



On the size of attractors in \mathbb{P}^k

Sandrine Daurat

► To cite this version:

| Sandrine Daurat. On the size of attractors in \mathbb{P}^k . 2013. <hal-00808654>

HAL Id: hal-00808654

<https://hal.science/hal-00808654v1>

Preprint submitted on 9 Apr 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

ON THE SIZE OF ATTRACTORS IN \mathbb{P}^k

Sandrine Daurat¹

RÉSUMÉ. Soit f un endomorphisme holomorphe de $\mathbb{P}^k(\mathbb{C})$ possédant un ensemble attractif \mathcal{A} . Dans cet article, nous nous intéressons à la “taille” de \mathcal{A} , au sens de la géométrie complexe et de la théorie du pluripotentiel. Nous introduisons un cadre conceptuellement simple permettant d’obtenir des ensembles attractifs non algébriques. Nous prouvons qu’en ajoutant une condition de dimension, ces ensembles supportent un courant positif fermé avec un quasi-potentiel borné (ce qui répond à une question de T.C. Dinh). Ils sont donc non pluripolaires. De plus, nous montrons que les exemples sont abondants dans \mathbb{P}^2 .

ABSTRACT. Let f be a holomorphic endomorphism of $\mathbb{P}^k(\mathbb{C})$ having an attracting set \mathcal{A} . In this paper, we address the question of the “size” of \mathcal{A} in a pluripolar sense. We introduce a conceptually simple framework to have non-algebraic attracting sets. We prove that adding a dimensional condition, these sets support a closed positive current with bounded quasi-potential (which answers a question from T.C. Dinh). Therefore, they are not pluripolar. Moreover, the examples are abundant on \mathbb{P}^2 .

INTRODUCTION

In this paper, we study the dynamics of holomorphic endomorphisms of the complex projective space $\mathbb{P}^k(\mathbb{C})$. Denote $\mathbb{P}^k(\mathbb{C})$ by \mathbb{P}^k from now on and let f be an endomorphism of \mathbb{P}^k of algebraic degree $d \geq 2$. Such a map admits a unique invariant probability measure μ of maximal entropy $k \log(d)$, called the equilibrium measure. The most chaotic part of the dynamics is concentrated on $\text{supp}(\mu)$. We refer to [DiS] for an introduction to this theme of research. However, as opposed to dimension 1, chaotic dynamics can also occur outside $\text{supp}(\mu)$. A basic non trivial dynamical phenomenon outside $\text{supp}(\mu)$ is that of attracting sets and attractors. We refer to [FS,FW,JW,R,Di] for some properties and examples of attracting sets and attractors for endomorphisms of \mathbb{P}^k and to [BDM] for a detailed study of a class of algebraic attractors. See also [Du2] for a basic structural description of the dynamics on the Julia set.

It is quite easy to find algebraic attracting set and attractors. On \mathbb{P}^1 , all attractors are algebraic. Let P, Q be homogeneous polynomials of degree $d > 2$ in \mathbb{C}^2 with a single common zero $(0, 0)$, then the line at infinity $\{[z : w : t]; t = 0\}$ is an attracting set for $f : [z : w : t] \mapsto [P(z, w) : Q(z, w) : t^d]$ and it is an attractor if the Julia set of $[z : w] \mapsto [P(z, w) : Q(z, w)]$ is the whole Riemann sphere, see [FS]. The first example of non algebraic attractor in \mathbb{P}^2 (resp. \mathbb{P}^k , with $k \geq 2$) was found by M. Jonsson and B. Weickert (resp. F. Rong), see [JW] (resp. [R]). So far, most previously known examples of non algebraic attracting sets are, in a sense, of codimension 1 and occur for maps of the form

$$f : [z : w : t] \mapsto [P(z, w) : Q(z, w) : t^d + \varepsilon R(z, w)],$$

¹CMLS, Ecole Polytechnique Route de Saclay 91120 Palaiseau
e-mail : sandrine.daurat@math.polytechnique.fr

see [FS,JW,R]. A precise notion of the dimension of an attracting set can easily be formalized, see Definition 1.3 below.

T.C. Dinh [Di] constructed, under some mild assumptions, a natural positive closed current supported on the attracting set. This will be referred as *the attracting current*. J. Taflin [T], under additional hypotheses, proved that this current is the unique positive closed current supported on the attracting set. It remains an interesting problem to understand the “size” of the attracting set in a potential theoretic sense. For instance, Dinh [Di] asks whether the quasi-potential of the attracting current is always unbounded. For the basic example $f : [z : w : t] \mapsto [P(z, w) : Q(z, w) : t^d]$, the attracting set is the line $\{t = 0\}$ thus it supports a unique positive closed current of bidegree (1,1) whose quasi-potential is unbounded.

In this paper, we address this problem, by introducing the concept of a mapping of small topological degree on an attracting set (see Definition 2.3). This notion was inspired by iteration theory of rational maps, see [DDG1-3]. The condition of being of small topological degree provides a conceptually simple framework to provide non algebraic attracting sets of any dimension (see Proposition 2.5).

In codimension 1, this condition implies that the attracting set is non pluripolar. For this we prove that it supports a positive closed current with bounded quasi-potential (see Theorem 2.7). This answers Dinh’s question by the negative. On the other hand, the condition of small topological degree is not sufficient in higher codimension to assure that the attracting set is non pluripolar (see Theorem 4.1).

The examples of small topological maps on attracting sets are abundant. More precisely we prove the following theorem :

Theorem 0.1. *Denote by \mathcal{F}_d the quasi-projective variety of triples (P, Q, R) of homogeneous polynomials of degree d in \mathbb{C}^2 , such that $(0,0)$ is the single common zero of P, Q . There exists a Zariski open set $\Omega \subset \mathcal{F}_d$ such that if $(P, Q, R) \in \Omega$ then for all ε small enough, $0 < |\varepsilon| < \varepsilon(P, Q, R)$, the endomorphism f of \mathbb{P}^2 defined by*

$$f : [z : w : t] \mapsto [P(z, w) : Q(z, w) : t^d + \varepsilon R(z, w)]$$

admits an attracting set \mathcal{A} on which f is of small topological degree. Moreover, \mathcal{A} support a positive closed current τ of bidegree (1,1) which admits a bounded quasi-potential.

Of course, this theorem cannot be true for every $(P, Q, R) \in \mathcal{F}_d$. For example, $[z : w : t] \mapsto [z^2 : w^2 : t^2 + \varepsilon z^2]$ is not of small topological degree on an attracting set, even for $\varepsilon \neq 0$. Indeed, the attracting set is a line in this case.

Denote by T the Green current of f , see [DiS]. With the notation of Theorem 0.1, we have that $\nu = T \wedge \tau$ is an invariant probability measure of maximal entropy supported in \mathcal{A} , see [Di]. Under the conditions of Theorem 0.1, we infer that ν puts no mass on pluripolar sets.

A recent work of N. Fakhruddin [Fa] gives alternate arguments for some of the results in this paper : genericity of non algebraic attracting sets, existence of Zariski dense attracting sets of higher codimension. Notice that we do not use the same notion of genericity. Fakhruddin proves that the set of holomorphic endomorphisms of \mathbb{P}^k which have no non trivial invariant set (i.e. not of zero dimension or not \mathbb{P}^k) contains a countable intersection of Zariski open sets. On the other hand, we work in a specific family \mathcal{F}_d and construct a Zariski open set of examples there. We also observe that the set of endomorphisms in \mathbb{P}^2 possessing an attracting set of small topological degree is open for the usual topology.

The outline of the paper is as follows. We start with some preliminaries. Then we introduce the notion of small topological degree on an attracting set. We show that it is a sufficient condition to have a non algebraic attracting set and under this hypothesis, in codimension one, the attracting set is non-pluripolar.

Section 3 is devoted to the proof of Theorem 0.1.

In section 4, we will exhibit examples in \mathbb{P}^3 of attracting sets of codimension 2. We will see, that the condition of small topological degree is not sufficient in higher codimension to assure that the attracting set is non pluripolar. Using Hénon-like maps of small topological degree, we will exhibit the first explicit example of a Zariski dense attracting set of codimension 2.

We finish with some remarks around Theorem 0.1 and some open questions.

1. PRELIMINARIES

1.1. Attracting sets and attractors. In this section we give some definitions and recall the framework of [Di].

Definition 1.1. A set \mathcal{A} is called an **attracting set** if there exists an open set U such that $f(U) \Subset U$ and $\mathcal{A} = \bigcap_{n \geq 0} f^n(U)$. We call such an open set U a **trapping region**. If moreover f is topologically mixing on \mathcal{A} then \mathcal{A} is called an **attractor**.

The following proposition is obvious.

Proposition 1.2. Let \mathcal{A} be an attracting set. Then \mathcal{A} is closed and $f(\mathcal{A}) = \mathcal{A}$.

Definition 1.3. An attracting set in \mathbb{P}^k is said to be of **dimension** $k - m$ if it supports a positive closed current of bidimension $(k - m, k - m)$ (i.e. of bidegree (m, m)) but none of bidimension (l, l) with $l > k - m$.

Remark.

- Equivalently, an attracting set $\mathcal{A} \subset \mathbb{P}^k$ is said to be of dimension $k - m$ if any (or one) trapping region of \mathcal{A} supports a (smooth) positive closed current of bidimension $(k - m, k - m)$ but none of bidimension (l, l) with $l > k - m$.
- If \mathcal{A} is of dimension $k - m$ then the Hausdorff dimension of \mathcal{A} is at least $2(k - m)$.

The proof of the following elementary proposition is left to the reader.

Proposition 1.4. If \mathcal{A} is an algebraic attracting set of dimension $k - m$ then Definition 1.3 is equivalent to the classical definition of dimension for algebraic sets.

Example 1.5. Let f be an endomorphism of \mathbb{P}^k and let U be an open set such that $f(U) \Subset U$. We assume that there exists two projective subspaces I and J of \mathbb{P}^k of dimension $m - 1$ and $k - m$, $1 \leq m \leq k - 1$, such that $J \subset U$ and $I \cap U = \emptyset$. Then $\mathcal{A} = \bigcap f^n(U)$ is of dimension $k - m$. In fact, U supports the positive closed current $[J]$ of bidimension $(k - m, k - m)$ but does not support a positive closed current of bidimension (l, l) with $l > k - m$. Indeed, the support of any positive closed current T of bidimension (l, l) with $l > k - m$ intersects I . \square

Before stating the main result of [Di], we need some notation. Let $\pi : \mathbb{P}^k \setminus I \rightarrow J$ be the projection of center I . More precisely, letting $I(x)$ be the projective space, of dimension m , containing I and passing through a point $x \in \mathbb{P}^k \setminus I$, then $\pi(x)$ is the unique intersection point between J and $I(x)$. We consider the point $\pi(x)$ as the origin of the complex vector space

$I(x) \setminus I \simeq \mathbb{C}^m$, where I is viewed as the hyperplane at infinity of $I(x) \simeq \mathbb{P}^m$. In other words, $\mathbb{P}^k \setminus I$ is viewed as a vector bundle over J . If $x \in J$, we have that $\pi(x) = x$. We suppose that

- (1) the open set $U \cap I(x)$ in $I(x) \setminus I \simeq \mathbb{C}^m$ is star-shaped at x for every $x \in J$.

If I is a point and J is a projective hyperplane, i.e. $m = 1$, the previous hypothesis is equivalent to the property that the open subset $\mathbb{P}^k \setminus U$ is star-shaped at I .

