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# Dimensional contraction via Markov transportation distance

François Bolley\*, Ivan Gentil† and Arnaud Guillin‡

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## Abstract

It is now well known that curvature conditions *à la* Bakry-Émery are equivalent to contraction properties of the heat semigroup with respect to the classical quadratic Wasserstein distance. However, this curvature condition may include a dimensional correction which up to now had not induced any strengthening of this contraction. We first consider the simplest example of the Euclidean heat semigroup, and prove that indeed it is so. To consider the case of a general Markov semigroup, we introduce a new distance between probability measures, based on the semigroup, and adapted to it. We prove that this Markov transportation distance satisfies the same properties for a general Markov semigroup as the Wasserstein distance does in the specific case of the Euclidean heat semigroup, namely dimensional contraction properties and Evolutional variational inequalities.

**Key words:** Diffusion equations, Wasserstein distance, Markov semigroups, Curvature-dimension bounds.

## 1 Introduction

Contraction properties of (Markov) semigroups are an important probabilistic and analytic tool: for instance they enable to study the existence of invariant probability measures, or the stability and long time behaviour of solutions to various linear (Fokker-Planck, kinetic Fokker-Planck, ...) or non linear (McKean-Vlasov, porous medium, Boltzmann, ...) partial differential equations. An important aspect is of course the distance in which we measure this contraction. If (weighted) total variation has often been the first choice of probabilists (see e.g. Meyn-Tweedie [MT93]),  $L^2$ , Sobolev or Fourier norms have more usually been chosen in PDE works (see e.g. [CT07]). However, recent progress has shown that the Kantorovich-Rubinstein-Wasserstein distance (Wasserstein distance in short) is a particularly relevant and natural choice, in particular, but not only, for dynamics which have been interpreted as gradient flows for this distance

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(see for example [Ott01, CT05, CMV06, CGM08, BGM10, NPS11, BGG12] and the reference books [AGS08, Vil09]). Here and below the Wasserstein distance between two Borel probability measures  $\nu$  and  $\mu$  on a Polish metric space  $(E, d)$  is defined by

$$W_2(\mu, \nu) = \inf \left( \int d^2(x, y) d\pi(x, y) \right)^{1/2},$$

where the infimum runs over all probability measures  $\pi$  in  $E \times E$  with marginals  $\mu$  and  $\nu$ . We refer again to [AGS08, Vil09] for a reference presentation of this distance, its interplay with the optimal transportation problem and many other issues.

Geometric properties of metric spaces are an important and vast topic with many diverse issues, and the Wasserstein distance has provided new insight on them, see [OW05, vRS05, Stu06, LV09, AGS12a, AGS12b, EKS13]. A particularly relevant notion is the one of curvature which has recently attracted much attention. It can be handled in terms of a contraction property in Wasserstein distance as follows.

Let  $(H_t)_{t \geq 0}$  denote the heat semigroup on a smooth and complete (and connex) Riemannian manifold  $(M, g)$ : it solves the heat equation  $\partial_t u = \Delta_g u$  where  $\Delta_g$  is the Laplace-Beltrami operator on  $M$ . Let also  $\mu$  be the Riemannian measure on  $(M, g)$  and  $d$  the associated Riemannian distance. Then a fundamental result, due to M. von Renesse and K.-T. Sturm in [vRS05], says that the Ricci curvature of the manifold is bounded from below by a constant  $R \in \mathbb{R}$  if and only if

$$W_2(H_t f \mu, H_t g \mu) \leq e^{-Rt} W_2(f \mu, g \mu) \quad (1)$$

for any  $t \geq 0$  and any probability densities  $f, g$  with respect to  $\mu$ . Diverse proofs and generalizations of this contraction result are given in [OW05, Wan04, BGL12].

A crucial challenging problem now consists in understanding the role of the dimension in the contraction property in Wasserstein distance. Indeed curvature and dimension are jointly considered in the synthetic definition by Lott-Sturm-Villani [Stu06, LV09], contraction properties, gradient commutation type properties or the Bakry-Émery curvature-dimension condition. It is for instance well known that, given  $R \in \mathbb{R}$  and  $n \geq 1$ , the  $CD(R, n)$  curvature-dimension condition proposed by D. Bakry and M. Émery in [BÉ85], see section 4, is satisfied for the Laplace-Beltrami operator on a Riemannian manifold if and only if the Ricci curvature of the manifold is uniformly bounded from below by  $R$  and the dimension of the manifold is smaller than  $n$ . The diverse notions of curvature have been proved to be equivalent for a very large panel of spaces, such as Alexandrov or Finsler spaces : we refer to the recent works [OS11, AGS, Pet11, AGS12b, AGS12a, EKS13] for a review on the subject.

This has been very recently performed in the following two remarkable results, by deriving an upper bound on the distance  $W_2(H_t f \mu, H_s g \mu)$  with two *different* times  $s, t > 0$ :

- The first result is due to K. Kuwada in [Kuw] : the Ricci curvature of the  $n$ -dimensional manifold  $M$  is bounded from below by a constant  $R \in \mathbb{R}$  if and only if

$$W_2^2(H_t f \mu, H_s g \mu) \leq A(s, t, R) W_2^2(f \mu, g \mu) + B(s, t, n, R), \quad (2)$$

for any  $s, t > 0$  and any probability densities  $f, g$  with respect to  $\mu$ , for appropriate functions  $A, B \geq 0$ . In the case  $R = 0$  the bound simplifies into

$$W_2^2(H_t f \mu, H_s g \mu) \leq W_2^2(f \mu, g \mu) + 2n(\sqrt{t} - \sqrt{s})^2, \quad (3)$$

stated independently in [BGL12].

• The second result is due to M. Erbar, K. Kuwada and K.-T. Sturm [EKS13] : the Ricci curvature of the  $n$ -dimensional manifold  $M$  is bounded from below by a constant  $R \in \mathbb{R}$  if and only if

$$s_{\frac{R}{n}} \left( \frac{1}{2} W_2(H_t f \mu, H_s g \mu) \right)^2 \leq e^{-R(t+s)} s_{\frac{R}{n}} \left( \frac{1}{2} W_2(f \mu, g \mu) \right)^2 + \frac{n}{R} (1 - e^{-R(s+t)}) \frac{(\sqrt{t} - \sqrt{s})^2}{2(t+s)}$$

for any  $s, t > 0$  and any probability densities  $f, g$  with respect to  $\mu$ . Here  $s_r(x) = \sin(\sqrt{r}x)/\sqrt{r}$  if  $r > 0$ ,  $s_r(x) = \sinh(\sqrt{-r}x)/\sqrt{-r}$  if  $r < 0$  and  $s_0(x) = x$ , hence recovering (3) for  $R = 0$ .

In these two results the dimension  $n$  appears only when the two solutions are considered at different times  $s$  and  $t$ . When  $t = s$  in (2) or (3) then the positive additional terms vanish and we only recover the classical contraction inequality (1).

A first aim of this paper is to take the dimension into account and to improve inequality (1) for solutions considered at the *same* time. For instance in section 2 we prove that

$$W_2^2(H_T f dx, H_T g dx) \leq W_2^2(f dx, g dx) - \frac{2}{n} \int_0^T (\text{Ent}_{dx}(H_t f) - \text{Ent}_{dx}(H_t g))^2 dt \quad (4)$$

for the heat semigroup on  $\mathbb{R}^n$ , any  $T \geq 0$  and any probability densities  $f, g$  with respect to the Lebesgue measure  $dx$ ; here  $\text{Ent}_\mu(g) = \int g \log g d\mu$  is the entropy. This inequality improves on (1) since the Euclidean space  $\mathbb{R}^n$  has null Ricci curvature and then satisfies (1) with  $R = 0$ . Let us observe that a dimensional contraction property in a Wasserstein distance with a modified cost was derived by F.-Y. Wang in [Wan11].

A second aim is to obtain dimensional contraction inequalities for *general* Markov semigroups. For that purpose we will work with a new distance called *Markov transportation distance*, based on the generator of the semigroup, and adapted to it and to the Bakry-Émery curvature-dimension condition formulation. It is defined by a modification of the following dynamical formulation of the Wasserstein distance proposed by J.-D. Benamou and Y. Brenier in [BB00] : for any probability densities  $f$  and  $g$  with respect to the Lebesgue measure in  $\mathbb{R}^n$ ,

$$W_2(f dx, g dx) = \inf \left( \int_0^1 \int \frac{|w_s|^2}{\rho_s} dx ds \right)^{1/2}$$

where the infimum runs over all paths  $(\rho_s)_{s \in [0,1]}$  and vector fields  $(w_s)_{s \in [0,1]}$  such that  $\partial_s \rho_s + \nabla \cdot w_s = 0$ ,  $\rho_0 = f$  and  $\rho_1 = g$ ; here  $\nabla \cdot$  stands for the divergence operator on  $\mathbb{R}^n$ . This dynamical approach is the starting point of the definition in [DNS09, DNS12] of generalized distances. Following [BB00], for any probability densities  $f, g$  with respect to the Lebesgue measure  $dx$  in  $\mathbb{R}^n$ , we define here the Markov transportation distance by

$$T_2(f dx, g dx) = \inf \left( \int_0^1 \int \frac{|\nabla h_s|^2}{\rho_s} dx ds \right)^{1/2}$$

where the infimum runs over all paths  $(\rho_s, h_s)_{s \in [0,1]}$  such that  $\partial_s \rho_s + \nabla \cdot (\nabla h_s) = 0$ ,  $\rho_0 = f$  and  $\rho_1 = g$ .

This distance can also be defined in the abstract setting of a Markov generator  $L$ , with carré du champ  $\Gamma$  and invariant measure  $\mu$ , by

$$T_2(f \mu, g \mu) = \inf \left( \int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds \right)^{1/2}$$

under the constraint  $\partial_s \rho_s + Lh_s = 0$ . In this abstract formulation, discrete and non-local operators can be studied in a similar way. A fundamental instance is that of  $L = \Delta - \nabla V \cdot \nabla$  on  $\mathbb{R}^n$ , with carré du champ  $\Gamma(f) = |\nabla f|^2$  and invariant measure  $\mu = e^{-V}$  : in this case  $W_2(f\mu, g\mu) \leq T_2(f\mu, g\mu)$  since the infima defining the distances run over a smaller set for  $T_2$  than for  $W_2$ .

The paper is organized as follows. In section 2 we show in a simple way how to reach the dimension dependent contraction property (4) in Wasserstein distance for the specific heat semigroup on  $\mathbb{R}^n$ . It will give a flavor of the results proved and the methods used below in the Markov transportation distance  $T_2$  for abstract Markov semigroups  $(P_t)_{t \geq 0}$ .

This distance is properly defined in section 3, for general Markov semigroups, together with fundamental properties and examples. In particular we derive an Otto-Villani theorem for  $T_2$  : a logarithmic Sobolev inequality implies a transportation Talagrand inequality.

Section 4 is devoted to our main application : the contraction property under the curvature-dimension condition  $CD(R, n)$  on our semigroup. Under this condition we prove that

$$T_2^2(P_T f\mu, P_T g\mu) \leq e^{-2RT} T_2^2(f\mu, g\mu) - \frac{2}{n} \int_0^T e^{-2R(T-t)} (\text{Ent}_\mu(P_t g) - \text{Ent}_\mu(P_t f))^2 dt,$$

for any  $T > 0$  and any probability densities  $f, g$  with respect to the invariant measure  $\mu$ .

In section 5 we briefly consider the so-called Evolutional variational inequalities (EVI in short). These inequalities say that if the Ricci curvature of a manifold is bounded from below by a constant  $R \in \mathbb{R}$ , then

$$W_2^2(f\mu, H_t g\mu) - W_2^2(f\mu, g\mu) \leq -\frac{e^{-2Rt} - 1 + 2Rt}{2Rt} W_2^2(f\mu, g\mu) + 2t(\text{Ent}_\mu(f) - \text{Ent}_\mu(H_t g)). \quad (5)$$

for the heat semigroup  $(H_t)_{t \geq 0}$ , any  $t \geq 0$  and any probability densities  $f, g$  with respect to  $\mu$ . This inequality characterizes  $(H_t)_{t \geq 0}$  as the gradient flow of the entropy with respect to the Wasserstein distance. This interpretation has been made by R. Jordan, D. Kinderlehrer and F. Otto in [JKO98], and has led to numerous developments, see in particular the seminal paper [OV00] and the huge contribution of [AGS08]. In section 5 we explain how to obtain a dimensional EVI for the Wasserstein distance and the Euclidean heat semigroup, and then in the general setting of the Markov transportation distance and a general Markov semigroup under a curvature-dimension condition.

