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About the transcendentality of some numbers

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Abstract

Everyone knows that π , e or C_n , the Champernowne number are transcendental, but what about $\pi + e$, $\pi + C_n$ or $e + C_n$? In this paper, we show a method in the goal to know if they are.

The approach

A number is transcendental if it is not the root of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$$

where a_i are rational not all equal to zero, else it is algebraic. We know that π , e and C_n , the Champernowne number are transcendental, but we do not know anything about $e + \pi$, $\pi + C_n$ or $e + C_n$. Effectively, if A is transcendental : B transcendental and we do not know the nature of $A + B$ or $A - B$. But if B is algebraic, then $A + B$ and $A - B$ are transcendental. And if A and B are algebraics, their sum and their difference are algebraics. Let us try to solve this problem. Let

$$C_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

And let

$$C'_1 = \begin{pmatrix} 0 & 5 & 10 & 15 & 20 \\ 10 & 15 & 20 & 0 & 5 \\ 20 & 0 & 5 & 10 & 15 \\ 5 & 10 & 15 & 20 & 0 \\ 15 & 20 & 0 & 5 & 10 \end{pmatrix}$$

Thus $C_1 + C'_1$ is a magic square. It contains all the numbers from 1 to 25.

Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

And

$$C_2 = \begin{pmatrix} C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A \\ C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A \\ C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 \\ C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A \\ C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A \end{pmatrix}$$

And

$$C'_2 = \begin{pmatrix} C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \end{pmatrix}$$

The recurrence is

$$C_{i+1} = \begin{pmatrix} C_i & C_i + 5^i A & C_i + 2.5^i A & C_i + 3.5^i A & C_i + 4.5^i A \\ C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A \\ C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 \\ C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A \\ C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A \end{pmatrix}$$

And

$$C'_{i+1} = \begin{pmatrix} C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \end{pmatrix}$$

And $C_i + C'_i$ is a magic square containing all the numbers from 1 to

$$\begin{aligned} 5 + 20(5 + 5^2 + \dots + 5^{i-1}) + 20 &= 5 + 100(1 + 5 + 5^2 + \dots + 5^{i-2}) + 20 \\ &= 5 + 100\left(\frac{5^{i-1} - 1}{4}\right) + 20 = 5 + 25(5^{i-1} - 1) + 20 = 5^{i+1} \end{aligned}$$

The sum of C_{i+1} is $S_{i+1} = 5S_i + 10.5^i + 5S'_i$ ($S_1 = 65$, $S'_i = 50.5^{i-1}$). Hence

$$S_{i+1} = 5S_i + 60.5^i$$

$$5S_i = 5^2 S_{i-1} + 60.5^i$$

$$\begin{aligned}
& \dots \\
& 5^{i-1}S_2 = 5^i S_1 + 60.5^i \\
& 5^i S_1 = 5^i(65) = 13.5^{i+1} \\
& S_{i+1} = (60i + 65)5^i = (12i + 13)5^{i+1}
\end{aligned}$$

If M is the greatest number of the square, S the sum, the value of the integer $\pi.10^{p_i}$ that approaches π in the square (example between 1 and 25, $\pi.10^{p_i} = 3$ and between 1 and 125, it is 31). The number $\pi(p_i)$ depends of M and S . This square contains p_i first digit of π without the dot, we note it : $\pi.10^{p_i}$. It contains also $e.10^{p_i}$ and $C_n.10^{p_i}$ (and also $\pi + e$, πe , $\pi + C_n$, $\pi C_n \dots$). For $i + 1$, the greatest number in the square is $M = 5^{i+2}$ which has $E\left(\frac{(i+2)\log 5}{\log 10}\right) = i + 1$ ciphers and begins by 1, 2, 3 and then $p_i = i$, or by a number geater and then $p_i = i + 1$. The sum is $S_{i+1} = (12i + 13)5^{i+1}$ which has less than $p_i + 2$ ciphers. The sum $S_i.10^{-p_i}$ tends, in the infinity, to a number S which has less than three ciphers! But S_i is rational, thus S is rational and algebraic.

Main results

$$\left\{ \begin{array}{l} S = 20 \\ A = S - \pi \\ B = S - e \\ C = S - C_n \\ A_1 = S - \pi - e \\ A_2 = S - \pi + e \\ A_3 = S - \pi e \\ A_4 = S - \pi^2 \end{array} \right.$$

We know that

$$S = \pi + A = e + B = \pi + e + A_1 = \pi - e + A_2 = \pi e + A_3$$

Definition A real number is composed if it is equal to $\pm p_1^{n_1} \dots p_i^{n_i}$ where p_j are integer prime numbers and n_j are rationals. We define other real prime numbers which can not be expressed like this : π , e , $\ln(2)$. Thus $\sqrt[q]{p} = p^{\frac{1}{q}}$ is composed.

