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About the transcendentality of some numbers

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Abstract

Everyone knows that π , e or C_n , the Champernowne number are transcendentals, but what about $\pi + e$, $\pi + C_n$ or $e + C_n$? In this paper, we show a method in the goal to know if they are.

The approach

A number is transcendental if it is not the root of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$$

where a_i are not all equal to zero, else it is algebraic. We know that π , e and C_n , the Champernowne number are transcendentals, but we do not know anything about $e + \pi$, $\pi + C_n$ or $e + C_n$. Effectively, if A is transcendental : B transcendental and we do not know the nature of $A + B$ or $A - B$. But if B is algebraic, then $A + B$ and $A - B$ are transcendentals. And if A and B are algebraics, their sum and their difference are algebraics. Let us try to solve this problem. Let

$$C_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

And let

$$C'_1 = \begin{pmatrix} 0 & 5 & 10 & 15 & 20 \\ 10 & 15 & 20 & 0 & 5 \\ 20 & 0 & 5 & 10 & 15 \\ 5 & 10 & 15 & 20 & 0 \\ 15 & 20 & 0 & 5 & 10 \end{pmatrix}$$

Thus $C_1 + C'_1$ is a magic square. It contains all the numbers from 1 to 25.

Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

And

$$C_2 = \begin{pmatrix} C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A \\ C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A \\ C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 \\ C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A \\ C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A \end{pmatrix}$$

And

$$C'_2 = \begin{pmatrix} C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \end{pmatrix}$$

The recurrence is

$$C_{i+1} = \begin{pmatrix} C_i & C_i + 5^i A & C_i + 2.5^i A & C_i + 3.5^i A & C_i + 4.5^i A \\ C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A \\ C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 \\ C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A \\ C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A \end{pmatrix}$$

And

$$C'_{i+1} = \begin{pmatrix} C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \end{pmatrix}$$

And $C_i + C'_i$ is a magic square containing all the numbers from 1 to

$$\begin{aligned} 5 + 20(5 + 5^2 + \dots + 5^{i-1}) + 20 &= 5 + 100(1 + 5 + 5^2 + \dots + 5^{i-2}) + 20 \\ &= 5 + 100\left(\frac{5^{i-1} - 1}{4}\right) + 20 = 5 + 25(5^{i-1} - 1) + 20 = 5^{i+1} \end{aligned}$$

The sum of C_{i+1} is $S_{i+1} = 5S_i + 10.5^i + 5S'_i$ ($S_1 = 65$, $S'_i = 50.5^{i-1}$). Hence

$$S_{i+1} = 5S_i + 60.5^i$$

$$5S_i = 5^2 S_{i-1} + 60.5^i$$

$$\begin{aligned}
& \dots \\
& 5^{i-1}S_2 = 5^i S_1 + 60.5^i \\
& 5^i S_1 = 5^i(65) = 13.5^{i+1} \\
& S_{i+1} = (60i + 65)5^i = (12i + 13)5^{i+1}
\end{aligned}$$

If M is the greatest number of the square, S the sum, the value of the integer $\pi.10^{p_i}$ that approaches π in the square (example between 1 and 25, $\pi.10^{p_i} = 3$ and between 1 and 125, it is 31). The number $\pi(p_i)$ depends of M and S . This square contains p_i first digit of π without the dot, we note it : $\pi.10^{p_i}$. It contains also $e.10^{p_i}$ and $C_n.10^{p_i}$ (and also $\pi + e$, πe , $\pi + C_n$, $\pi C_n \dots$). For $i + 1$, the greatest number in the square is $M = 5^{i+2}$ which has $E(\frac{(i+2)\log 5}{\log 10}) = i + 1$ ciphers and begins by 1, 2, 3 and then $p_i = i$, or by a number geater and then $p_i = i + 1$. The sum is $S_{i+1} = (12i + 13)5^{i+1}$ which has less than $p_i + 2$ ciphers. The sum $S_i.10^{-p_i}$ tends, in the infinity, to a number S which has less than three ciphers! But S_i is rational, thus S is rational and algebraic. By another way, we pose

$$\left\{ \begin{array}{l} S = 20 \\ A = S - \pi \\ B = S - e \\ C = S - C_n \\ A_1 = S - \pi - e \\ A_2 = S - \pi + e \\ A_3 = S - \pi e \\ A_4 = S - \pi^2 \end{array} \right.$$

We know that

$$S = \pi + A = e + B = \pi + e + A_1 = \pi - e + A_2 = \pi e + A_3 = \pi^2 + A_4$$

Definition : a real composed number is equal to $\pm p_1^{n_1} \dots p_i^{n_i}$ where p_j are prime numbers ans n_j are rationals. We define new real prime numbers like π , e , $\ln(2), \dots$. Thus $\sqrt[q]{p} = p^{\frac{1}{q}}$ is composed and not p of course. Let $T_1 = S - A$ and $T_2 = S - B$ (S algebraic) two real prime numbers which are transcendentals.