In Section 6 we briefly investigate natural generalizations of the Markov transportation distance.

Many questions are of course left aside in this work, such as the general existence of geodesics, dual formulations and further equivalence between the obtained contraction and curvature conditions : they will be further investigated elsewhere.

## 2 The heat equation on $\mathbb{R}^n$

This section is devoted to the simple derivation of a dimension dependent contraction property for the heat semigroup  $(H_t)_{t \geq 0}$  on  $\mathbb{R}^n$ . It is defined by

$$H_t f(x) = \int_{\mathbb{R}^n} f(y) \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{n/2}} dy$$

and is obtained as the solution of the heat equation  $\partial_t u = \Delta u$ ; here  $\Delta$  is the usual Laplace operator in  $\mathbb{R}^n$ .

For this semigroup, the bound (1) is classical with  $R = 0$  and  $\mu$  the Lebesgue measure  $dx$  on  $\mathbb{R}^n$ , and is optimal in the sense that equality holds for all  $t$  if  $g$  is obtained from  $f$  by a translation in  $\mathbb{R}^n$ . Let us see how to simply turn this classical bound into a more precise dimension dependent bound.

Following [DNS12], let  $(R_t)_{t \geq 0}$  be the heat semigroup acting on  $\mathbb{R}^n$ -valued maps, coordinate by coordinate. It satisfies

$$H_t(\nabla \cdot w) = \nabla \cdot (R_t w) \quad (6)$$

for all  $\mathbb{R}^n$ -valued functions  $w$ . This semigroup acting on vectors will be the main tool in our derivation. We omit regularity issues which are carefully considered in [DNS12].

As recalled in the introduction, the Benamou-Brenier Theorem ensures that

$$W_2^2(fdx, gdx) = \inf \int_0^1 \int \frac{|w_s|^2}{\rho_s} ds dx \quad (7)$$

for any probability measures  $fdx$  and  $gdx$  in  $\mathbb{R}^n$ ; here the infimum runs over all couples  $(\rho_s, w_s)_{s \in [0,1]}$  such that

$$\partial_s \rho_s + \nabla \cdot w_s = 0 \quad (8)$$

where, for all  $s \in [0,1]$ ,  $\rho_s$  is a probability density with respect to Lebesgue measure,  $\rho_0 = f$  and  $\rho_1 = g$ .

Let now  $(\rho_s, w_s)_{s \in [0,1]}$  interpolate the densities  $f$  and  $g$  with the constraint (8). Then  $(H_t(\rho_s))_{s \in [0,1]}$  interpolates the densities  $H_t f$  and  $H_t g$  and, by (6), the couple  $(H_t(\rho_s), R_t(w_s))_{s \in [0,1]}$  satisfies (8). Then, by (7),

$$W_2^2(H_T f dx, H_T g dx) \leq \int_0^1 \int \frac{|R_T(w_s)|^2}{H_T(\rho_s)} ds dx \quad (9)$$

for any  $T \geq 0$ . Moreover:

**Lemma 2.1** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g$  a positive probability density with respect to the Lebesgue measure, with  $F$  and  $g$  smooth. Then, for all  $T \geq 0$*

$$\int \frac{|R_T F|^2}{H_T g} dx \leq \int \frac{|F|^2}{g} dx - \frac{2}{n} \int_0^T \left( \int \frac{R_t F \cdot \nabla H_t g}{H_t g} dx \right)^2 dt.$$

**Proof**

◁ We let

$$\Lambda(t) = \int \frac{|R_t F|^2}{H_t g} dx$$

for  $t \geq 0$  and prove that

$$\Lambda'(t) \leq -\frac{2}{n} \left( \int \frac{R_t F \cdot \nabla H_t g}{H_t g} dx \right)^2,$$

which will prove the lemma by time integration. Indeed

$$\Lambda'(t) = \int \left( 2 \frac{R_t F \cdot \Delta R_t F}{H_t g} - \frac{\Delta H_t g |R_t F|^2}{(H_t g)^2} \right) dx.$$

For notational simplicity, we let  $\bar{F} = R_t F$ ,  $\bar{g} = H_t g$  and then  $\bar{G} = \log \bar{g}$ . Since

$$0 = \int \Delta \left( \frac{|\bar{F}|^2}{\bar{g}} \right) dx = \int 2 \nabla(|\bar{F}|^2) \cdot \nabla \left( \frac{1}{\bar{g}} \right) + \frac{1}{\bar{g}} \Delta(|\bar{F}|^2) + |\bar{F}|^2 \Delta \left( \frac{1}{\bar{g}} \right) dx,$$

we obtain

$$\begin{aligned} \Lambda'(t) &= - \int \frac{2}{\bar{g}} \left( \frac{1}{2} \Delta |\bar{F}|^2 - \bar{F} \cdot \Delta \bar{F} + \nabla(|\bar{F}|^2) \nabla \bar{G} + |\bar{F}|^2 |\nabla \bar{G}|^2 \right) dx \\ &= - \int \frac{2}{\bar{g}} \sum_{1 \leq i, j \leq n} \left( \partial_i \bar{F}_i + \bar{F}_i \partial_j \bar{G} \right)^2 dx \leq - \int \frac{2}{\bar{g}} \sum_{1 \leq i \leq n} \left( \partial_i \bar{F}_i + \bar{F}_i \partial_i \bar{G} \right)^2 dx \\ &= - \frac{2}{n} \int \bar{g} \left( \sum_{1 \leq i \leq n} \frac{\partial_i \bar{F}_i}{\bar{g}} + \frac{\bar{F}_i \partial_i \bar{G}}{\bar{g}} \right)^2 dx \\ &\leq - \frac{2}{n} \left( \sum_{1 \leq i \leq n} \int (\partial_i \bar{F}_i + \frac{\bar{F}_i \partial_i \bar{g}}{\bar{g}}) dx \right)^2 = - \frac{2}{n} \left( \int \frac{\bar{F} \cdot \nabla \bar{g}}{\bar{g}} dx \right)^2 \end{aligned}$$

by the Cauchy-Schwarz inequality, the Jensen inequality for the probability measure  $\bar{g} dx$  and the relation  $\int \sum_i \partial_i \bar{F}_i dx = 0$ .  $\triangleright$

Then, by Lemma 2.1 and the Cauchy-Schwarz inequality (with respect to the measure  $ds$ ),

$$\int_0^1 \int \frac{|R_T(w_s)|^2}{H_T(\rho_s)} ds dx \leq \int_0^1 \int \frac{|w_s|^2}{\rho_s} ds dx - \frac{2}{n} \int_0^T \left( \int \int_0^1 \frac{R_t(w_s) \cdot \nabla H_t(\rho_s)}{H_t(\rho_s)} dx ds \right)^2 dt.$$

Moreover the couple  $(H_t(\rho_s), R_t(w_s))$  satisfies (8), so

$$\int \frac{R_t(w_s) \cdot \nabla H_t(\rho_s)}{H_t(\rho_s)} dx = \partial_s \int H_t(\rho_s) \log H_t(\rho_s) dx,$$

and then

$$\int \int_0^1 \frac{R_t(w_s) \cdot \nabla H_t(\rho_s)}{H_t(\rho_s)} dx ds = \text{Ent}_{dx}(H_t f) - \text{Ent}_{dx}(H_t g).$$

Then inequality (9) leads to the following refined contraction inequality for the heat semigroup in  $\mathbb{R}^n$ :

**Proposition 2.2** *Let  $(H_t)_{t \geq 0}$  be the heat semigroup on  $\mathbb{R}^n$ . Then for any probability densities  $f$  and  $g$  in  $\mathbb{R}^n$  such that  $W_2(f dx, g dx) < \infty$ , for any  $T > 0$ ,*

$$W_2^2(H_T f dx, H_T g dx) \leq W_2^2(f dx, g dx) - \frac{2}{n} \int_0^T (\text{Ent}_{dx}(H_t f) - \text{Ent}_{dx}(H_t g))^2 dt. \quad (10)$$

**Remark 2.3** *By comparison with (3), the dimension brings a negative correction term in the contraction property. The bound (10) is again an equality if  $g$  is obtained from  $f$  by translation. Finally, a Taylor expansion of (10), for  $T$  close to 0 and  $g$  close to  $f$ , for any given  $f$ , implies back the curvature dimension  $CD(0, n)$  for the Laplace operator (see section 4 below for the precise definition of the curvature-dimension condition).*

**Remark 2.4** *Note that this result not only gives a correction for equal times, but also for different times  $s, t$  in the spirit of [BGL12], [EKS13] or [Kuw]. Let indeed  $s \leq t$ : then applying the contraction estimate (10) to  $P_{t-s}f$  and  $g$  and then using (3) lead to*

$$W_2^2(H_t f dx, H_s g dx) \leq W_2^2(f dx, g dx) + n(t-s) - \frac{2}{n} \int_0^s (\text{Ent}_{dx}(H_{t-s+u}f) - \text{Ent}_{dx}(H_u g))^2 du.$$

Our main goal is then the extension of this contraction result to general Markov semigroups satisfying a  $CD(R, n)$  condition, which will be given in Theorem 4.5 : there the Markov transportation distance will prove to be an adapted and efficient tool.

## 3 The Markov transportation distance

### 3.1 Definition

Let  $(E, \mu)$  be a Polish measure space and  $(P_t)_{t \geq 0}$  be a Markov semigroup on  $E$ , with invariant measure  $\mu$ , that is,

$$\int P_t f d\mu = \int f d\mu$$

for any  $f \in L^1(\mu)$ . Its Markov infinitesimal generator  $L$  is defined on a dense subspace  $\mathcal{D}(L)$  of  $L^2(\mu)$ . We assume that there exists an algebra  $\mathcal{A}$  of smooth and bounded functions, which is dense in all  $L^p(\mu)$  for  $1 \leq p < +\infty$  and stable by  $L$ ,  $P_t$  and the action of  $C^\infty$  functions which are zero at zero. Such a Markov semigroup admits a Markov probability kernel, that is for any function  $f \in L^2(\mu)$ ,  $t \geq 0$  and  $x \in E$ ,

$$P_t f(x) = \int_{\mathbb{R}^n} f(y) p_t(x, dy).$$

The carré du champ operator is defined on functions  $f, g \in \mathcal{A}$  by

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - f Lg - g Lf) \in \mathcal{A}.$$

For simplicity we shall let  $\Gamma(f) = \Gamma(f, f)$ . The Dirichlet form  $\mathcal{E}_\mu(f) = \int \Gamma(f) d\mu$  is defined on its domain  $\mathcal{D}(\mathcal{E}_\mu) \subset L^2(\mu)$ . We refer to [BGL13] for further details on Markov semigroups and to section 3.2 for examples.

Before giving the definition of the Markov transportation distance, we need to define the paths between probability densities. We let  $\mathcal{F}$  be the set of probability densities with respect to  $\mu$  and  $\mathcal{K}((0, 1) \times E)$  the set of measurable functions  $(s, x) \in [0, 1] \times E \mapsto \zeta_s(x)$  such that for every  $x \in E$ , the map  $s \mapsto \zeta_s(x)$  is a  $C^\infty$ -function with compact support in  $(0, 1)$ , and for every  $s \in [0, 1]$ ,  $\Gamma(\zeta_s)$  has a compact support in  $E$ .

**Definition 3.1** *Let  $(\rho_s)_{s \in [0, 1]}$  (resp.  $(h_s)_{s \in [0, 1]}$ ) be a family of functions in  $\mathcal{F}$  (resp.  $\mathcal{D}(\mathcal{E}_\mu)$ ) with  $(s, x) \mapsto (\rho_s(x), h_s(x))$  measurable. The couple  $(\rho_s, h_s)_{s \in [0, 1]}$  is a weak solution of*

$$\partial_s \rho_s + L h_s = 0, \tag{11}$$

*if*

$$\int_0^1 \int \Gamma(h_s) d\mu ds < \infty, \tag{12}$$



and for any  $\zeta_s \in \mathcal{K}((0, 1) \times E)$ ,

$$\int_0^1 \int \rho_s(x) \partial_s \zeta_s(x) d\mu ds + \int_0^1 \int \Gamma(h_s, \zeta_s) d\mu ds = 0. \quad (13)$$

Such a couple  $(\rho_s, h_s)_{s \in [0, 1]}$  is called an *admissible couple*.