Also $\sqrt[p]{p} + 1$ is prime, with p prime, hence $\sqrt[p]{p} - 1 = (p - 1)(\sqrt[p]{p} + 1)^{-1}$ is composed!

And π and e are primes in stead of π^{n_0} and e^{m_0} with $(n_0 - 1)(m_0 - 1) \neq 0$. We define the GCD of two numbers : If p_1 and p_2 are prime real numbers

$$p_1 \neq p_2 \Rightarrow GCD(p_1, p_2) = 1$$

$$n_1 n_2 < 0 \Rightarrow GCD(p_1^{n_1}, p_1^{n_2}) = 1$$

$$n_1 n_2 > 0 \Rightarrow GCD(p_1^{n_1}, p_1^{n_2}) = p_1^{\max(n_1, n_2) - \min(n_1, n_2)}$$

$$GCD(p_1^{n_1} p_2^{n_2} \dots p_i^{n_i}, p_1^{m_1} p_2^{m_2} \dots p_j^{m_j}) = \prod_{i \neq j} GCD(p_i^{n_i}, p_j^{m_j})$$

And if $x = p_1^{n_1} p_2^{n_2} \dots p_i^{n_i}$ and $y = p_1^{m_1} \dots p_{m_i}^{m_i}$ then y divides x if $1 < m_l < i$ for $1 \leq l \leq i$ and for $m_i = j, |n_{m_i}| < |j|$.

Theorem If T_1' and T_2' are transcendental prime real numbers, $T_1' T_2'$ and $T_1' + T_2'$ are transcendentals.

Proof of the theorem Let T_1 and T_2 two real composed transcendental numbers.

We have 4 possibilities

- 1)
 $T_1 T_2^n$ and $T_1 + m T_2$ are algebraics
- 2)
 $T_1 T_2^n$ is algebraic and $T_1 + m T_2$ is transcendental
- 3)
 $T_1 T_2^n$ is transcendental and $T_1 + m T_2$ is algebraic
- 4)
 $T_1 T_2^n$ and $T_1 + m T_2$ are transcendentals, $\forall m, n$

Thus

- 1)
 $T_1 T_2^n = A_A$ and $T_1 + m T_2 = A'_A$ are algebraics, then

$$T_1 T_2^n + m T_2^{n+1} = A_A + m T_2^{n+1} = A'_A T_2^n$$

And T_2 is solution of this algebraic equation : it is impossible !

- 2)
If $T_1 T_2^n$ and $T_1 T_2^{n'}$ are algebraics, then

$T_1 T_2^n (T_1 T_2^{n'})^{-1} = T_2^{n-n'}$ is algebraic and it is impossible with $n \neq n'$

There is only one $n = n_0$ for which $T_1 T_2^n$ is algebraic, all the others are transcendentals !

If we suppose that n_0 is not unique, there are three possibilities : $T_1 T_1' (T_2 T_2')^n$ is transcendental for all n or $T_1 T_1'^{-1} (T_2 T_2'^{-1})^n$ is transcendental for all n or there exists l_0 and l'_0 for which

$$\begin{cases} T_1 T_2^{n_0} = A_A \\ T_1' T_2^{m'_0} = A'_A \\ T_1 T_1' (T_2 T_2')^{l_0} = A''_A \\ T_1 T_1'^{-1} (T_2 T_2'^{-1})^{l'_0} = A'''_A \end{cases}$$

are algebraics, we have

$$\begin{cases} A_A^2 T_2^{l_0 + l'_0 - 2n_0} T_2^{l_0 - l'_0} = A''_A A'''_A \\ A_A'^2 T_2^{l_0 - l'_0} T_2^{l_0 + l'_0 - 2n'_0} = A''_A A'''_A^{-1} \end{cases}$$

