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The Hopf algebra of Fliess operators and its dual pre-Lie algebra

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ABSTRACT. We study the Hopf algebra H of Fliess operators coming from Control Theory in the one-dimensional case. We prove that it admits a graded, finite-dimensional, connected gradation. Dually, the vector space $\mathbb{R}\langle x_0, x_1 \rangle$ is both a pre-Lie algebra for the pre-Lie product dual of the coproduct of H , and an associative, commutative algebra for the shuffle product. These two structures admit a compatibility which makes $\mathbb{R}\langle x_0, x_1 \rangle$ a Com-pre-Lie algebra. We give a presentation of this object as a Com-pre-Lie algebra and as a pre-Lie algebra.

KEYWORDS. Fliess operators; pre-Lie algebras; Hopf algebras.

AMS CLASSIFICATION. 16W30, 17B60, 93B25, 05C05.

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Introduction

Right pre-Lie algebras, or shortly pre-Lie algebras [4, 1], are vector spaces with a bilinear product \bullet satisfying the following axiom:

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

Consequently, the antisymmetrization of \bullet is a Lie bracket. These objects are also called right-symmetric algebras or Vinberg algebra [11, 16]. If A is a pre-Lie algebra, then the symmetric algebra $S(A)$ inherits a product \star making it a Hopf algebra, isomorphic to the enveloping algebra of the Lie algebra A [12, 13]. Whenever it is possible, we can consider the dual Hopf algebra $S(A)^*$ and its group of characters G , which is the exponentiation, in a certain sense, of the Lie algebra A .

We here consider the inverse construction, departing from a group used in Control Theory, namely the group of Fliess operators [3, 5, 6]; this group is used to study the feedback product. We limit ourselves here to the one-dimensional case. This group is the set $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ of noncommutative formal series in two indeterminates, with a certain product generalizing the composition of formal series (definition 1). The Hopf algebra H of coordinates of this group is described in [5], where it is also proved that it is graded by the length of words; note that this gradation is not connected and not finite-dimensional. We first give a way to describe the composition in the group $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ and the coproduct of H by induction on the length of words (lemma 2 and proposition 3). We prove that H admits a second gradation, which is connected; the dimensions of this gradation are given by the Fibonacci sequence (proposition 8). As the product of $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ is left-linear, H is a commutative, right-sided combinatorial Hopf algebra [9], so, dually, $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ inherits a pre-Lie product \bullet , which is inductively defined in proposition 11. We prove that the words x_1^n , $n \geq 0$, form a minimal subset of generators of this pre-Lie algebra (theorem 12).

The pre-Lie algebra $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ has also an associative, commutative product, namely the shuffle product \sqcup [14]. We prove that the following axiom is satisfied (proposition 14):

$$(x \sqcup y) \bullet z = (x \bullet z) \sqcup y + x \sqcup (y \bullet z).$$

So $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ is a Com-pre-Lie algebra [10](definition 15). We give a presentation of this Com-pre-Lie algebra in theorem 27. We use for this a description of free Com-pre-Lie algebras in terms of partitioned trees (definition 17), which generalizes the construction of pre-Lie algebras in terms of rooted trees in [1]. We then deduce a presentation of $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ as a pre-Lie algebra in theorem 30. This presentation induces a new basis of $\mathbb{R}\langle\langle x_0, x_1 \rangle\rangle$ in terms of words with letters in \mathbb{N}^* , see corollary 31. The pre-Lie product of two elements of this basis uses a dendriform structure [2, 8] on the algebra of words with letters in \mathbb{N}^* (theorem 34). The study of this dendriform structure is postponed to the appendix, as well as the enumeration of partitioned trees; we also prove that free Com-pre-Lie algebras are free as pre-Lie algebras, using the rigidity theorem of [7].

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Notation. We denote by \mathbb{K} a commutative field of characteristic zero. All the objects (algebra, coalgebras, pre-Lie algebras...) in this text will be taken over \mathbb{K} .

1 Construction of the Hopf algebra

1.1 Definition of the composition

Let us consider an alphabet of two letters, denoted by x_0 and x_1 . We denote by $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ the completion of the free algebra generated by this alphabet. Note that $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ is an algebra

for the concatenation product and for the shuffle product, which we denote by \sqcup .

Examples. If $a, b, c, d \in \{x_0, x_1\}$:

$$\begin{aligned} abc \sqcup d &= abcd + abdc + adbc + dabc, \\ ab \sqcup cd &= abcd + acbd + cabd + acdb + cadb + cdab, \\ a \sqcup bcd &= abcd + bacd + bcad + bcda. \end{aligned}$$

The unit for both these products is the empty word, which we denote by \emptyset . The algebra $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ is given its usual ultrametric topology.

Definition 1 [5].

1. For any $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$, we define a continuous algebra map φ_d from $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ to $\text{End}(\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle)$ in the following way: for all $X \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$,

$$\varphi_d(x_0)(X) = x_0X, \quad \varphi_d(x_1)(X) = x_1X + x_0(d \sqcup X).$$

2. We define a composition \circ on $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ in the following way: for all $c, d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$, $c \circ d = \varphi_d(c)(\emptyset) + d$.

It is proved in [5] that this composition is associative.

Notation. For all $c, d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$, we put $c \tilde{\circ} d = c \circ d - d = \varphi_d(c)(\emptyset)$.

Remark. If $c_1, c_2, d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$, $\lambda \in \mathbb{K}$:

$$(c_1 + \lambda c_2) \tilde{\circ} d = \varphi_d(c_1 + \lambda c_2)(\emptyset) = (\varphi_d(c_1) + \lambda \varphi_d(c_2))(\emptyset) = \varphi_d(c_1)(\emptyset) + \lambda \varphi_d(c_2)(\emptyset) = c_1 \tilde{\circ} d + \lambda c_2 \tilde{\circ} d.$$

So the composition $\tilde{\circ}$ is linear on the left. As φ_d is continuous, the map $c \rightarrow c \tilde{\circ} d$ is continuous for any $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$. Hence, it is enough to know how to compute $c \tilde{\circ} d$ for any word c , which is made possible by the next lemma, using an induction on the length:

Lemma 2 For any word c , for any $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$:

1. $\emptyset \tilde{\circ} d = \emptyset$.
2. $(x_0c) \tilde{\circ} d = x_0(c \tilde{\circ} d)$.
3. $(x_1c) \tilde{\circ} d = x_1(c \tilde{\circ} d) + x_0(d \sqcup (c \tilde{\circ} d))$.

Proof. 1. $\emptyset \tilde{\circ} d = \varphi_d(\emptyset)(\emptyset) = Id(\emptyset) = \emptyset$.

$$2. (x_0c) \tilde{\circ} d = \varphi_d(x_0c)(\emptyset) = \varphi_d(x_0) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_0)(c \tilde{\circ} d) = x_0(c \tilde{\circ} d).$$

$$3. (x_1c) \tilde{\circ} d = \varphi_d(x_1c)(\emptyset) = \varphi_d(x_1) \circ \varphi_d(c)(\emptyset) = \varphi_d(x_1)(c \tilde{\circ} d) = x_1(c \tilde{\circ} d) + x_0(d \sqcup (c \tilde{\circ} d)). \quad \square$$

1.2 Dual Hopf algebra

We here give an inductive description of the Hopf algebra of the coordinates of the group $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ of [5].

For any word c , let us consider the map $X_c \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle^*$, such that for any $d \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$, $X_c(d)$ is the coefficient of c in d . We denote by V the subspace of A^* generated by these maps. Let $H = S(V)$, or equivalently the free commutative algebra generated by the X_c 's. The elements of

H are seen as polynomial functions on $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$; the elements of $H \otimes H$ are seen as polynomial functions on $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle \times \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$. Then H is given a multiplicative coproduct defined in the following way: for any word c , for any $f, g \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$,

$$\Delta(X_c)(f, g) = X_c(f \circ g).$$

As \circ is associative, Δ is coassociative, so H is a bialgebra.

Notations.

1. The space of words is a commutative algebra for the shuffle product \mathbb{W} . Dually, the space V inherits a coassociative, cocommutative coproduct, denoted by $\Delta_{\mathbb{W}}$. For example, if $a, b, c \in \{x_0, x_1\}$:

$$\begin{aligned} \Delta_{\mathbb{W}}(X_{\emptyset}) &= X_{\emptyset} \otimes X_{\emptyset}, \\ \Delta_{\mathbb{W}}(X_a) &= X_a \otimes X_{\emptyset} + X_{\emptyset} \otimes X_a, \\ \Delta_{\mathbb{W}}(X_{ab}) &= X_{ab} \otimes X_{\emptyset} + X_a \otimes X_b + X_b \otimes X_a + X_{\emptyset} \otimes X_{ab}, \\ \Delta_{\mathbb{W}}(X_{abc}) &= X_{abc} \otimes X_{\emptyset} + X_a \otimes X_{bc} + X_b \otimes X_{ac} + X_c \otimes X_{ab} \\ &\quad + X_{ab} \otimes X_c + X_{ac} \otimes X_b + X_{bc} \otimes X_a + X_{\emptyset} \otimes X_{abc}. \end{aligned}$$

2. We define two linear endomorphisms θ_0, θ_1 of V by $\theta_i(X_c) = X_{x_i c}$ for any word c .

The following proposition allows to compute $\Delta(X_c)$ for any word c by induction on the length of c .

Proposition 3 For all $x \in V$, we put $\tilde{\Delta}(x) = \Delta(x) - 1 \otimes x$.

1. $\tilde{\Delta}(X_{\emptyset}) = X_{\emptyset} \otimes 1$.
2. $\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes Id) \circ \tilde{\Delta} + (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}$.
3. $\tilde{\Delta} \circ \theta_1 = (\theta_1 \otimes Id) \circ \tilde{\Delta}$.

Proof. For any word c , for any $f, g \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$:

$$\tilde{\Delta}(X_c)(f, g) = \Delta(X_c)(f, g) - (1 \otimes X_c)(f, g) = X_c(f \circ g) - X_c(g) = X_c(f \otimes g - g) = X_c(f \tilde{\circ} g).$$

As $\tilde{\circ}$ is linear on the left, $\tilde{\Delta}(X_c) \in V \otimes H$, so formulas in 2. and 3. make sense.

Let $f \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$. It can be uniquely written as $f = x_0 f_0 + x_1 f_1 + \lambda \emptyset$, with $f_0, f_1 \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$, $\lambda \in K$. For all $g \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$:

$$\begin{aligned} f \tilde{\circ} g &= (x_0 f_0) \tilde{\circ} g + (x_1 f_1) \tilde{\circ} g + \lambda \emptyset \tilde{\circ} g \\ &= x_0(f_0 \tilde{\circ} g + g \mathbb{W}(f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda \emptyset. \end{aligned}$$

1. We obtain:

$$\tilde{\Delta}(X_{\emptyset})(f, g) = X_{\emptyset}(x_0(f_0 \tilde{\circ} g + g \mathbb{W}(f_1 \tilde{\circ} g)) + x_1(f_1 \tilde{\circ} g) + \lambda \emptyset) = 0 + 0 + \lambda = (X_{\emptyset} \otimes 1)(f, g).$$

So $\Delta(X_{\emptyset}) = X_{\emptyset} \otimes 1$.

2. Let c be a word.

$$\begin{aligned}
\tilde{\Delta} \circ \theta_0(X_c)(f, g) &= \tilde{\Delta}(X_{x_0c})(f, g) \\
&= X_{x_0c}(x_0(f_0\tilde{\circ}g + g\mathbb{W}(f_1\tilde{\circ}g)) + x_1(f_1\tilde{\circ}g) + \lambda\emptyset) \\
&= X_c(f_0\tilde{\circ}g + g\mathbb{W}(f_1\tilde{\circ}g)) + 0 + 0 \\
&= X_c(f_0\tilde{\circ}g + (f_1\tilde{\circ}g)\mathbb{W}g) + 0 + 0 \\
&= \tilde{\Delta}(X_c)(f_0, g) + (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)(f_1, g, g) \\
&= \tilde{\Delta}(X_c)(f_0, g) + (Id \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)(f_1, g) \\
&= (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_c)(f, g) + (\theta_1 \otimes Id) \circ (Id \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c)(f, g),
\end{aligned}$$

$$\text{so } \tilde{\Delta} \circ \theta_0(X_c) = (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_c) + (\theta_1 \otimes Id) \circ (Id \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbb{W}}(X_c).$$

3. Let c be a word.

$$\begin{aligned}
\tilde{\Delta} \circ \theta_1(X_c)(f, g) &= \tilde{\Delta}(X_{x_0c})(f, g) \\
&= X_{x_1c}(x_0(f_0\tilde{\circ}g + g\mathbb{W}(f_1\tilde{\circ}g)) + x_1(f_1\tilde{\circ}g) + \lambda\emptyset) \\
&= 0 + X_c(f_1\tilde{\circ}g) + 0 \\
&= \tilde{\Delta}(X_c)(f_1, g) \\
&= (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_c)(f, g),
\end{aligned}$$

$$\text{so } \tilde{\Delta} \circ \theta_1(X_c) = (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_c). \quad \square$$

Examples.

$$\begin{aligned}
\Delta(X_{x_0}) &= X_{x_0} \otimes 1 + 1 \otimes X_{x_0} + X_{x_1} \otimes X_{\emptyset}, \\
\Delta(X_{x_0^2}) &= X_{x_0^2} \otimes 1 + 1 \otimes X_{x_0^2} + X_{x_0x_1} \otimes X_{\emptyset} + X_{x_1x_0} \otimes X_{\emptyset} + X_{x_1x_1} \otimes X_{\emptyset}^2 + X_{x_1} \otimes X_{x_0}, \\
\Delta(X_{x_0x_1}) &= X_{x_0x_1} \otimes 1 + 1 \otimes X_{x_0x_1} + X_{x_1x_1} \otimes X_{\emptyset} + X_{x_1} \otimes X_{x_1}, \\
\Delta(X_{x_1x_0}) &= X_{x_1x_0} \otimes 1 + 1 \otimes X_{x_1x_0} + X_{x_1x_1} \otimes X_{\emptyset}.
\end{aligned}$$

Corollary 4 For all $n \geq 1$, $\tilde{\Delta}(X_{x_1^n}) = X_{x_1^n} \otimes 1$ and $\Delta(X_{x_1^n}) = X_{x_1^n} \otimes 1 + 1 \otimes X_{x_1^n}$.

