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# About the transcendentality of some numbers

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## Abstract

Everyone knows that  $\pi$ ,  $e$  or  $C_n$ , the Champernowne number are transcendental, but what about  $\pi + e$ ,  $\pi + C_n$  or  $e + C_n$ ? In this paper, we show a method in the goal to know if they are.

## The approach

A number is transcendental if it is not the root of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$$

where  $a_i$  are rational not all equal to zero, else it is algebraic. We know that  $\pi$ ,  $e$  and  $C_n$ , the Champernowne number are transcendental, but we do not know anything about  $e + \pi$ ,  $\pi + C_n$  or  $e + C_n$ . Effectively, if  $A$  is transcendental :  $B$  transcendental and we do not know the nature of  $A + B$  or  $A - B$ . But if  $B$  is algebraic, then  $A + B$  and  $A - B$  are transcendental. And if  $A$  and  $B$  are algebraics, their sum and their difference are algebraics. Let us try to solve this problem. Let

$$C_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

And let

$$C'_1 = \begin{pmatrix} 0 & 5 & 10 & 15 & 20 \\ 10 & 15 & 20 & 0 & 5 \\ 20 & 0 & 5 & 10 & 15 \\ 5 & 10 & 15 & 20 & 0 \\ 15 & 20 & 0 & 5 & 10 \end{pmatrix}$$

Thus  $C_1 + C'_1$  is a magic square. It contains all the numbers from 1 to 25.

Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

And

$$C_2 = \begin{pmatrix} C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A \\ C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A \\ C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 \\ C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A \\ C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A \end{pmatrix}$$

And

$$C'_2 = \begin{pmatrix} C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \end{pmatrix}$$

The recurrence is

$$C_{i+1} = \begin{pmatrix} C_i & C_i + 5^i A & C_i + 2.5^i A & C_i + 3.5^i A & C_i + 4.5^i A \\ C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A \\ C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 \\ C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A \\ C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A \end{pmatrix}$$

And

$$C'_{i+1} = \begin{pmatrix} C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \end{pmatrix}$$

And  $C_i + C'_i$  is a magic square containing all the numbers from 1 to

$$\begin{aligned} 5 + 20(5 + 5^2 + \dots + 5^{i-1}) + 20 &= 5 + 100(1 + 5 + 5^2 + \dots + 5^{i-2}) + 20 \\ &= 5 + 100\left(\frac{5^{i-1} - 1}{4}\right) + 20 = 5 + 25(5^{i-1} - 1) + 20 = 5^{i+1} \end{aligned}$$

The sum of  $C_{i+1}$  is  $S_{i+1} = 5S_i + 10.5^i + 5S'_i$  ( $S_1 = 65$ ,  $S'_i = 50.5^{i-1}$ ). Hence

$$S_{i+1} = 5S_i + 60.5^i$$

$$5S_i = 5^2 S_{i-1} + 60.5^i$$

$$\begin{aligned}
& \dots \\
& 5^{i-1}S_2 = 5^i S_1 + 60.5^i \\
& 5^i S_1 = 5^i(65) = 13.5^{i+1} \\
& S_{i+1} = (60i + 65)5^i = (12i + 13)5^{i+1}
\end{aligned}$$

If  $M$  is the greatest number of the square,  $S$  the sum, the value of the integer  $\pi.10^{p_i}$  that approaches  $\pi$  in the square (example between 1 and 25,  $\pi.10^{p_i} = 3$  and between 1 and 125, it is 31). The number  $\pi(p_i)$  depends of  $M$  and  $S$ . This square contains  $p_i$  first digit of  $\pi$  without the dot, we note it :  $\pi.10^{p_i}$ . It contains also  $e.10^{p_i}$  and  $C_n.10^{p_i}$  (and also  $\pi + e$ ,  $\pi e$ ,  $\pi + C_n$ ,  $\pi C_n \dots$ ). For  $i + 1$ , the greatest number in the square is  $M = 5^{i+2}$  which has  $E\left(\frac{(i+2)\log 5}{\log 10}\right) = i + 1$  ciphers and begins by 1, 2, 3 and then  $p_i = i$ , or by a number geater and then  $p_i = i + 1$ . The sum is  $S_{i+1} = (12i + 13)5^{i+1}$  which has less than  $p_i + 2$  ciphers. The sum  $S_i.10^{-p_i}$  tends, in the infinity, to a number  $S$  which has less than three ciphers! But  $S_i$  is rational, thus  $S$  is rational and algebraic.

### Main results

$$\left\{ \begin{array}{l}
S = 20 \\
A = S - \pi \\
B = S - e \\
C = S - C_n \\
A_1 = S - \pi - e \\
A_2 = S - \pi + e \\
A_3 = S - \pi e \\
A_4 = S - \pi^2
\end{array} \right.$$

We know that

$$S = \pi + A = e + B = \pi + e + A_1 = \pi - e + A_2 = \pi e + A_3$$

**Definition** A real number is composed if it is equal to  $\pm p_1^{n_1} \dots p_i^{n_i}$  where  $p_j$  are integer prime numbers and  $n_j$  are rationals. We define other real prime numbers which can not be expressed like this :  $\pi$ ,  $e$ ,  $\ln(2)$ . Thus  $\sqrt[q]{p} = p^{\frac{1}{q}}$  is composed.

Also  $\sqrt[p]{p} + 1$  is prime, with  $p$  prime, hence  $\sqrt[p]{p} - 1 = (p - 1)(\sqrt[p]{p} + 1)^{-1}$  is composed!