Theorem 1.6 (Dinh). *Let f and U be as above. Let S be a positive closed (m, m) -form of mass 1 with continuous coefficient and with compact support in U . Then the sequence $\left(\frac{1}{d^{k-m}}(f^n)_*S\right)$ converges to a positive closed current τ of mass 1, with support in $\mathcal{A} = \bigcap f^n(U)$. The current τ does not depend on S . Moreover, it is woven, forward invariant (i.e. under $\frac{1}{d^{k-m}}f_*$) and is extremal in the cone of invariant positive closed currents of bidegree (m, m) with support in \mathcal{A} .*

In the sequel we will refer to τ as *the attracting current* of \mathcal{A} .

1.2. Pluri-potential theory. Here we focus on the codimension 1 case. Let f be an endomorphism of \mathbb{P}^k and \mathcal{L} be the normalised push forward operator, i.e. $\mathcal{L} = \frac{1}{d^{k-1}}f_*$.

Let T be a positive closed current of bidegree $(1, 1)$ of mass 1. There exists a quasi-psh function u , i.e. u is locally the difference of a plurisubharmonic function (psh for short) and a smooth function, such that $T - \omega_{FS} = dd^c u$. We call such a function u a quasi-potential of T .

A dsh function is the difference of two quasi-psh functions, see Appendix A.4. of [DiS] for properties of dsh functions and [De] for basics on psh function. Let R, S be two positive closed currents on \mathbb{P}^k of bidegree $(1, 1)$ with the same mass then there exists a dsh function w such that $R - S = dd^c w$.

Notation 1.7. Let S, R be positive closed currents of bidegree $(1, 1)$ with mass 1. We denote by $u_{S,R}$ the unique dsh function such that $S - R = dd^c u_{S,R}$ and $\int_{\mathbb{P}^k} u_{S,R} \omega_{FS}^k = 0$.

Lemma 1.8. *If (S_i, R_i) converges, in the sense of currents, toward (S, R) then (u_{S_i, R_i}) converges, in sense of distributions, toward $u_{S,R}$.*

Proof. By compactness, $u_{S, \omega_{FS}}$ depends continuously on S and $u_{S,R} = u_{S, \omega_{FS}} - u_{R, \omega_{FS}}$ \square

Lemma 1.9. *Let R, S be as above, then $u_{\mathcal{L}S, \mathcal{L}R}(x) = \frac{1}{d^{k-1}}f_*u_{S,R}(x) = \frac{1}{d^{k-1}} \sum_{f(y)=x} u_{S,R}(y)$.*

Proof. We just need to prove this locally in \mathbb{P}^k . As f is finite, if V is a small enough open set there exists a psh function u , defined on $U = f^{-1}(V)$, such that $T|_U = dd^c u$.

Lemma 1.10. *With the previous notation, $v(x) = \frac{1}{d^{k-1}} \sum_{f(y)=x} u(y)$ is psh on V and $(\mathcal{L}T)|_V = dd^c v$.*

This is classical. We recall the proof for completeness.

Proof. Up to taking a decreasing regularisation, we may assume that u is smooth. Denote by C_f the critical set of f . As $f_*dd^c u = dd^c f_*u$, we have that

$$(2) \quad f_*u(x) = \sum_{f(y)=x} u(y)$$

on $V \setminus f(C_f)$, because f is a submersion on $U \setminus C_f$. Moreover, it is locally a biholomorphism so f_*u is psh on $V \setminus f(C_f)$. A locally bounded psh function on $V \setminus f(C_f)$ admits a unique psh extension on V and for all $x \in V$ we have that $f_*u(x) = \lim_{r \rightarrow 0} \int_{S^{k-1}} f_*u(x + r\xi) d\xi$. Thus (2) is true for all $x \in V$ \square

We now finish the proof of Lemma 1.9. There exists two psh functions u, v on U such that $R = dd^c u$ and $S = dd^c v$. So, on U , we have that $dd^c(u - v) = dd^c(u_{S,R})$, thus there exists a pluri-harmonic function h such that $u + h - v = u_{S,R}$. Then $u + h$ and v are psh functions, so by using the preceding lemma we have that $\frac{1}{d^{k-1}} f_*R - \frac{1}{d^{k-1}} f_*S = dd^c u_{\mathcal{L}S, \mathcal{L}R}$ on $f(U)$, with $u_{\mathcal{L}S, \mathcal{L}R}(x) = \frac{1}{d^{k-1}} \sum_{f(y)=x} u_{S,R}(y)$. \square

2. MAPPINGS OF SMALL TOPOLOGICAL DEGREE ON AN ATTRACTING SET

In this section, we introduce the notion of being of small topological degree on an attracting set in \mathbb{P}^k , prove that such an attracting set is never algebraic and that, in codimension 1, it is non pluripolar.

Definition 2.1. Let f be an endomorphism of \mathbb{P}^k of algebraic degree d and let U be a trapping region for an attracting set \mathcal{A} of dimension $k - m$. The endomorphism f is said to be of small topological degree on U if the number of preimages in U of any point belonging to $f(U)$ is strictly less than d^{k-m} . The endomorphism f is said to be **asymptotically of small topological degree** on U if for all $p \in f(U)$ we have that $\overline{\lim} (\text{card}(f^{-n}(p) \cap U))^{1/n} < d^{k-m}$.

Remark. The notion of being of small topological degree depends on the choice of U . We will encounter in Section 3 examples where f^3 is of small topological degree on $f(U)$ but not on U .

The proof of the following proposition is left to the reader.

Proposition 2.2. The definition of asymptotic small topological degree does not depend on the choice of the trapping region.

Moreover, if f is asymptotically of small topological degree then for each trapping region U there exists $n \geq 1$ such that f^n is of small topological degree on U .

Definition 2.3. We say that f is of **small topological degree** on an attracting set if f is asymptotically of small topological degree on some trapping region.

Sometimes, we will abbreviate this into “attracting set of small topological degree”.

Proposition 2.4. The property of being of small topological degree on some attracting set is open in the set of endomorphisms of degree d .

Proof. Let f be an endomorphism of \mathbb{P}^k of small topological degree on some attracting set of dimension $k - m$ and let U be a trapping region. Replacing f by an iterate we may assume that f is of small topological degree on U . Then for each $p \in f(U)$, we have that $\text{card}(f^{-1}(p) \cap U) < d^{k-m}$. Let g be close to f , there exists an open set V such that $g(V) \Subset f(U) \Subset V \Subset U$. Thus, the attracting set $\cap g^n(V)$ is of dimension $k - m$. For each $p \in g(V) \subset U$, $g^{-1}(p)$ is close to $f^{-1}(p)$ hence, if g is close enough to f , we have that $\text{card}(g^{-1}(p) \cap V) \leq \text{card}(f^{-1}(p) \cap U) < d^{k-m}$. Therefore, g is of small topological degree on V . \square

Let us prove that such attracting sets are non algebraic.

Proposition 2.5. *If a holomorphic endomorphism f of \mathbb{P}^k is of small topological degree on an attracting set \mathcal{A} , then \mathcal{A} is non algebraic.*

The proof relies on the following classical lemma.

Lemma 2.6. *Let f be an endomorphism of \mathbb{P}^k and $M \subset \mathbb{P}^k$ be an invariant algebraic set ($f(M) = M$) of pure dimension $k - m$, $1 \leq m \leq k - 1$. Then there exists $n \geq 1$ such that f^n fixed all irreducible components of M and the topological degree of $f^n|_M$ is $d^{n(k-m)}$.*

Proof. See [DiS, Lemma 1.48]. \square

Proof of Proposition 2.5. Let $k - m$ be the dimension of \mathcal{A} and let U be a trapping region. Up to replacing f by an iterate, we may assume that f is of small topological degree on U . Assume that \mathcal{A} is algebraic. By Proposition 1.4, \mathcal{A} is of dimension $k - m$. Let M be the (finite) union of irreducible components of pure dimension $k - m$ of \mathcal{A} . Then $f(M) = M$, because $f(\mathcal{A}) = \mathcal{A}$ and f does not contract any algebraic subvariety on an algebraic subvariety of lower dimension. Hence, by Lemma 2.6, f cannot be of small topological degree on M . \square

We now proceed with the non pluripolarity (in codimension 1). The corresponding result in higher codimension fails (see below Theorem 4.1).

Theorem 2.7. *Let f be an holomorphic endomorphism of \mathbb{P}^k and let \mathcal{A} be an attracting set of dimension $k - 1$. Assume f is of small topological degree on some trapping region U of \mathcal{A} . Let T be a closed positive current of bidegree $(1,1)$ of mass 1, with support in U , admitting a bounded quasi-potential. Then each cluster value of $\left(\frac{1}{d^{n(k-1)}}(f^n)_*T\right)$ has a bounded quasi-potential. In particular, \mathcal{A} is non pluripolar.*

Moreover, under the assumptions of Theorem 1.6, the attracting current τ has a bounded quasi-potential.

The proof of this theorem is essentially contained in the following lemma. We recall that $\mathcal{L} = \frac{1}{d^{k-1}}f_*$. We denote by d_t the maximum number of preimages in U of a point in $f(U)$ under f .

Lemma 2.8. *For every $U' \Subset U$, there exists a constant $c > 0$ such that if S, R are positive closed currents of bidegree $(1,1)$ with mass 1, supported in U and with bounded quasi-potentials then on U' we have that*

$$\|u_{\mathcal{L}S,R}\|_\infty \leq \frac{d_t}{d^{k-1}}\|u_{S,R}\|_\infty + \|u_{\mathcal{L}R,R}\|_\infty + c.$$

Proof. As expected, the proof just consists in making precise the idea that since $u_{S,R}$ is pluri-harmonic on $U \setminus U'$, we do not need to focus on preimages belonging to $U \setminus U'$.

Fix $U' \Subset U$ such that in addition $f(U) \Subset U'$. As f is finite, we may choose for any $p \in f(U)$ a neighbourhood V_p small enough such that each connected component of $f^{-1}(V_p)$ contains a unique preimage of p . (Of course V_p cannot be uniform with respect to p .) Denote by C_q the connected component of $f^{-1}(V_p)$ which contains $q \in f^{-1}(p)$. We have that $u_{\mathcal{L}S,R} = \frac{1}{d^{k-1}}f_*u_{S,R} + u_{\mathcal{L}R,R}$ and $f_*u_{S,R}(x) = \sum_{f(y)=x} u_{S,R}(y)$. We denote by $U_{1,p} = \cup_{q \in f^{-1}(p) \cap U} C_q$ and

by $U_{2,p} = f^{-1}(V_p) \setminus U_{1,p}$. We can restrict more V_p , so that $U_{1,p} \subset U$, $f(U_{1,p}) = V_p$ and $U_{2,p} \cap U' = \emptyset$, thus the number of preimages (with multiplicity) of a point of V_p in $U_{1,p}$ is fixed. Up to replacing U by U' , we may assume that $f(U)$ is covered by a finite number of such neighbourhoods V_p .

Let x belonging to V_p . We know that $\tilde{u}_{S,R}(x) = \frac{1}{d^{k-1}} \sum_{f(y)=x} u_{S,R}(y)$ is a dsh function such that $\mathcal{L}(S - R) = \text{dd}^c \tilde{u}_{S,R}$. We define, $v_{S,R,p}$ by : for all $x \in V_p$

$$v_{S,R,p}(x) = \frac{1}{d^{k-1}} \sum_{\substack{f(y)=x \\ y \in U_{1,p}}} u_{S,R}(y).$$

As $\text{supp}(S - R) \subset U'$, $u_{S,R}$ is pluri-harmonic outside U' thus $\text{dd}^c v_{S,R,p} = \text{dd}^c \tilde{u}_{S,R} = \mathcal{L}(S - R)$ on V_p and $\|v_{S,R,p}\|_\infty \leq \frac{d_t}{d^{k-1}} \|u_{S,R}\|_\infty$. As $\text{dd}^c u_{\mathcal{L}S,R} = \mathcal{L}S - R = \mathcal{L}(S - R) + (\mathcal{L}R - R) = \text{dd}^c(v_{S,R,p} + u_{\mathcal{L}R,R})$ on V_p , we just have to prove that on V_p there exists c independent of S, R such that $\|u_{\mathcal{L}S,R} - (v_{S,R,p} + u_{\mathcal{L}R,R})\|_\infty \leq c$.