Proof. Easy induction on n . □

1.3 gradation

It is proved in [5] that the Hopf algebra H is graded by the length of words, but this gradation is not connected, that is to say that the homogeneous component of degree 0 is not (0) , as it contains X_{\emptyset} . We here define another gradation, which is connected.

Definition 5 Let $c = c_1 \dots c_k$ be a word. We put:

$$\text{deg}(c) = \text{lg}(c) + 1 + \#\{i \in \{1, \dots, k\} \mid c_i = x_0\}.$$

For all $k \geq 1$, we put:

$$V_k = \text{Vect}(X_c \mid \text{deg}(x) = k).$$

This define a connected gradation of V , that is to say:

$$V = \bigoplus_{k \geq 1} V_k.$$

This gradation induces a connected gradation of the algebra H :

$$H = \bigoplus_{k \geq 0} H_k, \text{ and } H_0 = \mathbb{K};$$

Proposition 6 *If c is a word of degree n , then:*

$$\tilde{\Delta}(X_c) \in \bigoplus_{i+j=n} V_i \otimes H_j.$$

So the gradation $(V_k)_{k \geq 1}$ is a gradation of the Hopf algebra H .

Proof. Let us start by the following observations:

1. Let c be a word of degree k . Then x_0c is a word of degree $k+2$. Hence, θ_0 is homogeneous of degree 2 on V .
2. Let c be a word of degree k . Then x_1c is a word of degree $k+1$. Hence, θ_1 is homogeneous of degree 1 on V .
3. Let c and d be two words of respective degrees k and l . Then any word obtained by shuffling c and d is of degree $k+l-1$: its length is the sum of the length of c and d , and the number of x_0 in it is the sum of the numbers of x_0 in c and d . Hence, the coproduct $\Delta_{\mathfrak{u}}$ is homogeneous of degree 1 from V to $V \otimes V$.

Let us prove the result by induction on the length k of c . If $k=0$, then $c = \emptyset$ so $n=1$, and $\tilde{\Delta}(X_c) = X_c \otimes 1 \in V_1 \otimes H_0$. Let us assume the result for all words of length $< k-1$. Two cases can occur.

1. If $c = x_0d$, then $\deg(d) = n-2$. we put $\Delta_{\mathfrak{u}}(X_d) = \sum x'_i \otimes x''_i$. By the preceding third observation, we can assume that for all i , x'_i, x''_i are homogeneous elements of V , with $\deg(x'_i) + \deg(x''_i) = n-2+1 = n-1$. Then:

$$\tilde{\Delta}(X_c) = (\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) + \sum_i (\theta_1 \otimes m) \circ (\tilde{\Delta}(x'_i) \otimes x''_i).$$

By the induction hypothesis, $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$. By the second observation, $(\theta_0 \otimes Id) \circ \tilde{\Delta}(X_d) \in (V \otimes H)_n$. By the induction hypothesis applied to x'_i , for all i , $(\tilde{\Delta}(x'_i) \otimes x''_i) \in (V \otimes H \otimes V)_{n-1}$, so by the first observation, $(\theta_1 \otimes m) \circ (\tilde{\Delta}(x'_i) \otimes x''_i) \in (V \otimes H)_{n-1+1} \subseteq (V \otimes H)_n$. So $\Delta(X_c) \in (V \otimes H)_n$.

2. $c = x_1d$, then $\deg(d) = n-1$. Moreover, $\tilde{\Delta}(X_c) = (\theta_1 \otimes Id) \circ \tilde{\Delta}(X_d)$. By the induction hypothesis, $\tilde{\Delta}(X_d) \in (V \otimes H)_{n-1}$. By the second observation, $\tilde{\Delta}(X_c) \in (V \otimes H)_n$.

So the result holds for any word c . □

Corollary 7 *For all $n \geq 0$:*

$$\Delta(H_n) \subseteq \bigoplus_{i+j=n} H_i \otimes H_j.$$

Proof. Comes from the multiplicativity of Δ . □

Let us now study the formal series of V and H .

Proposition 8 *1. For all k , let us put $p_k = \dim(V_k)$ and $F_V = \sum_{k=1}^{\infty} p_k X^k$. Then:*

$$F_V = \frac{X}{1 - X - X^2},$$

and for all $k \geq 1$:

$$p_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right).$$

This is the Fibonacci sequence (A000045 in [15]).

2. We put $F_H = \sum_{k=0}^{\infty} \dim(H_k)X^k$. Then:

$$F_H = \prod_{k=1}^{\infty} \frac{1}{(1 - X^k)^{p_k}}.$$

Proof. Let us consider the formal series:

$$F(X_0, X_1) = \sum_{i,j \geq 0} \#\{\text{words in } x_0, x_1 \text{ with } i \text{ } x_0 \text{ and } j \text{ } x_1\} X_0^i X_1^j.$$

Then $F(X_0, X_1) = \frac{1}{1 - X_0 - X_1}$. Moreover, by definition of the degree of a word:

$$F_V = XF(X^2, X) = \frac{X}{1 - X - X^2}.$$

As H is the symmetric algebra generated by V , its formal series is given by the second point. \square

Examples. We obtain:

k	0	1	2	3	4	5	6	7	8	9	10
$\dim(V_k)$	0	1	1	2	3	5	8	13	21	34	55
$\dim(H_k)$	1	1	2	4	8	15	30	56	108	203	384

The third row is sequence A166861 of [15].

Remark. Consequently, the space V inherits a bigradation:

$$V_{k,n} = \text{Vect}(X_c \mid \text{deg}(c) = k \text{ and } \text{lg}(c) = n).$$

If c is a word of length n and of degree k , denoting by a the number of its letters equal to x_0 and by b the number of its letters equal to x_1 , then:

$$\begin{cases} a + b = n, \\ 2a + b + 1 = k, \end{cases}$$

so $a = k - n - 1$. Hence:

$$\dim(V_{k,n}) = \binom{n}{k - n - 1},$$

and the formal series of this bigradation is:

$$\sum_{k,n \geq 0} \dim(V_{k,n})X^k Y^n = \frac{X}{1 - XY - X^2 Y}.$$

2 Pre-Lie structure on $\mathbb{K}\langle x_0, x_1 \rangle$

2.1 pre-Lie coproduct on V

As the composition \circ is linear on the left, the dual coproduct satisfies $\tilde{\Delta}(V) \subseteq V \otimes H$, so H is a commutative right-sided Hopf algebra in the sense of [9], and V inherits a right pre-Lie coproduct: if π is the canonical projection from $H = S(V)$ onto V ,

$$\delta = (\pi \otimes \pi) \circ \Delta = (Id \otimes \pi) \circ \tilde{\Delta}.$$

It satisfies the right pre-Lie coalgebra axiom:

$$(23).((\delta \otimes Id) \circ \delta - (Id \otimes \delta) \circ \delta) = 0.$$

The following proposition allows to compute $\delta(X_c)$ by induction on the length of c .

Proposition 9 1. $\delta(X_\emptyset) = 0$.

2. $\delta \circ \theta_0 = (\theta_0 \otimes Id) \circ \delta + (\theta_1 \otimes Id) \circ \Delta_{\mathbf{w}}$.

3. $\delta \circ \theta_1 = (\theta_1 \otimes Id) \circ \delta$.

Proof. The first point comes from $\Delta(X_\emptyset) = X_\emptyset \otimes 1 + 1 \otimes X_\emptyset$. Let $x \in V$. We put $\Delta_{\mathbf{w}}(x) = x' \otimes x'' \in V \otimes V$. For any $y \in V$, we put $\tilde{\Delta}(y) - y \otimes 1 = y^{(1)} \otimes y^{(2)} \in V \otimes H_+$. Then:

$$\begin{aligned} (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbf{w}}(x) &= (\theta_1 \otimes m)(x' \otimes 1 \otimes x'' + x'^{(1)} \otimes x'^{(2)} \otimes x'') \\ &= \theta_1(x') \otimes \underbrace{x''}_{\in V} + x'^{(1)} \otimes \underbrace{x'^{(2)} x''}_{\in Ker(\pi)}. \end{aligned}$$

Applying $Id \otimes \pi$, it remains:

$$(Id \otimes \pi) \circ (\theta_1 \otimes m) \circ (\tilde{\Delta} \otimes Id) \circ \Delta_{\mathbf{w}}(x) = (\theta_1 \otimes Id) \circ \Delta_{\mathbf{w}}(x).$$

Let $i = 0$ or 1 . Then:

$$(Id \otimes \pi) \circ (\theta_i \otimes Id) \circ \tilde{\Delta} = (\theta_i \otimes Id) \circ (Id \otimes \pi) \circ \tilde{\Delta} = (\theta_i \otimes Id) \circ \delta.$$

The result is induced by these remarks, combined with proposition 3. \square

Examples.

$$\begin{aligned} \delta(X_{x_0}) &= X_{x_1} \otimes X_\emptyset, \\ \delta(X_{x_0^2}) &= X_{x_0 x_1} \otimes X_\emptyset + X_{x_1 x_0} \otimes X_\emptyset + X_{x_1} \otimes X_{x_0}, \\ \delta(X_{x_0 x_1}) &= X_{x_1 x_1} \otimes X_\emptyset + X_{x_1} \otimes X_{x_1}, \\ \delta(X_{x_1 x_0}) &= X_{x_1 x_1} \otimes X_\emptyset. \end{aligned}$$

Proposition 10 $Ker(\delta) = Vect(X_{x_1^n}, n \geq 0)$.

Proof. The inclusion \supseteq is trivial by corollary 4. Let us prove the other inclusion.

First step. Let us prove the following property: if $x \in V_k$ is such that

$$\delta(x) = \lambda \sum_{i+j=k-2} \frac{(k-2)!}{i!j!} X_{x_1^i} \otimes X_{x_1^j},$$

then there exists $\mu \in \mathbb{K}$ such that $x = \mu x_1^{k-1}$. It is obvious if $k = 1$, as then $x = \mu \emptyset$. Let us assume the result at all ranks $< k$. We put $x = x_1^\alpha (x_0 f_0 + x_1 f_1)$, where $\alpha \geq 0$, f_0 is homogeneous of degree $k - 2 - \alpha$ and f_1 is homogeneous of degree $k - 1 - \alpha$.

$$\delta(x) = (\theta_1^\alpha \otimes Id) ((\theta_0 \otimes Id) \circ \delta(f_0) + (\theta_1 \otimes Id) \circ \delta(f_1) + (\theta_1 \otimes Id) \circ \Delta_{\mathbf{w}}(f_0)).$$

Let us consider the terms in this expression of the form $X_\emptyset \otimes X_c$, with c a word. This gives:

$$\lambda X_\emptyset \otimes X_{x_1^{k-2}} = 0,$$

so $\lambda = 0$ and $\delta(x) = 0$. Let us now consider the terms of the form $X_{x_1^\alpha x_0 c} \otimes X_d$, with c, d words. We obtain:

$$0 = (\theta_1^\alpha \circ \theta_0 \otimes Id) \circ \delta(f_0).$$

As both θ_0 and θ_1 are injective, we obtain $\delta(f_0) = 0$. By the induction hypothesis, $f_0 = \nu X_1 x_1^l$, with $l = k - 2 - \alpha < k$. Hence:

$$\Delta_{\mathbf{w}}(f_0) = \nu \sum_{i+j=l} \frac{l!}{i!j!} X_{x_1^i} \otimes X_{x_1^j},$$

and:

$$(\theta_1^{\alpha+1} \otimes Id) \left(\delta(f_1) + \nu \sum_{i+j=l} \frac{l!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) = 0.$$

As θ_1 is injective, we obtain with the induction hypothesis that $f_1 = \mu X_{x_1^{k-2-\alpha}}$, so:

$$x = \mu X_{x_1^{k-1}} + \nu X_{x_1^\alpha x_0 x_1^{k-\alpha-2}}.$$

This gives:

$$\begin{aligned} \delta(x) &= \nu(\theta_1^{\alpha+1} \otimes Id) \left(\sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^i} \otimes X_{x_1^j} \right) \\ &= \nu \sum_{i+j=k-\alpha-2} \frac{(k-\alpha-2)!}{i!j!} X_{x_1^{i+\alpha}} \otimes X_{x_1^j} \\ &= 0, \end{aligned}$$

so necessarily $\nu = 0$ and $x = \mu X_{x_1^{k-1}}$.

Second step. Let $x \in Ker(\delta)$. As δ is homogeneous of degree 0, the homogeneous components of x are in $Ker(\delta)$. By the first step, with $\lambda = 0$, these homogeneous components, hence x , belong to $Vect(X_{x_1^k}, k \geq 0)$. \square

2.2 Dual pre-Lie algebra

As V is a graded pre-Lie coalgebra, its graded dual is a pre-Lie algebra. We identify this graded dual with $\mathbb{K}\langle x_0, x_1 \rangle \subseteq \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$; for any words c, d , $X_c(d) = \delta_{c,d}$. The pre-Lie product of $\mathbb{K}\langle x_0, x_1 \rangle$ is denoted by \bullet . Dualizing proposition 9, we obtain:

Proposition 11 1. For all word c , $\emptyset \bullet c = 0$.

