport. A detailed introduction to the ∞ -Laplacian can be found in [3].The infinity PDE infinity

Laplacian equation that can be expressed as

$$-\Delta_\infty f = 0 \quad (1)$$

where $\Delta_\infty f = \sum_{i,j}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_i \partial x_j}$ denotes the infinity Laplacian on Euclidean domain for smooth functions in some open set $\Omega \subseteq \mathbb{R}^n$. Aronson [1] interpreted formally $\Delta_\infty f = 0$ as the limit of the problem of minimization $\int \|\nabla f\|^p dx$ as $p \rightarrow \infty$, under given boundary conditions.

In this paper, we consider a graph $G_w = (V, E, w)$, consisting in a finite set of vertices V , a set of edges $E \subset V \times V$, and a similarity weight function w , which encodes local and nonlocal interactions, defined on edges. The main contributions of our paper can be summarized as follows.

As presented above, in a continuous setting, finding $f_p^* = \min \int \|\nabla f\|_2^p dx$ as $p \rightarrow \infty$ comes to find f^* that is solution of $\Delta_\infty f = 0$. In a discrete setting, with our propositions, finding $f_p^* = \min \sum \|\nabla_w f\|_p^p$ as $p \rightarrow \infty$ comes to find f^* that is solution of $L(f) = \|(\nabla_w^- f)(u)\|_\infty - \|(\nabla_w^+ f)(u)\|_\infty = 0$ where $\|(\nabla_w^+ f)(u)\|_\infty$ and $\|(\nabla_w^- f)(u)\|_\infty$ are the infinity norms of two families of discrete gradients on graph introduced in the next Sections. In the sequel, we will refer to $L(f)$ as the infinity Laplacian. This new definition of the nonlocal discrete infinity Laplacian is an extension of the well-known finite difference approximation of the infinity Laplacian. We also prove the existence and the uniqueness of solutions of the ∞ -Laplacian equation on graphs and we provide fast and simple digital algorithms for the solution.

We will use the infinity Laplacian equation as a framework for interpolation in image and manifold processing.

2 Partial difference operators on graphs

Let us consider the general situation where any discrete domain can be viewed as a weighted graph. A weighted graph $G_w = (V, E)$ consists in a finite set V of N vertices and in a finite set $E \subseteq V \times V$ of edges. Let (u, v) be the edge that connect vertices u and v . An undirected graph is weighted if it is associated with a weight function $w : E \rightarrow \mathbb{R}^+$ satisfying $w(u, v) = w(v, u)$, for all $(u, v) \in E$ and $w(u, v) = 0$, if $(u, v) \notin E$. The weight function represents a similarity measure between two vertices of the graph. We use the notation $u \sim v$ to denote two adjacent vertices. In this paper, the considered graphs are connected, undirected, with no self-loops nor multiple edge. Data to be processed are represented by real-valued functions, $f : V \rightarrow \mathbb{R}$, which assign a real value $f(u)$ to each vertex $u \in V$. These functions form a finite N -dimensional space and can be represented by vectors of \mathbb{R}^N .

We recall several definition of gradient and p -Laplacian on graphs. For details see [5, 6].

2.1 Gradient and norms

The \mathcal{L}_p -norm of this vector represents the *local variation* of a function $f \in \mathcal{H}(V)$ at a vertex of the graph. It is defined by [5, 6] for $0 < p < +\infty$:

$$\|(\nabla_w f)(u)\|_p = \left[\sum_{v \sim u} w(u, v)^{p/2} |f(v) - f(u)|^p \right]^{1/p}. \quad (2)$$

For the \mathcal{L}_∞ -norm, we have:

$$\|(\nabla_w f)(u)\|_\infty = \max_{v \sim u} w(u, v)^{1/2} |f(v) - f(u)|. \quad (3)$$

On the same way, and by considering two upwind discrete gradients on graphs, the \mathcal{L}_p norm of these gradients is given by

$$\|(\nabla_w^\pm f)(u)\|_p = \left[\sum_{v \sim u} w(u, v)^{p/2} ((f(v) - f(u))^\pm)^p \right]^{1/p}. \quad (4)$$

with $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0) = \max(-x, 0)$. For the \mathcal{L}_∞ -norm, we have

$$\|(\nabla_w^\pm f)(u)\|_\infty = \max_{v \sim u} \left(w(u, v)^{1/2} (f(v) - f(u))^\pm \right). \quad (5)$$