**Definition 3.2** Let  $f, g \in \mathcal{F}$  and let  $(\rho_s, h_s)_{s \in [0, 1]}$  be a weak solution of (11) such that  $\rho_0 = f$  and  $\rho_1 = g$ . For such a couple we define

$$\varphi(\rho_s, h_s) = \int \frac{\Gamma(h_s)}{\rho_s} d\mu \in [0, \infty]$$

for  $s \in [0, 1]$ , and the action

$$\Phi(\rho, h) = \int_0^1 \varphi(\rho_s, h_s) ds = \int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds \in [0, \infty].$$

An admissible couple  $(\rho_s, h_s)_{s \in [0, 1]}$  satisfying  $\rho_0 = f$  and  $\rho_1 = g$  is called an *admissible path* between  $f$  and  $g$ . We let  $\mathcal{A}(f, g)$  denote the set of such admissible paths between  $f$  and  $g$ .

**Definition 3.3** The Markov transportation distance is defined for  $f, g \in \mathcal{F}$  by

$$T_2(f\mu, g\mu) = \inf \left( \int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds \right)^{1/2} \in [0, \infty],$$

where the infimum runs over all admissible paths  $(\rho_s, h_s)_{s \in [0, 1]} \in \mathcal{A}(f, g)$ . We agree that  $T_2(f\mu, g\mu) = +\infty$  if the set  $\mathcal{A}(f, g)$  is empty.

For instance let  $f, g \in \mathcal{F}$  for which there exists  $h \in \mathcal{D}(L)$  such that  $Lh = f - g$ . Then, assuming that  $\int \Gamma(h) d\mu < \infty$ , the quantity  $T_2(f\mu, g\mu)$  is well defined since the path  $\rho_s = sf + (1-s)g$  associated with the function  $h$  (independent of  $s$ ) is an admissible path. Moreover it is bounded from above by

$$\int_0^1 \varphi(sf + (1-s)g, h_s) ds = \int_0^1 \int \frac{\Gamma(h)}{sf + (1-s)g} ds d\mu = \int \Gamma(h) \frac{\log(f) - \log(g)}{f - g} d\mu.$$

In particular, if for instance  $f, g > \eta$  for some  $\eta > 0$ , then

$$T_2^2(f\mu, g\mu) \leq \frac{1}{\eta} \int \Gamma(h) d\mu.$$

## 3.2 Examples

The Markov transportation distance heavily depends on both the reference measure  $\mu$  and the generator  $L$ . For instance, changing  $L$  into  $\kappa L$  for  $\kappa > 0$  multiplies the Markov transportation distance by  $\kappa^{-1/2}$ . Moreover, for two generators  $L_1$  and  $L_2$  with same carré du champ  $\Gamma$  but diverse invariant measures  $\mu_1$  and  $\mu_2$  with for instance  $\mu_1$  absolutely continuous with respect to  $\mu_2$ , then a priori  $T_2(f\mu_1, g\mu_1)$ , defined from  $L_1$  and  $\mu_1$ , differs from  $T_2(f \frac{d\mu_1}{d\mu_2} \mu_2, g \frac{d\mu_1}{d\mu_2} \mu_2)$ , defined from  $L_2$  and  $\mu_2$ .

## Diffusion case

As it has been presented in the introduction, the Markov transportation distance is a generalization of the Benamou-Brenier dynamical formulation ([BB00]). For instance, for  $L = \Delta - \nabla V \cdot \nabla$ ,  $\mu = e^{-V}$  and  $\Gamma(\cdot) = |\nabla \cdot|^2$ , then

$$W_2^2(f\mu, g\mu) = \inf \int_0^1 \int \frac{|w_s|^2}{\rho_s} d\mu ds$$

where the infimum runs over all paths  $(\rho_s, w_s)_{s \in [0,1]}$  satisfying

$$\partial_s \rho_s + \nabla \cdot w_s - \nabla V \cdot w_s = 0. \quad (14)$$

By comparison, in this setting, we have that

$$T_2^2(f\mu, g\mu) = \inf \int_0^1 \int \frac{|\nabla h_s|^2}{\rho_s} d\mu ds$$

where the infimum runs over all paths  $(\rho_s, h_s)_{s \in [0,1]}$  satisfying

$$\partial_s \rho_s + \Delta h_s - \nabla V \cdot \nabla h_s = 0. \quad (15)$$

But  $(\rho_s, w_s = \nabla h_s)$  satisfies (14) for any  $(\rho_s, h_s)_{s \in [0,1]} \in \mathcal{A}(f, g)$ , so  $T_2^2(f\mu, g\mu) \geq W_2^2(f\mu, g\mu)$ .

In this setting, in dimension 1, equality holds between  $T_2$  and  $W_2$ . Equality also holds in specific instances, such as  $L = \Delta$  in  $\mathbb{R}^n$ ,  $\mu$  is the Lebesgue measure and  $g$  is obtained from  $f$  by dilation.

More generally, if  $\sigma = \sigma(x)$  are symmetric matrices for  $x \in \mathbb{R}^n$ , then the generator defined by  $Lf = \nabla \cdot (\sigma \nabla f) - \sigma \nabla V \cdot \nabla f$  on  $\mathbb{R}^n$  has reversible measure  $\mu = e^{-V}$  and carré du champ  $\Gamma(f) = \sigma \nabla f \cdot \nabla f$ ; then  $T_2^2(f\mu, g\mu) \geq c W_2^2(f\mu, g\mu)$  as soon as the matrices  $\sigma(x)$  are positive with  $h \cdot \sigma(x)^{-1} h \geq c|h|^2$  for a constant  $c$  and all  $x$  and  $h$ .

## Discrete case

In the case of a countable state space  $E$ , a Markov semigroup  $(P_t)_{t \geq 0}$  is described by an infinite matrix of positive kernels  $(p_t(x, y))_{(x, y) \in E \times E}$ ,  $t \geq 0$ , such that for all  $t \geq 0$  and  $x \in E$ , and any positive function  $f$  on  $E$ ,

$$P_t f(x) = \sum_{y \in E} f(y) p_t(x, y).$$

For any  $x \in E$ ,  $p_t(x, \cdot)$  is a probability measure on  $E$ . The generator  $L$  is given by an infinite matrix  $(L(x, y))_{(x, y) \in E \times E}$ , where for any finitely supported function  $f$  on  $E$ ,

$$Lf(x) = \sum_{y \in E} L(x, y) f(y).$$

For the matrix  $L$  to be a generator, it is required that  $L(x, y) \geq 0$  whenever  $x \neq y$ , and  $\sum_y L(x, y) = 0$  for every  $x \in E$ . The carré du champ operator is defined on finitely supported functions  $f$  by

$$\Gamma(f)(x) = \frac{1}{2} \sum_{y \in E} L(x, y) [f(x) - f(y)]^2, \quad x \in E.$$

The measure is  $\mu$  reversible if

$$\mu(x)L(x, y) = \mu(y)L(y, x).$$

Let us illustrate the discrete setting with the two point space  $\{a, b\}$ . The generator is unique up to a multiplicative factor, and is given by  $Lf(a) = \kappa(f(b) - f(a))$  and  $Lf(b) = \kappa(f(a) - f(b))$  for a nonnegative constant  $\kappa$ ; moreover the carré du champ is constant, equal to

$$\Gamma(f) = \frac{\kappa}{2}(f(b) - f(a))^2,$$

and the reversible measure is  $\mu = \frac{1}{2}(\delta_a + \delta_b)$ . There one can simply and explicitly compute a geodesic curve between  $\delta_a$  and  $\delta_b$  for the  $T_2$  distance:

**Proposition 3.4** *In the above notation, we have*

$$T_2(2\mathbb{I}_a\mu, 2\mathbb{I}_b\mu) = \frac{\pi}{\sqrt{2\kappa}}$$

and the minimum is achieved by the curve

$$\begin{cases} \rho_s = 2\sin^2(\frac{\pi}{2}s)\mathbb{I}_b + 2\cos^2(\frac{\pi}{2}s)\mathbb{I}_a \\ h_s = \frac{\pi}{\kappa}\sin(\pi s)\mathbb{I}_b \end{cases} \quad s \in [0, 1].$$

By comparison, the Wasserstein distance between  $\delta_a$  and  $\delta_b$  is  $W_2(\delta_a, \delta_b) = d(a, b) = \sqrt{2/\kappa}$  if the distance on the space is chosen as the intrinsic distance  $d(a, b) = \sup_{\Gamma(f) \leq 1} (f(b) - f(a))$  defined by  $\Gamma$ .

**Proof**

$\triangleleft$  Let  $(\rho_s, h_s)_{s \in [0, 1]}$  be an admissible path between  $2\mathbb{I}_a$  and  $2\mathbb{I}_b$ . Then there exists a map  $\varphi : [0, 1] \rightarrow [0, 1]$  such that  $\varphi(0) = 0$  and  $\varphi(1) = 1$ , and

$$\rho_s = 2\varphi(s)\mathbb{I}_b + 2(1 - \varphi(s))\mathbb{I}_a.$$

The map  $h_s$  has to satisfy  $2\varphi'(s)(\mathbb{I}_b - \mathbb{I}_a) = -Lh_s$  for  $s \in [0, 1]$ , that is  $\varphi'(s) = (h_s(b) - h_s(a))\kappa/2$ . It remains to minimize

$$\int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds = \frac{\kappa}{2} \int_0^1 \int \frac{(h_s(b) - h_s(a))^2}{\rho_s} d\mu ds = \frac{1}{\kappa} \int_0^1 \varphi'(s)^2 \left( \frac{1}{\rho_s(a)} + \frac{1}{\rho_s(b)} \right) ds$$

Since  $\rho_s(a) = 2(1 - \varphi(s))$  and  $\rho_s(b) = 2\varphi(s)$  we need to minimize

$$\frac{1}{2\kappa} \int_0^1 \varphi'(s)^2 \left( \frac{1}{\varphi(s)} + \frac{1}{1 - \varphi(s)} \right) ds$$

over all functions  $\varphi$  such that  $\varphi(0) = 0$  and  $\varphi(1) = 1$ . The Euler-Lagrange equation is

$$2\varphi'' \left( \frac{1}{\varphi} + \frac{1}{1 - \varphi} \right) = \varphi'^2 \left( \frac{1}{\varphi^2} - \frac{1}{(1 - \varphi)^2} \right).$$

It implies that  $\varphi'^2 = a\varphi(1 - \varphi)$  for some  $a > 0$ . This solves into  $\varphi(s) = \sin^2(\frac{\pi}{2}s)$  for  $s \in [0, 1]$ , and then  $a = \pi^2$ . For such a solution,

$$\varphi'(s)^2 \left( \frac{1}{\varphi(s)} + \frac{1}{1 - \varphi(s)} \right) ds = \frac{\varphi'^2}{\varphi(1 - \varphi)} = \pi^2, \quad s \in (0, 1).$$

This implies that  $T_2^2(2\mathbf{I}_a\mu, 2\mathbf{I}_b\mu) = \pi^2/(2\kappa)$ . Moreover

$$\int \frac{\Gamma(h_s)}{\rho_s} d\mu = \frac{\pi^2}{2\kappa}$$

for all  $s \in (0, 1)$ , and then  $[0, 1]$ , so the path  $(\rho_s, h_s)$  is a geodesic between  $2\mathbf{I}_a$  and  $2\mathbf{I}_b$ .  $\triangleright$

### 3.3 General properties of $T_2$

We first exhibit  $\varepsilon$ -geodesics for the  $T_2$  distance. We will study the existence of geodesics in the specific Gaussian case in section 5, and in the general case in a forthcoming paper. Actually, we shall see below how properties on the distance and curvature-dimension bounds can be obtained without geodesics.