$$\begin{aligned}
& \Rightarrow \begin{cases} T_2^A = A'T_2^B \\ A = l_0 + l'_0 - 2n_0 \\ B = l'_0 - l_0 \\ A' = A''_A A'''_A A_A^{-2} \\ T_2^{-B} = B'T_2^C \\ C = -(l_0 + l_0 - 2n'_0) \\ B' = A_A'^2 A_A''^{-1} A_A''' \end{cases} \\
& \Rightarrow \begin{cases} T_2^{-AB} = A'^{-B} T_2'^{-B^2} \\ = T_2^{-AB} = B'^A T_2'^{AC} \\ T_2'^{BC} = A'^{-C} T_2^{AC} \\ = T_2'^{BC} = B'^{-B} T_2^{-B^2} \\ A = l_0 + l'_0 - 2n_0 \\ B = l'_0 - l_0 \\ A' = A''_A A'''_A A_A^{-2} \\ C = -(l_0 + l_0 - 2n'_0) \\ B' = A_A'^2 A_A''^{-1} A_A''' \end{cases} \\
& \Rightarrow \begin{cases} -B^2 = AC = -(l_0 - l'_0)^2 = -(l_0 + l'_0 - 2n_0)(l_0 + l'_0 - 2n'_0) \\ B'^{-B^2} = B'^{AC} = A'^{AB} = A'^{-BC} \\ A = l_0 + l'_0 - 2n_0 \\ B = l'_0 - l_0 \\ A' = A''_A A'''_A A_A^{-2} \\ C = -(l_0 + l_0 - 2n'_0) \\ B' = A_A'^2 A_A''^{-1} A_A''' \end{cases} \\
& \Rightarrow \begin{cases} -B^2 = AC = -(l_0 - l'_0)^2 = -(l_0 + l'_0 - 2n_0)(l_0 + l'_0 - 2n'_0) \\ (A + C)B = 0 = 2(n'_0 - n_0)(l'_0 - l_0) \\ A = l_0 + l'_0 - 2n_0 \\ B = l'_0 - l_0 \\ A' = A''_A A'''_A A_A^{-2} \\ C = -(l_0 + l_0 - 2n'_0) \\ B' = A_A'^2 A_A''^{-1} A_A''' \end{cases} \\
& \Rightarrow \begin{cases} (l'_0 - l_0)^2 = (l_0 + l'_0 - 2n_0)(l_0 + l'_0 - 2n'_0) = 0 \\ l_0 - l'_0 = l_0 + l'_0 - 2n_0 = l_0 + l'_0 - 2n'_0 = 0 \\ \Rightarrow l_0 = l'_0 = n'_0 = n_0 \end{cases}
\end{aligned}$$

n_0 is unique

3)

$(T_1 + nT_2) - (T_1 + n'T_2) = (n - n')T_2 = A_A$ is algebraic, T_2 can not be solution of this algebraic equation, it means that it is impossible and $n = n'$.

There is only one $n = m_0$ for which $T_1 + nT_2$ is algebraic, all the others are transcendental!

If m_0 is not unique, there are three possibilities : $T_1 + T'_1 + m(T_2 + T'_2)$ is

transcendental for all m or $T_1 - T_1' + m(T_2 - T_2')$ is transcendental for all m or there exists l_0, l_0' for which

$$\begin{cases} T_1 + m_0 T_2 = A_A \\ T_1' + m_0' T_2' = A_A' \\ T_1 + T_1' + l_0(T_2 + T_2') = A_A'' \\ T_1 - T_1' + l_0'(T_2 - T_2') = A_A''' \end{cases}$$

algebraics, then

$$\begin{cases} 2T_1 + l_0(T_2 + T_2') + l_0'(T_2 - T_2') = A_A'' + A_A''' \\ = 2T_1 + 2m_0 T_2 + l_0(T_2 + T_2') + l_0'(T_2 - T_2') - 2m_0 T_2 \\ = 2A_A + (l_0 + l_0' - 2m_0)T_2 + (l_0 - l_0')T_2' \\ 2T_1' + l_0(T_2 + T_2') - l_0'(T_2 - T_2') = A_A'' - A_A''' \\ 2A_A' + (l_0 - l_0')T_2 + (l_0 + l_0' - 2m_0')T_2' \end{cases}$$

$$\Rightarrow \begin{cases} AT_2 + BT_2' = C \\ A = l_0 + l_0' - 2m_0 \\ B = l_0 - l_0' \\ C = A_A'' + A_A''' - 2A_A \\ BT_2 + B'T_2' = C' \\ B' = l_0 + l_0' - 2m_0' \\ C' = A_A'' - A_A''' - 2A_A' \end{cases}$$

$$\Rightarrow \begin{cases} ABT_2 + B^2T_2' = BC \\ B'AT_2 + BB'T_2' = B'C \\ ABT_2 + AB'T_2' = AC' \\ B^2T_2 + BB'T_2' = B'C' \\ A = l_0 + l_0' - 2m_0 \\ B = l_0' - l_0 \\ C = A_A'' + A_A''' - 2A_A \\ B' = l_0 + l_0' - 2m_0' \\ C' = A_A'' - A_A''' - 2A_A' \end{cases}$$