2. For all words c, d , $(x_0 c) \bullet d = x_0(c \bullet d)$.

3. For all words c, d , $(x_1 c) \bullet d = x_1(c \bullet d) + x_0(c \sqcup d)$.

Proof. Let u, v, w be words. Then $X_w(u \bullet v) = \delta(X_w)(u \otimes v)$. Hence, if d is a word:

$$\begin{aligned} X_\emptyset(u \bullet v) &= 0, \\ X_{x_0 d}(u \bullet v) &= (\theta_0 \otimes Id) \circ \delta(X_d)(u \otimes v) + (\theta_1 \otimes Id) \circ \Delta_{\sqcup}(X_d)(u \otimes v) \\ &= X_d(\theta_0^*(u) \bullet v + \theta_1^*(u) \sqcup v), \\ X_{x_1 d}(u \bullet v) &= (\theta_1 \otimes Id) \circ \delta(X_d)(u \otimes v) \\ &= X_d(\theta_1^*(u) \bullet v). \end{aligned}$$

Moreover, for all word c :

$$\begin{aligned} \theta_0^*(\emptyset) &= 0, & \theta_0^*(x_0 c) &= c, & \theta_0^*(x_1 c) &= 0 \\ \theta_1^*(\emptyset) &= 0, & \theta_1^*(x_0 c) &= 0, & \theta_1^*(x_1 c) &= c. \end{aligned}$$

Hence, for any words c, d :

$$\begin{aligned} X_{x_0d}(x_0c \bullet v) &= X_d(c \bullet v) \\ &= X_{x_0d}(x_0(x \bullet v)), \\ X_{x_1d}(x_0c \bullet v) &= 0 \\ &= X_{x_1d}(x_0(x \bullet v)); \end{aligned}$$

$$\begin{aligned} X_{x_0d}(x_1c \bullet v) &= X_d(c \sqcup v) \\ &= X_{x_0d}(x_1(c \bullet v) + x_0(c \sqcup v)), \\ X_{x_1d}(x_1c \bullet v) &= X_d(c \bullet v) \\ &= X_{x_1d}(x_1(c \bullet v) + x_0(c \sqcup v)). \end{aligned}$$

Hence, for any w , $X_w(x_0c \bullet v) = X_w(x_0(x \bullet v))$ and $X_w(x_1c \bullet v) = X_w((x_1(c \bullet v) + x_0(c \sqcup v)))$. \square

Examples.

$x_0 \bullet x_0 = 0$	$x_0 \bullet x_0x_0 = 0$	$x_1 \bullet x_0x_0 = x_0x_0x_0$
$x_0 \bullet x_1 = 0$	$x_0 \bullet x_0x_1 = 0$	$x_1 \bullet x_0x_1 = x_0x_0x_1$
$x_1 \bullet x_0 = x_0x_0$	$x_0 \bullet x_1x_0 = 0$	$x_1 \bullet x_1x_0 = x_0x_1x_0$
$x_1 \bullet x_1 = x_0x_1$	$x_0 \bullet x_1x_1 = 0$	$x_1 \bullet x_1x_1 = x_0x_1x_1$
$x_0x_0 \bullet x_0 = 0$	$x_0x_0 \bullet x_1 = 0$	$x_0x_0 \bullet x_1 = 0$
$x_0x_1 \bullet x_0 = x_0x_0x_0$	$x_0x_1 \bullet x_1 = x_0x_0x_1$	$x_0x_1 \bullet x_1 = x_0x_0x_1$
$x_1x_0 \bullet x_0 = 2x_0x_0x_0$	$x_1x_0 \bullet x_1 = x_0x_0x_1 + x_0x_1x_0$	$x_1x_0 \bullet x_1 = x_0x_0x_1 + x_0x_1x_0$
$x_1x_1 \bullet x_0 = x_1x_0x_0 + x_0x_1x_0 + x_0x_0x_1$	$x_1x_1 \bullet x_1 = x_1x_0x_1 + 2x_0x_1x_1$	$x_1x_1 \bullet x_1 = x_1x_0x_1 + 2x_0x_1x_1$

Dualizing proposition 10:

Theorem 12 $\mathbb{K}\langle x_0, x_1 \rangle = Vect(x_1^n, n \geq 0) \oplus (\mathbb{K}\langle x_0, x_1 \rangle \bullet \mathbb{K}\langle x_0, x_1 \rangle)$. Hence, $(x_1^n)_{n \geq 0}$ is a minimal system of generators of the pre-Lie algebra $\mathbb{K}\langle x_0, x_1 \rangle$.

Proof. As $\bullet = \delta^*$, $Im(\bullet) = Ker(\delta)^\perp = Vect(X_{x_1^n}, n \geq 0)^\perp$. The first assertion is then immediate. As $\mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ is a graded, connected pre-Lie coalgebra, $\mathbb{K}\langle x_0, x_1 \rangle$ is a graded, connected pre-Lie algebra. The result then comes from the next lemma. \square

Lemma 13 Let A be a graded, connected pre-Lie algebra, and V be a graded subspace of A .

1. V generates A if, and only if, $A = V + A \bullet A$.
2. V is a minimal subspace of generators of A if, and only if, $A = V \oplus A \bullet A$.

Proof. 1. \implies . Let $x \in A$. Then it can be written as an element of the pre-Lie subalgebra generated by v , so as the sum of an element of V and of iterated pre-Lie products of elements of V . Hence, $x \in V + A \bullet A$. Note that we did not use the gradation of A to prove this.

1. \impliedby . Let B be the pre-Lie subalgebra generated by V . Let $x \in A_n$, let us prove that $x \in B$ by induction on n . As $A_0 = (0)$, it is obvious if $n = 0$. Let us assume the result at all ranks $< n$. We obtain, by the gradation:

$$A_n = V_n \oplus \sum_{i=1}^{n-1} A_i \bullet A_{n-i}.$$

So we can write $x = \lambda x_1^{n-1} + \sum x_i \bullet y_i$, where x_i, y_i are homogeneous of degree $< n$. By the induction hypothesis, these elements belong to B , so $x \in B$.

2. \implies . By 1. \implies , $A = V + A \bullet A$. If $V \cap A \bullet A \neq (0)$, we can choose a graded subspace $W \subsetneq V$, such that $A = W \oplus A \bullet A$. By 1. \Leftarrow , W generates A , so V is not a minimal system of generators of A : contradiction. So $A = V \oplus A \bullet A$.

2. \Leftarrow . By 1. \Leftarrow , V is a space of generators of A . If $W \subsetneq V$, then $W \oplus A \bullet A \subsetneq A$. By 1. \implies , W does not generate V . So V is a minimal subspace of generators. \square

Proposition 14 For all $x, y, z \in \mathbb{K}\langle x_0, x_1 \rangle$,

$$(x \sqcup y) \bullet z = (x \bullet z) \sqcup y + x \sqcup (y \bullet z).$$

Proof. We prove it if x, y, z are words. If $x = \emptyset$, then:

$$(\emptyset \sqcup y) \bullet z = y \bullet z = (\emptyset \bullet z) \sqcup y + \emptyset \sqcup (y \bullet z).$$

If $y = \emptyset$, the result is also true, using the commutativity of \sqcup . We can now consider that x, y are nonempty words.

Let us proceed by induction on $k = lg(x) + lg(y)$. If $k = 0$ or 1 , there is nothing to prove. Let us assume the result at all rank $< k$. Four cases can occur.

First case. $x = x_0a$ and $y = x_0b$. Then:

$$\begin{aligned} (x \sqcup y) \bullet z &= (x_0(a \sqcup x_0b)) \bullet z + (x_0(x_0a \sqcup b)) \bullet z \\ &= x_0((a \sqcup x_0b) \bullet z) + x_0((x_0a \sqcup b) \bullet z) \\ &= x_0((a \bullet z) \sqcup x_0b) + x_0(a \sqcup ((x_0b) \bullet z)) + x_0(((x_0a) \bullet z) \sqcup b) + x_0(x_0a \sqcup (b \bullet z)) \\ &= x_0((a \bullet z) \sqcup x_0b) + x_0(a \sqcup (x_0(b \bullet z))) + x_0((x_0(a \bullet z)) \sqcup b) + x_0(x_0a \sqcup (b \bullet z)) \\ &= x_0(a \bullet z) \sqcup x_0b + x_0a \sqcup x_0(b \bullet z) \\ &= (x \bullet z) \sqcup y + x \sqcup (y \bullet z). \end{aligned}$$

Second case. $x = x_1a$ and $y = x_0b$. This gives:

$$\begin{aligned} (x \sqcup y) \bullet z &= (x_1(a \sqcup x_0b)) \bullet z + (x_0(x_1a \sqcup b)) \bullet z \\ &= x_1((a \bullet z) \sqcup x_0b) + x_1(a \sqcup x_0(b \bullet z)) \\ &\quad + x_0(a \sqcup x_0b \sqcup z) + x_0(((x_1a) \bullet z) \sqcup b) + x_0(x_1a \sqcup (b \bullet z)) \\ &= x_1((a \bullet z) \sqcup x_0b) + x_1(a \sqcup x_0(b \bullet z)) \\ &\quad + x_0(a \sqcup x_0b \sqcup z) + x_0((x_1(a \bullet z)) \sqcup b) + x_0((x_0(a \sqcup z)) \sqcup b) + x_0(x_1a \sqcup (b \bullet z)), \end{aligned}$$

$$\begin{aligned} (x \bullet z) \sqcup y &= (x_1(a \bullet z)) \sqcup x_0b + (x_0(a \sqcup z)) \sqcup (x_0b) \\ &= x_1((a \bullet z) \sqcup (x_0b)) + x_0(x_1(a \bullet z) \sqcup b) \\ &\quad + x_0(a \sqcup z \sqcup x_0b) + x_0((x_0(a \sqcup z)) \sqcup b), \end{aligned}$$

$$\begin{aligned} x \sqcup (y \bullet z) &= x_1a \sqcup x_0(b \bullet z) \\ &= x_1(a \sqcup x_0(b \bullet z)) + x_0(x_1a \sqcup (b \bullet z)). \end{aligned}$$

These computations imply the required equality.

Third case. $x = x_0a$ and $y = x_1b$. This is a consequence of the second case, using the commutativity of \sqcup .

Last case. $x = x_1a$ and $y = x_1b$. Similar computations give:

$$(x \sqcup y) \bullet z = x_1((a \bullet z) \sqcup x_1b) + x_1(a \sqcup x_1(b \bullet w)) + x_1(a \sqcup x_0(b \sqcup z)) + x_0(a \sqcup x_1b \sqcup z) \\ + x_1(x_1a \sqcup (b \bullet z)) + x_1((x_1(a \bullet z)) \sqcup b) + x_1((x_0(a \sqcup z)) \sqcup b) + x_0(a \sqcup x_1b \sqcup z),$$

$$(x \bullet z) \sqcup y = x_1((a \bullet z) \sqcup x_1b) + x_1((x_1(a \bullet z)) \sqcup b) + x_0(a \sqcup x_1b \sqcup z) + x_1((x_0(a \sqcup z)) \sqcup b),$$

$$x \sqcup (y \bullet z) = x_1(a \sqcup x_1(b \bullet w)) + x_1(a \sqcup x_0(b \sqcup z)) + x_1(x_1a \sqcup (b \bullet z)) + x_0(a \sqcup x_1b \sqcup z).$$

So the result holds in all cases. \square

3 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-pre-Lie algebra

Proposition 14 motivates the following definition:

Definition 15 [10] *A Com-pre-Lie algebra is a triple (V, \bullet, \sqcup) , such that:*

1. (V, \bullet) is a pre-Lie algebra.
2. (V, \sqcup) is a commutative, associative algebra (non necessarily unitary).
3. For all $a, b, c \in V$, $(a \sqcup b) \bullet c = (a \bullet c) \sqcup b + a \sqcup (b \bullet c)$.

For example, $\mathbb{K}\langle x_0, x_1 \rangle$ is a Com-pre-Lie algebra. See [10] for an example of Com-pre-Lie algebra based on rooted trees.

3.1 Free Com-pre-Lie algebras

Definition 16 1. *A partitioned forest is a pair (F, I) such that:*

- (a) F is a rooted forest (the edges of F being oriented from the leaves to the roots).
 - (b) I is a partition of the vertices of F with the following condition: if x, y are two vertices of F which are in the same part of I , then either they are both roots, or they have the same direct descendant.
2. We shall say that a partitioned forest is a partitioned tree if all the roots are in the same part of the partition.
 3. Let \mathcal{D} be a set. A partitioned tree decorated by \mathcal{D} is a pair (t, d) , where t is a partitioned tree and d is a map from the set of vertices of t into \mathcal{D} . For any vertex x of t , $d(x)$ is called the decoration of x .
 4. The set of isoclasses of partitioned trees will be denoted by \mathcal{PT} . For any set \mathcal{D} , the set of isoclasses of partitioned trees decorated by \mathcal{D} will be denoted by $\mathcal{PT}(\mathcal{D})$.

Examples. We represent partitioned trees by the Hasse graph of the underlying rooted forest, the partition being represented by horizontal edges. Here are partitioned trees with ≤ 4 vertices:

$$\bullet, \uparrow, \dashv; \vee, \nabla, \updownarrow, \perp = \lrcorner, \llcorner; \mathbb{V}, \mathbb{V} = \mathbb{V}, \mathbb{V}, \downarrow = \downarrow, \downarrow = \downarrow, \Upsilon, \nabla, \updownarrow, \\ \mathbb{V} = \mathbb{V}, \perp = \lrcorner, \nabla = \nabla, \mathbb{V}, \perp = \lrcorner = \llcorner, \dots$$

Definition 17 Let $t = (t, I)$ and $t' = (t', J) \in \mathcal{PT}$.

1. Let s be a vertex of t' . The partitioned tree $t \bullet_s t'$ is defined as follows:

- (a) As a rooted forest, $t \bullet_s t'$ is obtained by grafting all the roots of t' on the vertex s of t .
- (b) We put $I = \{I_1, \dots, I_k\}$ and $J = \{J_1, \dots, J_l\}$. The partition of the vertices of this rooted forest is $I \sqcup J = \{I_1, \dots, I_k, J_1, \dots, J_l\}$.

2. The partitioned tree $t \sqcup t'$ is defined as follows:

- (a) As a rooted forest, $t \sqcup t'$ is tt' .
- (b) We put $I = \{I_1, \dots, I_k\}$ and $J = \{J_1, \dots, J_l\}$ and we assume that the set of roots of t is I_1 and the set of roots of t' is J_1 . The partition of the vertices of $t \sqcup t'$ is $\{I_1 \sqcup J_1, I_2, \dots, I_k, J_1, \dots, J_l\}$.

Examples.

1. Here are the three possible graftings $\nabla \bullet_s \cdot$: ∇ , \downarrow and \downarrow .

2. Here are the two possible graftings $\downarrow \bullet_s \dashv$: ∇ and ∇ .

These operations can also be defined for decorated partitioned trees.