**Proposition 3.5 ( $\varepsilon$ -geodesics)** *Let  $f, g \in \mathcal{F}$  such that  $T_2(f\mu, g\mu) < +\infty$  and let  $\varepsilon > 0$ . Then there exists an  $\varepsilon$ -geodesic map, that is an admissible path  $(\rho_s, h_s)_{s \in [0, 1]} \in \mathcal{A}(f, g)$  such that for all  $s \in [0, 1]$ ,*

$$\varphi(\rho_s, h_s) = \int \frac{\Gamma(h_s)}{\rho_s} d\mu \leq T_2^2(f\mu, g\mu) + \varepsilon.$$

**Proof**

$\triangleleft$  Let  $\varepsilon > 0$  and an admissible path  $(\rho_s, h_s) \in \mathcal{A}(f, g)$  such that

$$\Phi(\rho, h) = \int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds \leq T_2^2(f\mu, g\mu) + \varepsilon.$$

It is easy to see that there exists  $a > 0$  such that

$$\int_0^1 \sqrt{\varphi(\rho_u, h_u) + a} du = \sqrt{\Phi(\rho, h) + \varepsilon}.$$

Then let  $\beta : [0, 1] \rightarrow [0, 1]$  be defined by

$$s = \frac{\int_0^{\beta(s)} \sqrt{\varphi(\rho_u, h_u) + a} du}{\sqrt{\Phi(\rho, h) + \varepsilon}}$$

for  $s \in [0, 1]$ . The function  $\beta$  is increasing and differentiable in  $[0, 1]$  and satisfies  $\beta(0) = 0$  and  $\beta(1) = 1$ , so  $(\rho_{\beta(s)}, \beta'(s)h_{\beta(s)}) \in \mathcal{A}(f, g)$ . Moreover

$$\begin{aligned} \varphi(\rho_{\beta(s)}, \beta'(s)h_{\beta(s)}) &= \beta'(s)^2 \varphi(\rho_{\beta(s)}, h_{\beta(s)}) = (\Phi(\rho, h) + \varepsilon) \frac{\varphi(\rho_{\beta(s)}, h_{\beta(s)})}{\varphi(\rho_{\beta(s)}, h_{\beta(s)}) + a} \\ &\leq \Phi(\rho, h) + \varepsilon \leq T_2^2(f\mu, g\mu) + 2\varepsilon \end{aligned}$$

for any  $s \in [0, 1]$ . This means that the couple  $(\rho_\beta, \beta'h_\beta)$  is a  $2\varepsilon$ -geodesic.  $\triangleright$

**Proposition 3.6** *The space  $(\mathcal{F}, T_2)$  is a pseudometric space.*

**Remark 3.7** Assume  $T_2 \geq cW_2$  for a positive constant  $c$ : as seen above this is for instance the case on  $\mathbb{R}^n$  for  $L$  defined by  $Lf = \nabla \cdot (\sigma \nabla f) - \sigma \nabla V \cdot \nabla f$  and  $\mu = e^{-V}$  as soon as  $\sigma = \sigma(x)$  are positive symmetric matrices with  $h \cdot \sigma(x)^{-1} h \geq \sqrt{c}|h|^2$  for all  $x$  and  $h$ .

Then  $T_2$  defines a metric on the set where it is finite. Indeed in this case  $T_2(f\mu, g\mu) = 0$  implies  $W_2(f\mu, g\mu) = 0$ , which implies  $f = g$ .

**Proof**

$\triangleleft$  For any  $f \in \mathcal{F}$  it is clear that  $T_2(f\mu, f\mu) = 0$  by choosing  $\rho_s$  constant equal to  $f$ ; moreover  $T_2$  is a symmetric function with respect to the two densities.

Let now  $f, g$  and  $h$  three probability densities with respect to  $\mu$ . Let  $(\rho_s^1, h_s^1)$  (resp.  $(\rho_s^2, h_s^2)$ ) be an  $\varepsilon$ -geodesic map between  $f$  and  $g$  (resp.  $g$  and  $h$ ). Let  $\alpha \in (0, 1)$  and define  $(\rho_s, h_s)_{s \in [0, 1]}$  by

$$\rho_s = \begin{cases} \rho_{s/\alpha}^1, & \text{if } s \in [0, \alpha], \\ \rho_{(s-\alpha)/(1-\alpha)}^2, & \text{if } s \in [\alpha, 1]. \end{cases} \quad h_s = \begin{cases} \frac{1}{\alpha} h_{s/\alpha}^1, & \text{if } s \in [0, \alpha], \\ \frac{1}{1-\alpha} h_{(s-\alpha)/(1-\alpha)}^2, & \text{if } s \in [\alpha, 1]. \end{cases}$$

Then the couple  $(\rho_s, h_s)$  is an admissible path between  $f$  and  $h$ . Moreover

$$T_2^2(f\mu, h\mu) \leq \Phi(\rho, h) = \int_0^\alpha \left(\frac{1}{\alpha}\right)^2 \varphi(\rho_{s/\alpha}^1, h_{s/\alpha}^1) ds + \int_\alpha^1 \left(\frac{1}{1-\alpha}\right)^2 \varphi(\rho_{(s-\alpha)/(1-\alpha)}^2, h_{(s-\alpha)/(1-\alpha)}^2) ds.$$

Now  $\rho_s^1$  and  $\rho_s^2$  are  $\varepsilon$ -geodesics, so

$$\varphi(\rho_{s/\alpha}^1, h_{s/\alpha}^1) \leq T_2^2(f\mu, g\mu) + \varepsilon$$

and

$$\varphi(\rho_{(s-\alpha)/(1-\alpha)}^2, h_{(s-\alpha)/(1-\alpha)}^2) \leq T_2^2(g\mu, h\mu) + \varepsilon.$$

Hence

$$T_2^2(f\mu, h\mu) \leq \frac{1}{\alpha}(T_2^2(f\mu, g\mu) + \varepsilon) + \frac{1}{1-\alpha}(T_2^2(g\mu, h\mu) + \varepsilon).$$

Now, choose

$$\alpha = \frac{T_2(f\mu, g\mu)}{T_2(f\mu, g\mu) + T_2(g\mu, h\mu)}$$

and let  $\varepsilon$  go to 0 to obtain the triangular inequality

$$T_2(f\mu, h\mu) \leq T_2(f\mu, g\mu) + T_2(g\mu, h\mu).$$

$\triangleright$

**Proposition 3.8 (Tensorization)** Let  $(P_t^i)_{t \geq 0}$ ,  $i \in \{1, \dots, N\}$  be  $N$  Markov semigroups on probability Polish spaces  $(E_i, \mu_i)$ ,  $i \in \{1, \dots, N\}$ , with generators  $L_i$  and carrés du champ  $\Gamma_i$ . Then one can define a product semigroup  $P_t = \otimes_{i=1}^N P_t^i$  on the product space  $(E, \mu) = (\times_{i=1}^N E_i, \otimes_{i=1}^N \mu_i)$  with generator  $L = \oplus_{i=1}^N L_i$  and carré du champ  $\Gamma = \oplus_{i=1}^N \Gamma_i$ .

Then, for any densities  $f(x) = \prod_{i=1}^N f_i(x_i)$  and  $g(x) = \prod_{i=1}^N g_i(x_i)$  ( $x = (x_1, \dots, x_N)$ ) in  $\mathcal{F}$ ,

$$T_2^2(f\mu, g\mu) \geq \sum_{i=1}^N T_{2,i}^2(f_i\mu_i, g_i\mu_i). \quad (16)$$

**Proof**

◁ For simplicity we prove the result for  $N = 2$ . Let  $(\rho_s, h_s)$  be an admissible path between the densities  $f_1(x)f_2(y)$  and  $g_1(x)g_2(y)$ . Let  $\rho_s^1(x) = \int \rho_s(x, y)d\mu_2(y)$  and  $h_s^1(x) = \int h_s(x, y)d\mu_2(y)$ , and let  $\rho_s^2$  and  $h_s^2$  similarly defined. Then

$$\varphi(\rho_s, h_s) = \int \frac{\Gamma(h_s)}{\rho_s} d\mu_1 d\mu_2 = \int \frac{\Gamma_1(h_s)}{\rho_s} d\mu_1 d\mu_2 + \int \frac{\Gamma_2(h_s)}{\rho_s} d\mu_1 d\mu_2. \quad (17)$$

Let us first prove that

$$\int \frac{\Gamma_1(h_s)}{\rho_s} d\mu_2 \geq \frac{\Gamma_1(h_s^1)}{\rho_s^1}, \quad (18)$$

and similarly for the second coordinate. Since

$$\Gamma_1(f)(x) = \lim_{t \rightarrow 0} \frac{1}{2t} \int \int (f(y_1) - f(y_2))^2 p_t^1(x, dy_1) p_t^1(x, dy_2) \quad (19)$$

for every function  $f$ , and for the Markov kernel  $p_t^1$  of the semigroup  $(P_t^1)_{t \geq 0}$  (see for instance [BGL13]), then for all  $x$

$$\begin{aligned} \int \frac{\Gamma_1(h_s)(x, y)}{\rho_s(x, y)} d\mu_2(y) &= \\ \rho_s^1(x) \lim_{t \rightarrow 0} \frac{1}{2t} \int \int \int \left( \frac{h_s(z_1, y) - h_s(z_2, y)}{\rho_s(x, y)} \right)^2 \frac{\rho_s(x, y)}{\rho_s^1(x)} d\mu_2(y) p_t^1(x, dz_1) p_t^1(x, dz_2) \\ &\geq \frac{1}{\rho_s^1(x)} \lim_{t \rightarrow 0} \frac{1}{2t} \int \int \left( \int (h_s(z_1, y) - h_s(z_2, y)) d\mu_2(y) \right)^2 p_t^1(x, dz_1) p_t^1(x, dz_2) = \frac{\Gamma_1(h_s^1)(x)}{\rho_s^1(x)} \end{aligned}$$

by the Cauchy-Schwarz inequality for the probability measure  $\frac{\rho_s(x, y)}{\rho_s^1(x)} d\mu_2(y)$ .

By (17) and (18) written for both variables we obtain

$$\varphi(\rho_s, h_s) \geq \varphi_1(\rho_s^1, h_s^1) + \varphi_2(\rho_s^2, h_s^2).$$

After integration over  $s \in [0, 1]$ , we get, for any admissible path  $(\rho_s, h_s) \in \mathcal{A}(f_1 f_2, g_1 g_2)$  :

$$\Phi(\rho, h) = \int_0^1 \varphi(\rho_s, h_s) ds \geq \Phi_1(\rho^1, h^1) + \Phi_2(\rho^2, h^2) \geq T_2^2(f_1 \mu_1, g_1 \mu_1) + T_2^2(f_2 \mu_2, g_2 \mu_2) \quad (20)$$

since  $(\rho_s^i, h_s^i)$  is an admissible path between  $f_i$  and  $g_i$ , for  $i = 1, 2$ . The result follows by optimizing over  $(\rho_s, h_s)$ . ▷

### 3.4 First application : the Talagrand inequality

As explained in the introduction, we will recover classical bounds as the contraction properties in the Markov transportation distance. Actually, such results are not adapted to the discrete setting and in all what follows we will make the following *diffusion property* assumption on the Markov semigroups:

**Definition 3.9** *A Markov semigroup with generator  $L$  satisfies the diffusion property if for any smooth function  $\Phi$  and any function  $g \in \mathcal{A}$ ,*

$$L\Phi(g) = \Phi'(g)Lg + \Phi''(g)\Gamma(g). \quad (21)$$

The definition implies in particular that  $\Gamma(\Phi'(g)) = \Phi''(g)\Gamma(g)$ , and the carré du champ can be seen as a derivation operator for each variables.

Some of our results will also require the semigroup to be *reversible*, in the sense that

$$\int f P_t g d\mu = \int g P_t f d\mu \quad \text{or equivalently} \quad \int f L g d\mu = \int g L f d\mu \quad (22)$$

for all  $f, g \in \mathcal{A}$  and  $t \geq 0$ .

For instance if the generator of a Markov semigroup on  $\mathbb{R}^n$  is given by  $L = \Delta - \nabla V \cdot \nabla$  where  $V$  is a smooth function in  $\mathbb{R}^n$ , then  $\Gamma(\cdot) = |\nabla \cdot|^2$  and the semigroup is diffusive and reversible. This is also the case for the Laplace-Beltrami operator  $\Delta_g$  on a Riemannian manifold. In this case the carré du champ operator is  $\Gamma(f) = |\nabla f|^2$  where  $|\nabla f|$  stands for the length of vector  $\nabla f$ . On the other side, no semigroup coming from a jump process satisfies the diffusion property.