We know that $u_{\mathcal{L}S,R} - (v_{S,R,p} + u_{\mathcal{L}R,R})$ is pluri-harmonic on V_p , hence reducing a little bit V_p , if necessary, there exists $c_{S,R}$ such that $\|u_{\mathcal{L}S,R} - (v_{S,R,p} + u_{\mathcal{L}R,R})\|_\infty \leq c_{S,R}$ on V_p . Let us show that $c_{S,R}$ is uniform with R and S . Assume by contradiction that there exists a sequence (S_i, R_i) such that $\|u_{\mathcal{L}S_i,R_i} - (v_{S_i,R_i} + u_{\mathcal{L}R_i,R_i})\|_\infty \rightarrow \infty$. We can extract a converging subsequence, still denoted by (S_i, R_i) . Let (S, R) be its limit. By Lemma 1.8, the sequence of pluri-harmonic functions $(u_{\mathcal{L}S_i,R_i} - (v_{S_i,R_i} + u_{\mathcal{L}R_i,R_i}))$ converges, in sense of distributions, toward $u_{\mathcal{L}S,R} - (v_{S,R} + u_{\mathcal{L}R,R})$, therefore the convergence is uniform (see corollary 3.1.4 in [H]). Thus $(\|u_{\mathcal{L}S_i,R_i} - (v_{S_i,R_i} + u_{\mathcal{L}R_i,R_i})\|_\infty)$ is bounded because its limit $\|u_{\mathcal{L}S,R} - (v_{S,R} + u_{\mathcal{L}R,R})\|_\infty$ is bounded. we have a contradiction. Therefore, there exists c_1 independent of (S, R) such that $\|u_{\mathcal{L}S,R} - (v_{S,R} + u_{\mathcal{L}R,R})\|_\infty \leq c_1$ hence

$$\|u_{\mathcal{L}S,R}\|_\infty \leq \frac{d_t}{d^{k-1}} \|u_{S,R}\|_\infty + \|u_{\mathcal{L}R,R}\|_\infty + c_1 \text{ on } V_p.$$

Since $f(U)$ is covered by finitely many such V_p , we infer that there exists c_2 such that $\|u_{\mathcal{L}S,R}\|_\infty \leq \frac{d_t}{d^{k-1}} \|u_{S,R}\|_\infty + \|u_{\mathcal{L}R,R}\|_\infty + c_2$ on $f(U)$. Finally, as $u_{\mathcal{L}S,R}$ is pluri-harmonic outside $f(U')$, $u_{\mathcal{L}S,R}$ is bounded by $c_{U,U'}$ on $U' \setminus f(U)$, we obtain that

$$\|u_{\mathcal{L}S,R}\|_\infty \leq \frac{d_t}{d^{k-1}} \|u_{S,R}\|_\infty + \|u_{\mathcal{L}R,R}\|_\infty + c_2 + c_{U,U'} \text{ on } U'.$$

For similar reasons as in the case of $c_{S,R}$, we may assume that $c_{U,U'}$ is independent of S, R . Therefore, the proof of the lemma is complete. \square

Proof of Theorem 2.7. Denote by u_n the dsh function such that $\mathcal{L}^n T - T = \text{dd}^c u_n$ and $\int u_n \omega_{FS}^k = 0$, i.e. $u_n = u_{\mathbb{L}^n T, T}$. Then u_1 is bounded. By Lemma 2.8, with $S = \mathcal{L}^n T$ and $R = T$, we have

$$\|u_{n+1}\|_\infty \leq \frac{d_t}{d^{k-1}} \|u_n\|_\infty + \|u_1\|_\infty + c \text{ on } U.$$

It is classical that a sequence (x_n) satisfying $0 \leq x_{n+1} \leq \alpha x_n + c$ with $0 < \alpha < 1$ is bounded. Therefore, each cluster value of $(\mathcal{L}^n T)$ has a bounded quasi-potential. If in addition, f satisfies (1) then $(\mathcal{L}^n T)$ converges toward τ , thus τ has also a bounded quasi-potential. \square

Remark. I do not know how to prove the continuity of the quasi-potential. A reason for this is that the proof does not yield the convergence of the sequence of potentials. Note that the same difficulty appears in Dinh's Theorem 1.6.

3. A CLASS OF ATTRACTING SETS OF SMALL TOPOLOGICAL DEGREE IN \mathbb{P}^2

Let \mathcal{F}_d be the set of triples (P, Q, R) of homogeneous polynomials of degree $d \geq 2$ in \mathbb{C}^2 , such that $(0, 0)$ is the single common zero of P, Q . Clearly, \mathcal{F}_d is a quasi-projective variety. In this section, we consider a particular class of endomorphisms of \mathbb{P}^2 given by the formula

$$(3) \quad f : [z : w : t] \rightarrow [P(z, w) : Q(z, w) : t^d + \varepsilon R(z, w)]$$

where $(P, Q, R) \in \mathcal{F}_d$. Attractors for mappings of this form were studied by J.E. Fornæss and N. Sibony in [FS], as well as F. Rong [R].

These endomorphisms preserve the pencil of lines passing through $[0 : 0 : 1]$. Such a line is determined by a point of the form $[z : w : 0]$, thus we identify the line passing through $[0 : 0 : 1]$ and $[z : w : 0]$ with the point of the line at infinity $[z : w : 0]$. The dynamics on this pencil of lines is that of $f_\infty : [z : w] \rightarrow [P(z, w) : Q(z, w)]$ on \mathbb{P}^1 .

Theorem 3.1. *There exists a Zariski open set $\Omega \subset \mathcal{F}_d$ such that if $(P, Q, R) \in \Omega$ then for small enough ε , $0 < |\varepsilon| < \varepsilon(P, Q, R)$, the endomorphism f of \mathbb{P}^2 defined by*

$$f : [z : w : t] \mapsto [P(z, w) : Q(z, w) : t^d + \varepsilon R(z, w)]$$

admits an attracting set \mathcal{A} of small topological degree. In particular, \mathcal{A} is non pluripolar. Moreover, f satisfies the condition of Theorem 1.6 and the attracting current τ has a bounded quasi-potential.

Remark. Since having an attracting set of small topological degree is an open condition, by perturbing in the set of all endomorphisms of \mathbb{P}^2 , we obtain maps which do not preserve a pencil but also satisfy the conclusion of Theorem 3.1.

The proof gives specific examples in any degree and we can arrange for f to be topologically mixing on \mathcal{A} , therefore we obtain the following :

Proposition 3.2. *There exists an attractor $\mathcal{A} \subset \mathbb{P}^2$ of small topological degree.*

We now proceed to the proof of Theorem 3.1.

Proof of theorem 3.1. The proof will proceed in four steps. First we show that any f of the form (3) admits an attracting set and satisfies the conditions of Theorem 1.6. Next we give a sufficient condition ensuring that f^3 is of small topological degree on some trapping region. The third step is to show that this condition is algebraic. Finally, we give examples in any degree, therefore showing that the condition is generically satisfied.

Step 1. The map f admits an attracting set and satisfies the conditions of Theorem 1.6.

As R is an homogeneous polynomial of degree d , there exists $\beta > 0$ such that for all $(z, w) \in \mathbb{C}^2$ we have that $|R(z, w)| \leq \beta \max(|z|, |w|)^d$. Up to multiply ε by β we may assume that $\beta = 1$. Denote by $\alpha = \inf_{\max(|z|, |w|)=1} \max(|P(z, w)|, |Q(z, w)|)$, as $(0, 0)$ is the single common zero of P, Q , we have that $\alpha > 0$.

Let U_ρ be the open set of $[z : w : t] \in \mathbb{P}^2$ such that $|t| < \rho \max(|z|, |w|)$. Then we have $|t^d + \varepsilon R(z, w)| < (\rho^d + \varepsilon) \max(|z|, |w|)^d < \frac{1}{\alpha}(\rho^d + \varepsilon) \max(|P(z, w)|, |Q(z, w)|)$, by definition of α . To assume that $f(U_\rho) \Subset U_\rho$, all we need is that $\frac{1}{\alpha}(\rho^d + \varepsilon) < \rho$. We choose $\varepsilon \ll 1$. We let the reader check that the choice

$$(4) \quad \rho = 4 \frac{\varepsilon}{\alpha}$$

is convenient.

Thus $\mathcal{A} = \cap_{n \in \mathbb{N}} f^n(U_\rho)$ is an attracting set. Denote by $l_{[z:w]}$ the line passing through $[0 : 0 : 1]$ and $[z : w : 0]$ and by L_∞ the line at infinity. The line $l_{[z:w]}$ intersects U_ρ into an open disc centered at $[z : w : 0]$. Thus, by taking $I = \{[0 : 0 : 1]\}$, $J = L_\infty$ and $U = U_\rho$, f satisfies the hypothesis (1).

Step 2. A sufficient condition for being of small topological degree.

Throughout the end of the proof, by a line we mean a line passing through $[0 : 0 : 1]$, unless for the line at infinity L_∞ .

Here we will introduce for $(P, Q, R) \in \mathcal{F}_d$ subsets X and Y of $L_\infty \simeq \mathbb{P}^1$ and a condition on these sets insuring that for all $\varepsilon \neq 0$ small enough f^3 is of small topological degree on $f(U_\rho)$, where f is defined by (3), (see Proposition 3.3 and the comments that precede it).

For maps of the form (3), a generic line has d preimages and a generic point has d^2 preimages, with d in each line. We leave the reader check that the image of a disc in $l_{[z:w]}$ centered at $[z : w : 0]$ is a disc in $l_{f_\infty([z:w])}$ centered at $f([z : w : 0])$. A line l having d preimages (with multiplicity), $f(U_\rho) \cap l$ is made of at most d discs which might intersect.

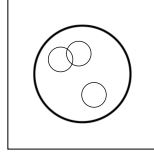


FIGURE 1. An example of the intersection of a line with U_ρ (large circle) and $f(U_\rho)$ (small circle). Here f is of degree 3.

If they are disjoint, the preimages of a point of $f(U_\rho) \cap l$, which are in U_ρ , are contained in a single line. Denote by A the set of lines l such that $f(U_\rho) \cap l$ is made of d disjoint discs. Thus if $l \in A$, for each point $p \in l \cap f(U_\rho)$ there exists a unique line l_{-1} such that the preimages of p in U_ρ are all in l_{-1} .

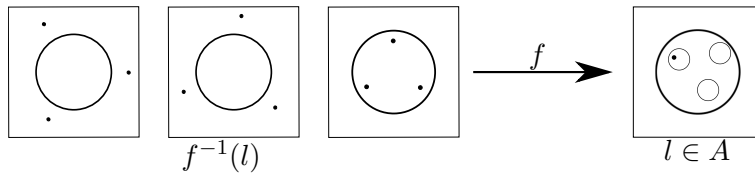


FIGURE 2. The intersection of a line $l \in A$ with U_ρ and $f(U_\rho)$, on the right, and the preimages of l and of a point in $f(U_\rho) \cap l$, on the left.