Proposition 18 Let \mathcal{D} be a set. We denote by $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ the vector space generated by $\mathcal{PT}(\mathcal{D})$. We extend \sqcup by bilinearity on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ and we define a second product \bullet on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ in the following way: if $t, t' \in \mathcal{PT}(\mathcal{D})$,

$$t \bullet t' = \sum_{s \in V(t)} t \bullet_s t'.$$

Then $(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}, \bullet, \sqcup)$ is a Com-pre-Lie algebra.

Proof. Let t, t', t'' be three partitioned trees.

If s', s'' are two vertices of t , we define by $t \bullet_{s', s''} (t', t'')$ the partitioned trees obtained by grafting the roots of t' on s' , the roots of t'' on s'' , the partition of the vertices of the obtained rooted forest being the union of the partitions of t, t' and t'' . Then:

$$\begin{aligned} (t \bullet t') \bullet t'' &= \sum_{s' \in V(t)} (t \bullet_{s'} t') \bullet t'' \\ &= \sum_{s', s'' \in V(t)} (t \bullet_{s'} t') \bullet_{s''} t'' + \sum_{s' \in V(t), s'' \in V(t')} (t \bullet_{s'} t') \bullet_{s''} t'' \\ &= \sum_{s', s'' \in V(t)} t \bullet_{s' s''} (t', t'') + \sum_{s' \in V(t), s'' \in V(t')} t \bullet_{s'} (t' \bullet_{s''} t'') \\ &= \sum_{s', s'' \in V(t)} t \bullet_{s' s''} (t', t'') + t \bullet (t' \bullet t''). \end{aligned}$$

So $(t \bullet t') \bullet t'' - t \bullet (t' \bullet t'')$ is clearly symmetric in t and t' , and \bullet is pre-Lie.

Moreover, $(t \sqcup t') \sqcup t'' = t \sqcup (t' \sqcup t'')$ is the rooted forest $tt't''$, the partition being $\{I_1 \cup J_1 \cup K_1, I_2, \dots, I_k, J_2, \dots, J_l, K_2, \dots, K_m\}$, with immediate notations; $t \sqcup t' = t' \sqcup t$ is the rooted forest $t't$, the partition being $\{I_1 \cup J_1, I_2, \dots, I_k, J_2, \dots, J_l\}$. So \sqcup is an associative, commutative product.

Finally:

$$\begin{aligned} (t \sqcup t') \bullet t'' &= \sum_{s \in V(t)} (t \sqcup t') \bullet_s t'' + \sum_{s' \in V(t')} (t \sqcup t') \bullet_{s'} t'' \\ &= \sum_{s \in V(t)} (t \bullet_s t'') \sqcup t' + \sum_{s' \in V(t')} t \sqcup (t' \bullet_{s'} t'') \\ &= (t \bullet t') \sqcup t'' + t \sqcup (t' \bullet t''). \end{aligned}$$

So $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is Com-pre-Lie. □

In particular, $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is pre-Lie. Let us use the extension of the pre-Lie product \bullet to $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$ defined by Oudom and Guin [12, 13]:

1. If $t_1, \dots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $t_1 \dots t_k \bullet 1 = t_1 \dots t_k$.
2. If $t, t_1, \dots, t_k \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $t \bullet t_1 \dots t_k = (t \bullet t_1 \dots t_{k-1}) \bullet t_k - t \bullet (t_1 \dots t_{k-1} \bullet t_k)$.
3. If $a, b, c \in S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$, $ab \bullet c = (a \bullet c^{(1)})(b \bullet c^{(2)})$, where $\Delta(c) = c^{(1)} \otimes c^{(2)}$ is the usual coproduct of $S(\mathfrak{g}_{\mathcal{PT}(\mathcal{D})})$. In particular, if $t_1, \dots, t_k, t \in \mathcal{PT}(\mathcal{D})$:

$$t_1 \dots t_k \bullet t = \sum_{i=1}^k t_1 \dots (t_i \bullet t) \dots t_k.$$

Lemma 19 *Let $t = (t, I), t_1 = (t_1, I^{(1)}), \dots, t_k = (t_k, I^{(k)})$ be partitioned trees ($k \geq 1$). Let $s_1, \dots, s_k \in V(t)$. The partitioned tree $t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)$ is obtained by grafting the roots of t_i on s_i for all i , the partition being $I \sqcup I^{(1)} \sqcup \dots \sqcup I^{(k)}$. Then:*

$$t \bullet t_1 \dots t_k = \sum_{s_1, \dots, s_k \in V(t)} t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k).$$

Proof. By induction on k . This is obvious if $k = 1$. Let us assume the result at rank k .

$$\begin{aligned} t \bullet t_1 \dots t_{k+1} &= (t \bullet t_1 \dots t_k) \bullet t_{k+1} - \sum_{i=1}^k t \bullet (t_1 \dots (t_i \bullet t_{k+1}) \dots t_k) \\ &= \sum_{s_1, \dots, s_k \in V(t)} (t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)) \bullet t_{k+1} - \sum_{i=1}^k \sum_{s \in V(t_i)} t \bullet (t_1 \dots (t_i \bullet_s t_{k+1}) \dots t_i) \\ &= \sum_{s_1, \dots, s_{k+1} \in V(t)} (t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)) \bullet_{s_{k+1}} t_{k+1} \\ &\quad + \sum_{i=1}^k \sum_{s \in V(t_i)} (t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_k)) \bullet_s t_{k+1} \\ &\quad - \sum_{i=1}^k \sum_{s_1, \dots, s_k \in V(t)} \sum_{s \in V(t_i)} t \bullet_{s_1, \dots, s_k} (t_1, \dots, t_i \bullet_s t_{k+1}, \dots, t_i) \\ &= \sum_{s_1, \dots, s_{k+1} \in V(t)} t \bullet_{s_1, \dots, s_{k+1}} (t_1, \dots, t_{k+1}). \end{aligned}$$

Hence, the result holds for all k . □

Theorem 20 *Let \mathcal{D} be a set, let A be a Com-pre-Lie algebra, and let $a_d \in A$ for all $d \in \mathcal{D}$. There exists a unique morphism of Com-pre-Lie algebra $\phi : \mathfrak{g}_{\mathcal{PT}(\mathcal{D})} \rightarrow A$, such that $\phi(\cdot_d) = a_d$ for all $d \in \mathcal{D}$. In other words, $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is the free Com-pre-Lie algebra generated by \mathcal{D} .*

Proof. *Unicity.* Let $t \in \mathcal{T}^d$. We denote by r_1, \dots, r_n its roots. For all $1 \leq i \leq n$, let $t_{i,1}, \dots, t_{i,k_i}$ be the partitioned trees born from r_i and let d_i be the decoration of r_i . Then:

$$t = (\cdot_{d_1} \bullet t_{1,1} \dots t_{1,k_1}) \sqcup \dots \sqcup (\cdot_{d_n} \bullet t_{n,1} \dots t_{n,k_n}).$$

So ϕ is inductively defined by:

$$\phi(t) = (a_{d_1} \bullet \phi(t_{1,1}) \dots \phi(t_{1,k_1})) \sqcup \dots \sqcup (a_{d_n} \bullet \phi(t_{n,1}) \dots \phi(t_{n,k_n})). \quad (1)$$

Existence. As the product \sqcup of A is commutative and associative, (1) defines inductively a morphism ϕ from $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ to A . By definition, it is compatible with the product \sqcup . Let us prove the compatibility with the product \bullet . Let t, t' be two partitioned trees, let us prove that $\phi(t \bullet t') = \phi(t) \bullet \phi(t')$ by induction on the number N of vertices of t . If $N = 1$, then $t = \cdot_d$ and:

$$\phi(t \bullet t') = a_d \bullet \phi(t') = \phi(t) \bullet \phi(t'),$$

by definition of t' . If $N > 1$, two cases are possible.

First case. If t has only one roots, then $t = \cdot_d \bullet t_1 \dots t_k$, and:

$$t \bullet t' = \cdot_d \bullet t_1 \dots t_k t' + \sum_{i=1}^k \cdot_d \bullet t_1 \dots t_i \circ t' \bullet t_k.$$

Using the induction hypothesis on t_1, \dots, t_k :

$$\begin{aligned} \phi(t \bullet t') &= a_d \bullet \phi(t_1) \dots \phi(t_k) \phi(t') + \sum_{i=1}^k a_d \bullet \phi(t_1) \dots \phi(t_i \circ t') \dots \phi(t_k) \\ &= a_d \bullet \phi(t_1) \dots \phi(t_k) \phi(t') + \sum_{i=1}^k a_d \bullet (\phi(t_1) \dots \phi(t_i) \circ \phi(t') \dots \phi(t_k)) \\ &= (a_d \bullet \phi(t_1) \dots \phi(t_k)) \bullet \phi(t') \\ &= \phi(t) \bullet \phi(t'). \end{aligned}$$

Second case. If t has $k > 1$ roots, we put $t = t_1 \sqcup \dots \sqcup t_k$. The induction hypothesis holds for t_1, \dots, t_k , so:

$$\begin{aligned} \phi(t \circ t') &= \sum_{i=1}^k \phi(t_1 \sqcup t_i \bullet t' \sqcup \dots \sqcup t_k) \\ &= \sum_{i=1}^k \phi(t_1) \sqcup \phi(t_i \bullet t') \sqcup \dots \sqcup \phi(t_k) \\ &= \sum_{i=1}^k \phi(t_1) \sqcup \phi(t_i) \bullet \phi(t') \sqcup \dots \sqcup \phi(t_k) \\ &= (\phi(t_1) \sqcup \dots \sqcup \phi(t_k)) \bullet \phi(t') \\ &= \phi(t) \bullet \phi(t'). \end{aligned}$$

Hence, ϕ is a morphism of Com-pre-Lie algebras. \square

3.2 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a Com-pre-Lie algebra

Proposition 21 *As a Com-pre-Lie algebra, $\mathbb{K}\langle x_0, x_1 \rangle$ is generated by \emptyset and x_1 .*

Proof. Let A be the Com-pre-Lie subalgebra of $\mathbb{K}\langle x_0, x_1 \rangle$ generated by \emptyset and x_1 . For all $n \geq 1$, it contains $x_1^{\sqcup n} = n!x_1^n$, so it contains x_1^n for all $n \geq 0$. As $\mathbb{K}\langle x_0, x_1 \rangle$ is generated by these elements as a pre-Lie algebra, $A = \mathbb{K}\langle x_0, x_1 \rangle$. \square

We denote by $\phi_{CPL} : \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \longrightarrow \mathbb{K}\langle x_0, x_1 \rangle$ the unique morphism of Com-pre-Lie algebras which sends \cdot_1 to \emptyset and \cdot_2 to \cdot_2 . By proposition 21, it is surjective.

Lemma 22 *Let $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$.*

1. $\phi_{CPL}(\cdot_1 \bullet t_1 \dots t_k) = 0$ if $k \geq 1$.

2. $\phi_{CPL}(\bullet_2 \bullet t_1 \dots t_k) = 0$ if $k \geq 2$.

3. If $t \in \mathcal{PT}(\{1, 2\})$, $\phi_{CPL}(\bullet_2 \bullet t) = x_0 \phi_{CPL}(t)$.

Proof. We prove 1.-3. by induction on k . If $k = 1$:

$$\begin{aligned}\phi_{CPL}(\bullet_1 \bullet t) &= \emptyset \bullet \phi_{CPL}(t) = 0, \\ \phi_{CPL}(\bullet_2 \bullet t) &= x_1 \bullet \phi_{CPL}(t) = x_0 \phi_{CPL}(t).\end{aligned}$$

Let us assume the results at rank $k - 1 \geq 1$. Then:

$$\begin{aligned}\phi_{CPL}(\bullet_1 \bullet t_1 \dots t_k) &= \emptyset \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_k) \\ &= (\emptyset \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_{k-1})) \bullet \phi_{CPL}(t_k) \\ &\quad - \sum_{i=1}^k \emptyset \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_i \bullet t_k) \dots \phi_{CPL}(t_{k-1}) \\ &= 0 - 0, \\ \phi_{CPL}(\bullet_2 \bullet t_1 \dots t_k) &= x_1 \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_k) \\ &= (x_1 \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_{k-1})) \bullet \phi_{CPL}(t_k) \\ &\quad - \sum_{i=1}^k x_1 \bullet \phi_{CPL}(t_1) \dots \phi_{CPL}(t_i \bullet t_k) \dots \phi_{CPL}(t_{k-1}).\end{aligned}$$

If $k \geq 3$, the induction hypothesis immediately allows to conclude that $\phi_{CPL}(\bullet_2 \bullet t_1 \dots t_k) = 0 - 0 = 0$. If $k = 2$, this gives:

$$\begin{aligned}\phi_{CPL}(\bullet_2 \bullet t_1 t_2) &= (x_1 \bullet \phi_{CPL}(t_1)) \bullet \phi_{CPL}(t_2) - x_1 \bullet \phi_{CPL}(t_1 \bullet t_2) \\ &= (x_0 \phi_{CPL}(t_1)) \bullet \phi_{CPL}(t_2) - x_0 \phi_{CPL}(t_1 \bullet t_2) \\ &= x_0 (\phi_{CPL}(t_1) \bullet \phi_{CPL}(t_2)) - x_0 \phi_{CPL}(t_1 \bullet t_2) \\ &= 0.\end{aligned}$$

Hence, the result holds for all $k \geq 1$. □

Lemma 23 For all $t \in \mathcal{PT}(\{1, 2\})$, $\phi_{CPL}(t)$ is a linear span of words of length the number of vertices of t decorated by 2.

Proof. By induction on the number of vertices N of t . If $N = 1$, then $t = \bullet_1$ or \bullet_2 and the result is obvious. Let us assume the result at all rank $< N$.

First case. If t has only one root, we put $t = \bullet_i \bullet t_1 \dots t_k$. By the preceding lemma, we can assume that $i = 2$ and $k = 1$. Then $\phi_{CPL}(t) = x_0 \phi_{CPL}(t_1)$ and the result is obvious.

Second case. If t has $k > 1$ roots, we put $t = t_1 \sqcup \dots \sqcup t_k$. Then $\phi_{CPL}(t)$ is equal to $\phi_{CPL}(t_1) \sqcup \dots \sqcup \phi_{CPL}(t_k)$ and the result is immediate. □

Lemma 24 We define inductively a family F of elements of $\mathcal{PT}(\{1, 2\})$ by:

1. $F(1) = \{\bullet_1, \bullet_2\}$.
2. $F(n+1) = (\bullet_2 \bullet F(n)) \cup \bigcup_{i=1}^n (F(i) \sqcup F(n+1-i))$.
3. $F = \bigcup_{n \geq 1} F(n)$.