The so-called Otto-Villani Theorem says that a logarithmic Sobolev inequality with constant  $C$  (see (26) below) implies the Talagrand inequality

$$W_2^2(f\mu, \mu) \leq 4C \text{Ent}_\mu(f)$$

for all probability densities  $f$ . This inequality has first been derived by M. Talagrand in [Tal96] and linked with the logarithmic Sobolev inequality in [OV00] (see also [BGL01]). The proofs in [OV00, GL13] are based on the inequality

$$W_2^2(P_t f \mu, f \mu) \leq t(\text{Ent}_\mu(f) - \text{Ent}_\mu(P_t f)) \quad (23)$$

(see also [GKO13]).

**Proposition 3.10** *Let  $(P_t)_{t \geq 0}$  be a diffusion semigroup and  $f \in \mathcal{F}$ . Then*

$$T_2^2(P_t f \mu, f \mu) \leq t(\text{Ent}_\mu(f) - \text{Ent}_\mu(P_t f)) \quad (24)$$

for every  $t \geq 0$ , and in particular

$$\limsup_{t \rightarrow 0^+} \frac{T_2(P_t f \mu, f \mu)}{t} \leq \sqrt{\int \frac{\Gamma(f)}{f} d\mu}. \quad (25)$$

**Proof**

$\triangleleft$  Let  $f$  be a probability density with respect to  $\mu$ . Then  $(P_{st}f, -tP_{st}f)_{s \in [0,1]}$  is an admissible couple between  $f$  and  $P_t f$ . By definition of the distance  $T_2$ , it implies that

$$T_2^2(P_t f \mu, f \mu) \leq t \int_0^t \int \frac{\Gamma(P_r f)}{P_r f} d\mu dr$$

by change of time variable. Moreover

$$\frac{d}{dr} \text{Ent}_\mu(P_r f) = \frac{d}{dr} \int P_r f \log P_r f d\mu = - \int \frac{\Gamma(P_r f)}{P_r f} d\mu$$

by diffusion property of the semigroup. This leads to (24) by integrating in  $r$ .  $\triangleright$

In the case where  $W_2 \leq T_2$ , for instance when  $\Gamma(\cdot) = |\nabla \cdot|^2$ , then inequality (24) implies the classical inequality (23).

**Corollary 3.11** *Let  $(P_t)_{t \geq 0}$  be a diffusion semigroup with invariant measure  $\mu$  which is a probability measure satisfying a logarithmic Sobolev inequality with constant  $C$ , that is,*

$$\text{Ent}_\mu(f) \leq C \int \frac{\Gamma(f)}{f} d\mu, \quad (26)$$

*for every function  $f \geq 0$  with finite Fisher information. Then*

$$T_2^2(f\mu, P_T f\mu) \leq 4C \text{Ent}_\mu(f)$$

*for any probability density  $f \in \mathcal{A}$ . In particular, if for instance  $T_2$  is lower semicontinuous with respect to narrow convergence, then  $\mu$  satisfies a Talagrand type inequality for the distance  $T_2$ , namely*

$$T_2^2(f\mu, \mu) \leq 4C \text{Ent}_\mu(f) \quad (27)$$

*for any probability density  $f \in \mathcal{F}$ .*

**Proof**

$\triangleleft$  Let  $f$  be a probability density with respect to  $\mu$ , and let  $\varphi(t) = \text{Ent}_\mu(P_t f)$ . Then (25) and semigroup properties imply

$$\frac{d^+}{dt} T_2(P_t f\mu, f\mu) \leq \sqrt{-\varphi'(t)}.$$

Moreover the logarithmic Sobolev inequality for  $\mu$  ensures that  $\varphi'(t) \leq -\varphi(t)/C$ , and thus

$$\sqrt{-\varphi'(t)} \leq -\sqrt{4C} \left( \sqrt{\varphi(t)} \right)'.$$

Let now  $T > 0$ . Then

$$\begin{aligned} T_2(P_T f\mu, f\mu) &\leq \int_0^T \frac{d^+}{dt} T_2(P_t f\mu, f\mu) dt \leq -\sqrt{4C} \int_0^T \left( \sqrt{\varphi(t)} \right)' dt \\ &= \sqrt{4C} \left( \sqrt{\text{Ent}_\mu(f)} - \sqrt{\text{Ent}_\mu(P_T f)} \right) \leq \sqrt{4C} \sqrt{\text{Ent}_\mu(f)}. \end{aligned}$$

Moreover  $P_T f\mu$  narrowly converges to  $f\mu$ . Hence

$$T_2(\mu, f\mu) \leq \liminf_{T \rightarrow \infty} T_2(P_T f\mu, f\mu) \leq \sqrt{4C} \sqrt{\text{Ent}_\mu(f)}$$

if  $T_2$  is lower semicontinuous with respect to narrow convergence.  $\triangleright$

Again, in the case where  $W_2 \leq T_2$ , for instance where  $\Gamma(\cdot) = |\nabla \cdot|^2$  on  $\mathbb{R}^n$ , then the Talagrand inequality for the  $T_2$  distance implies the classical Talagrand inequality for the distance  $W_2$ , with the same constant.

This inequality will be useful in the following section in the derivation of refined convergence rates.

## 4 Contraction property under the curvature-dimension condition $CD(R, n)$

In this section we prove a dimension dependent contraction property in the Markov transportation distance. We will see that the  $\Gamma_2$ -calculus is a well adapted and efficient tool.



## Curvature condition, examples and useful commutation properties

The  $\Gamma_2$ -operator, or iterated carré du champ operator, is defined on functions  $f \in \mathcal{A}$  by

$$\Gamma_2(f) = \frac{1}{2} \left( L\Gamma(f) - 2\Gamma(f, Lf) \right).$$

**Definition 4.1** A Markov semigroup  $(P_t)_{t \geq 0}$  is said to satisfy a curvature-dimension condition  $CD(R, n)$  for  $R \in \mathbb{R}$  and  $n \geq 1$  if

$$\Gamma_2(f) \geq \rho \Gamma(f) + \frac{1}{n} (Lf)^2$$

for all functions  $f \in \mathcal{A}$ .

This criterion has been introduced in [BÉ85] by D. Bakry and M. Émery. Here are three main examples.

**Example 1.** A fundamental example is the heat semigroup on  $\mathbb{R}^n$ , considered in section 2. Its generator is the Laplacian, its carré du champ is  $\Gamma(f) = |\nabla f|^2$  and  $\Gamma_2(f) = \sum_{i,j=1}^n (\partial_{ij}^2 f)^2$ . It satisfies the  $CD(0, n)$  criterion by the Cauchy-Schwarz inequality.

**Example 2.** Another fundamental example is the Ornstein-Uhlenbeck semigroup defined by

$$P_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \gamma(dy)$$

where  $\gamma(dy) = (2\pi)^{-n/2} e^{-|y|^2/2} dy$  is the standard Gaussian measure on  $\mathbb{R}^n$ . Its generator, carré du champ and  $\Gamma_2$  operators are given by

$$Lf(x) = \Delta f(x) - x \cdot \nabla f(x), \quad \Gamma(f) = |\nabla f|^2, \quad \Gamma_2(f) = \sum_{i,j=1}^n (\partial_{ij}^2 f)^2 + |\nabla f|^2.$$

The Ornstein-Uhlenbeck semigroup satisfies the  $CD(1, \infty)$  criterion.

**Example 3.** The Laplace-Beltrami operator on the sphere  $S^n \subset \mathbb{R}^{n+1}$  satisfies a  $CD(n-1, n)$ .

More generally, for a complete Riemannian manifold  $M$  with dimension  $n$ , equipped with the Laplace-Beltrami operator  $\Delta_g$  and the Riemannian measure  $\mu_g$ , the curvature-dimension condition  $CD(R, n)$  holds for  $\Delta_g$  if the Ricci curvature of  $M$  is uniformly bounded from below by  $R$ . This example gives a way to prove that a generator of the form

$$L = \sum_{i,j} \sigma_{i,j}(x) \partial_{ij}^2 + \sum_i a_i(x) \partial_i$$

satisfies a  $CD(R, n)$  criterion.

One of the main results concerning the curvature-dimension condition  $CD(R, \infty)$  is a regularity property of the Markov semigroup. The  $CD(R, \infty)$  conditions holds for a diffusive Markov semigroup if and only if for any function  $f \in \mathcal{A}$

$$\Gamma(P_t(f)) \leq e^{-2Rt} \left( P_t \sqrt{\Gamma(f)} \right)^2. \quad (28)$$

This result, proved in [Bak94], is the key point for many applications such as logarithmic Sobolev inequalities, Harnack parabolic inequalities, etc. (see [BGL13]). Gradient bounds, in a weaker form, also hold under the  $CD(R, n)$  condition with finite  $n$  (see [BL06] and [Wan11]). Here is a new such bound which will be the key point for the Markov transportation distance.

**Lemma 4.2** *Let  $(P_t)_{t \geq 0}$  be a diffusion Markov semigroup, and let  $R \in \mathbb{R}$  and  $n \geq 1$ . The following assertions are equivalent :*

- (i) *The Markov semigroup satisfies a  $CD(R, n)$  condition.*
- (ii) *For all functions  $f, g \in \mathcal{A}$  with  $g > 0$  and all  $t \geq 0$*

$$\frac{\Gamma(P_t f)}{P_t g} \leq e^{-2Rt} P_t \left( \frac{\Gamma(f)}{g} \right) - \frac{2}{n} \int_0^t e^{-2Ru} \frac{[LP_t f - P_u(\Gamma(P_{t-u} f, \log P_{t-u} g))]^2}{P_t g} du. \quad (29)$$

*In particular, under the  $CD(R, n)$  condition and for any  $t \geq 0$  and probability density  $g$  with respect to the invariant measure  $\mu$ ,*

$$\int \frac{\Gamma(P_t f)}{P_t g} d\mu \leq e^{-2Rt} \int \frac{\Gamma(f)}{g} d\mu - \frac{2}{n} \int_0^t e^{-2R(t-u)} \left( \int \frac{\Gamma(P_u f, P_u g)}{P_u g} d\mu \right)^2 du. \quad (30)$$

**Proof**

$\triangleleft$  Let us first prove that (i) implies (ii). We let  $t > 0$  and  $f, g \in \mathcal{A}$  be fixed, with  $g > 0$ . Then we define

$$\Lambda(s) = P_s \left( \frac{\Gamma(P_{t-s} f)}{P_{t-s} g} \right)$$

for  $s \in [0, t]$ , and then  $F = P_{t-s} f$  and  $G = P_{t-s} g$ . Then

$$\Lambda'(s) = P_s \left( -2 \frac{\Gamma(F, LF)}{G} + \Gamma(F) \frac{LG}{G^2} + L \left( \frac{\Gamma(F)}{G} \right) \right).$$

But

$$L(hk) = 2\Gamma(h, k) + hLk + kLh$$

for any function  $h, k \in \mathcal{A}$ , so the diffusion property (21) and the definition of  $\Gamma_2$  lead to

$$\Lambda'(s) = 2P_s \left( \frac{1}{G} [\Gamma_2(F) - \Gamma(\Gamma(F), \log G) + \Gamma(F)\Gamma(\log G)] \right).$$

Now Lemma 4.3 below, applied with  $f = F$  and  $g = -\log G$ , ensures that

$$\Gamma_2(F) - \Gamma(\Gamma(F), \log G) + \Gamma(F)\Gamma(\log G) \geq R\Gamma(F) + \frac{1}{n}(LF - \Gamma(F, \log G))^2.$$

Since  $G \geq 0$ , this gives

$$\Lambda'(s) \geq 2R\Lambda(s) + \frac{2}{n} P_s \left( \frac{[LF - \Gamma(F, \log G)]^2}{G} \right) \geq 2R\Lambda(s) + \frac{2}{n} \frac{[LP_t f - P_s(\Gamma(P_{t-s} f, \log P_{t-s} g))]^2}{P_t g}$$

by the Cauchy-Schwarz inequality for the Markov kernel of  $P_s$  and semigroup properties. Inequality (29) follows by integration over  $s \in [0, t]$ .