And  $\pi$  and  $e$  are primes in stead of  $\pi^{n_0}$  and  $e^{m_0}$  with  $(n_0 - 1)(m_0 - 1) \neq 0$ . We define the GCD of two numbers : If  $p_1$  and  $p_2$  are prime real numbers

$$p_1 \neq p_2 \Rightarrow GCD(p_1, p_2) = 1$$

$$n_1 n_2 < 0 \Rightarrow GCD(p_1^{n_1}, p_1^{n_2}) = 1$$

$$n_1 n_2 > 0 \Rightarrow GCD(p_1^{n_1}, p_1^{n_2}) = p_1^{\max(n_1, n_2) - \min(n_1, n_2)}$$

$$GCD(p_1^{n_1} p_2^{n_2} \dots p_i^{n_i}, p_1^{m_1} p_2^{m_2} \dots p_j^{m_j}) = \prod_{i \neq j} GCD(p_i^{n_i}, p_j^{m_j})$$

And if  $x = p_1^{n_1} p_2^{n_2} \dots p_i^{n_i}$  and  $y = p_{m_1}^{m_1} \dots p_{m_i}^{m_i}$  then  $y$  divides  $x$  if  $1 < m_l < i$  for  $1 \leq l \leq i$  and for  $m_i = j$ ,  $|n_{m_i}| < |j|$ .

**Theorem** If  $T'_1$  and  $T'_2$  are transcendental prime real numbers,  $T'_1 T'_2$  and  $T'_1 + T'_2$  are transcendentals.

**Proof of the theorem** Let  $T_1$  and  $T_2$  ( $S$  algebraic) two real composed transcendental numbers. We have

$$nT_i^2 - mT_i + (-nT_i + m)T_i = 0$$

$n$  is algebraic,  $i \in \{1, 2\}$ , as  $T_i$  is transcendental, it can not be solution of a polynomial equation, thus  $(-nT_i + m)T_i$  is transcendental  $\forall m, n$  algebraic. We have 4 possibilities

- 1)  
 $T_1 T_2^n$  and  $T_1 + mT_2^m$  are algebraics
- 2)  
 $T_1 T_2^n$  is algebraic and  $T_1 + mT_2^m$  is transcendental
- 3)  
 $T_1 T_2^n$  is transcendental and  $T_1 + mT_2^m$  is algebraic
- 4)  
 $T_1 T_2^n$  and  $T_1 + mT_2^m$  are transcendentals,  $\forall m, n$

Thus

- 1)  
 $m = n$  implies that  $nT_1 T_2^n$  and  $T_1 + nT_2^n$  are algebraics, but

$$T_1^2 - (T_1 + nT_2^n)T_2 + nT_1 T_2^n = 0$$

It is impossible : it means that they can not be algebraics simultaneously !  
 $m \neq n$  then  $T_1 T_2^n = A_A$  and  $T_1 + mT_2^m = A'_A$  are algebraics, then

$$T_1 T_2^n + mT_2^{m+n} = A_A + mT_2^{n+m} = A'_A T_2^n$$

And  $T_2$  is solution of this algebraic equation : it is impossible !

- 2)  
If  $T_1 T_2^n$  and  $T_1 T_2^{n'}$  are algebraics, then

$T_1 T_2^n (T_1 T_2^{n'})^{-1} = T_2^{n-n'}$  is algebraic and it is impossible with  $n \neq n'$

There is only one  $n = n_0$  for which  $T_1 T_2^n$  is algebraic, all the others are transcendentals !

- 3)  
If  $T_1 + nT_2^n$  and  $T_1 + n'T_2^{n'}$  are algebraics, then  $(T_1 + nT_2^n) - (T_1 + n'T_2^{n'}) =$

$nT_2^n - n'T_2^{n'} = A_A$  is algebraic,  $T_2$  can not be solution of this algebraic equation, it means that it is impossible and  $n = n'$ .

There is only one  $n = m_0$  for which  $T_1 + nT_2^n$  is algebraic, all the others are transcendental!

Thus there are two possibilities :

I)

$T_1T_2^n$  is transcendental  $\forall n \neq n_0$  and  $T_1 + mT_2^m$  is transcendental  $\forall m$ , in this case  $T_1^2T_2^{m_0}$  is transcendental, we chose  $T_1 = T_1'^{\frac{1}{2}}$  and  $T_2 = T_2'^{\frac{1}{m_0}}$  and  $m = 1$ , thus  $T_1'T_2'$  and  $t_1' + T_2'$  are transcendentals!

II)

$T_1T_2^n$  is transcendental  $\forall n$  and  $T_1 + mT_2^m, \forall m \neq m_0$ , in this case  $2T_1 + m_0T_2^{m_0}$  is transcendental, we chose  $n = 1$  and  $T_1 = \frac{T_1'}{2}$  and  $T_2 = \frac{T_2'}{m_0}$ . And  $T_1' + T_2'$  is transcendental!

**Application of the theorem**  $T_1' = \pi$  and  $T_2' = e$  are transcendental prime real numbers means  $e\pi, e + \pi$  and  $e - \pi$ , are transcendentals for example!  $\pi^m e^n$  and  $\pi^m + e^n$  are transcendental for all  $m, n$ .

Also, by the same way, we prove that  $\pi^n, e^m, C_n^r, \pi^n e^m, C_n^r \pi^n$  and  $C_n^r e^m$  are transcendentals for all  $n, m, r$ !

## Conclusion

We did not presented like this, but we have given a method to find the nature of several numbers, we have shown the nature of  $\pi^n e^m, \pi^n C_n^p, e^m C_n^p, \pi^n, e^m$  and  $C_n^p \dots$

## Références

- [1] Alan Baker, Transcendental number theory *Cambridge University Press*, (1975).