Let $t \in \mathcal{PT}(\{1, 2\})$. Then $\phi_{CPL}(t) \neq 0$ if, and only if, $t \in F$.

Proof. \implies . We proceed by induction on the number N of vertices of t . This is obvious if $N = 1$. Let us assume the result at all rank $< N$.

First case. If N has only one root, we put $N = \bullet_i \bullet t_1 \dots t_k$. By lemma 22, $i = 2$ and $k = 1$. Then $\phi_{CPL}(t) = x_0 \phi_{CPL}(t_1)$. By the induction hypothesis, $t_1 \in F$, so $t \in F$.

Second case. If N has $k > 1$ roots, we put $t = t_1 \sqcup \dots \sqcup t_k$. Then:

$$\phi_{CPL}(t) = \phi_{CPL}(t_1) \sqcup \phi_{CPL}(t_2 \sqcup \dots \sqcup t_k) \neq 0,$$

so by the induction hypothesis, t_1 and $t_2 \sqcup \dots \sqcup t_k \in F$, and $t \in F$.

\Leftarrow . Let $t \in T(n)$. We proceed by induction on n . It $n = 1$, this is obvious. If $n > 1$ then $t = \bullet_2 \bullet t'$, with $t' \in F(n-1)$, or $t = t' \sqcup t''$, with $t' \in F(i)$, $t'' \in F(n-i)$. In the first case, by the induction hypothesis, $\phi_{CPL}(t') \neq 0$ and $\phi_{CPL}(t) = x_0 \phi_{CPL}(t') \neq 0$. In the second case, $\phi_{CPL}(t'), \phi_{CPL}(t'') \neq 0$ by the induction hypothesis, so $\phi_{CPL}(t) = \phi_{CPL}(t') \sqcup \phi_{CPL}(t'') \neq 0$. \square

We define a second family of elements of $\mathcal{PT}(\{1, 2\})$ in the following way:

1. $F'(1) = \{\bullet_1, \bullet_2\}$.
2. $F'(2) = \{\bullet_2 \sqcup \bullet_2, \mathbf{1}_2^2, \mathbf{1}_2^1\}$.
3. $F'(n+1) = (\bullet_2 \bullet F'(n)) \cup \bigcup_{i=2}^{n-1} (F'(i) \sqcup F'(n+1-i)) \cup (\bullet_2 \sqcup F'(n))$ if $n \geq 2$.
4. $F' = \bigcup_{n \geq 1} F'(n)$.

We define a map π from F to $\mathcal{PT}(\{1, 2\})$ in the following way:

1. $\pi(\bullet_i) = \bullet_i$ if $i = 1, 2$.
2. $\pi(\bullet_1 \sqcup \dots \sqcup \bullet_1) = \bullet_1$.
3. If $t = \bullet_1 \sqcup \dots \sqcup \bullet_1 \sqcup t_1 \sqcup \dots \sqcup t_k$, $k \geq 1$, with $t_1, \dots, t_k \neq \bullet_1$, then $\pi(t) = \pi(t_1) \sqcup \dots \sqcup \pi(t_k)$.
4. If $t = \bullet_2 \bullet t_1 \dots t_k$, then $\pi(t) = \bullet_2 \bullet \pi(t_1) \dots \pi(t_k)$.

Lemma 25 π is a projection on F' and $\phi_{CPL} \circ \pi = \phi_{CPL}|_F$.

Proof. Let $t \in F$. Let us prove by induction on the number N of vertices of t that:

1. $\pi(t) \in F'$.
2. If $t \in F'$, $\pi(t) = t$.
3. $\phi_{CPL} \circ \pi(t) = \phi_{CPL}(t)$.
4. If $\pi(t) = \bullet_1$, then $t = \bullet_1 \sqcup^n$ for a particular n .

All these points are immediate if $N = 1$. Let us assume the result at all rank $< N$, $N \geq 2$. We put $t = \bullet_1 \sqcup \dots \sqcup \bullet_1 \sqcup t_1 \sqcup \dots \sqcup t_k$, $k \geq 0$, with $t_1, \dots, t_k \neq \bullet_1$.

First case. If $k \geq 2$, then $\pi(t) = \pi(t_1) \sqcup \dots \sqcup \pi(t_k)$. By the induction hypothesis, $\pi(t_1), \dots, \pi(t_k) \in F'$ and are not equal to \cdot_1 , so $\pi(t) \in F'$. By the induction hypothesis, $\pi(t_1) \neq \cdot_1$, so $\pi(t) \neq \cdot_1$. Moreover:

$$\begin{aligned}\phi_{CPL}(t) &= \phi_{CPL}(\cdot_1) \sqcup \dots \sqcup \phi_{CPL}(\cdot_1) \sqcup \phi_{CPL}(t_1) \sqcup \dots \sqcup \phi_{CPL}(t_k) \\ &= \emptyset \sqcup \dots \sqcup \emptyset \sqcup \phi_{CPL} \circ \pi(t_1) \sqcup \dots \sqcup \phi_{CPL} \circ \pi(t_k) \\ &= \phi_{CPL}(\pi(t_1)) \sqcup \dots \sqcup \phi_{CPL}(\pi(t_k)) \\ &= \phi_{CPL} \circ \pi(t).\end{aligned}$$

If $t \in F'$, necessarily $t = t_1 \sqcup \dots \sqcup t_k$, and $t_1, \dots, t_k \in F'$. By the induction hypothesis, $\pi(t_1) = t_1, \dots, \pi(t_k) = t_k$, so $\pi(t) = t$.

Second case. If $k = 1$, as $t_1 \in F$, we put $t_1 = \cdot_2 \bullet s$. Then $\pi(t) = \cdot_2 \bullet \pi(s)$. By the induction hypothesis, $\pi(s) \in F'$, so $\pi(t) \in F'$. Moreover:

$$\begin{aligned}\phi_{CPL}(t) &= \phi_{CPL}(\cdot_1) \sqcup \dots \sqcup \phi_{CPL}(\cdot_1) \sqcup (\phi_{CPL}(\cdot_2) \bullet \phi_{CPL}(s)) \\ &= \emptyset \sqcup \dots \sqcup \emptyset \sqcup (\phi_{CPL}(\cdot_2) \bullet \phi_{CPL}(s)) \\ &= \phi_{CPL} \circ \pi(\cdot_2) \bullet \phi_{CPL} \circ \pi(s) \\ &= \phi_{CPL} \circ \pi(t).\end{aligned}$$

If $t' \in F'$, then $s \in F'$, and $t = \cdot_2 \bullet s$. Then $\pi(t) = \cdot_2 \bullet \pi(s) = \cdot_2 \bullet s = t$.

Last case. If $k = 0$, all the results are obvious. □

Lemma 26 *Let $t, t' \in \mathcal{PT}(\{1, 2\})$. Then:*

$$\phi_{CPL}((\cdot_2 \bullet t) \sqcup (\cdot_2 \bullet t')) = \phi_{CPL}(\cdot_2 \bullet ((\cdot_2 \bullet t) \sqcup t' + t \sqcup (\cdot_2 \bullet t'))).$$

Proof. Indeed, putting $w = \phi_{CPL}(t)$ and $w' = \phi_{CPL}(t')$:

$$\begin{aligned}\phi_{CPL}((\cdot_2 \bullet t) \sqcup (\cdot_2 \bullet t')) &= x_0 w \sqcup x_0 w' \\ &= x_0(w \sqcup x_0 w') + x_0(x_0 w \sqcup w') \\ &= \phi_{CPL}(\cdot_2 \bullet ((\cdot_2 \bullet t) \sqcup t' + t \sqcup (\cdot_2 \bullet t'))).\end{aligned}$$

We used lemma 22 for the first and third equalities. □

Theorem 27 *The kernel of ϕ_{CPL} is the Com-pre-Lie ideal generated by the elements:*

1. $\cdot_1 \bullet t_1 \dots t_k$, where $k \geq 1$, $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$.
2. $\cdot_2 \bullet t_1 \dots t_k$, where $k \geq 2$, $t_1, \dots, t_k \in \mathcal{PT}(\{1, 2\})$.
3. $\cdot_1 \sqcup t - t$, where $t \in \mathcal{PT}(\{1, 2\})$.
4. $(\cdot_2 \bullet t) \sqcup (\cdot_2 \bullet t') - \cdot_2 \bullet ((\cdot_2 \bullet t) \sqcup t' - t \sqcup (\cdot_2 \bullet t'))$, where $t, t' \in \mathcal{PT}(\{1, 2\})$.

Proof. Let I be the ideal generated by these elements. Lemmas 22 and 26 prove that the elements 1., 2. and 4. belong to $\text{Ker}(\phi_{CPL})$. Moreover, for all $t \in \mathcal{PT}(\{1, 2\})$, $\pi(\cdot_1 \sqcup t) = \pi(t)$. For all $t \in \mathcal{PT}(\{1, 2\})$:

$$\phi_{CPL}(\cdot_1 \sqcup t) = \emptyset \sqcup \phi_{CPL}(t) = \phi_{CPL}(t),$$

so elements 3. also belong to $\text{Ker}(\phi_{CPL})$. Hence, $I \subseteq \text{Ker}(\phi_{CPL})$.

Let $h = \mathfrak{g}_{\mathcal{PT}(\{1, 2\})}/I$. As the elements 1. and 2. belong to I , h is linearly spanned by the elements \bar{t} , $t \in F$. As the elements 3. belong to I , for all $t \in F$, $\overline{\pi(t)} = \bar{t}$. As π is a projection on F' , h is linearly spanned by the elements \bar{t} , $t \in F'$.

We now define inductively two families of partitionned trees in the following way:

1. $T''(1) = \{\cdot_2\}$ and $F''(1) = \{\cdot_1, \cdot_2\}$.
2. $\mathcal{T}''(n+1) = \cdot_2 \bullet F''(n)$.
3. $F''(n+1) = \bigcup_{i=1}^{n+1} T''(i) \sqcup \cdot_2 \mathfrak{W}^{(n+1-i)}$.
4. $F'' = \bigcup_{n \geq 1} F''(n)$.

Let us prove that for all $t \in F'$, there exists $t' \in Vect(F'')$ such that $\bar{t} = \bar{t}'$. We proceed by induction on the number N of vertices of t . If $N = 1$, then $t = \cdot_1$ or \cdot_2 and we take $t' = t$. Let us assume the result at all rank $< N$. We put $t = t_1 \sqcup \dots \sqcup t_k \sqcup \cdot_2 \sqcup \dots \sqcup \cdot_2$, with $t_i = \cdot_2 \bullet s_i$, $s_i \neq 1$, for all $1 \leq i \leq k$. We proceed by induction on k . If $k = 0$, we take $t' = t = \cdot_2 \sqcup \dots \sqcup \cdot_2$. If $k = 1$, then, by the induction hypothesis on N applied to s_1 :

$$\bar{t} = (\overline{\cdot_2 \bullet s_1}) \sqcup \overline{\cdot_2} \sqcup \dots \sqcup \overline{\cdot_2} = (\overline{\cdot_2 \bullet s'_1}) \sqcup \overline{\cdot_2} \sqcup \dots \sqcup \overline{\cdot_2} = \overline{(\cdot_2 \bullet s'_1) \sqcup \cdot_2 \sqcup \dots \sqcup \cdot_2}.$$

We take $t' = (\cdot_2 \bullet s'_1) \sqcup \cdot_2 \sqcup \dots \sqcup \cdot_2$, which clearly belongs to $Vect(F'')$, as $s'_1 \in Vect(F'')$. Let us assume the result at all rank $< k$. Then, as the elements 4. belong to I :

$$\overline{t_1 \sqcup t_2} = \overbrace{(\cdot_2 \bullet (t_1 \sqcup s_2))}^{t'_1} + \overbrace{(\cdot_2 \bullet (s_1 \bullet t_2))}^{t'_1},$$

so:

$$\bar{t} = \overline{t'_1 \sqcup t_3 \sqcup \dots \sqcup t_k \sqcup \cdot_2 \sqcup \dots \sqcup \cdot_2} + \overline{t'_1 \sqcup t_3 \sqcup \dots \sqcup t_k \sqcup \cdot_2 \sqcup \dots \sqcup \cdot_2}.$$

By the induction hypothesis on k applied to these two partitionned trees, there exists x'_1 and $x''_1 \in Vect(F'')$, such that $\bar{t} = \overline{x'_1} + \overline{x''_1}$. We take $t' = x'_1 + x''_1$. Consequently, the elements \bar{t} , $t \in F''$, linearly span h .

Let $t \in F''(n)$. Then it has n vertices, and at most one of them is decorated by 1. We denote by $F''_1(n)$ the set of elements of $F''(n)$ with one vertex decorated by 1, and we put $F''_2(n) = F''(n) \setminus F''_1(n)$. Let us prove that for all $n \geq 1$, $|F''_1(n+1)| \leq 2^{n-1}$ and $|F''_2(n)| \leq 2^{n-1}$. For $n = 0$, as $F''_1(2) = \{\mathbf{!}\frac{1}{2}\}$ and $F''_2(1) = \{\cdot_2\}$, this is immediate. Let us assume the result at all rank $\leq n$. Then:

$$\begin{aligned} F''_2(n+1) &= \bigcup_{i=1}^{n+1} \cdot_2 \mathfrak{W}^{(n+1-i)} \sqcup T''(i) \cap F''_2(i) \\ &= \{\cdot_2 \mathfrak{W}^{(n+1)}\} \cup \bigcup_{i=1}^n \cdot_2 \mathfrak{W}^{(n+1-i)} \sqcup \cdot_2 \bullet F''_2(i). \end{aligned}$$

Hence, $|F''_2(n+1)| \leq 1 + 1 + 2 + \dots + 2^{n-1} = 2^n$.

$$\begin{aligned} F''_1(n+2) &= \bigcup_{i=1}^{n+2} \cdot_2 \mathfrak{W}^{(n+2-i)} \sqcup T''(i) \cap F''_1(i) \\ &= \bigcup_{i=2}^{n+2} \cdot_2 \mathfrak{W}^{(n+2-i)} \sqcup \cdot_2 \bullet F''_1(i-1) \end{aligned}$$

Hence, $|F''_1(n+2)| \leq 1 + 1 + \dots + 2^{n-1} = 2^n$.