The points $[z : w : e^{\frac{2ik\pi}{d}}t]$ all have the same image and are either all in U_ρ , or all outside U_ρ . Therefore, for any line l and any point $p \in l \cap f(U_\rho)$, if we fix a preimage $l_{[z:w]}$ of l , the point p has d preimages with multiplicity in the line $l_{[z:w]}$, they are of the form $[z : w : e^{\frac{2ik\pi}{d}}t]$. Thus, p has 0 or d preimages (with multiplicity) in $l_{[z:w]} \cap U_\rho$. To further reduce the number of preimages we must see if they are or not in $f(U_\rho)$.

Denote by B_{-1} the set of lines $l_{[z:w]}$ such that for all $|t| < \rho \max(|z|, |w|)$ there exists a unique $k \in \{0, \dots, d-1\}$ such that $[z : w : e^{\frac{2ik\pi}{d}}t] \in f(U_\rho)$ and denote by B the set of lines whose preimages all of lie in B_{-1} .

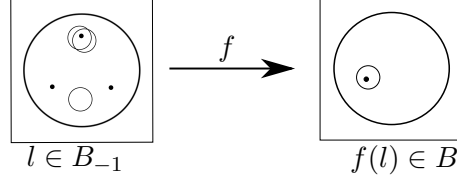


FIGURE 3. The intersection of the lines l and $f(l)$ with U_ρ and $f(U_\rho)$, and, the preimages of a point in $f(l) \cap f(U_\rho)$, when $l \in B_{-1}$.

In this way, if $l \in B$ and $p \in l \cap f^2(U_\rho)$, for any preimage l_{-1} of l , p has only one preimage in $l_{-1} \cap f(U_\rho)$. If in addition $l \in B \cap A$ then a point $p \in l \cap f^2(U_\rho)$ has a single preimage in $f(U_\rho)$.

We need to understand the complementary of $A \cap B$.

Denote by $l_{[z_1:w_1]}, \dots, l_{[z_d:w_d]}$ the preimages (with possible repetitions) of a line $l_{[z:w]}$.

A sufficient condition for a line $l_{[z:w]}$ not to be in A is that there exists $i \neq j$ in $\{1, \dots, d\}$ such that $f([z_i : w_i : 0]) = f([z_j : w_j : 0])$. Denote by X_{-1} the set of lines $l_{[z_i:w_i]}, l_{[z_j:w_j]}$ which satisfy this condition and let $X = f(X_{-1})$. Therefore, $l_{[z:w]} \in X$ iff there exists $i \neq j$ in $\{1, \dots, d\}$ such that $f(l_{[z_i:w_i]} \cap U_\rho) = f(l_{[z_j:w_j]} \cap U_\rho)$.

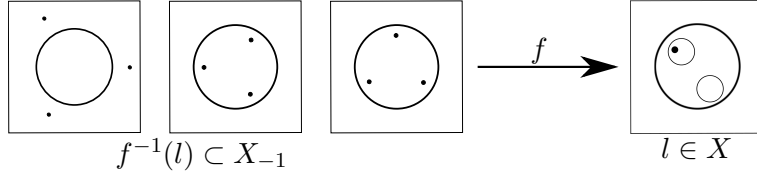


FIGURE 4. The problem that can occur when a line is in X .

A point in $f(U_\rho)$ is of the form $[P(z, w) : Q(z, w) : t^d + \varepsilon R(z, w)]$ with $|t| < \rho \max(|z|, |w|)$. A sufficient condition for a line $l_{[z:w]}$ not to be in B_{-1} is that there exists $i, j \in \{1, \dots, d\}$ and $k \in \{1, \dots, d-1\}$ such that $[P(z_i, w_i) : Q(z_i, w_i) : \varepsilon R(z_i, w_i)] = [P(z_j, w_j) : Q(z_j, w_j) : \varepsilon e^{\frac{2ik\pi}{d}} R(z_j, w_j)]$. Possibly $i = j$ in which case this correspond to $R(z_i, w_i) = 0$ and necessarily, as in $l_{[z:w]} \cap U_\rho$, there will be points in $f(l_{[z:w]})$ that have d preimages in $l_{[z:w]} \cap f(U_\rho)$. Thus, two problems can occur (when $i = j$ or not). Denote by Y_{-2} the set of lines $l_{[z_i:w_i]}, l_{[z_j:w_j]}$ which satisfy this condition and let $Y_{-1} = f(Y_{-2})$ and $Y = f^2(Y_{-2})$.

By definition, we have that $X \cup Y \subset (A \cap B)^c$. Denote by $f_\infty^+([z : w]) = [P(z, w) : Q(z, w) : R(z, w)]$ and $pr_1 : (l, l') \mapsto l$. As $(l, l') \in X_{-1}$ (resp. Y_{-2}) iff $(l', l) \in X_{-1}$ (resp. Y_{-2}), we will, sometimes, identify X_{-1}, Y_{-2} with $pr_1(X_{-1}), pr_1(Y_{-2})$, particularly in the examples.

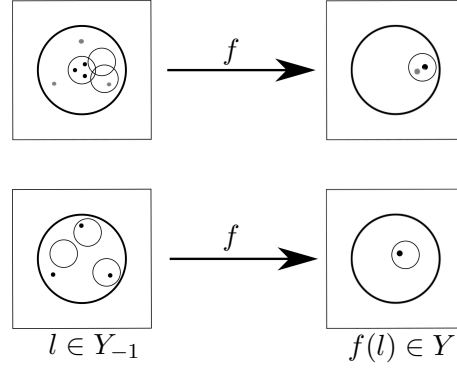


FIGURE 5. The two problems that can occur when a line is in Y . The black (resp. grey) points in l are the preimages of the black (resp. grey) point in $f(l)$.

We note for further reference that, by identifying $l_{[z:w]}$ to $[z:w] \in \mathbb{P}^1$, X_{-1} and Y_{-2} may be written formally as :

$$(5) \quad \begin{cases} X_{-1} = \{(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1; p \neq q, f_{\infty}^+(p) = f_{\infty}^+(q)\} \\ \quad \cup \{(p, p) \in \mathbb{P}^1 \times \mathbb{P}^1; p \text{ is a critical point of } f_{\infty}\} \\ Y_{-2} = \{(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1; f_{\infty}^+(p) \neq f_{\infty}^+(q), f_{\infty}^{+2}(p) = f_{\infty}^{+2}(q), f_{\infty}(p) = f_{\infty}(q)\} \\ \quad \cup \{(p, p) \in \mathbb{P}^1 \times \mathbb{P}^1; R(p) = 0\} \\ X = f(pr_1(X_{-1})) \\ Y = f^2(pr_1(Y_{-2})). \end{cases}$$

The next proposition is a sufficient condition to be of small topological degree on the attracting set \mathcal{A} . Recall that A, B, X, Y are sets of lines passing through $[0 : 0 : 1]$, that we identify such a line with a point in $L_{\infty} \simeq \mathbb{P}^1$ and that the dynamics of the pencil of these lines is the same as that of $f_{\infty} : [z : w] \rightarrow [P(z, w) : Q(z, w)]$. Note that X_{-1}, Y_{-2}, X, Y only depend on the choice of $(P, Q, R) \in \mathcal{F}_d$, not on ε .

Proposition 3.3. *Denote by $Z = X \cup Y$ and $\mathcal{Z} = (f_{\infty}^{-1}(Z)) \cap Z$. If $(P, Q, R) \in \mathcal{F}_d$ are such that*

- (1) *X and Y are disjoint*
- (2) *$(f_{\infty}^{-1}(\mathcal{Z})) \cap Z = \emptyset$, or equivalently $f_{\infty}^{-2}(Z) \cap f_{\infty}^{-1}(Z) \cap Z = \emptyset$*

then for small enough $\varepsilon \neq 0$ there exists ρ such that f^3 is of small topological degree on $f(U_{\rho})$, with f define by (3).

See Section 5.1 for a simplified version of these conditions.

Proof of Proposition 3.3. For ε as before, we fix $\rho = 4\frac{\varepsilon}{\alpha}$, as in (4), so that $f(U_{\rho}) \Subset U_{\rho}$. Denote by X^r (resp. Y^r, Z^r, \mathcal{Z}^r) the r -neighbourhood of X (resp. Y, Z, \mathcal{Z}) in L_{∞} . We choose r small enough such that X^r (resp. Y^r, Z^r, \mathcal{Z}^r) is the union of disjoint open discs centered at the points of X (resp. Y, Z, \mathcal{Z}) and such that we have that :

- $X^r \cap Y^r = \emptyset$, by hypothesis (1),

- $f_\infty^{-1}(Z^r) \cap Z^r \subset \mathcal{Z}^r$, since $(f_\infty^{-1}(Z)) \cap Z = \mathcal{Z}$,
- and $f_\infty^{-1}(\mathcal{Z}^r) \cap Z^r = \emptyset$, by hypothesis (2).

Lemma 3.4. *Fix ρ and r as above. Then if $\varepsilon \neq 0$ is small enough,*

$$X \cup Y = Z \subset (A \cap B)^c \subset Z^r.$$

Proof. Indeed, the first inclusion is due to the definition of X and Y . Regarding the second inclusion, recall that $l_{[z_1:w_1]}, \dots, l_{[z_d:w_d]}$ denoted the preimages (with possible repetitions) of a line $l_{[z:w]}$. Through the end of the proof, for each $i \in \{1, \dots, d\}$, we normalize (z_i, w_i) so that

$$P(z_i, w_i) = z \text{ and } Q(z_i, w_i) = w.$$

The line $l_{[z:w]}$ is in A iff for $i \neq j$ in $\{1, \dots, d\}$ and t_i, t_j such that $[z_i : w_i : t_i], [z_j : w_j : t_j] \in U_\rho$ we have that $t_i^d + \varepsilon R(z_i, w_i) \neq t_j^d + \varepsilon R(z_j, w_j)$. By definition of U_ρ , for this, it suffices that

$$(6) \quad |\varepsilon R(z_i, w_i) - \varepsilon R(z_j, w_j)| > \rho^d (\max(|z_i|, |w_i|) + \max(|z_j|, |w_j|))^d.$$

The line $l_{[z:w]}$ is in B_{-1} iff for all $i, j \in \{1, \dots, d\}$, all t_i, t_j such that $[z_i : w_i : t_i], [z_j : w_j : t_j] \in U_\rho$ and all $k \in \{1, \dots, d-1\}$ we have that $t_i^d + \varepsilon R(z_i, w_i) \neq e^{\frac{2ik\pi}{d}} (t_j^d + \varepsilon R(z_j, w_j))$. For this, it suffices that for each $k \in \{1, \dots, d-1\}$

$$(7) \quad |\varepsilon R(z_i, w_i) - e^{\frac{2ik\pi}{d}} \varepsilon R(z_j, w_j)| > \rho^d (\max(|z_i|, |w_i|) + \max(|z_j|, |w_j|))^d.$$

By definition of X and Y , there exists $i \neq j$ in $\{1, \dots, d\}$ such that $R(z_i, w_i) - R(z_j, w_j) = 0$ iff $l_{[z_i:w_i]}, l_{[z_j:w_j]} \in X_{-1}$ (iff $l_{[z:w]} \in X$) and there exists $i, j \in \{1, \dots, d\}$ and $k \in \{1, \dots, d-1\}$ such that $R(z_i, w_i) - e^{\frac{2ik\pi}{d}} R(z_j, w_j) = 0$ iff $l_{[z_i:w_i]}, l_{[z_j:w_j]} \in Y_{-2}$ (iff $f(l_{[z:w]}) \in Y$). Since $\mathbb{P}^1 \setminus (f^{-1}(X^r) \cup f^{-2}(Y^r))$ is compact and $X_{-1} \subset f_\infty^{-1}(X^r)$ and $Y_{-2} \subset f_\infty^{-2}(Y^r)$, there exists $\gamma > 0$ such that if $l_{[z_i:w_i]}, l_{[z_j:w_j]} \notin f^{-1}(X^r)$ with $i \neq j$ then

$$|R(z_i, w_i) - R(z_j, w_j)| > \gamma (\max(|z_i|, |w_i|) + \max(|z_j|, |w_j|))^d$$

and if $l_{[z_i:w_i]}, l_{[z_j:w_j]} \notin f^{-2}(Y^r)$, (i, j can be equal here) then for each $k \in \{1, \dots, d-1\}$

$$|R(z_i, w_i) - e^{\frac{2ik\pi}{d}} R(z_j, w_j)| > \gamma (\max(|z_i|, |w_i|) + \max(|z_j|, |w_j|))^d.$$