$$\Rightarrow \begin{cases} B^2 = AB' \\ ABT_2 + AB'T_2' = BC = AC' \\ BC = AC' \\ B'C' = B'C \\ A = l_0 + l_0' - 2m_0 \\ B = l_0' - l_0 \\ C = A_A'' + A_A''' - 2A_A \\ B' = l_0 + l_0' - 2m_0' \\ C' = A_A'' - A_A''' - 2A_A' \end{cases}$$

$$\Rightarrow \begin{cases} (l_0 - l'_0)^2 = (l_0 + l'_0 - 2m_0)(l_0 + l'_0 - 2m'_0) \\ (l'_0 - l_0)C = (l_0 + l'_0 - 2m_0)C' \\ (l_0 + l'_0 - 2m'_0)(C' - C) = 0 \\ A = l_0 + l'_0 - 2m_0 \\ B = l'_0 - l_0 \\ C = A''_A + A'''_A - 2A_A \\ B' = l_0 + l'_0 - 2m'_0 \\ C' = A''_A - A'''_A - 2A'_A \end{cases}$$

$$\Rightarrow l_0 = l'_0 = m_0 = m'_0$$

m_0 is unique !

Thus there are finally two possibilities :

I)

There are three sub-possibilities :

$T_1 T_1'' (T_2 T_2'')^n$ is transcendental for all n , for all T_1', T_2' transcendental prime numbers, there exists T_1, T_1'', T_2, T_2'' for which $T_1 = T_1' T_1''^{-1}$ and $T_2 = T_2' T_2''^{-1}$, in this case $T_1' T_2'$ is always transcendental and of course $T_1' + T_2'$ too.

If $T_1 T_1''^{-1} (T_2 T_2''^{-1})^n$ is transcendental for all n , for all T_1', T_2' transcendental prime numbers, there exists T_1, T_1'', T_2, T_2'' for which $T_1 = T_1' T_1''$ and $T_2 = T_2' T_2''$, in this case $T_1' T_2'$ is always transcendental and of course $T_1' + T_2'$ too.

$T_1 T_2^n$ is transcendental $\forall n \neq n_0$ and $T_1 + m T_2$ is transcendental $\forall m$, in this case $T_1^2 T_2^{n_0}$ is transcendental, and $T_1^2 T_2^{n_0}$ and $T_1 + m T_2$ are always transcendentals. For all T_1', T_2' prime real numbers exist $T_1 = T_1'^{\frac{1}{2}}$ and $T_2 = T_2'^{\frac{1}{n_0}}$ and $m = 1$, thus $T_1' T_2'$ and $T_1' + T_2'$ are transcendentals !

II)

There are three sub-possibilities :

$T_1 + T_1'' + m(T_2 + T_2'')$ is transcendental for all m , for all T_1', T_2' transcendental prime numbers, there exists T_1, T_1'', T_2, T_2'' for which $T_1 = T_1' - T_1''$ and $T_2 = T_2' - T_2''$, in this case $T_1' + T_2'$ is always transcendental and of course $T_1' T_2'$ too.

$T_1 - T_1'' + m(T_2 - T_2'')$ is transcendental for all m , for all T_1', T_2' transcendental prime numbers, there exists T_1, T_1'', T_2, T_2'' for which $T_1 = T_1' + T_1''$ and $T_2 = T_2' + T_2''$, in this case $T_1' + T_2'$ is always transcendental and of course $T_1' T_2'$ too.

$T_1 T_2^n$ is transcendental $\forall n$ and $T_1 + m T_2, \forall m \neq m_0$, in this case $2T_1 + m T_2$ is always transcendental, for all T_1', T_2' prime real numbers exist $T_1 = \frac{T_1'}{2}$ and $T_2 = \frac{T_2'}{m_0}$ and $n = 1$, thus $T_1' + T_2'$ and $T_1' T_2'$ are transcendentals !

Application of the theorem $T_1' = \pi$ and $T_2' = e$ are transcendental prime real numbers means $e\pi, e + \pi$ and $e - \pi$, are transcendentals for example ! $\pi^m e^n$ and $\pi^m + e^n$ are transcendental for all m, n .

Also, by the same way, we prove that $\pi^n, e^m, C_n^r, \pi^n e^m, C_n^r \pi^n$ and $C_n^r e^m$ are transcendentals for all n, m, r !

Conclusion

We did not presented like this, but we have given a method to find the nature of several numbers, we have shown the nature of $\pi^n e^m$, $\pi^n C_n^p$, $e^m C_n^p$, π^n , e^m and C_n^p ...

Références

- [1] Alan Baker, *Transcendental number theory Cambridge University Press*, (1975).