Let $\bar{\phi}_{APL}$ be the linear map induced by ϕ_{CPL} on h . If $t \in F_1''(n)$, by lemma 23, $\bar{\phi}_{APL}(\bar{t})$ is a linear span of word of length $n - 1$. If $t \in F_2''(n)$, by lemma 23, $\bar{\phi}_{APL}(\bar{t})$ is a linear span of word of length n . Hence, for all $n \geq 0$:

$$\bar{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) \subseteq Vect(\text{words of length } n).$$

As ϕ_{CPL} is surjective, we obtain:

$$\bar{\phi}_{APL}(Vect(F_2''(n)) + Vect(F_1''(n+1))) = Vect(\text{words of length } n).$$

Moreover, as $\dim(Vect(\text{words of length } n)) = 2^n$ and $\dim(Vect(F_2''(n)) + Vect(F_1''(n+1))) \leq |F_2''(n)| + |F_1''(n)| \leq 2^{n-1} + 2^{n-1} = 2^n$, the restriction of $\bar{\phi}_{APL}$ to $Vect(F_2''(n)) + Vect(F_1''(n+1))$ is injective. Finally, $\bar{\phi}_{APL}$ is injective, so $Ker(\phi_{CPL}) = I$. \square

4 Presentation of $\mathbb{K}\langle x_0, x_1 \rangle$ as a pre-Lie algebra

4.1 A surjective morphism

Let $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ be the free pre-Lie algebra generated by \mathbb{N}^* , as described in [1]. It can be seen as the subspace of $\mathfrak{g}_{\mathcal{PT}(\mathbb{N}^*)}$ generated by rooted trees (which are seen as partitioned trees such that any part of the partition is a singleton), with the restriction of the pre-Lie product \bullet defined by graftings. For example, in $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, if $a, b, c, d > 0$:

$$\mathbf{!}_a^b \bullet \mathbf{!}_c^d = \mathbf{!}_a^b \downarrow_a^c \uparrow_a^d + \mathbf{!}_a^c \downarrow_a^b \uparrow_a^d.$$

This pre-Lie algebra is graded, the degree of a tree being the sum of its decorations.

By theorem 12, there exists a unique surjective map of pre-Lie algebras $\Phi_{PL} : \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} \longrightarrow \mathbb{K}\langle x_0, x_1 \rangle$, sending \bullet_n to x_1^{n-1} for all $n \geq 1$. As x_1^{i-1} is homogeneous of degree i for all i , this morphism is homogeneous of degree 0.

Notation. If $t_1 \dots t_k \in \mathcal{T}(\mathbb{N}^*)$ and $n \in \mathbb{N}^*$, we put:

$$B_n(t_1 \dots t_k) = \bullet_n \bullet t_1 \dots t_k.$$

This is the tree obtained by grafting t_1, \dots, t_k on a common root decorated by n .

Proposition 28 *Let $t = B_n(t_1 \dots t_k) \in \mathcal{T}(\mathbb{N}^*)$. We put $\phi_{PL}(t_i) = w_i$ for all $1 \leq i \leq k$. Then:*

$$\phi_{PL}(t) = \begin{cases} x_0 w_1 \sqcup \dots \sqcup x_0 w_k \sqcup x_1^{n-1-k} & \text{if } k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As $\mathfrak{g}_{\mathcal{PT}(\{1,2\})}$ is pre-Lie, there exists a unique morphism of pre-Lie algebras:

$$\psi : \begin{cases} \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} & \longrightarrow & \mathfrak{g}_{\mathcal{PT}(\{1,2\})} \\ \bullet_n & \longrightarrow & \frac{1}{(n-1)!} \bullet_2 \sqcup^{(n-1)}. \end{cases}$$

Then $\phi_{APL} \circ \psi$ is a pre-Lie algebra morphism sending \bullet_n to $\frac{1}{(n-1)!} x_1^{\sqcup^{(n-1)}} = x_1^{n-1}$ for all $n \geq 1$, so $\phi_{APL} \circ \psi = \phi_{PL}$. We obtain, by lemma 19:

$$\begin{aligned} \psi(\bullet_n \bullet t_1 \dots t_k) &= \frac{1}{(n-1)!} \bullet_2 \sqcup^{(n-1)} \bullet (\psi(t_1) \dots \psi(t_k)) \\ &= \frac{1}{(n-1)!} \sum_{I_1 \sqcup \dots \sqcup I_n = \{1, \dots, k\}} \bullet_2 \bullet \left(\prod_{i \in I_1} t_i \right) \sqcup \dots \sqcup \bullet_2 \bullet \left(\prod_{i \in I_k} t_i \right) \end{aligned}$$

Let us apply ϕ_{APL} to this expression. If $|I_j| \geq 2$, by theorem 27:

$$\phi_{APL}(\bullet \bullet \left(\prod_{i \in I_j} t_i \right)) = 0.$$

Consequently, if $k \geq n$, at least one of the I_j contains two elements, so $\phi_{APL} \circ \psi(t) = \phi_{PL}(t) = 0$. Let us assume that $k < n$. Hence, using the commutativity of \boxplus :

$$\begin{aligned} \phi_{PL}(\bullet_n \bullet t_1 \dots t_k) &= \frac{1}{(n-1)!} \sum_{I_1 \boxplus \dots \boxplus I_n = \{1, \dots, k\}, |I_j| \leq 1} x_1 \bullet \left(\prod_{i \in I_1} w_i \right) \boxplus \dots \boxplus x_1 \bullet \left(\prod_{i \in I_k} w_i \right) \\ &= \frac{1}{(n-1)!} \sum_{\iota: \{1, \dots, k\} \rightarrow \{1, \dots, n-1\}, \text{injective}} x_1 \bullet w_1 \boxplus \dots \boxplus x_1 \bullet w_k \boxplus x_1^{\boxplus(n-1-k)} \\ &= \frac{1}{(n-1)!} \sum_{\iota: \{1, \dots, k\} \rightarrow \{1, \dots, n-1\}, \text{injective}} x_0 w_1 \boxplus \dots \boxplus x_0 w_k \boxplus x_1^{\boxplus(n-1-k)} \\ &= \frac{(n-1) \dots (n-k)}{(n-1)!} x_0 w_1 \boxplus \dots \boxplus x_0 w_k \boxplus x_1^{\boxplus(n-1-k)} \\ &= \frac{(n-1) \dots (n-k)(n-1-k)!}{(n-1)!} x_0 w_1 \boxplus \dots \boxplus x_0 w_k \boxplus x_1^{n-1-k} \\ &= x_0 w_1 \boxplus \dots \boxplus x_0 w_k \boxplus x_1^{n-1-k}, \end{aligned}$$

which is the announced result. \square

Corollary 29 *Let $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\{N^*\})$, $k, l \geq 0$. For all $i, j, n \geq 1$:*

$$\begin{aligned} &\phi_{PL}(B_{n+1}((B_i(s_1 \dots s_k)B_j(t_1 \dots t_l))) \\ &= \phi_{PL}(B_n(B_{i+1}(s_1 \dots s_k)B_j(t_1 \dots t_l))) + \phi_{PL}(B_n(B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l))). \end{aligned}$$

Proof. We note:

$$\begin{aligned} T_1 &= B_{n+1}((B_i(s_1 \dots s_k)B_j(t_1 \dots t_l))) = \bullet_{n+1} \bullet ((\bullet_i \bullet s_1 \dots s_k)(\bullet_j \bullet t_1 \dots t_l)), \\ T_2 &= B_n(B_{i+1}(s_1 \dots s_k)B_j(t_1 \dots t_l)) = \bullet_n \bullet (\bullet_{i+1} \bullet (s_1 \dots s_k)(\bullet_j \bullet t_1 \dots t_l)), \\ T_3 &= B_n(B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l)) = \bullet_n \bullet (\bullet_{j+1} \bullet ((\bullet_i \bullet s_1 \dots s_k)t_1 \dots t_l)). \end{aligned}$$

If $k \geq i$, or $l \geq j$, or $n = 1$, all these elements are sent to zero by ϕ_{PL} by proposition 28. Let us assume now that $k < i$, $l < j$, $n < 1$. We put $v_i = \phi_{PL}(s_i)$ and $w_i = \phi_{PL}(t_i)$. Then:

$$\begin{aligned} \phi_{PL}(T_1) &= x_0 \underbrace{(x_0 v_1 \boxplus \dots \boxplus x_0 v_k \boxplus x_1^{i-1-k})}_X \boxplus x_0 \underbrace{(x_0 w_1 \boxplus \dots \boxplus x_0 w_l \boxplus x_1^{j-1-l})}_Y \boxplus x_1^{n-2} \\ &= x_0 X \boxplus x_0 Y \boxplus x_1^{n-2}, \\ \phi_{PL}(T_2) &= x_0 (x_0 v_1 \boxplus \dots \boxplus x_0 (x_0 w_1 \boxplus \dots \boxplus x_0 w_l \boxplus x_1^{j-1-l})) \boxplus x_1^{i-1-k} \boxplus x_1^{n-2} \\ &= x_0 (X \boxplus x_0 Y) \boxplus x_1^{n-2}, \\ \phi_{PL}(T_3) &= x_0 (x_0 (x_0 v_1 \boxplus \dots \boxplus x_0 v_k \boxplus x_1^{i-1-k})) \boxplus x_0 w_1 \boxplus x_0 w_l \boxplus x_1^{j-1-l} \boxplus x_1^{n-2} \\ &= x_0 (x_0 X \boxplus Y) \boxplus x_1^{n-2}. \end{aligned}$$

As $x_0 X \boxplus x_0 Y = x_0 (X \boxplus x_0 Y) + x_0 (x_0 X \boxplus Y)$, we obtain the result. \square

Theorem 30 *The kernel of ϕ_{PL} is the pre-Lie ideal generated by:*

1. $B_1(t_1 \dots t_k)$, where $k \geq 1$, $t_1, \dots, t_k \in \mathcal{T}(N^*)$.

2. $B_{n+1}(B_i(s_1 \dots s_k)B_j(t_1 \dots t_l)) - B_n(B_{i+1}(s_1 \dots s_k)B_j(t_1 \dots t_l)) - B_{j+1}(B_i(s_1 \dots s_k)t_1 \dots t_l)$,
where $k, l \geq 0$, $s_1, \dots, s_k, t_1, \dots, t_l \in \mathcal{T}(\mathbb{N}^*)$.

Proof. Let I be the ideal generated by these elements. By proposition 28 and corollary 29, $I \subseteq \text{Ker}(\phi_{PL})$. We put $h = \mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}/I$. Applying repeatedly the relation given by elements of the second form, it is not difficult to prove that for any $t \in \mathcal{T}(\mathbb{N}^*)$, there exists a linear span of ladders t' such that $\bar{t} = \bar{t}'$ in h . Moreover, by the relation given by elements 1., if one of the vertices of a ladder t which is not the leaf is decorated by 1, then $\bar{t} = 0$. Let us denote by $L(n)$ the set of ladders decorated by \mathbb{N}^* , of weight n , such that all the vertices which are not the leaf are decorated by integer > 1 . It turns out that h is generated by the elements \bar{t} , $t \in L = \bigcup L(n)$.

Let $\bar{\phi}_{PL}$ be the morphism from h to $\mathbb{K}\langle x_0, x_1 \rangle$ induced by ϕ_{PL} . By homogeneity, as ϕ_{PL} is surjective, for all $n \geq 1$:

$$\bar{\phi}_{PL}(\text{Vect}(L(n))) = \text{Vect}(\text{words of degree } n).$$

In order to prove that $I = \text{Ker}(\phi_{PL})$, it is enough to prove that $\bar{\phi}_{PL}$ is injective. By homogeneity, it is enough to prove that $\bar{\phi}|_{\text{Vect}(L(n))}$ is injective for all $n \geq 1$. Hence, it is enough to prove that for all $n \geq 1$,

$$|L(n)| = \dim(\text{Vect}(\text{words of degree } n)) = p_n,$$

where the p_n are the integers defined in proposition 8. Let $l_n = |L(n)|$ and q_n be the number of $t \in L(n)$ with no vertex decorated by 1. Then for all $n \geq 2$, $l_n = q_n + q_{n-1}$, and $l_1 = 1$. We put:

$$L = \sum_{n=1}^{\infty} l_n X^n, \quad Q = \sum_{n=1}^{\infty} q_n X^n.$$

We obtain $P = X + Q + XQ$. Moreover:

$$Q = \frac{1}{1 - \sum_{i \geq 2} X^i} - 1 = \frac{1}{1 - \frac{X^2}{1-X}} - 1 = \frac{X^2}{1 - X - X^2},$$

Finally:

$$L = \frac{X}{1 - X - X^2} = F.$$

So, for all $n \geq 1$, $|L(n)| = p_n$. □

As an immediate corollary, a basis of h is given by the classes of the elements of L . Turning to $\mathbb{K}\langle x_0, x_1 \rangle$, we obtain:

Corollary 31 *Let $w = a_1 \dots a_k$ be a word with letters in \mathbb{N}^* .*

1. *We put:*

$$m_w = x_1^{a_1-1} \bullet (x_1^{a_1-1} \bullet (\dots (x_1^{a_{k-1}-1} \bullet x_1^{a_k}) \dots)).$$

2. *We shall say that w is admissible if $a_1, \dots, a_{k-1} > 1$. The set of admissible words is denoted by Adm .*

Then $(m_w)_{w \in \text{Adm}}$ is a basis of $\mathbb{K}\langle x_0, x_1 \rangle$.

Remark. If w is not admissible, that is to say if there exists $1 \leq i < k$, such that $a_i = 1$, then $m_w = 0$ by proposition 28.

We extend the map $w \rightarrow m_w$ by linearity.

4.2 Pre-Lie product in the basis of admissible words

Notations.