Since γ is independent of ρ and ε , $\rho = 4\frac{\varepsilon}{\alpha}$ and $d \geq 2$, if ε is small enough then $\varepsilon\gamma > \rho^d$, which implies that if $l_{[z:w]} \notin X^r$ and i, j are distinct then

$$(8) \quad |\varepsilon R(z_i, w_i) - \varepsilon R(z_j, w_j)| > \rho^d (\max(|z_i|, |w_i|) + \max(|z_j|, |w_j|))^d$$

and if $l_{[z:w]} \notin f^{-1}(Y^r)$, (i, j can be equal here) then for all $k \in \{1, \dots, d-1\}$

$$(9) \quad |\varepsilon R(z_i, w_i) - e^{\frac{2ik\pi}{d}} \varepsilon R(z_j, w_j)| > \rho^d (\max(|z_i|, |w_i|) + \max(|z_j|, |w_j|))^d.$$

If $l_{[z:w]} \notin X^r$ then for all i, j , $l_{[z_i:w_i]}, l_{[z_j:w_j]} \notin f^{-1}(X^r)$ with $i \neq j$, thus, from (6) and (8), $l_{[z:w]} \in A$. If $l_{[z:w]} \notin f^{-1}(Y^r)$ then for all i, j , we have that $l_{[z_i:w_i]}, l_{[z_j:w_j]} \notin f^{-2}(Y^r)$, thus, from (7) and (9), $l_{[z:w]} \in B_{-1}$. Consequently, if $l_{[z:w]} \notin Y^r$, no preimage of $l_{[z:w]}$ is in $f^{-1}(Y^r)$, thus all preimages of $l_{[z:w]}$ are in B_{-1} , i.e. $l_{[z:w]} \in B$. Therefore, if $l_{[z:w]} \notin Z^r$ then $l_{[z:w]} \in A \cap B$, i.e. $(A \cap B)^c \subset Z^r$.

This finish the proof of the lemma. \square

To finish the proof of Proposition 3.3 we have to count the number of preimages of a point $p \in f^4(U_\rho)$ under f^3 . Denote by $\pi : \mathbb{P}^2 \setminus \{[0 : 0 : 1]\} \rightarrow L_\infty$ the projection defined by $\pi([z : w : t]) = [z : w]$.

A point in $\pi^{-1}((Z^r)^c) \cap f^2(U_\rho)$ has one preimage in $f(U_\rho)$. Moreover, the complementary of B is contained in Y^r and the complementary of A is contained in X^r . Since $X^r \cap Y^r = \emptyset$, we infer that a point in $\pi^{-1}(Z^r) \cap f^2(U_\rho)$ has at most d preimages in $f(U_\rho)$. Thus, a point has at most d preimages under f in $f(U_\rho)$.

For the sake of simplicity, until the end of the proof, without precision by preimage we mean preimage in $f(U_\rho)$. Pick a point $p \in f^4(U_\rho)$.

Case 1. If $p \notin \pi^{-1}(Z^r) \cap f(U_\rho)$ then $p \in \pi^{-1}(A \cap B)$, so p has one preimage under f and $\text{card}\{f^{-3}(p) \cap U_\rho\} \leq d^2$.

Case 2. If $p \in \pi^{-1}(Z^r) \cap f(U_\rho)$ then p has at most d preimages under f . For each preimage of p we have two cases :

Case 2.1. Either it is still in $\pi^{-1}(Z^r) \cap f(U_\rho)$. Thus it has d preimages under f which are in $\pi^{-1}(\mathcal{Z}^r)$ (because $f_\infty^{-1}(Z^r) \cap Z^r \subset \mathcal{Z}^r$), but since $f_\infty^{-1}(\mathcal{Z}^r) \cap Z^r = \emptyset$, these preimages leave $\pi^{-1}(Z^r) \cap f(U_\rho)$ and thus have only one preimage under f . It follows that this preimage of p has at most d preimages under f^2 .

Case 2.2. Or it is outside $\pi^{-1}(Z^r) \cap f(U_\rho)$, then it has one preimage. It follows that this preimage of p has at most d preimages under f^2 .

Therefore, $\text{card}\{f^{-3}(p) \cap f(U_\rho)\} \leq d^2$ in all cases. Consequently, f^3 is of small topological degree on $f(U_\rho)$. This concludes the proof of the Proposition. \square

Denote by Ω the subset of $(P, Q, R) \in \mathcal{F}_d$ satisfying the hypotheses of Proposition 3.3.

At this point to finish the proof of Theorem 3.1 it only remains to prove that Ω is a Zariski open set and that it is not empty.

Step 3. The subset Ω of \mathcal{F}_d is a Zariski open set.

This is simply an exercise in complex geometry. Recall from (5) that $X = pr_1 \circ (f_\infty \otimes f_\infty)(X_{-1})$, $Y = pr_1 \circ (f_\infty^2 \otimes f_\infty^2)(Y_{-2})$, $Z = X \cup Y$ and $\mathcal{Z} = (f_\infty^{-1}(Z)) \cap Z$, where $pr_1(p, q) = p$. As already observed, X, Y, Z only depend on the choice of $(P, Q, R) \in \mathcal{F}_d$ and not on the choice of $\varepsilon \neq 0$. Recall that the conditions of Proposition 3.3 are

- (1) X and Y are disjoint
- (2) $(f_\infty^{-1}(\mathcal{Z})) \cap Z = \emptyset$ or equivalently $f_\infty^{-2}(Z) \cap f_\infty^{-1}(Z) \cap Z = \emptyset$.

Set

$$\begin{aligned} \tilde{X}_{-1} = & \{(p, q, (P, Q, R)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{F}_d; p \neq q, f_\infty^+(p) = f_\infty^+(q)\} \\ & \cup \{(p, p, (P, Q, R)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{F}_d; p \text{ is a critical point of } f_\infty\}, \end{aligned}$$

$$\begin{aligned} \tilde{Y}_{-2} = & \{(p, q, (P, Q, R)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{F}_d; f_\infty^+(p) \neq f_\infty^+(q), f_\infty^{+2}(p) = f_\infty^{+2}(q), f_\infty(p) = f_\infty(q)\} \\ & \cup \{(p, p, (P, Q, R)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{F}_d; R(p) = 0\}, \end{aligned}$$

and denote by $\Phi : (p, q, (P, Q, R)) \mapsto (f_\infty(p), f_\infty(q), (P, Q, R))$, $\tilde{X} = \Phi(\tilde{X}_{-1})$, $\tilde{Y} = \Phi^2(\tilde{Y}_{-2})$ and $\tilde{Z} = \tilde{X} \cup \tilde{Y}$. Thus $\{(p, q, f_\infty(p), f_\infty(q), (P, Q, R)); (p, q, (P, Q, R)) \in \tilde{X}_{-1}\}$ is an algebraic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{F}_d$ as well as $\{(p, q, f_\infty^2(p), f_\infty^2(q), (P, Q, R)); (p, q, (P, Q, R)) \in \tilde{Y}_{-2}\}$.

Therefore, \tilde{X}, \tilde{Y} and \tilde{Z} are algebraic subvarieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{F}_d$, and $\tilde{X} \cap \tilde{Y}$ and $\Phi^{-2}(\tilde{Z}) \cap \Phi^{-1}(\tilde{Z}) \cap \tilde{Z}$ are algebraic subvarieties too. The complementary of Ω is in the image of $(\tilde{X} \cap \tilde{Y}) \cup (\Phi^{-2}(\tilde{Z}) \cap \Phi^{-1}(\tilde{Z}) \cap \tilde{Z})$ by the projection $(p, q, (P, Q, R)) \mapsto (P, Q, R)$, thus it is a Zariski closed subset in \mathcal{F}_d .

Step 4. The Zariski open set Ω defined in step 3 is not empty.

Let us show that $f = [w^d + az^d : bw^d + z^d : t^d + \varepsilon(z^d + z^{d-1}w)]$ satisfies the hypotheses of Proposition 3.3 for almost every $a, b \in \mathbb{C}$ small enough. We first assume $|a|, |b| > 0$.

If $l_{[z:w]}$ is a preimage of a line, the other preimages are $l_{[z:e^{\frac{2ik\pi}{d}}w]}$ with $k \in \{1, \dots, d-1\}$. For $l \in \{1, \dots, d-1\}, k \in \{0, \dots, d-1\}$ put

$$z_{k,l} = [e^{\frac{2i(k+l)\pi}{d}} - 1 : 1 - e^{\frac{2il\pi}{d}}].$$

We let the reader check that $X_{-1} = \{[0 : 1], [1 : 0]\}$ and

$$\begin{aligned} Y_{-2} &= \{[e^{\frac{2i(k+l)\pi}{d}} - 1 : 1 - e^{\frac{2il\pi}{d}}]; l \in \{1, \dots, d-1\}, k \in \{0, \dots, d-1\}\} \\ &= \{[0 : 1]\} \cup \{z_{k,l}; l \in \{1, \dots, d-1\}, k \in \{0, \dots, d-1\}, k+l \neq d\}. \end{aligned}$$

Thus

$$\begin{aligned} X &= \{[1 : b], [a : 1]\}, \\ Y &= f_\infty^2(Y_{-2}) = \{f_\infty([1 : b])\} \cup \{f_\infty^2(z_{k,l}); l \in \{1, \dots, d-1\}, k \in \{0, \dots, d-1\}, k+l \neq d\}. \end{aligned}$$

Suppose that $a, b \in \mathbb{C}^*$ are small and $a \sim b$.

If $k+l = d$ then $z_{k,l} = 0 = [0 : 1]$, $f_\infty(z_{k,l}) = [1 : b]$ and $f_\infty^2(z_{k,l}) = [b^d + a : b^{d+1} + 1] \approx [a : 1]$, otherwise $f_\infty(z_{k,l}) \approx z_{k,l}^{-d}$ and $f_\infty^2(z_{k,l}) \approx z_{k,l}^{2d}$.

If $k+l \neq d$ then $z_{k,l}$, $f_\infty(z_{k,l})$ and $f_\infty^2(z_{k,l})$ are far from $[0 : 1], [1 : 0]$. In particular, $\{f_\infty(z_{k,l}), f_\infty^2(z_{k,l})\} \cap \{[0 : 1], [1 : 0], [1 : b], [a : 1], [b^d + a : b^{d+1} + 1], [1 + a^{d+1} : b + a^d]\} = \emptyset$. We conclude that, if $a, b \in \mathbb{C}^*$ are small and $a \sim b$ then X and Y are disjoint, i.e. f satisfies the hypothesis (1) of Proposition 3.3. To check assumption (2), write

$$Z = X \cup Y = \{[1 : b], [a : 1], f_\infty([1 : b])\} \cup \{f_\infty^2(z_{k,l}); k+l \neq d\},$$

so that

$$f_\infty^{-1}(Z) = \{[0 : 1], [1 : 0], [1 : e^{\frac{2in\pi}{d}}b], e^{\frac{2in\pi}{d}}f_\infty(z_{k,l}) | n \in \mathbb{N}\}.$$

The $z_{k,l}$ are independent of a, b , so we can choose a, b so that $\{e^{\frac{2in\pi}{d}}f_\infty(z_{k,l})\} \cap \{f_\infty^2(z_{k,l})\} = \emptyset$, thus $f_\infty^{-1}(Z) \cap Z = \{[1 : b]\}$. Since $f_\infty^{-1}([1 : b]) = \{[0 : 1]\} \notin Z$, we conclude that f satisfies the second assumption of Proposition 3.3.