Let us now assume (ii) and let  $g = 1$ . Then inequality (29) writes

$$\Gamma(P_t f) \leq e^{-2Rt} P_t \Gamma(f) - \frac{2}{n} \int_0^t e^{-2Ru} (L P_u f)^2 du.$$

Taking the time derivative at  $t = 0$  implies back the  $CD(R, n)$  condition.

Let us finally prove (30): integrating (29) with respect to  $\mu$  gives

$$\int \frac{\Gamma(P_t f)}{P_t g} d\mu \leq e^{-2Rt} \int \frac{\Gamma(f)}{g} d\mu - \frac{2}{n} \int_0^t e^{-2Ru} \left[ \int \frac{[L P_u f - P_u(\Gamma(P_{t-u} f, \log P_{t-u} g))]^2}{P_u g} d\mu \right] du$$

by invariance property of  $\mu$ . Then the Cauchy-Schwarz inequality for the measure  $\mu$  implies (30) by recalling that  $\int g d\mu = 1$ , invariance property of  $\mu$  and change of time variable.  $\triangleright$

**Lemma 4.3** *For a diffusion Markov semigroup, under the curvature-dimension condition  $CD(R, n)$  (with  $R \in \mathbb{R}$ ), for all functions  $f, g \in \mathcal{A}$*

$$\Gamma_2(f) + \Gamma(\Gamma(f), g) + \Gamma(f)\Gamma(g) \geq R\Gamma(f) + \frac{1}{n}(Lf + \Gamma(f, g))^2. \quad (31)$$

**Proof**

$\triangleleft$  The proof is inspired from Lemma 5.4.4, p. 83 of [ABC<sup>+</sup>00]. Let  $f, g \in \mathcal{A}$  and  $x_0 \in E$ . Let  $\Phi$  be a smooth map on  $\mathbb{R}^2$  such that

$$\partial_2 \Phi = \partial_{11}^2 \Phi = \partial_{22}^2 \Phi = 0, \quad \partial_1 \Phi = 1 \quad \text{and} \quad \partial_{12}^2 \Phi = \frac{1}{2}$$

at the point  $(f(x_0), g(x_0))$ . Then the  $CD(R, n)$  condition applied to the function  $\Phi(f, g)$  at the point  $x_0$  yields

$$\Gamma_2(\Phi(f, g)) \geq R\Gamma\Phi(f, g) + \frac{1}{n}(L\Phi(f, g))^2.$$

The usual change of variable rules for the  $\Gamma$  and  $\Gamma_2$  operators (see for instance [ABC<sup>+</sup>00, p. 83]) imply

$$\Gamma_2(f) + \Gamma(\Gamma(f), g) + \frac{1}{2}[\Gamma(f, g)^2 + \Gamma(f)\Gamma(g)] \geq R\Gamma(f) + \frac{1}{n}(Lf + \Gamma(f, g))^2.$$

The result follows since  $\Gamma(f, g)^2 \leq \Gamma(f)\Gamma(g)$ .  $\triangleright$

**Remark 4.4** *Without the dimension, namely under the curvature-dimension condition  $CD(R, \infty)$ , inequality (29) is a direct consequence of inequality (28). Indeed (28) implies*

$$\frac{\Gamma(P_t f)}{P_t g} \leq e^{-2Rt} \frac{P_t(\sqrt{\Gamma(f)})^2}{P_t g} \leq e^{-2Rt} P_t \left( \frac{\Gamma(f)}{g} \right),$$

by the Cauchy-Schwarz inequality for the Markov kernel of  $P_t$ .

## Contraction property under $CD(R, n)$

The following result is a generalization of Proposition 2.2 for the heat semigroup on  $\mathbb{R}^n$  to arbitrary Markov semigroups with the general curvature-dimension condition  $CD(R, n)$ . For  $R = 0$  we precisely recover the bound obtained in Wasserstein distance for the heat semigroup on  $\mathbb{R}^n$ .

**Theorem 4.5** *Let  $(P_t)_{t \geq 0}$  be a reversible diffusion Markov semigroup satisfying a  $CD(R, n)$  condition with  $R \in \mathbb{R}$  and  $n \geq 1$ . Then, for all  $f, g \in \mathcal{F}$  and  $T \geq 0$ ,*

$$T_2^2(P_T f \mu, P_T g \mu) \leq e^{-2RT} T_2^2(f \mu, g \mu) - \frac{2}{n} \int_0^T e^{-2R(T-t)} (\text{Ent}_\mu(P_t g) - \text{Ent}_\mu(P_t f))^2 dt. \quad (32)$$

### Proof

$\triangleleft$  Let  $(\rho_s, h_s)$  be an admissible path between  $f$  and  $g$ . Then  $(P_t(\rho_s), P_t(h_s))_{s \in [0,1]}$  is also an admissible path between  $P_t f$  and  $P_t g$ .

Then inequality (30) of Lemma 4.2, applied at time  $T$  to the functions  $\rho_s$  and  $h_s$ , implies

$$\int \frac{\Gamma(P_T h_s)}{P_T \rho_s} d\mu \leq e^{-2RT} \int \frac{\Gamma(h_s)}{\rho_s} d\mu - \frac{2}{n} \int_0^T e^{-2R(T-t)} \left( \int \frac{\Gamma(P_t(\rho_s), P_t(h_s))}{P_t(\rho_s)} d\mu \right)^2 dt.$$

Integrating over  $s \in [0, 1]$  and the Cauchy-Schwarz inequality imply

$$\begin{aligned} \int_0^1 \int \frac{\Gamma(P_T h_s)}{P_T \rho_s} d\mu ds &\leq e^{-2RT} \int_0^1 \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds \\ &\quad - \frac{2}{n} \int_0^T e^{-2R(T-t)} \left( \int_0^1 \int \frac{\Gamma(P_t(\rho_s), P_t(h_s))}{P_t(\rho_s)} d\mu ds \right)^2 dt. \end{aligned}$$

We finally obtain (32) since, letting  $\varphi(s) = \text{Ent}_\mu(P_t \rho_s)$ ,

$$\int_0^1 \int \frac{\Gamma(P_t(\rho_s), P_t(h_s))}{P_t(\rho_s)} d\mu ds = \int_0^1 \varphi'(s) ds = \varphi(1) - \varphi(0) = \text{Ent}_\mu(P_t g) - \text{Ent}_\mu(P_t f)$$

by the reversibility property (22) of the semigroup.  $\triangleright$

**Remark 4.6** *As noted in the introduction, this result has a particular new flavor. Indeed the recent results (2)-(3) in [BGL12], [EKS13] and [Kuw] present a dimensional correction term for the contraction property, but for solutions at different times only. If the approaches for these inequalities are slightly different, it would be of interest to obtain a dimensional correction term for our contraction also in different times. A possible approach could be through Evolution variational inequalities, as studied in the next section, as the contraction result in [EKS13] is deduced from these inequalities.*

Assuming that  $\mu$  is a probability measure, and taking  $g = 1$ , under the  $CD(R, \infty)$  condition, the following bound

$$T_2^2(P_T f \mu, \mu) \leq e^{-2Rt} T_2^2(f \mu, \mu)$$

holds for the  $T_2$  distance as it does for the  $W_2$  distance. The following corollary gives a more precise bound under the  $CD(R, n)$  condition:

**Corollary 4.7** *Let  $(P_t)_{t \geq 0}$  be a reversible diffusion Markov semigroup satisfying a  $CD(R, n)$  condition with  $R > 0$  and  $n \geq 1$ , and assume that  $\mu$  is a probability measure. Then, in the framework of Corollary 3.11, for all  $f \in \mathcal{F}$  and  $T \geq 0$ ,*

$$T_2^2(P_T f \mu, \mu) \leq e^{-2RT} T_2^2(f \mu, \mu) \frac{1}{1 + n R T_2^2(f \mu, \mu) \frac{1 - e^{-2RT}}{4(n-1)^2}}.$$

**Proof**

◁ Taking  $g = 1$  in (32), the map  $\Lambda(t) = e^{2Rt} T_2^2(P_T f \mu, \mu)$  satisfies

$$\Lambda'(t) \leq -\frac{nR^2}{2(n-1)^2} e^{-2Rt} \Lambda(t)^2.$$

Observe indeed that, under the  $CD(R, n)$  condition, the measure  $\mu$  satisfies the logarithmic Sobolev inequality (26) with constant  $C = \frac{n-1}{2Rn}$  (see [BGL13]), whence a Talagrand inequality (27) with constant  $4C$  by Corollary 3.11. The conclusion follows by time integration. ▷

## 5 Evolution variational inequalities

Evolution variational inequalities (EVI in short) have recently been developed as a connection between curvature conditions  $CD(R, \infty)$  (usually in the sense of the commutation of the semigroup and the carré du champ), heat semigroups and the notion of curvature bound introduced by J. Lott, K.-T. Sturm and C. Villani (see [Stu06] and [LV09]). We refer to the recent work [AGS12a] for a nearly complete picture in this dimensionless setting. However, no dimensional EVI, namely related to a  $CD(R, n)$  curvature-dimension condition, were known until M. Erbar, K. Kuwada and K.-T. Sturm very recently proved in [EKS13] that  $CD(R, n)$  is (roughly) equivalent to

$$\frac{d}{dt} s_{\frac{R}{n}} \left( \frac{1}{2} W_2(f \mu, H_t g \mu) \right)^2 \leq -R s_{\frac{R}{n}} \left( \frac{1}{2} W_2(f \mu, H_t g \mu) \right)^2 + \frac{n}{2} \left( 1 - e^{-\frac{1}{n}(\text{Ent}_\mu(f) - \text{Ent}_\mu(H_t g))} \right) \quad (33)$$

on the heat semigroup. Here  $s_r(x) = \sin(\sqrt{r}x)/\sqrt{r}$  if  $r > 0$  and  $s_r(x) = \sinh(\sqrt{-r}x)/\sqrt{-r}$  if  $r < 0$ . Forgetting for a time the map  $s_r$ , which is equivalent to  $x$  for small  $x$ , and using that  $1 - e^{-x} \leq x$ , this inequality clearly appears to improve the classical EVI obtained under a  $CD(R, \infty)$  type condition.

The main goal of this section is twofold. First, we will give a (time integrated) EVI for the Markov transportation distance  $T_2$  (rather than the usual Wasserstein distance), under the  $CD(R, \infty)$  condition. Then we will see how the possible existence of geodesics can lead to a dimensional EVI, here with a negative corrective term in the spirit of the contraction result in Theorem 4.5. As in section 2 we will start with the Euclidean heat equation, i.e. under a  $CD(0, n)$  condition, obtaining a dimensional EVI in Wasserstein distance. Assuming the existence of geodesics for  $T_2$ , we will see that one can obtain a statement under the  $CD(R, n)$  curvature condition for a general Markov semigroup.

### Evolution variational inequality for $T_2$ under $CD(R, \infty)$

**Theorem 5.1** *Let  $(P_t)_{t \geq 0}$  be a reversible diffusion Markov semigroup satisfying a  $CD(R, \infty)$  condition with  $R \in \mathbb{R}$ . Then, for all  $f, g \in \mathcal{F}$ ,*

$$T_2^2(f \mu, P_t g \mu) - T_2^2(f \mu, g \mu) \leq -\frac{e^{-2Rt} - 1 + 2Rt}{2Rt} T_2^2(f \mu, g \mu) + 2t(\text{Ent}_\mu(f) - \text{Ent}_\mu(P_t g)).$$

**Proof**

$\triangleleft$  Let  $(\rho_s, h_s)$  be an admissible path between  $f$  and  $g$ . Then  $(P_{ts}(\rho_s), P_{ts}(h_s - t\rho_s))$  is an admissible path between  $f$  and  $P_t g$ , so

$$\begin{aligned} T_2^2(f\mu, P_t g\mu) &\leq \int_0^1 \int \frac{\Gamma(P_{ts}(h_s - t\rho_s))}{P_{ts}(\rho_s)} d\mu ds \\ &= \int_0^1 \int \frac{\Gamma(P_{ts}(h_s))}{P_{ts}(\rho_s)} d\mu ds - 2t \int_0^1 \int \frac{\Gamma(P_{ts}(h_s - t\rho_s), P_{ts}(\rho_s))}{P_{ts}(\rho_s)} d\mu ds - t^2 \int_0^1 \int \frac{\Gamma(P_{ts}(\rho_s))}{P_{ts}(\rho_s)} d\mu ds. \end{aligned} \quad (34)$$

By Lemma 4.2 under the  $CD(R, \infty)$  curvature condition, then for all  $s$

$$\int \frac{\Gamma(P_{ts}(h_s))}{P_{ts}(\rho_s)} d\mu \leq e^{-2Rts} \int \frac{\Gamma(h_s)}{\rho_s} d\mu \leq e^{-2Rts} (T_2^2(f\mu, g\mu) + \varepsilon)$$

if the admissible path  $(\rho_s, h_s)$  is an  $\varepsilon$ -geodesic, for given  $\varepsilon > 0$ . Moreover

$$\int \frac{\Gamma(P_{ts}(h_s - t\rho_s), P_{ts}(\rho_s))}{P_{ts}(\rho_s)} d\mu = \frac{d}{ds} \int P_{ts}(\rho_s) \log P_{ts}(\rho_s) d\mu$$

by reversibility. Hence, forgetting the last term in (34) and integrating in  $s \in [0, 1]$  conclude the argument by letting  $\varepsilon$  go to 0.  $\triangleright$

**Remark 5.2** Combining the contraction result of Theorem 4.5 under a  $CD(R, n)$  condition and this EVI (of course valid under  $CD(R, n)$ ) we may get a contraction type result in the  $T_2$  distance for  $s < t$ , in the spirit of [EKS13] (see Remark 2.4).