1. For all k, l , we denote by $Sh(k, l)$ the set of (k, l) -shuffles, that is to say permutations $\zeta \in \mathfrak{S}_{k+l}$ such that $\zeta(1) < \dots < \zeta(k)$, $\zeta(k+1) < \dots < \zeta(k+l)$.
2. For all k, l we denote by $Sh_{\prec}(k, l)$ the set of (k, l) -shuffles ζ such that $\zeta^{-1}(k+l) = k$.
3. For all k, l we denote by $Sh_{\succ}(k, l)$ the set of (k, l) -shuffles ζ such that $\zeta^{-1}(k+l) = k+l$.
4. The symmetric group \mathfrak{S}_n acts on the set of words with letters in \mathbb{N}^* of length n by permutation of the letters:

$$\sigma.(a_1 \dots a_n) = a_{\sigma^{-1}(1)} \dots a_{\sigma^{-1}(n)}.$$

Proposition 32 *Let $\mathbb{K}\langle \mathbb{N}^* \rangle$ be the space generated by words with letters in \mathbb{N}^* . We define a dendriform structure on this space by:*

$$\begin{aligned} (a_1 \dots a_k) \prec (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\prec}(k, l)} \zeta.a_1 \dots a_k b_1 \dots b_{k-1} (b_k + 1) \\ (a_1 \dots a_k) \succ (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\succ}(k, l)} \zeta.a_1 \dots a_{k-1} (a_k + 1) b_1 \dots b_k. \end{aligned}$$

The associative product $\prec + \succ$ is denoted by \star .

Proof. We denote by $Sh(k, l, m)$ the set of $k+l+m$ -permutations such that $\zeta(1) < \dots < \zeta(k)$, $\zeta(k+1) < \dots < \zeta(k+l)$, $\zeta(k+l+1) < \dots < \zeta(k+l+m)$. Then:

$$\begin{aligned} &(a_1 \dots a_k \prec b_1 \dots b_l) \prec c_1 \dots c_m = a_1 \dots a_k \prec (b_1 \dots b_l \star c_1 \dots c_m) \\ &= \sum_{\zeta \in Sh(k, l, m), \zeta^{-1}(k+l+m)=k} \zeta.a_1 \dots a_k b_1 \dots (b_l + 1) c_1 \dots (c_m + 1); \\ &(a_1 \dots a_k \succ b_1 \dots b_l) \prec c_1 \dots c_m = a_1 \dots a_k \succ (b_1 \dots b_l \prec c_1 \dots c_m) \\ &= \sum_{\zeta \in Sh(k, l, m), \zeta^{-1}(k+l+m)=k+l} \zeta.a_1 \dots (a_k + 1) b_1 \dots b_l c_1 \dots (c_m + 1); \\ &(a_1 \dots a_k \star b_1 \dots b_l) \succ c_1 \dots c_m = a_1 \dots a_k \succ (b_1 \dots b_l \succ c_1 \dots c_m) \\ &= \sum_{\zeta \in Sh(k, l, m), \zeta^{-1}(k+l+m)=k+l+m} \zeta.a_1 \dots (a_k + 1) b_1 \dots (b_l + 1) c_1 \dots c_m. \end{aligned}$$

So $\mathbb{K}\langle \mathbb{N}^* \rangle$ is a dendriform algebra. □

We postpone the study of this dendriform algebra to section 5.2.

Notations. For all $a_1, \dots, a_k \in \mathbb{N}^*$, we denote by $l(a_1 \dots a_k) = B_{a_1} \circ \dots \circ B_{a_k}(1)$ the ladder decorated from the root to the leaf by a_1, \dots, a_k . Note that $m_{a_1 \dots a_k} = \phi_{PL}(l(a_1 \dots a_k))$.

Lemma 33 *Let $k, l \geq 1$ and let $a_1, \dots, a_l, b_1, \dots, b_l \in \mathbb{N}^*$. Then:*

$$\phi_{PL}(B_{a_1+1}(l(a_2 \dots a_k)l(b_1 \dots b_l)) + B_{b_1+1}(l(a_1 \dots a_k)l(b_2 \dots b_l))) = m_{a_1 \dots a_k \star b_1 \dots b_l}.$$

Proof. By induction on $k+l$. If $k=l=1$, then:

$$\phi_{PL}(\mathbf{1}_{a_1+1}^{b_1} + \mathbf{1}_{b_1+1}^{a_1}) = m_{(a_1+1)b_1+(b_1+1)a_1} = m_{a_1 \star b_1}.$$

Let us assume the result at all ranks $< k + l$. If $k = 1$, then:

$$\begin{aligned}
&= \phi_{PL}(B_{a_1+1}(l(b_2 \dots b_l)) + B_{b_1+1}(l(a_1)l(b_2 \dots b_l))) \\
&= \phi_{PL}(\bullet_{a_1+1} \bullet l(b_2 \dots b_l) + \bullet_{b_1+1} \bullet (l(a_1)l(b_2 \dots b_l))) \\
&= \phi_{PL}(l((a_1 + 1)b_2 \dots b_l)) + \phi_{PL}(\bullet_{b_1} \bullet (l((a_1 + 1)b_2 \dots b_l) + \bullet_{b_2+1} \bullet (l(a_1)l(b_3 \dots b_l)))) \\
&= m_{(a_1+1)b_2 \dots b_l} + m_{b_1(a_1 \star b_2 \dots b_l)} \\
&= m_{(a_1+1)b_2 \dots b_l} + \sum_{i=1}^{l-1} m_{b_1 \dots b_i(a_1+1) \dots b_l} + m_{b_1 \dots (b_l+1)a_1} \\
&= m_{a_1 \star b_1 \dots b_l}.
\end{aligned}$$

If $l = 1$, a similar computation, permuting the a_i 's and the b_j 's, proves the result. If $k, l > 1$, then:

$$\begin{aligned}
&\phi_{PL}(B_{a_1+1}(l(a_2 \dots a_k)l(b_1 \dots b_l)) + B_{b_1+1}(l(a_1 \dots a_k)l(b_2 \dots b_l))) \\
&= \phi_{PL}(\bullet_{a_1} \bullet (\bullet_{a_2+1} \bullet l(a_3 \dots a_k)l(b_1 \dots b_l)) + \bullet_{b_1+1} \bullet l(a_1 \dots a_k)l(b_2 \dots b_l)) \\
&\quad + \phi_{PL}(\bullet_{b_1} \bullet (\bullet_{a_1+1} \bullet l(a_2 \dots a_k)l(b_2 \dots b_l)) + \bullet_{b_2+1} \bullet l(a_1 \dots a_k)l(b_3 \dots b_l)) \\
&= m_{a_1(a_2 \dots a_k \star b_1 \dots b_l) + b_1(a_1 \dots a_k \star b_2 \dots b_l)} \\
&= m_{a_1 \dots a_k \star b_1 \dots b_l}.
\end{aligned}$$

Hence, the result holds for all $k, l \geq 1$. \square

Theorem 34 For all $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{N}^*$:

$$m_{a_1 \dots a_k} \bullet m_{b_1 \dots b_l} = \sum_{i=1}^{k-1} m_{a_1 \dots a_{i-1}(a_i-1)(a_{i+1} \dots a_k \star b_1 \dots b_l)} + m_{a_1 \dots a_k b_1 \dots b_l}.$$

Proof. By definition of $m_{a_1 b_1 \dots b_l}$, if $k = 1$, $m_{a_1} \bullet m_{b_1 \dots b_l} = m_{a_1 b_1 \dots b_l}$. So the result holds if $k = 1$. Let us assume that $k \geq 2$. In $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, we have:

$$l(a_1 \dots a_k) \bullet l(b_1 \dots b_l) = \bullet_{a_1} \bullet (l(a_2 \dots a_k) \bullet l(b_1 \dots b_l)) + \bullet_{a_1} \bullet l(a_2 \dots a_k)l(b_1 \dots b_l).$$

Applying ϕ_{PL} :

$$\begin{aligned}
m_{a_1 \dots a_k} \bullet m_{b_1 \dots b_l} &= m_{a_1(a_2 \dots a_k) \bullet (b_1 \dots b_l)} \\
&\quad + \phi_{PL}(\bullet_{a_1-1} \bullet (\bullet_{a_2+1} l(a_3 \dots a_k)l(b_1 \dots b_l)) + \bullet_{b_1+1} \bullet l(a_1 \dots a_k)l(b_2 \dots b_l)) \\
&= m_{a_1(a_2 \dots a_k) \bullet (b_1 \dots b_l)} + m_{(a_1-1)(a_2 \dots a_k \star b_1 \dots b_l)},
\end{aligned}$$

by the preceding lemma. The result follows from an easy induction. \square

Remark. In particular, $m_1 \circ m_{b_1 \dots b_l} = 0$.

Corollary 35 Let $a_1 \dots a_k, b_1 \dots b_l$ be two words with letters in \mathbb{N}^* . Then $m_{a_1 \dots a_k} \bullet m_{b_1 \dots b_l}$ is a span of m_w , where w is a word with $k + l$ letters and of weight $a_1 + \dots + a_k + b_1 + \dots + b_l$.

Hence, $\mathbb{K}\langle x_0, x_1 \rangle$ is a bigraded pre-Lie algebra, with:

$$\mathbb{K}\langle x_0, x_1 \rangle_{n,k} = Vect(m_{a_1 \dots a_k} \mid a_1 + \dots + a_k = n).$$

We put:

$$G = \sum_{k,n \geq 0} dim(\mathbb{K}\langle x_0, x_1 \rangle_{n,k}) X^n Y^k.$$

Proposition 36 $G = \frac{XY}{1-X-X^2Y} = \sum_{k=1}^{\infty} \sum_{l=2k-1}^{\infty} \binom{l-k}{k-1} X^l Y^k.$

Proof. Note that $\dim(\mathbb{K}\langle x_0, x_1 \rangle_{n,k})$ is the number of words $a_1 \dots a_k$ of length k , such that $a_1, \dots, a_{k-1} \geq 2$, and $a_1 + \dots + a_k = n$. Hence:

$$G = \sum_{k=1}^{\infty} \left(\frac{X^2Y}{1-X} \right)^{k-1} \frac{XY}{1-X} = \frac{XY}{1-X} \frac{1}{1 - \frac{X^2Y}{1-X}} = \frac{XY}{1-X-X^2Y}.$$

An easy development in formal series gives the second formula. □

4.3 An associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$

We now define an associative product on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, in such a way that ϕ_{PL} becomes a morphism of Com-pre-Lie algebras.

Proposition 37 *We define a product \sqcup on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ by:*

$$B_p(s_1 \dots s_k) \sqcup B_q(t_1 \dots t_l) = \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l).$$

Then $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ is a Com-pre-Lie algebra and ϕ_{PL} is a morphism of Com-pre-Lie algebras.

Proof. As $\binom{p+q-k-l-2}{p-k-1} = \binom{p+q-k-l-2}{q-l-1}$, \sqcup is commutative. Let $t = B_p(s_1 \dots s_k)$, $t' = B_q(\bullet t_1 \dots t_l)$ and $t'' = B_r(u_1 \dots u_m)$. Then:

$$\begin{aligned} t \sqcup (t' \sqcup t'') &= \underbrace{\binom{q+r-l-m-2}{q-l-1} \binom{p+q+r-k-l-m-3}{q+r-l-m-2}}_A B_{p+q+r-2}(s_1 \dots s_k t_1 \dots t_l u_1 \dots u_m), \\ (t \sqcup t') \sqcup t'' &= \underbrace{\binom{p+q-k-l-2}{p-k-1} \binom{p+q+r-k-l-m-3}{p+q-k-l-2}}_B B_{p+q+r-2}(s_1 \dots s_k t_1 \dots t_l u_1 \dots u_m). \end{aligned}$$

If $p \leq k$ or $q \leq l$ or $r \leq m$, then $A = B = 0$. If $p > k$ and $q > l$ and $r > m$, then:

$$A = B = \frac{(p+q+r-k-l-m-3)!}{(p-k-1)!(q-l-1)!(r-m-1)!}.$$

So \sqcup is associative.

Let $t_1 = B_p(s_1 \dots s_k)$, $t_2 = B_q(t_1 \dots t_l)$ and $t \in \mathcal{T}(\mathbb{N}^*)$. Then:

$$\begin{aligned} (t_1 \sqcup t_2) \circ T &= \binom{p+q-k-l-2}{m-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t) \\ &+ \sum_{i=1}^k \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\ &+ \sum_{j=1}^l \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l), \end{aligned}$$

$$\begin{aligned}
(t_1 \bullet t) \sqcup t_2 &= \left(\sum_{i=1}^k B_p(s_1 \dots (s_i \bullet t) \dots s_k) + B_p(s_1 \dots s_k t) \right) \sqcup t_2 \\
&= \sum_{i=1}^k \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots (s_i \bullet t) \dots s_k t_1 \dots t_l) \\
&\quad + \binom{p+q-k-l-3}{p-k-2} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t), \\
t_1 \sqcup (t_2 \bullet t) &= t_1 \sqcup \left(\sum_{j=1}^l B_q(t_1 \dots (t_j \bullet t) \dots t_l) + B_q(t_1 \dots t_l t) \right) \\
&= \sum_{j=1}^l \binom{p+q-k-l-2}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots (t_j \bullet t) \dots t_l) \\
&\quad + \binom{p+q-k-l-3}{p-k-1} B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l t).
\end{aligned}$$

As $\binom{p+q-k-l-3}{p-k-2} + \binom{p+q-k-l-3}{p-k-1} = \binom{p+q-k-l-2}{p-k-1}$, $(t_1 \sqcup t_2) \bullet t = (t_1 \bullet t) \sqcup t_2 + t_1 \sqcup (t_2 \bullet t)$. So $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$ is Com-pre-Lie.