2.2 The anisotropic p -Laplacian with $1 \leq p < \infty$

The *weighted p -Laplace anisotropic operator* of a function $f \in \mathcal{H}(V)$, noted $\Delta_{w,p} : \mathcal{H}(V) \rightarrow \mathcal{H}(V)$, is defined by:

$$(\Delta_{w,p} f)(u) = \frac{1}{2} d_w^* (|(d_w f)(u, v)|^{p-2} (d_w f)(u, v)). \quad (6)$$

The anisotropic p -Laplace operator of $f \in \mathcal{H}(V)$, at a vertex $u \in V$, can be computed by:

$$(\Delta_{w,p} f)(u) = \sum_{v \sim u} (\gamma_{w,p} f)(u, v) (f(u) - f(v)) \quad (7)$$

with

$$(\gamma_{w,p} f)(u, v) = w(u, v)^{p/2} |f(u) - f(v)|^{p-2}. \quad (8)$$

This operator is nonlinear if $p \neq 2$. In this latter case, it corresponds to the *combinatorial graph Laplacian*, as the isotropic 2-Laplacian. To avoid zero denominator in (7) when $p \leq 1$, $|f(u) - f(v)|$ is replaced by $|f(u) - f(v)|_\epsilon = |f(u) - f(v)| + \epsilon$, where $\epsilon \rightarrow 0$ is a small fixed constant.

One can remark that all these definitions do not depend on the graph structure.

3 The ∞ -Laplacian Equation on Weighted Graphs

let us show that there is a relation between the anisotropic Laplacian and morphological gradients presented above.

Proposition 3.1 *For $1 \leq p < +\infty$, at a vertex $u \in V$*

$$(\Delta_{w,(p+1)}f)(u) = \|(\nabla_{w'}^- f)(u)\|_p^p - \|(\nabla_{w'}^+ f)(u)\|_p^p \quad (9)$$

with $w'(u, v) = w(u, v)^{\frac{p+1}{p}}$.

Proof 1 *From (7) and (8), we have*

$$(\Delta_{w,p}f)(u) = \sum_{v \sim u} w(u, v)^{p/2} |f(u) - f(v)|^{p-2} (f(u) - f(v)).$$

Since $|x| = x^+ + x^-$ and $x = x^+ - x^-$, one has, with $A = (f(v) - f(u))$:

$$\begin{aligned} (\Delta_{w,(p+1)}f)(u) = \\ \sum_{v \sim u} w(u, v)^{\frac{p+1}{2}} (A^- - A^+) (A^+ + A^-)^{p-1}. \end{aligned} \quad (10)$$

Then, by developing $(A^+ + A^-)^{p-1}$, since $A^+ A^- = 0$, it is easy to obtain (9).

3.1 The nonlocal infinity Laplacian

We consider the following minimization problem: $f_p^* = \min \sum \|\nabla_w f\|_p^p$. By standard convex analysis, this is equivalent to find f^* that is solution of $\Delta_{w,p}f = 0$.

From proposition 3.1, we have

$$(\Delta_{w,p}f)(u) = \|(\nabla_{w'}^- f)(u)\|_{p-1}^{p-1} - \|(\nabla_{w'}^+ f)(u)\|_{p-1}^{p-1} = 0$$

with $w'(u, v) = w(u, v)^{\frac{p}{p-1}}$ that we can easily simplify into

$$(\Delta_{w,p}f)(u) = \|(\nabla_{w'}^- f)(u)\|_{p-1} - \|(\nabla_{w'}^+ f)(u)\|_{p-1} = 0. \quad (11)$$

By substituting and letting $p \rightarrow \infty$ in (11), one has

$$\begin{aligned} \lim_{p \rightarrow \infty} (\|(\nabla_{w'}^- f)(u)\|_{p-1} - \|(\nabla_{w'}^+ f)(u)\|_{p-1}) = \\ \|(\nabla_w^- f)(u)\|_\infty - \|(\nabla_w^+ f)(u)\|_\infty = 0. \end{aligned} \quad (12)$$

Based on proposition 3.1, and the previous limit, we propose a new definition of the ∞ -Laplacian:

$$(\Delta_{w,\infty}f)(u) = \|(\nabla_w^- f)(u)\|_\infty - \|(\nabla_w^+ f)(u)\|_\infty \quad (13)$$

3.2 Uniqueness and existence of the infinity Laplacian equation on graphs

Theorem 3.2 *Given $G_w = (V, E, w)$ a finite weighted graph, let A a subset of V , ∂A the boudary of A and a function $g : V \rightarrow \mathbb{R}$. There exist a unique function f on A wich satsfies*

$$\begin{cases} \Delta_{\infty, w} f(u) = 0 & \text{if } u \in A \\ f(u) = g & \text{if } u \in \partial A. \end{cases}$$