This completes the proof of Theorem 3.1. \square

We now prove Proposition 3.2.

Proof of Proposition 3.2. We first show that $f = [(z - 2w)^2 : z^2 : t^2 + \varepsilon(z^2 + zw)]$ satisfies the hypotheses of Proposition 3.3. The preimages of $L_{[\alpha:\beta]}$ are $L_{[2\sqrt{\beta}:\sqrt{\beta}-\sqrt{\alpha}]}$ and $L_{[2\sqrt{\beta}:\sqrt{\beta}+\sqrt{\alpha}]}$. We let the reader check that $X = \{[0 : 1], [1 : 0]\}$ and $Y = \{f_\infty([1 : -1]), f_\infty([1 : 9])\} = \{[9 : 1], [17^2 : 1]\}$, which are disjoint.

Thus $Z = \{[0 : 1], [1 : 0], [9 : 1], [17^2 : 1]\}$ and $f_\infty^{-1}(Z) = \{[2 : 1], [0 : 1], [1 : 2], [1 : -1], [1 : 9], [1 : 8]\}$, therefore $\mathcal{Z} = (f_\infty^{-1}(Z)) \cap Z = \{[0 : 1], [1 : 9]\}$. Finally, $f_\infty^{-1}(\mathcal{Z}) = \{[2 : 1], [1 : 2], [1 : -1]\}$ and f satisfies the hypotheses of Proposition 3.3.

Thus, by Proposition 2.7 and 3.3, f has an attracting set $\mathcal{A} \subset \mathbb{P}^2$ supporting a positive closed current of bidegree (1,1) which admits a bounded quasi potential.

From the work of [FS] Lemma 2.12 and [R] corollary 4.15, the natural extension \hat{f}_∞ of f_∞ is semi-conjugate to f restricted to \mathcal{A} . Since $J_\infty = L_\infty$, f_∞ is topologically mixing on L_∞ , thus f is topologically mixing on \mathcal{A} and \mathcal{A} is an attractor. \square

4. NON ALGEBRAIC ATTRACTING SETS IN HIGHER CODIMENSION

In this part, we will explain why the condition of being of small topological degree is not enough to ensure non pluripolarity in higher codimension. For this we exhibit attracting sets of small topological degree in \mathbb{P}^3 (of codimension 2) which are contained in an hyperplane (see Theorem 4.1). We also construct the first explicit example of a Zariski dense attracting set of higher codimension (see Theorem 4.7).

Theorem 4.1. *For a generic choice of $(P, Q, R) \in \mathcal{F}_d$, there exists $\varepsilon_1(P, Q, R), \varepsilon_2(P, Q, R) > 0$ such that for all $0 < |\varepsilon_1| < \varepsilon_1(P, Q, R)$ and $0 < |\varepsilon_2| < \varepsilon_2(P, Q, R)$, the mapping*

$$\begin{aligned} f : \quad \mathbb{P}^3 &\longrightarrow \mathbb{P}^3 \\ [z : w : t : u] &\mapsto [P(z, w) : Q(z, w) : t^d + \varepsilon_1 R(z, w) : u^d + \varepsilon_2 Q(z, w) - \varepsilon_2^d w^d] \end{aligned}$$

is of small topological degree on an attracting set \mathcal{A} of dimension 1 which is contained in $\{[z : w : t : u]; u = \varepsilon_2 w\} \simeq \mathbb{P}^2$.

In particular, \mathcal{A} is non algebraic nevertheless it is contained in a hyperplane.

Remark. If $f : [z : w : t] \mapsto [P_0(z, w, t) : P_1(z, w, t) : P_2(z, w, t)]$ admits an attracting set \mathcal{A} of dimension 1 then $\tilde{\mathcal{A}} = \{[z : w : t : u] \mid [z : w : t] \in \mathcal{A}, u = 0\}$ is an attracting set for $\tilde{f} : [z : w : t : u] \mapsto [P_0(z, w, t) : P_1(z, w, t) : P_2(z, w, t) : u^d]$. But \tilde{f} is not of small topological degree on $\tilde{\mathcal{A}}$ even if f is of small topological degree on \mathcal{A} . This is why need a non-trivial construction in Theorem 4.1.

Proof. Denote by f_1, f_2 the maps

$$\begin{aligned} f_1 : [z : w : t] &\mapsto [P(z, w) : Q(z, w) : t^d + \varepsilon_1 R(z, w)], \\ f_2 : [z : w : u] &\mapsto [P(z, w) : Q(z, w) : u^d + \varepsilon_2 Q(z, w) - \varepsilon_2^d w^d] \end{aligned}$$

and denote by U_1, U_2 the open sets

$$U_1 = \{[z : w : t]; |t| < \rho_1 \max(|z|, |w|)\}, U_2 = \{[z : w : u]; |u - \varepsilon_2 w| < \rho_2 \max(|z|, |w|)\}.$$

First we choose (P, Q, R) in the non empty Zariski open set of \mathcal{F}_d such that the map f_1 satisfies the hypotheses of Proposition 3.3 and fix $\varepsilon_1(P, Q, R) > |\varepsilon_1| > 0$ small enough and $\rho_1 > 0$ such that $f_1(U_1) \subseteq U_1$, see Step 1 of the proof of Theorem 3.1. By modifying the end of Step 2 of the proof of Theorem 3.1, a point in $f_1^7(U_1)$ has at most d^4 preimages in $f_1(U_1)$ under f_1^6 .

It is clear that the hyperplane $\{[z : w : t : u]; u = \varepsilon_2 w\}$ is invariant under f and we see that the dynamics on it is the same as that of f_1 . Denote by $\rho_2 = c\varepsilon_2$, with $c > 0$, and

$$U = \{[z : w : t : u]; |t| < \rho_1 \max(|z|, |w|), |u - \varepsilon_2 w| < \rho_2 \max(|z|, |w|)\}.$$

We choose $\varepsilon_2(P, Q, R)$ small enough such that $((1+c)^d + 1)\varepsilon_2(P, Q, R)^d < c\varepsilon_2(P, Q, R)$ and we will fix c later. We let the reader check that $f(U) \subseteq U$ and denote by $\mathcal{A} = \bigcap f^n(U)$, hence \mathcal{A} is an attracting set of dimension 1.

We now prove that, for a generic choice of $(P, Q, R) \in \mathcal{F}_d$, f^6 is of small topological degree on $f(U)$.

Let us further assume that for all $i \in \{1, \dots, 6\}$ $f_\infty^i([1 : 0]) \neq [1 : 0]$ or equivalently $[1 : 0] \notin f_\infty^{-i}([1 : 0])$ where $f_\infty : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is defined by $f_\infty([z, w]) = [P(z, w) : Q(z, w)]$. Arguing as in Step 3 of the proof of Theorem 3.1, we see that this condition is algebraic. To show that it is generically satisfied, we need to find an example. Let us give an example. We know by Step 4 of the proof of Theorem 3.1 that for almost every $(a, b) \in \mathbb{C}^2$ with $|a|, |b| > 0$ small enough $f_1 : [z : w : t] \mapsto [w^d + az^d : bw^d + z^d : t^d + \varepsilon_1(z^d + z^{d-1}w)]$ satisfies the hypotheses of Proposition 3.3. For all $(a, b) \in \mathbb{R}^2$ such that $a, b > 0$, we have that for each $i \in \mathbb{N}^*$ $f_1^i([1 : 0]) \neq [1 : 0]$, because if $(z, w) \neq (0, 0)$ are non negative real numbers then $w^d + az^d, bw^d + z^d$ are positive. Therefore, it is a generic condition.

Fix D a disc centered at $[1 : 0]$ in \mathbb{P}^1 such that for all $i \in \{1, \dots, 6\}$ $f_\infty^{-i}(D) \cap D = \emptyset$, fix $c > 0$ such that if $[z : w] \notin D$ then for each $l \in \{1, \dots, d-1\}$

$$|e^{\frac{2il\pi}{d}}w - w| > 3c \max(|z|, |w|).$$

This implies that if $[z : w : u] \in U_2$ and $[z : w] \notin D$ then for each $l \in \{1, \dots, d-1\}$,

$$|e^{\frac{2il\pi}{d}}u - \varepsilon_2 w| > \rho_2 \max(|z|, |w|), \text{ i.e. } [z : w : e^{\frac{2il\pi}{d}}u] \notin U_2.$$

Let $[z : w : u] \in f_2^2(U_2)$ and $[z_{-1} : w_{-1}]$ such that $f_\infty([z_{-1} : w_{-1}]) = [z : w]$. If $[z_{-1} : w_{-1}] \notin D$ there exists a unique u_{-1} , such that $f([z_{-1} : w_{-1} : u_{-1}]) = [z : w : u]$ and $[z_{-1} : w_{-1} : u_{-1}] \in f_2(U_2)$. Otherwise, if $[z_{-1} : w_{-1}] \in D$ there exists at most d such values u_{-1} . Thus, if we fix $[z_{-6} : w_{-6}]$ such that $f_\infty^6([z_{-6} : w_{-6}]) = [z : w]$, since for all $i \in \{1, \dots, 6\}$ $f_\infty^{-i}(D) \cap D = \emptyset$, there exists at most one $l \in \{0, \dots, 6\}$ such that $f^l([z_{-6} : w_{-6}]) \in D$. So there exists at most d values u_{-6} , such that $f_2^6([z_{-6} : w_{-6} : u_{-6}]) = [z : w : u]$ and $[z_{-6} : w_{-6} : u_{-6}] \in f_2(U_2)$.

Let $[z : w : t : u] \in f^7(U)$, then $[z : w : t] \in f_1^7(U_1)$, so there exists at most d^4 preimages $[z_{-6} : w_{-6} : t_{-6}]$ in $f_1(U_1)$ such that $f_1^6([z_{-6} : w_{-6} : t_{-6}]) = [z : w : t]$. For each choice $[z_{-6} : w_{-6}]$, there exists at most d values u_{-6} , such that $f_2^6([z_{-6} : w_{-6} : u_{-6}]) = [z : w : u]$ and $[z_{-6} : w_{-6} : u_{-6}] \in f(U_2)$. Then for each choice $[z_{-6} : w_{-6} : t_{-6}] \in f_1(U_1)$, there exists at most d values u_{-6} , such that $f^6([z_{-6} : w_{-6} : t_{-6} : u_{-6}]) = [z : w : t : u]$ and $[z_{-6} : w_{-6} : t_{-6} : u_{-6}] \in f(U)$. In conclusion, a point in $f^7(U)$ has at most d^5 preimages under f^6 in $f(U)$. Thus f^6 is of small topological degree on $f(U)$. \square

We now turn to the Zariski dense example of codimension 2 in \mathbb{P}^3 . We first recall some results about Hénon-like maps.

Let W be an open set and D an open disc, denote by $\mathbb{B} = W \times D$ and $\partial_h \mathbb{B} = W \times \partial D$, $\partial_v \mathbb{B} = (\partial W) \times D$, $N(\mathbb{B})$ a neighbourhood of \mathbb{B} and d_t the maximum number of preimages in \mathbb{B} of a point in $f(\mathbb{B}) \cap \mathbb{B}$.