To obtain dimension dependent bounds under the  $CD(R, n)$  condition we will need geodesics. As above for the contraction property, let us see first which additional term coming from the dimension appears for the Euclidean heat equation.

## A dimensional EVI in Wasserstein distance for the heat equation in $\mathbb{R}^n$

For simplicity the EVI is here described in its time derivative form, but it may be easily justified by first considering an integrated form of the EVI.

As in section 2, the Benamou-Brenier formulation (7) is the starting point. Let  $(\rho_s, w_s)$  be an admissible path between  $f$  and  $g$ , satisfying the constraint (8). Then  $(H_{ts}(\rho_s), R_{ts}(w_s) - t\nabla H_{ts}(\rho_s))$  is an admissible path between  $f$  and  $H_t g$ , and satisfying (8), so by (7),

$$W_2^2(fdx, H_t gdx) \leq \int_0^1 \int \frac{|R_{ts}(w_s) - t\nabla H_{ts}(\rho_s)|^2}{H_{ts}(\rho_s)} ds dx. \quad (35)$$

Assume further that  $(\rho_s, w_s)$  is a minimizer in the Benamou-Brenier formulation, that is,

$$W_2^2(fdx, gdx) = \int_0^1 \int \frac{|w_s|^2}{\rho_s} ds dx.$$

The path  $(\rho_s)$  is then a geodesic path between  $fdx$  and  $gdx$  with respect to the Wasserstein distance (see [DNS09] for more details). In particular inequality (35) is an equality at time  $t = 0$  and, formally, the time derivative of (35) at  $t = 0$  implies

$$\left. \frac{d}{dt} W_2^2(fdx, H_t gdx) \right|_{t=0} \leq \left. \frac{d}{dt} \int_0^1 \int \frac{|R_{ts}(w_s) - t\nabla H_{ts}(\rho_s)|^2}{H_{ts}(\rho_s)} ds dx \right|_{t=0}. \quad (36)$$

The term on the right-hand side is controlled by the following lemma.

**Lemma 5.3** *Let  $(\rho_s, w_s)_{s \in [0,1]}$  be a couple satisfying the constraint (8), where  $\rho_s$  is a probability density with respect to Lebesgue measure. Letting*

$$\Lambda(t) = \int_0^1 \int \frac{|R_{ts}(w_s) - t \nabla H_{ts}(\rho_s)|^2}{H_{ts}(\rho_s)} ds dx$$

for  $t \geq 0$ , then

$$\Lambda'(0) \leq -\frac{2}{n} \int_0^1 s \left( \int \frac{w_s \cdot \nabla \rho_s}{\rho_s} dx \right)^2 ds - 2 \int_0^1 \int \frac{w_s \cdot \nabla \rho_s}{\rho_s} ds dx.$$

We skip the proof since it is almost the same as for Lemma 2.1.

Now Lemma 5.3 and (36) imply

$$\left. \frac{d}{dt} W_2^2(f dx, H_t g dx) \right|_{t=0} \leq -\frac{2}{n} \int_0^1 s \left( \int \frac{w_s \cdot \nabla \rho_s}{\rho_s} dx \right)^2 ds - 2 \int_0^1 \int \frac{w_s \cdot \nabla \rho_s}{\rho_s} ds dx.$$

Letting  $\varphi(s) = \int \rho_s \log \rho_s dx$ , the relation (8) between  $w_s$  and  $\rho_s$  implies

$$\varphi'(s) = \int \frac{w_s \cdot \nabla \rho_s}{\rho_s} dx,$$

so that

$$\left. \frac{d}{dt} W_2^2(f dx, H_t g dx) \right|_{t=0} \leq -\frac{2}{n} \int_0^1 s (\varphi'(s))^2 ds - 2 \int_0^1 \varphi'(s) ds.$$

Then the Jensen inequality for the measure  $2s ds$  and an integration by parts give

$$\left. \frac{d}{dt} W_2^2(f dx, H_t g dx) \right|_{t=0} \leq -\frac{4}{n} \left( \varphi(1) - \int_0^1 \varphi(s) ds \right)^2 - 2(\varphi(1) - \varphi(0)).$$

We have obtained the following result :

**Proposition 5.4** *Let  $(H_t)_{t \geq 0}$  be the heat semigroup on  $\mathbb{R}^n$ . Then, for any probability densities  $f$  and  $g$  in  $\mathbb{R}^n$  such that  $W_2(f dx, g dx) < \infty$ ,*

$$\left. \frac{1}{2} \frac{d}{dt} W_2^2(f dx, H_t g dx) \right|_{t=0} \leq -\frac{2}{n} \left( \text{Ent}_{dx}(g) - \int_0^1 \text{Ent}_{dx}(\rho_s) ds \right)^2 + \text{Ent}_{dx}(f) - \text{Ent}_{dx}(g) \quad (37)$$

where  $(\rho_s)_{s \in [0,1]}$  is a geodesic path between  $f$  and  $g$  for the Wasserstein distance.

At the time being, we have not been able to get a “geodesic” free version of this dimensional EVI. Note however, once again, that the correction term is quite different in nature from the one obtained for example in [EKS13]. We will see in the next subsection that we obtain results in the same flavor with the  $T_2$  distance.

## A dimension dependent EVI for $T_2$ in the geodesic case

In this subsection we will assume the existence of geodesics for the  $T_2$  distance. Therefore, this section is a bit formal, as also we should first consider the integrated form of EVI (which can be obtained, but is however quite difficult to read). We will closely follow the approach used above for the heat equation and the Wasserstein distance.

We then consider a reversible diffusion Markov semigroup  $(P_t)_{t \geq 0}$  and, for  $f, g \in \mathcal{F}$ , a geodesic path  $(\rho_s, h_s)$  between  $f$  and  $g$  for the associated  $T_2$  distance. Then  $(P_{ts}(\rho_s), P_{ts}(h_s - t\rho_s))$  is an admissible path between  $f$  and  $P_t g$ . In particular

$$T_2^2(f\mu, P_t g\mu) \leq \int_0^1 \int \frac{\Gamma(P_{ts}(h_s - t\rho_s))}{P_{ts}(\rho_s)} d\mu ds. \quad (38)$$

We will use the following adaptation of Lemma 5.3 to our setting, which does not need  $(\rho_s, h_s)$  to be a geodesic path:

**Lemma 5.5** *Let  $(\rho_s)_{s \in [0,1]}$  be a path of probability densities,  $(h_s)_{s \in [0,1]}$  be a smooth path, and*

$$\Lambda(t) = \int_0^1 \int \frac{\Gamma(P_{ts}(h_s - t\rho_s))}{P_{ts}(\rho_s)} d\mu ds$$

*for  $t \geq 0$ . Then, under the  $CD(R, n)$  condition,*

$$\Lambda'(0) \leq -2R \int_0^1 s \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds - \frac{2}{n} \int_0^1 s \left( \int \frac{\Gamma(h_s, \rho_s)}{\rho_s} d\mu \right)^2 ds - 2 \int_0^1 \int \frac{\Gamma(h_s, \rho_s)}{\rho_s} d\mu ds.$$

Since  $(\rho_s, h_s)$  is a geodesic path, then (38) is an equality at time  $t = 0$ , so, taking the time derivative at  $t = 0$  and using Lemma 5.5,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} T_2^2(f\mu, P_t g\mu) \Big|_{t=0} &\leq -R \int_0^1 s \int \frac{\Gamma(h_s)}{\rho_s} d\mu ds \\ &\quad - \frac{1}{n} \int_0^1 s \left( \int \frac{\Gamma(h_s, \rho_s)}{\rho_s} d\mu \right)^2 ds - \int_0^1 \int \frac{\Gamma(h_s, \rho_s)}{\rho_s} d\mu ds. \end{aligned}$$

Again  $(\rho_s, h_s)$  is a geodesic path, so

$$\int \frac{\Gamma(h_s)}{\rho_s} d\mu = T_2^2(f\mu, g\mu)$$

for all  $s$ . Letting again  $\varphi(s) = \int \rho_s \log \rho_s d\mu$ , the inequality may then be rewritten as

$$\frac{1}{2} \frac{d}{dt} T_2^2(f\mu, P_t g\mu) \Big|_{t=0} \leq -\frac{R}{2} T_2^2(f\mu, g\mu) - \frac{1}{n} \int_0^1 s \varphi'(s)^2 ds - \int_0^1 \varphi'(s) ds.$$

Hence, by the Jensen inequality for the measure  $2s ds$  and an integration by parts, the  $CD(R, n)$  condition and the existence of geodesics ensure the following dimensional EVI:

$$\frac{1}{2} \frac{d}{dt} T_2^2(f\mu, P_t g\mu) \Big|_{t=0} \leq -\frac{R}{2} T_2^2(f\mu, g\mu) - \frac{2}{n} \left( \text{Ent}_\mu(g) - \int_0^1 \text{Ent}_\mu(\rho_s) ds \right)^2 + \text{Ent}_\mu(f) - \text{Ent}_\mu(g), \quad (39)$$

where  $(\rho_s)$  is a geodesic path between  $f\mu$  and  $g\mu$  for the  $T_2$  distance.

## Discussions about geodesics for the $T_2$ distance

We consider here the problem of existence of geodesics for the  $T_2$  distance. We will develop further on this issue in a future work. Nevertheless, let us consider some toy cases where one



can check the existence of such geodesics.

For that purpose we will give us more stringent conditions for  $T_2$ , considering the case where  $\mu$  is the Gaussian measure on  $\mathbb{R}^n$ , though in fact any measure satisfying a Poincaré inequality would work. Then we let  $m \in (0, 1)$  be fixed,  $F_m$  be the set of probability densities  $f$  with respect to  $\mu$ , bounded from below by  $m$  and from above by  $1/m$ , and define  $T_2$  by taking the infimum over all paths  $(\rho_s, h_s)$  with  $\rho_s \in F_m$  for all  $s$ , and  $L = \Delta - x \cdot \nabla$ .

This assumption has the following important consequences:

- following the argument in [DNS09], we then have that the distance  $T_2$  has geodesics ;
- the  $T_2$  distance does not tensorize, namely there may be a strict inequality in the tensorization inequality (16) of Proposition 3.8.

Indeed consider,  $f, g$  and  $h$  probability densities with respect to the Gaussian measure  $\gamma$  on  $\mathbb{R}$ , and send  $f \otimes g$  onto  $f \otimes h$  in  $\mathbb{R}^2$ . Assume that (16) is an equality: then for a geodesic  $(\rho_s)$  there is also an equality in (18). It means that there exist two functions  $\alpha_s$  and  $\beta_s$  such that

$$\begin{cases} \frac{\partial_x h_s(x, y)}{\rho_s(x, y)} = \alpha_s(x) \frac{\partial_x h_s^1(x)}{\rho_s^1(x)} \\ \frac{\partial_y h_s(x, y)}{\rho_s(x, y)} = \beta_s(y) \frac{\partial_y h_s^2(y)}{\rho_s^2(y)} \end{cases}$$

Since  $T_2^2(f d\gamma, f d\gamma) = 0$ , then  $\partial_x h_s^1(x) = 0$  and  $\rho_s^1(x) = f(x)$ , which implies from the first equation that  $h_s(x, y)$  only depends on the variable  $y$ . Then the second equation proves that  $\rho_s(x, y)$  depends only on the variable  $y$ . Since  $\int \rho_s(x, y) d\gamma(y) = f(x)$ , this is impossible if  $f$  is a non-constant function. In other words (16) is a strict inequality.