$$T_i^2 - nT_i - (T_i - n)T_i = 0$$

As T_i is transcendental, it can not be solution of a polynomial equation, thus $(T_i - n)T_i$ is transcendental $\forall n$ algebraic. Also if $T_1 T_2$ is algebraic, as T_2 is transcendental, $T_2(T_1 + n)$ is transcendental, $\forall n$ algebraic, thus $T_1 T_2$ or (and) $T_2(T_1 \pm n)$ is transcendental. Also $T_1 T_2$ or (and) $T_1(T_2 \pm n)$ is transcendental, $\forall n$ algebraic. Let us prove now that $T_1 T_2$ is transcendental. Let us suppose that $\exists n_0, m_0$ integers for which $T_2 T_2^{m_0}$ and $T_1 T_2^{m_0}$ are algebraics for $m_0, n_0 \neq 0$ and $T_2 T_1^n$ and $T_1 T_2^m$ are transcendentals for $n \neq n_0, m \neq m_0$, thus

$$T_2^{m_0+n_0} T_1^{n_0+1} = (T_2 T_1^{\frac{n_0+1}{m_0+n_0}})^{m_0+n_0}$$

is algebraic it means that

$$T_2 T_1^{n_0} T_1^{\frac{n_0^2+1}{n_0+m_0}-n_0} = T_2 T_1^{n_0} T_1^{\frac{1-n_0 m_0}{n_0+m_0}}$$

is algebraic and $T_1^{(1-n_0 m_0)}$ is algebraic then $n_0 = m_0 = \pm 1$ and $T_1 T_2$ and $\frac{T_2}{T_1}$ are algebraics! But

$$n T_1^2 - \left(\frac{T_2}{T_1} + n\right) T_1^2 + T_2 T_1 = 0$$

And

$$n T_1^{-2} - (T_2 T_1 + n) T_1^{-2} + \frac{T_2}{T_1} = 0$$

T_1 and T_1^{-1} are transcendentals, they can not be solutions of polynomial equations, thus one at least of the numbers $\frac{T_2}{T_1}, T_2 T_1, \frac{T_1}{T_2}$ is transcendental! The initial hypothesis is false: m_0, n_0 do not exist, $e\pi^n$ or πe^m is transcendental for all n, m ! Then $m = 1$ or $n = 1$ implies that $T_2 T_1$ is transcendental! Let us prove now that $T_2 + T_1$ is transcendental, we suppose it algebraic: $T_2 + T_1 = A_A$, it means that $A + B = 2S - T_2 - T_1 = A'_A$ is algebraic, but

$$T_2 T_1 = (S - A)(S - B) = S^2 - (A + B)S + AB$$

But

$$\begin{aligned} T_1(T_2 + T_1) &= A_A T_1 \\ \Rightarrow T_2 T_1 &= (T_2 + T_1)T_1 - T_1^2 = A_A T_1 - T_1^2 = S^2 - (A + B)S + AB \\ T_1^2 - A_A T_1 + S^2 - (A + B)S + AB &= 0 \\ &= (S - A)^2 - A_A(S - A) + S^2 - (A + B)S + AB = 0 \\ &= S^2 - 2AS + A^2 - A_A S + A_A A + S^2 - (A + B)S + AB = 0 \end{aligned}$$

And

$$A^2 + (B + A_A - 2S)A + 2S^2 - (A + B)S - A_A S = 0 = A^2 + (B + A_A - 2S)A + A''_A$$

Or

$$A = \frac{-B - A_A + 2S \pm \sqrt{(B + A_A - 2S)^2 - 4A''_A}}{2}$$

Or

$$(2A + B + A_A - 2S)^2 = A^2 = B^2 + 2(A_A - 2S)B + (A_A - 2S)^2 - 4A''_A$$

And

$$\begin{aligned} A^2 - B^2 &= A'_A(A - B) = 2(A_A - 2S)B + (A_A - 2S)^2 - 4A''_A \\ &= A'_A(-2A + A_A) = 2(A_A - 2S)(A_A - A) + (A_A - 2S)^2 - 4A''_A \end{aligned}$$

And

$$(-2A'_A + 2A_A - 4S)A = (-8S + 4AA)A = 2A_A(A_A - 2S) - A'_A + (A_A - 2S)^2 - 4A''_A \Rightarrow A = S - T_1 = A'''_A$$

It means that T_1 is algebraic and it is impossible, thus $T_2 + T_1$ is transcendental! If $T_1 = \pi$ and $T_2 = e$, we have $e + \pi$ transcendental! Also, by the same way, we prove that π^n , e^m , C_n^r , $\pi^n e^m$, $C_n^r \pi^n$ and $C_n^r e^m$ are transcendentals for all n, m, p integers! Hence $\pi + A$ and $C_n + C$ have the same nature : they are algebraics, with A and C transcendentals (because $\pi = S - A$ and $C_n = S - C$ are!), also $\pi + C$ and $C_n + A$ have the same nature, thus $2\pi + A + C = 4S - 2C_n - (A + C) = \pi + 2S - C_n$ has the same nature than $2C_n + A + C$. If $A + C$ is algebraic, then $\pi + C_n$ will be algebraic and $\pi - C_n = 2S - (A + C) - 2C_n$ will be transcendental! If $A + C$ is transcendental, then $\pi + C_n$ is! By the same reasoning, if $A + B$ is transcendental, $\pi + e = 2S - (A + B)$ will be transcendental and if $A + B$ is algebraic, then $\pi + e$ is algebraic and $\pi - e = 2S - (A + B) - 2e$ is transcendental. The nature of $e + C_n$ depends of $B + C$. Let $\pi + e + C_n + A + B + C = 3S$, it is algebraic : $\pi + e + A + B$ has the same nature than $C_n + C$ (they are algebraics) and $\pi + e + C$ has the same nature than $C_n + A + B$ hence, by the same reasoning, if $A + B + C$ is transcendental then $\pi + e + C_n$ is transcendental, if $A + B + C$ is algebraic, then $\pi + e$, $\pi + C_n$ and $e + C_n$ are transcendentals.

Conclusion

We did not presented like this, but we have given a method to find the nature of several numbers, we have shown the nature of $\pi^n e^m$, $\pi^n C_n^p$, $e^m C_n^p$, π^n , e^m and C_n^p ...

Références

- [1] Alan Baker, Transcendental number theory *Cambridge University Press*, (1975).