Definition 4.2. *The map $f : N(\mathbb{B}) \rightarrow \mathbb{C}^2$ is called a horizontal-like mapping if*

- (1) $f(\partial_v \mathbb{B}) \cap \mathbb{B} = \emptyset$
- (2) $f(\mathbb{B}) \cap \partial \mathbb{B} \subset \partial_v \mathbb{B}$,

If, furthermore, f is injective, f is said to be Hénon-like.

See [Du1] for some proprieties of Hénon-like and horizontal-like mappings. A current T is said to be horizontal if its support is contained in $W \times D_{1-\varepsilon}$.

Proposition 4.3. *Let f be a horizontal-like map. Then there exists an integer $d \geq 1$ such that for every normalized horizontal positive closed current T in \mathbb{B} , $\frac{1}{d}f_*T$ is normalized, horizontal, positive and closed in \mathbb{B} .*

The integer d is called the degree of f

Proof. See [DDS] or [Du1]. □

Theorem 4.4. *Let f be a Hénon-like map in \mathbb{B} . If T is an invariant current by $\frac{1}{d}f_*$, then the potentials of T are Hölder continuous.*

Proof. This is identical to Theorem 2.12 in [Du1]. □

Definition 4.5. *Let f be a horizontal-like map of degree d . If the number of preimages in \mathbb{B} of a point in $f(\mathbb{B}) \cap \mathbb{B}$ is strictly less than d , f is said to be of small topological degree.*

Lemma 4.6. *For ε small enough, there exists ρ such that the map*

$$f[z : w : t] \mapsto [z^2 + 0.1w^2 : w^2 : t^2 + \varepsilon(z^2 + zw)]$$

is a Hénon-like map on $f(\mathbb{B})$ where $\mathbb{B} = \{[z : w : t]; 0.8|w| < |z| < 1.2|w|, |t| < \rho \max(|z|, |w|)\}$. Furthermore, denoting by $\mathcal{A} = \bigcap f^n(U_\rho)$, a point in $\mathcal{A} \cap \mathbb{B}$ has a unique preimage in \mathcal{A} and this preimage is still in $\mathcal{A} \cap \mathbb{B}$.

Proof. Denote by $U_\rho = \{[z : w : t]; |t| < \rho \max(|z|, |w|)\}$, by the above, it follows that for ε small enough, there exists ρ such that $f(U_\rho) \subseteq U_\rho$. Denote by $W = \{[z : w]; 0.8|w| < |z| < 1.2|w|\}$, an easy computation shows that $f_\infty^{-1}(W) \subseteq W \subseteq f_\infty(W)$, where $f_\infty[z : w] \mapsto [z^2 + 0.1w^2 : w^2]$. Thus f is a horizontal-like map on \mathbb{B} . With notation as in Step 2 of Theorem 3.1, we get that $X = \{[0.1 : 1], [1 : 0]\}$ and $Y = [1.31 : 1]$. Reducing ε if necessary, we get that $W \subset A \cap B$ (see Step 2 of Theorem 3.1), thus f is injective on $f(\mathbb{B})$.

As f is Hénon-like on $f(\mathbb{B})$, if $[z : w : t] \in \mathcal{A} \cap \mathbb{B}$ then $[z : w : t] \in \mathcal{A} \cap f(\mathbb{B})$ and $[z : w] \in W$ thus there exists a unique $[z_{-1} : w_{-1} : t_{-1}] \in \mathcal{A}$ such that $f([z_{-1} : w_{-1} : t_{-1}]) = [z : w : t]$. Moreover, as $f_\infty^{-1}(W) \subseteq W$, $[z_{-1} : w_{-1}]$ is still in W . □

Theorem 4.7. *There exists $\varepsilon_1, \varepsilon_2 \neq 0$ such that the map*

$$f : [z : w : t_1 : t_2] \mapsto [z^2 + 0.1w^2 : w^2 : t_1^2 + \varepsilon_1(z^2 + zw) : t_2^d + \varepsilon_2(z^2 + zw)]$$

admits a Zariski dense attracting set \mathcal{A} close to the line $\{[z : w : t_1 : t_2]; t_1 = 0; t_2 = 0\}$.

Notation 4.8. Denote by pr_1, pr_2 the projection defined by $pr_1([z : w : t_1 : t_2]) = [z : w : t_1]$ and $pr_2([z : w : t_1 : t_2]) = [z : w : t_2]$.

Proof. For $i = 1, 2$ denote by f_i the map $f_i : [z : w : t_i] \mapsto [z^2 + 0.1w^2 : w^2 : t_i^2 + \varepsilon_i(z^2 + zw)]$ and by $U_i = \{[z : w : t_i]; |t_i| < \rho_i \max(|z|, |w|)\}$. We choose ε_i and ρ_i such that $f_i(U_i) \subseteq U_i$. Denote by $\mathcal{A}_i = \bigcap f_i^n(U_i)$. Denote by τ_1 the positive closed current given by Theorem 1.6 for f_1 .

By Theorem 4.4 and Lemma 4.6, there exists $\theta > 0$ such that, in any neighbourhood of a point in $V_1 = \{[z : w : t_1]; 0.8|w| < |z| < 1.2|w|, |t_1| < \rho_1 \max(|z|, |w|)\}$, $\text{supp}(\tau_1) \subset \mathcal{A}_1$ has Hausdorff dimension at least $2 + \theta$. On the other hand, J.E. Fornæss and N. Sibony [FS, Prop. 2.17] proved that if ε_2 is small enough then \mathcal{A}_2 has Hausdorff dimension less than $2 + \frac{\theta}{2}$.

Denote by $U = \{[z : w : t_1 : t_2]; |t_1| < \rho_1 \max(|z|, |w|), |t_2| < \rho_2 \max(|z|, |w|)\}$, thus $f(U) \subseteq U$. The attracting sets $\mathcal{A} = \bigcup f^n(U)$ is of dimension 1, i.e. of codimension 2.

we have that $pr_1(\mathcal{A}) = \mathcal{A}_1$ and $pr_2(\mathcal{A}) = \mathcal{A}_2$, thus \mathcal{A} is non algebraic. We will use the discrepancy between the Hausdorff dimension of \mathcal{A}_1 and \mathcal{A}_2 to show that \mathcal{A} must be Zariski dense.

Denote by $(\hat{\mathbb{P}}^1, \hat{f}_\infty)$ the natural extension of (\mathbb{P}^1, f_∞) where $f_\infty : [z : w] \mapsto [P(z, w) : Q(z, w)]$, see section 3 of [R] for an introduction. Let $p = [z : w] \in \mathbb{P}^1$ and denote by l_p the line (resp. hyperplane) passing through $[0 : 0 : 1]$ and $[z : w : 0]$ (resp. $[0 : 0 : 1 : 0]$, $[0 : 0 : 0 : 1]$ and $[z : w : 0 : 0]$). We already seen that $l_p \cap U_i$ is a disk and its image is relatively compact in $l_{f_\infty(p)} \cap U_i$, see Step 1 of Theorem 3.1. Thus if $(p_{-n}) \in \hat{\mathbb{P}}^1$ is a sequence of preimages of p then $\bigcap_{n \geq 0} f_i^n(l_{f_\infty(p_{-n})} \cap U_i)$ is a unique point in $l_p \cap \mathcal{A}_i$. Therefore, we have defined a continuous and onto map $\pi_i : \hat{\mathbb{P}}^1 \rightarrow \mathcal{A}_i$, see [R, Prop. 4.13] or [FS, Lemma 2.8]. We have that if $p_{-n} = [z_{-n} : w_{-n}]$ then $\pi_i((p_{-n})) = \lim_{n \rightarrow \infty} f_i^n([z_{-n} : w_{-n} : 0])$. We can also define a continuous and onto maps $\pi : \hat{\mathbb{P}}^1 \rightarrow \mathcal{A}$ by replacing disks by balls in the previous argument. We have the three commutative diagrams :

$$\begin{array}{ccc} \hat{\mathbb{P}}^1 & \xrightarrow{\hat{f}_\infty} & \hat{\mathbb{P}}^1 \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ \mathcal{A}_1 & \xrightarrow{f_1} & \mathcal{A}_1 \end{array} \quad \begin{array}{ccc} \hat{\mathbb{P}}^1 & \xrightarrow{\hat{f}_\infty} & \hat{\mathbb{P}}^1 \\ \downarrow \pi_2 & & \downarrow \pi_2 \\ \mathcal{A}_2 & \xrightarrow{f_2} & \mathcal{A}_2 \end{array} \quad \begin{array}{ccc} \hat{\mathbb{P}}^1 & \xrightarrow{\hat{f}_\infty} & \hat{\mathbb{P}}^1 \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{f} & \mathcal{A} \end{array}$$

As f_i is Hénon-like on $f_i(V_i)$, π_i restricted to $\pi_i^{-1}(V_i \cap \mathcal{A}_i)$ is a bijection, thus there is a bijection $\varphi : \mathcal{A}_2 \cap V_2 \rightarrow \mathcal{A}_1 \cap V_1$. Let $[\alpha : \beta : \gamma_2] \in \mathcal{A}_2$ and $(p_{-n}) \in \hat{\mathbb{P}}^1$ the unique sequence such that $[\alpha : \beta : \gamma_2] = \pi_2((p_{-n})) = \lim_{n \rightarrow \infty} f_2^n([z_{-n} : w_{-n} : 0])$. Thus there exists a unique point in the intersection $\mathcal{A} \cap pr_2^{-1}([\alpha : \beta : \gamma_2])$, because if $(q_{-n}) \in \hat{\mathbb{P}}^1$ such that $pr_2(\pi((p_{-n}))) = [\alpha : \beta : \gamma_2]$ then $[\alpha : \beta : \gamma_2] = \pi_2((q_{-n}))$. As $\varphi([\alpha : \beta : \gamma_2]) = \pi_1((p_{-n})) = pr_1(\pi((p_{-n})))$, we have that :

$$(10) \quad \forall [\alpha : \beta : \gamma_2] \in \mathcal{A}_2 \cap V_2, \exists! \gamma_1 : [\alpha : \beta : \gamma_1 : \gamma_2] \in \mathcal{A} \text{ and } \varphi([\alpha : \beta : \gamma_2]) = [\alpha : \beta : \gamma_1].$$

In other words, $\varphi([\alpha : \beta : \gamma_2])$ is the image by pr_1 of the unique point in the intersection of \mathcal{A} and $pr_2^{-1}([\alpha : \beta : \gamma_2])$.

Suppose that the Zariski closure of \mathcal{A} is an hypersurface M . As $pr_2(\mathcal{A}) = \mathcal{A}_2$ is non algebraic, M cannot be the union of hyperplane passing through $[0 : 0 : 0 : 1]$

Let $[\alpha : \beta] \in \{[z : w] ; 0.8|w| < |z| < 1.2|w|\}$ and $[\alpha : \beta : \gamma_1 : \gamma_2] \in \mathcal{A}$, such that there exists an open neighbourhood N of $[\alpha : \beta : \gamma_2]$ include in V_2 such that $M \cap pr_2^{-1}(N)$ is a biholomorphic copy of N . As $N \cap \mathcal{A}_2$ is of Hausdorff dimension less than $2 + \frac{\theta}{2}$, $\mathcal{A} \cap pr_2^{-1}(N) \subset M \cap pr_2^{-1}(N)$ is of Hausdorff dimension less than $2 + \frac{\theta}{2}$. As pr_1 is Lipschitz, $pr_1(\mathcal{A} \cap pr_2^{-1}(N))$ is of Hausdorff dimension less than $2 + \frac{\theta}{2}$. Moreover, $pr_1(pr_2^{-1}(N))$ is open and $pr_1(\mathcal{A} \cap pr_2^{-1}(N)) = pr_1(pr_2^{-1}(N)) \cap \mathcal{A}_1$, thus $pr_1(\mathcal{A} \cap pr_2^{-1}(N))$ must be of Hausdorff dimension at least $2 + \theta$. We have a contradiction.