- in  $\mathbb{R}^n$  with  $n > 1$ , the  $T_2$  distance can be strictly larger than the Wasserstein  $W_2$  distance: they are indeed equal in dimension 1, and  $W_2^2$  does tensorize.

## 6 $\Phi$ -entropies versus usual entropy

There are many ways of extending the Markov transportation distance. Here we present the one associated with  $\Phi$ -entropies, well adapted to the  $\Gamma_2$ -calculus. For the Wasserstein distance this generalization has been formulated in [DNS09, DNS12].

Let again  $(P_t)_{t \geq 0}$  be a diffusion Markov semigroup with invariant measure  $\mu$ . Let  $\xi$  be a  $\mathcal{C}^2$  positive function on  $(0, +\infty)$  with  $1/\xi$  concave. Let also

$$\text{Ent}_\mu^\Phi(f) = \int \Phi(f) d\mu - \Phi\left(\int f d\mu\right)$$

be the  $\Phi$ -entropy of a positive map  $f$ , with  $\Phi'' = \xi$ . The  $\Phi$ -entropies have been studied for instance in [Bak94, Cha04, AD05, BG10]. By analogy with Definition 3.2, for  $\Phi(x) = x \log x$  and  $\xi(x) = 1/x$ :

**Definition 6.1** For  $f, g \in \mathcal{F}$  we let

$$T_\xi(f\mu, g\mu) = \inf \left( \int_0^1 \int \Gamma(h_s) \xi(\rho_s) d\mu ds \right)^{1/2} \in [0, +\infty]$$

where the infimum runs over all admissible paths  $(\rho_s, h_s)_{s \in [0, 1]} \in \mathcal{A}(f, g)$ .

For a general map  $\xi$  this distance shares the same properties of existence of  $\varepsilon$ -geodesics and tensorization as the distance  $T_2$ , which can be proved as in section 3. For instance:

**Proposition 6.2 (Tensorization)** *Let  $\xi$  be a  $\mathcal{C}^2$  positive function on  $(0, +\infty)$  with  $1/\xi$  concave. Let  $(P_t^i)_{t \geq 0}$ ,  $i \in \{1, \dots, N\}$  be  $N$  Markov semigroups on probability Polish spaces  $(E_i, \mu_i)$ ,  $i \in \{1, \dots, N\}$  with generators  $L_i$  and carrés du champ  $\Gamma_i$ . Then one can define a product semigroup  $P_t = \otimes_{i=1}^N P_t^i$  on the product space  $(E, \mu) = (\times_{i=1}^N E_i, \otimes_{i=1}^N \mu_i)$  with generator  $L = \oplus_{i=1}^N L_i$  and carré du champ  $\Gamma = \oplus_{i=1}^N \Gamma_i$ .*

*Then, for any densities  $f(x) = \prod_{i=1}^N f_i(x_i)$  and  $g(x) = \prod_{i=1}^N g_i(x_i)$  ( $x = (x_1, \dots, x_N)$ ) in  $\mathcal{F}$ ,*

$$T_\xi^2(f\mu, g\mu) \geq \sum_{i=1}^N T_\xi^2(f_i\mu_i, g_i\mu_i). \quad (40)$$

**Proof**

◁ The argument follows the proof of Proposition 3.8. The key bound (18) is replaced by

$$\int \Gamma_1(h_s)\xi(\rho_s)d\mu_2 \geq \Gamma_1(h_s^1)\xi(\rho_s^1).$$

This is a consequence of the definition (19) of the carré du champ and of the fact that the map  $(x, y) \mapsto x^2\xi(y)$  is convex under our assumption on  $\xi$ , see [Cha04]. ▷

## Contraction and evolution variational inequalities

**Theorem 6.3** *Let  $(P_t)_{t \geq 0}$  be a reversible diffusion Markov semigroup satisfying a  $CD(R, \infty)$  condition with  $R \in \mathbb{R}$ . Then for any  $f, g \in \mathcal{F}$  and  $t \geq 0$ , the contraction property*

$$T_\xi^2(P_t f\mu, P_t g\mu) \leq e^{-2Rt} T_\xi^2(f\mu, g\mu) \quad (41)$$

*holds, as well as the Evolution Variational Inequality*

$$T_\xi^2(f\mu, P_t g\mu) - T_\xi^2(f\mu, g\mu) \leq -\frac{e^{-2Rt} - 1 + 2Rt}{2Rt} T_\xi^2(f\mu, g\mu) + 2t (\text{Ent}_\mu^\Phi(f) - \text{Ent}_\mu^\Phi(P_t g)), \quad (42)$$

*where  $\Phi'' = \xi$ .*

**Proof**

◁ The proof follows the idea of the classical case of Theorems 4.5 and 5.1. It uses the relation

$$\int_0^1 \int \Gamma(P_{ts}(h_s - t\rho_s), P_{ts}(\rho_s))\xi(P_{ts}(\rho_s)) d\mu ds = \text{Ent}_\mu^\Phi(P_t g) - \text{Ent}_\mu^\Phi(f).$$

for any admissible path  $(\rho_s, h_s)$  between  $f$  and  $g$ , and in particular

$$\int_0^1 \int \Gamma(\rho_s, h_s)\xi(\rho_s) d\mu ds = \text{Ent}_\mu^\Phi(g) - \text{Ent}_\mu^\Phi(f),$$

and the following lemma. ▷

**Lemma 6.4** *Let  $f, g \in \mathcal{A}$  with  $g > 0$  and let*

$$\Lambda(t) = \int \Gamma(P_t f) \xi(P_t g) d\mu$$

*for  $t \geq 0$ . Then, under the curvature condition  $CD(R, \infty)$ ,*

$$\Lambda'(t) \leq -2R\Lambda(t) - \int \xi^2(P_t g) \left(-\frac{1}{\xi}\right)''(P_t g) \Gamma(P_t f) \Gamma(P_t g) d\mu \quad (43)$$

**Proof**

$\triangleleft$  We only briefly check the proof since it follows the one of Lemma 4.2. For any  $t \geq 0$ ,

$$\Lambda'(t) = \int \left[ 2\Gamma(P_t f, LP_t f) \xi(P_t g) + \Gamma(P_t f) \xi'(P_t g) LP_t g \right] d\mu.$$

In the notation  $G = P_t g$  and  $F = P_t f$ , the invariance property  $\int L[\Gamma(F) \xi(G)] d\mu = 0$  and the diffusion property of  $L$  give that

$$\int \Gamma(F) \xi'(G) LG d\mu = - \int \left[ 2\xi'(G) \Gamma(\Gamma(F), G) + \xi''(G) \Gamma(G) \Gamma(F) + \xi(G) L\Gamma(F) \right] d\mu.$$

Hence, using the definition of the  $\Gamma_2$  operator,

$$\begin{aligned} \Lambda'(t) &= -2 \int \xi(G) \left[ \Gamma_2(F) + \Gamma(\Gamma(F), \log \xi(G)) + \frac{\xi''(G) \xi(G)}{2\xi'(G)^2} \Gamma(F) \Gamma(\log \xi(G)) \right] d\mu \\ &= -2 \int \xi(G) \left[ \Gamma_2(F) + \Gamma(\Gamma(F), \log \xi(G)) + \Gamma(F) \Gamma(\log \xi(G)) \right] d\mu - \int \xi^2(G) \left(-\frac{1}{\xi}\right)''(G) \Gamma(F) \Gamma(G) d\mu. \end{aligned}$$

Then Lemma 4.3 for  $n = \infty$ , applied to  $f = F$  and  $g = \log \xi(G)$ , implies inequality (43).  $\triangleright$

### The particular case of power functions

Poincaré and logarithmic Sobolev inequalities belong to the family of  $\Phi$ -entropy inequalities, namely for  $\Phi(x) = x^2/2$  and  $\Phi(x) = x \log x$  respectively (see [Cha04]). An interpolation family of inequalities between them consist in the Beckner inequalities, for  $\Phi_p(x) = \frac{x^p}{p(p-1)}$ . It has been proved in [AD05, BG10] how to refine these Beckner inequalities under the curvature-dimension condition  $CD(R, \infty)$ . In the same way, the contraction inequalities proved in [DNS12] and in (41) for a general  $\Phi$  can be made more precise for these power functions, as follows.

For  $p \in (1, 2)$  we let  $\xi_p(x) = x^{p-2}$  and  $\Phi_p(x) = \frac{x^p}{p(p-1)}$  for  $x > 0$ , so that  $\Phi_p'' = \xi_p$ .

**Theorem 6.5 (Refined contraction inequality)** *Let  $(P_t)_{t \geq 0}$  be a reversible diffusion Markov semigroup satisfying a  $CD(R, \infty)$  condition with  $R \in \mathbb{R}$ . Then, for any  $f, g \in \mathcal{F}$  and  $t \geq 0$ ,*

$$\begin{aligned} T_{\xi_p}^2(P_t f \mu, P_t g \mu) &\leq e^{-2Rt} T_{\xi_p}^2(f \mu, g \mu) \\ &\quad - 4 \frac{2-p}{p^2(p-1)} \int_0^t e^{-2R(t-u)} \left( \sqrt{\int (P_u f)^p d\mu} - \sqrt{\int (P_u g)^p d\mu} \right)^2 du. \quad (44) \end{aligned}$$

**Proof**

$\triangleleft$  Let  $(\rho_s, h_s)$  be an admissible path between  $f$  and  $g$ , and  $\Lambda(t, s) = \int \Gamma(P_t(h_s)) \xi_p(P_t(\rho_s)) d\mu$ . Then inequality (43) for  $h_s$  and  $\rho_s$  writes

$$\partial_t \Lambda(t, s) \leq -2R\Lambda(t, s) - (2-p)(p-1) \int (P_t(\rho_s))^{p-4} \Gamma(P_t(\rho_s)) \Gamma(P_t(h_s)) d\mu.$$

But

$$\int (P_t(\rho_s))^{p-4} \Gamma(P_t(\rho_s)) \Gamma(P_t(h_s)) d\mu \geq \frac{\left( \int (P_t(\rho_s))^{p-2} \Gamma(P_t(\rho_s), P_t(h_s)) d\mu \right)^2}{\int P_t(\rho_s)^p d\mu}$$

by the Cauchy-Schwarz inequality, so

$$\partial_t \Lambda(t, s) \leq -2R\Lambda(t, s) - \frac{2-p}{p} \frac{[\partial_s \varphi(t, s)]^2}{\varphi(t, s)}$$

where

$$\varphi(t, s) = \frac{1}{p(p-1)} \int (P_t(\rho_s))^p d\mu.$$

Integrating over  $s \in [0, 1]$  and applying the Gronwall inequality in  $t$ , we obtain

$$\int_0^1 \Lambda(t, s) ds \leq e^{-2Rt} \int_0^1 \Lambda(0, s) ds - \frac{2-p}{p} \int_0^t \int_0^1 e^{-2R(t-u)} \frac{[\partial_s \varphi(u, s)]^2}{\varphi(u, s)} ds du.$$

But

$$\int_0^1 \frac{[\partial_s \varphi(u, s)]^2}{\varphi(u, s)} ds \geq \left( \int_0^1 \frac{\partial_s \varphi(u, s)}{\sqrt{\varphi(u, s)}} ds \right)^2 = 4(\sqrt{\varphi(u, 1)} - \sqrt{\varphi(u, 0)})^2$$

again by the Cauchy-Schwarz inequality. The result follows by optimizing over  $(\rho_s, h_s)$ .  $\triangleright$

In the limit case where  $p = 2$ , the improvement in the contraction inequality disappears in (44), as observed in the refined Beckner inequalities of [AD05, BG10]. Moreover this improvement goes to 0 when  $p$  goes to 1, hence recovering the classical contraction inequality (32)-(41) under the curvature condition  $CD(R, \infty)$ .

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