The proof is based on the introduction of the following nonlocal average operator $\text{NLA}(f(u)) = \max_{v \sim u}(\sqrt{w(u, v)}(f(v) - f(u)) + \min_{v \sim u}(\sqrt{w(u, v)}(f(v) - f(u))$. It's easy to show that $\Delta_{\infty, w} f(u) = 0$ can be expressed by $f(u) = \text{NLA}(f(u))$. In that case, the Uniqueness of the laplacian equation is proved by the comparison principled applied to $\text{NLA}(f(u)) = f(u)$. The Existence is proved by considering the sequence $f^{n+1}(u) = \text{NLA}(f^n(u))$ and its limit. For details see [4].

4 Inverse problems on weighted graphs with the infinity Laplacian

Many tasks in image processing and computer vision can be formulated as interpolation problems. Image inpainting is a typical example of such an interpolation problem. Inpainting data consists in constructing new values for missing data in coherence with a set of known data. This problem can be formalized as follows: We consider that data are defined on a general domain represented on a graph $G_w = (V, E)$. Let $g : V \rightarrow \mathbb{R}$ be a function. Let $A \subset V$ be the subset of vertices with unknown values and ∂A its external boundary (such as g is only known on ∂A). The purpose of inpainting is to find a function f approximating g in V minimizing the following energy:

$$\begin{cases} \Delta_{\infty, w} f(u) = 0 & \forall u \in A \\ f(u) = g(u) & \forall u \in \partial A. \end{cases} \quad (14)$$

Figure 1 shows an example of such image inpainting results. One can see that results are similar with some little improvement in texture restoration for the ∞ -Laplacian.

5 Conclusion

In this paper an adaptation of the infinity Laplacian equation to the case of weighted graphs was proposed. We have proved the existence and the uniqueness of this equation. We show that this equation can be used as a framework

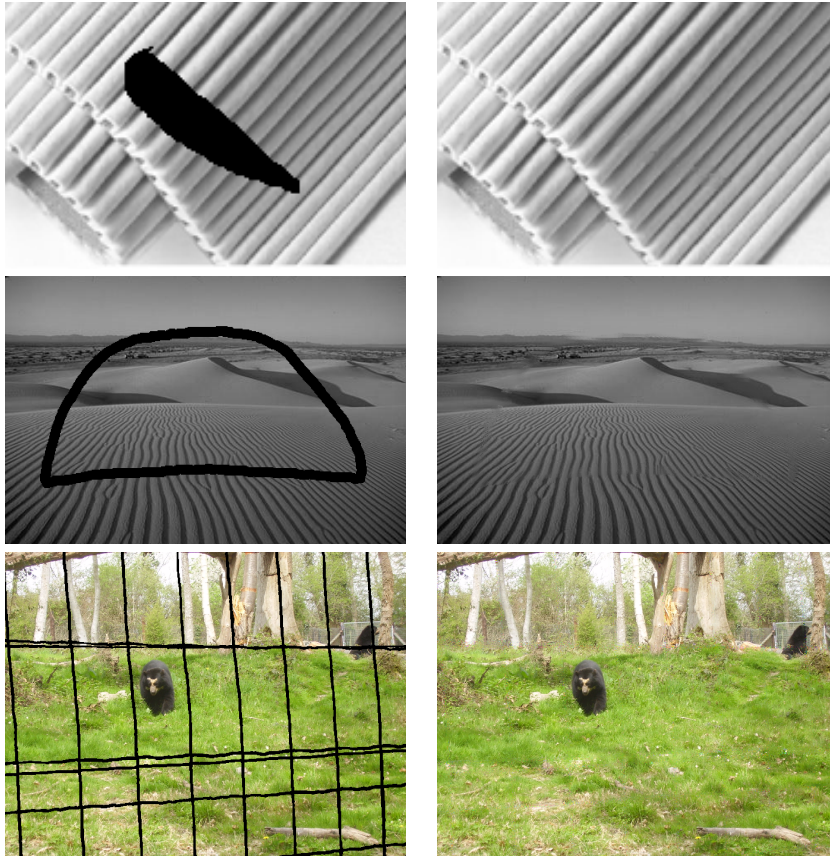


Figure 1: Image inpainting illustration. From left to right: image with a zone to inpaint, image inpainted with $p = 2$, image inpainted with $p = \infty$, where graph's weight depend on distance between patches defined on image.

for interpolation on graphs. Experimental results with the proposed numerical scheme have illustrated the interest of the approach.

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