Therefore, \mathcal{A} is Zariski dense in \mathbb{P}^3 . □

Remark. For each $n \in \mathbb{N}$ the point $[1 : 0]$ has multiplicity 2^n for f_∞^n thus f is not of small topological degree on \mathcal{A} .

5. FURTHER RESULTS AND OPEN PROBLEMS

5.1. A simpler version of Proposition 3.3. The original generic condition that we had in mind for small topological degree map on an attracting set (Proposition 3.3) was somewhat simpler. Unfortunately, we were only able to find corresponding examples in degree 2.

Indeed, Condition 2 of Proposition 3.3 can be replaced by the algebraic condition

$$2'. f^{-1}(Z) \cap Z = \emptyset.$$

It is clear from the proof of Proposition 3.3 that this implies that for all $|\varepsilon| > 0$ small enough there exists ρ such that f^2 is of small topological degree on $f(U_\rho)$.

Example 5.1. The map $f = [w^2 + az^2 : bw^2 + z^2 : t^2 + \varepsilon(z^2 + 2zw + w^2)]$ satisfies the hypothesis 1 of the Proposition 3.3 and 2' for almost every $(a, b) \in \mathbb{C}^2$. In fact, we have that $X_{-1} = \{[0 : 1], [1 : 0]\}$ thus $X = \{[1 : b], [a : 1]\}$ and $Y_{-2} = \{[i : 1], [i : -1], [1 : -1]\}$ thus

$$Y = f_\infty^2(Y_{-2}) = \{[(1+b)^2 + a(1+a)^2 : b(b+1)^2 + (1+a)^2], [(b-1)^2 + a(1-a)^2 : b(b-1)^2 + (1-a)^2]\}.$$

Therefore, for almost every $(a, b) \in \mathbb{C}^2$, X and Y are disjoint, i.e. f satisfies the hypothesis 1 of 3.3 and as $Z = X \cup Y$

$$Z = \{[1 : b], [a : 1], [(1+b)^2 + a(1+a)^2 : b(b+1)^2 + (1+a)^2], [(b-1)^2 + a(1-a)^2 : b(b-1)^2 + (1-a)^2]\}$$

and

$$f_\infty^{-1}(X \cup Y) = \{[0 : 1], [1 : 0], [1 + a : b + 1], [1 - a : b - 1], [1 + a : -b - 1], [1 - a : -b + 1]\},$$

for almost every $(a, b) \in \mathbb{C}^2$, we have that $f^{-1}(Z) \cap Z = (f_\infty^{-1}(X \cup Y)) \cap (X \cup Y) = \emptyset$, thus f satisfies the condition 2'.

5.2. Generalisation of Theorem 3.1 to higher dimension. To extend Theorem 3.1 to higher dimension, we consider the family of endomorphisms of \mathbb{P}^k of the form

$$f : [z_0 : \dots : z_k] \mapsto [P_0(z_0 : \dots : z_{k-1}) : \dots : P_{k-1}(z_0 : \dots : z_{k-1}) : z_k^d + \varepsilon P_k(z_0 : \dots : z_{k-1})],$$

where P_0, \dots, P_k are homogeneous polynomials of degree $d \geq 2$ such that $(0, \dots, 0)$ is the single common zero of P_0, \dots, P_{k-1} . Let $U_\rho = \{[z_0 : \dots : z_k]; |z_k| < \rho \max(|z_0|, \dots, |z_k|)\}$, then if ε, ρ are small enough we have that $f(U_\rho) \subseteq U_\rho$. Denote by

$$f_\infty : [z_0 : \dots : z_{k-1}] \mapsto [P_0(z_0 : \dots : z_{k-1}) : \dots : P_{k-1}(z_0 : \dots : z_{k-1})],$$

$$f_\infty^+ : [z_0 : \dots : z_{k-1}] \mapsto [P_0(z_0 : \dots : z_{k-1}) : \dots : P_{k-1}(z_0 : \dots : z_{k-1}) : \varepsilon P_k(z_0 : \dots : z_{k-1})].$$

We define Y as before Proposition 3.3 and

$$X = \left\{ p \in \mathbb{P}^{k-1} \mid \text{card}\{f_\infty^+(p_{-1}) \mid p_{-1} \in f_\infty^{-1}(p)\} \leq d^{k-2} \right\}.$$

These are algebraic set of \mathbb{P}^{k-1} . Moreover, for a generic choice of P_0, \dots, P_k , the set Y is of codimension 1 and X is of dimension 0. We may replace the conditions of Proposition 3.3 by

- (1) $X \cap Y = \emptyset$,
- (2) and $(f_\infty^{-(k-1)}(X \cup Y)) \cap \dots \cap (f_\infty^{-1}(X \cup Y)) \cap (X \cup Y) = \emptyset$.

Following the idea of the proof of Proposition 3.3, we show that this implies that f^k is of small topological degree on $f(U_\rho)$. Therefore, we get a Zariski open set Ω of mappings asymptotically of small topological degree on attracting sets in \mathbb{P}^k . Nevertheless, it is unclear how to find explicit examples of parameters in Ω to ensure that it is not empty.

5.3. Non-pluripolar attracting sets with unbounded potentials. The last remark around Theorem 3.1 is that it is not necessary to be of small topological degree “everywhere” on an attracting set in order for it to be non pluripolar. Furthermore, there exists non-pluripolar attracting sets that cannot support a current of bidegree (1,1) of mass 1 with a bounded quasi-potential.

Let f be an endomorphism of the form

$$(11) \quad f([z : w : t]) = [w^d P\left(\frac{z}{w}\right) : w^d : t^d + \varepsilon R(z, w)]$$

where P is a polynomial of degree d and R is a homogeneous polynomial of degree d . In particular, it is of the form (3). On the other hand, f is not asymptotically of small topological degree and the attracting current τ has a unbounded quasi-potential because $f^{-1}(l_{[1:0]}) = \{l_{[1:0]}\}$, hence $\mathcal{A} \cap l_{[1:0]}$ is reduced to a point.

Nevertheless, there exists a bidisk of the form $\{[z : w : t]; |z| \leq R|w|, |t| < \rho \max(|z|, |w|)\}$ where f is horizontal-like. Moreover, we can adapt the proof of Theorem 3.1 to arrange that this horizontal-like map is of small topological degree. For this, we replace the generic conditions of Proposition 3.3 by the following :

- (1) $X \cap Y = \emptyset$
- (2) and $f_\infty^{-2}(Z) \cap f_\infty^{-1}(Z) \cap Z = \{[1 : 0]\}$, with $Z = X \cup Y$.

These properties are clearly satisfied outside some algebraic set and it can be shown that for $c \neq 0$ small enough, $f = [w^d + cz^d : w^d : t^d + \varepsilon(z^d + z^{d-1}w)]$ satisfies these conditions.

Finally, we adapt the proof of Theorem 2.7 to horizontal-like maps using the canonical potential. We recall that it is defined as follows (see [DDS] for details). If S is a horizontal current, i.e. with support in $D \times D_{1-\varepsilon}$, its canonical potential is defined by

$$u_S(z, w) = \int_{\{z\} \times D} \log |w - s| dm^z(w)$$

with $m^z = S \wedge [\{z\} \times D]$. Thus we finally get :

Theorem 5.2. *If f is of the form (11), then for a generic choice of P, R there exists $\varepsilon(P, R) > 0$ such that if $\varepsilon(P, R) > |\varepsilon| > 0$ then f admits a non pluripolar attracting set.*

5.4. Other open questions. There are still two questions left related to Theorem 2.7.

- In the light of Theorem 4.1, it would be interesting to find the right hypothesis ensuring non pluripolarity of attracting set in higher codimension.
- Does an attracting set of codimension 1 support a current with continuous quasi-potential ? Can we prove the convergence of the quasi-potentials ?

REFERENCES

- [BDM] A. Bonifant, M. Dabija, J. Milnor ; *Elliptic curves as attractors in \mathbb{P}^2 . I. Dynamics*. Experiment. Math. 16 (2007), no. 4, 385–420.
- [BD] J.-Y. Briend; J. Duval *Deux caractérisations de la mesure d’équilibre d’un endomorphisme de $\mathbb{P}^k(\mathbb{C})$* . Publ. Math. Inst. Hautes Études Sci. No. 93 (2001), 145–159.
- [De] J.P. Demailly *Complex analytic and differential geometry*. Free access at (<http://www-fourier.ujf-grenoble.fr/~demailly/books.html>)

- [DDG1] J. Diller, R. Dujardin, V. Guedj ; *Dynamics of meromorphic maps with small topological degree I: from cohomology to currents*. Indiana Univ. Math. J. 59 (2010), no. 2, 521-561.
- [DDG2] J. Diller, R. Dujardin, V. Guedj ; *Dynamics of meromorphic mappings with small topological degree II: Energy and invariant measure*. Comment. Math. Helv. 86 (2011), no. 2, 277-316.
- [DDG3] J. Diller, R. Dujardin, V. Guedj ; *Dynamics of meromorphic maps with small topological degree III: geometric currents and ergodic theory*. Ann. Sci. Éc. Norm. Supér. (4) 43 (2010), no. 2, 235-278.
- [Di] T.C. Dinh ; *Attracting current and equilibrium measure for attractors on \mathbb{P}^k* . J. Geom. Anal. 17 (2007), no. 2, 227-244.
- [DDS] T.C. Dinh ; R. Dujardin ; N. Sibony ; *On the dynamics near infinity of some polynomial mappings in \mathbb{C}^2* . Math. Ann. 333 (2005), no. 4, 703-739.
- [DiS] T.C. Dinh ; N. Sibony *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*. Holomorphic dynamical systems, 165-294, Lecture Notes in Math., 1998, Springer, Berlin, 2010.
- [Du1] R. Dujardin ; *Hénon-like mappings*. Amer. J. Math. 126 (2004), no. 2, 439-472.
- [Du2] R. Dujardin ; *Fatou directions along the Julia set for endomorphisms of \mathbb{CP}^k* . J. Math. Pures Appl. (9) 98 (2012), no. 6, 591-615.
- [Fa] N. Fakhruddin ; *The algebraic dynamics of generic endomorphisms of \mathbb{P}^n* . Preprint, arxiv.org/abs/1211.7198
- [FS II] J.E. Fornæss ; N. Sibony ; *Complex dynamics in higher dimension. II. Modern methods in complex analysis* (Princeton, NJ, 1992), 135-182, Ann. of Math. Stud., 137, Princeton Univ. Press, Princeton, NJ, 1995.
- [FS] J.E. Fornæss ; N. Sibony ; *Dynamics of \mathbb{P}^2 (examples). Laminations and foliations in dynamics, geometry and topology* (Stony Brook, NY, 1998), 47-85, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001.
- [FW] J.E. Fornæss ; B. Weickert ; *Attractors in \mathbb{P}^2 . Several complex variables* (Berkeley, CA, 1995-1996), 297-307, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
- [JW] M. Jonsson ; B. Weickert ; *A nonalgebraic attractor in \mathbb{P}^2* . Proc. Amer. Math. Soc. 128 (2000), no. 10, 2999-3002.
- [H] Hörmander ; *Notion of convexity*. Progress in Mathematics, 127. Birkhäuser Boston, Inc., Boston, MA, 1994. viii+414 pp.
- [R] F. Rong ; *Non-algebraic attractors on \mathbb{P}^k* . Discrete Contin. Dyn. Syst. 32 (2012), no. 3, 977-989.
- [T] J. Taflin ; *Speed of convergence towards attracting sets for endomorphisms of \mathbb{P}^k* . to appear in Indiana Univ. Math. Journal.