Let $t_1 = B_p(s_1 \dots s_k)$ and $t_2 = B_q(t_1 \dots t_l)$. If $k \geq p$, then $\binom{p+q-k-l-2}{p-k-1} = 0$, so $t_1 \sqcup t_2 = 0$. By proposition 28, $\phi_{PL}(t_1) = 0$, so $\phi_{PL}(t_1 \sqcup t_2) = \phi_{PL}(t_1) \sqcup \phi_{PL}(t_2) = 0$. Similarly, if $l \geq q$, $\phi_{PL}(t_1 \sqcup t_2) = \phi_{PL}(t_1) \sqcup \phi_{PL}(t_2) = 0$. If $k < p$ and $l < q$, we put $w_i = \phi_{PL}(s_i)$ and $w'_j = \phi_{PL}(t_j)$. Then:

$$\begin{aligned}
\phi_{PL}(t_1) \sqcup \phi_{PL}(t_2) &= x_0 w_1 \sqcup \dots \sqcup x_0 w_k \sqcup x_1^{p-1-k} \sqcup x_0 w'_1 \sqcup \dots \sqcup x_0 w'_l \sqcup x_1^{q-1-l} \\
&= \binom{p+q-k-l-2}{p-k-1} x_0 w_1 \sqcup \dots \sqcup x_0 w'_l \sqcup x_1^{p+q-k-l-2} \\
&= \binom{p+q-k-l-2}{p-k-1} \phi_{PL}(B_{p+q-1}(s_1 \dots s_k t_1 \dots t_l)) \\
&= \phi_{PL}(t_1 \sqcup t_2).
\end{aligned}$$

So ϕ_{PL} is a Com-pre-Lie algebra morphism. \square

Remark. By the proof of proposition 28, we have a commutative diagram of prelie algebra morphisms:

$$\begin{array}{ccc}
\mathfrak{g}_{\mathcal{PT}(\{1,2\})} & \xrightarrow{\phi_{CPL}} & \mathbb{K}\langle x_0, x_1 \rangle \\
\psi \uparrow & \nearrow \phi_{PL} & \\
\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)} & &
\end{array}$$

Moreover, ϕ_{CPL} is a morphism of Com-pre-Lie algebra. With the commutative, associative product previously defined on $\mathfrak{g}_{\mathcal{T}(\mathbb{N}^*)}$, ϕ_{PL} is now a morphism of Com-pre-Lie algebra. However, ψ is not compatible with \sqcup . Indeed, $\psi(\mathbf{1}_2) = \psi(\bullet_2) \bullet \psi(\bullet_1) = \mathbf{1}_2$, so:

$$\psi(\mathbf{1}_2) \sqcup \psi(\mathbf{1}_2) = \mathbf{1}_2 \sqcup \mathbf{1}_2 = \mathbf{1}_2 \mathbf{1}_2.$$

Moreover, $\mathbf{1}_2 \sqcup \mathbf{1}_2 = \mathbf{1} \mathbf{V}_3^1$, so:

$$\psi(\mathbf{1}_2 \sqcup \mathbf{1}_2) = \psi(\bullet_3) \bullet \psi(\bullet_1) \psi(\bullet_1) = \frac{1}{2} \mathbf{2} \bullet \bullet \bullet_1 \bullet_1 = \frac{1}{2} \mathbf{1}_2 \mathbf{1}_2 + \frac{1}{2} \mathbf{1} \mathbf{V}_2^1.$$

5 Appendix

5.1 Enumeration of partitioned trees

Let $d \geq 1$. For all $n \geq 1$, let f_n be the number of partitioned trees decorated by $\{1, \dots, d\}$ with n vertices and let t_n be the number of partitioned trees decorated by $\{1, \dots, d\}$ with n vertices and one root. By convention, $f_0 = 1$. We put:

$$T = \sum_{n=1}^{\infty} t_n X^n, \quad F = \sum_{n=0}^{\infty} f_n X^n.$$

Let V_T be the vector space generated by the set of partitioned trees decorated by $\{1, \dots, d\}$ and V_F be the vector space generated by the set of partitioned trees decorated by $\{1, \dots, d\}$ with only one root. There is a bijection:

$$\begin{cases} S(V_T) & \longrightarrow V_F \\ t_1 \dots t_k & \longrightarrow t_1 \sqcup \dots \sqcup t_k. \end{cases}$$

Hence:

$$F = \prod_{i=1}^{\infty} \frac{1}{(1 - X^i)^{t_i}}. \quad (2)$$

There is a bijection:

$$\begin{cases} \bigoplus_{i=1}^d S(V_F) & \longrightarrow V_T \\ (F_{1,1} \dots, F_{1,k_1}, \dots, F_{d,1} \dots F_{d,k_d}) & \longrightarrow \sum_{i=1}^d \bullet_i \bullet (F_{i,1} \dots F_{i,k_i}). \end{cases}$$

This gives:

$$T = dX \prod_{i=1}^{\infty} \frac{1}{(1 - X^i)^{f_{i-1}}}. \quad (3)$$

Formulas (2) and (3) allow to compute inductively f_k and t_k for all $k \geq 1$. This gives for example:

$$\begin{cases} f_1 = d \\ f_2 = \frac{d(3d+1)}{2} \\ f_3 = \frac{d(19d^2+9d+2)}{6} \\ f_4 = \frac{d(63d^2+34d^2+13d+2)}{8} \\ f_5 = \frac{d(644d^4+400d^3+175d^2+35d+6)}{30} \end{cases}$$

Here are examples of f_n for $d = 1$ or 2 :

n	1	2	3	4	5	6	7	8	9	10
$d = 1$	1	2	5	14	42	134	444	1518	5318	18989
$d = 2$	2	7	32	167	952	5759	36340	236498	1576156	10702333

The row $d = 1$ is sequence A035052 of [15].

5.2 Study of the dendriform structure on admissible words

We here study the dendriform algebra $\mathbb{K}\langle\mathbb{N}^*\rangle$ of proposition 32. It is clearly commutative, via the bijection from $Sh_{\prec}(k, l)$ to $Sh_{\succ}(l, k)$ given by the composition (on the left) by the permutation $(l + 1 \dots l + k \ 1 \dots l)$.

Let V be a vector space. The shuffle dendriform algebra $Sh(V)$ is $T_+(V)$, with the products given by:

$$\begin{aligned} (a_1 \dots a_k) \prec (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\prec}(k, l)} \zeta \cdot a_1 \dots a_k b_1 \dots b_{k-1} b_k \\ (a_1 \dots a_k) \succ (b_1 \dots b_l) &= \sum_{\zeta \in Sh_{\succ}(k, l)} \zeta \cdot a_1 \dots a_{k-1} a_k b_1 \dots b_k. \end{aligned}$$

Moreover, this is the free commutative dendriform algebra generated by V , that is to say if A is a commutative dendriform algebra and $f : V \rightarrow A$ is any linear map, there exists a morphism of dendriform algebras $\phi : Sh(V) \rightarrow A$ such that $\phi|_V = f$. As $a_1 \dots a_k \succ b = a_1 \dots a_k b$ in $Sh(V)$ for all $a_1, \dots, a_k, b \in V$, this morphism ϕ is defined by:

$$\phi(a_1 \dots a_k) = (\dots (a_1 \succ a_2) \succ a_3) \dots \succ a_k.$$

Proposition 38 1. *Let V be the space generated by the words $1^k i$, $k \in \mathbb{N}$, $i \geq 1$. Then $K\langle\mathbb{N}^*\rangle$ is isomorphic, as a dendriform algebra, to $Sh(V)$.*

2. *Let A be the subspace of $K\langle\mathbb{N}^*\rangle$ generated by admissible words. Then it is a dendriform subalgebra of $K\langle\mathbb{N}^*\rangle$. Moreover, if W is the space generated by the letters i , $i \geq 1$, then A is isomorphic, as a dendriform algebra, to $Sh(W)$.*

Proof. Let $w = a_1 \dots a_k$ be a word with letters in \mathbb{N}^* . We denote by $o(w)$ the sequence of indices $j \in \{1, \dots, k-1\}$ such that $a_j \neq 1$. This sequences are totally ordered in this way: $(j_1, \dots, j_k) < (j'_1, \dots, j'_l)$ if there exists a p such that $j_k = j'_l$, $j_{k-1} = j'_{l-1}$, \dots , $j_{k-p+1} = j'_{l-p+1}$, $j_{k-p} < j'_{l-p}$, with the convention $j_0 = j_{-1} = \dots = j'_0 = j'_{-1} = \dots = 0$.

Let $\phi : Sh(V) \rightarrow K\langle\mathbb{N}^*\rangle$ be the unique morphism of dendriform algebras which extends the identity of V . Then:

$$\begin{aligned} \phi((1^{k_1-1} a_1) \dots (1^{k_n-1} a_n)) &= 1^{k_1-1} (a_1 + 1) \dots 1^{k_n-1} (a_{n-1} + 1) 1^{k_n-1} a_n \\ &\quad + \text{words } w' \text{ such that } o(w') > (k_1, \dots, k_{n-1}). \end{aligned}$$

By thriangularity, ϕ is an isomorphism. Moreover, for all $a_1, \dots, a_n \geq 1$:

$$\phi(a_1 \dots a_n) = (a_1 + 1) \dots (a_{n-1} + 1) a_n.$$

Consequently, $\phi(Sh(W)) = A$, so A is a dendriform subalgebra of $K\langle\mathbb{N}^*\rangle$ and is isomorphic to $Sh(W)$. \square

5.3 Freeness of the pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$

Notations. Let $k \geq 1$, $d_1, \dots, d_k \in \mathcal{D}$ and let F_1, \dots, F_k be decorated partitioned forests. We put:

$$B_{d_1, \dots, d_k}(F_1, \dots, F_k) = (\bullet_{d_1} \bullet F_1) \sqcup \dots \sqcup (\bullet_{d_k} \bullet F_k).$$

Note that any partitioned tree can be written under the form $B_{d_1, \dots, d_k}(F_1, \dots, F_k)$. This writing is unique up to a permutation of the d_i 's and the F_i 's.

Proposition 39 We define a coproduct δ on $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ in the following way: for any decorated partitioned tree $t = B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{k,1} \dots t_{k,n_k})$,

$$\delta(t) = \frac{1}{k} \sum_{i=1}^k \sum_{j=1}^{n_i} B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j-1} t_{i,j+1} \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k}) \otimes t_{i,j}.$$

1. For all $x \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $(\delta \otimes Id) \circ \delta(x) = (23)(\delta \otimes Id) \circ \delta(x)$.
2. For all $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$, $\delta(x \bullet y) = x \otimes y + \delta(x) \bullet y$.

Proof. 1. Let $t = B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{k,1} \dots t_{k,n_k})$. For all i, j , we put:

$$t/t_{i,j} = B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j-1} t_{i,j+1} \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k}).$$

Then:

$$\delta(t) = \frac{1}{k} \sum_{i,j} t/t_{i,j} \otimes t_{i,j}.$$

Hence:

$$(\delta \otimes Id) \circ \delta(t) = \sum_{(i,j) \neq (i',j')} (t/t_{i,j})/t_{i',j'} \otimes t_{i',j'} \otimes t_{i,j}$$

As $(t/t_{i,j})/t_{i',j'}$ and $(t/t_{i',j'})/t_{i,j}$ are both the partitioned tree obtained by cutting $t_{i,j}$ and $t_{i',j'}$ in t , they are equal, so $(\delta \otimes Id) \circ \delta(t)$ is invariant under the action of (23).

2. Let t' be a decorated partitioned tree.

$$\begin{aligned} \delta(t \bullet t') &= \sum_{i=1}^k \delta(B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,n_i} t', \dots, t_{k,1} \dots t_{k,n_k})) \\ &\quad + \sum_{i,j} \delta(B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j} \bullet t' \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k})) \\ &= \frac{1}{k} k t \otimes t' + \frac{1}{k} \sum_i \sum_{i',j'} B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,n_i} t', \dots, t_{k,1} \dots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\ &\quad + \frac{1}{k} \sum_{(i,j) \neq (i',j')} B_{d_1, \dots, d_k}(t_{1,1} \dots t_{1,n_1}, \dots, t_{i,1} \dots t_{i,j} \bullet t' \dots t_{i,n_i}, \dots, t_{k,1} \dots t_{k,n_k})/t_{i',j'} \otimes t_{i',j'} \\ &\quad + \frac{1}{k} \sum_{i,j} t/t_{i,k} \otimes t_{i,j} \bullet t' \\ &= t \otimes t' + \sum t^{(1)} \otimes t^{(2)} \bullet t' + \sum t^{(1)} \otimes t^{(2)} \bullet t'. \end{aligned}$$

So $\delta(t \bullet t') = t \otimes t' + \delta(t) \bullet t'$. □

By Muriel Livernet's pre-Lie rigidity theorem [7]:

Corollary 40 The pre-Lie algebra $\mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$ is freely generated by $\text{Ker}(\delta)$.

Remarks.

1. It is not difficult to prove that for any $x, y \in \mathfrak{g}_{\mathcal{PT}(\mathcal{D})}$:

$$\delta(x \sqcup y) = \sum x^{(1)} \otimes x^{(2)} \sqcup y + \sum y^{(1)} \otimes x \sqcup y^{(2)}.$$

Hence, $\text{Ker}(\delta)$ is an algebra for the product \sqcup .

2. Here are elements of $Ker(\delta)$ in the non decorated case. Let t_1, t_2, t_3, t_4 be partitioned trees.

$$\begin{aligned}
X &= B(t_1 t_2, 1) - B(t_1, t_2), \\
Y &= B(t_1 t_2 t_3, 1, 1) - B(t_1 t_2, t_3, 1) - B(t_1 t_3, t_2, 1) - B(t_2 t_3, t_1, 1) + 2B(t_1, t_2, t_3), \\
Z &= B(t_1 t_2 t_3 t_4, 1) - B(t_1 t_2 t_3, t_4) - B(t_1 t_2 t_4, t_3) - B(t_1 t_3 t_4, t_2) - B(t_2 t_3 t_4, t_1) \\
&\quad + B(t_1 t_2, t_3 t_4) + B(t_1 t_3, t_2 t_4) + b(t_1 t_4, t_2 t_3), \\
T &= B(t_1 t_2, t_3 t_4, 1, 1) + B(t_1 t_3, t_2 t_4, 1, 1) + B(t_1 t_4, t_2 t_3, 1, 1) - B(t_1 t_2, t_3, t_4, 1) \\
&\quad - B(t_1 t_3, t_2, t_4, 1) - B(t_1 t_4, t_2, t_3, 1) - B(t_2 t_3, t_1, t_4, 1) - B(t_2 t_4, t_1, t_3, 1) \\
&\quad - B(t_3 t_4, t_1, t_2, 1) + 3B(t_1, t_2, t_3, t_4).
\end{aligned}$$

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