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About the transcendentality of some numbers

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Abstract

Everyone knows that π , e or C_n , the Champernowne number are transcendentals, but what about $\pi + e$, $\pi + C_n$ or $e + C_n$? In this paper, we show a method in the goal to know if they are.

The approach

An number is transcendental if it is not the rooth of a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$$

where a_i are not all equal to zero, else it is algebraic. We know that π , e and C_n , the Champernowne number are transcendentals, but we do not know anything about $e + \pi$, $\pi + C_n$ or $e + C_n$. Effectively, if A is transcendental : B transcendental and we do not know the nature of $A + B$ or $A - B$. But if B is algebraic, then $A + B$ and $A - B$ are transcendentals. And if A and B are algebraics, their sum and their difference are algebraics. Let us try to solve this problem. Let

$$C_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

And let

$$C'_1 = \begin{pmatrix} 0 & 5 & 10 & 15 & 20 \\ 10 & 15 & 20 & 0 & 5 \\ 20 & 0 & 5 & 10 & 15 \\ 5 & 10 & 15 & 20 & 0 \\ 15 & 20 & 0 & 5 & 10 \end{pmatrix}$$

Thus $C_1 + C'_1$ is a magic square. It contains all the numbers from 1 to 25.

Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

And

$$C_2 = \begin{pmatrix} C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A \\ C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A \\ C_1 + 25A & C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 \\ C_1 + 100A & C_1 & C_1 + 25A & C_1 + 50A & C_1 + 75A \\ C_1 + 50A & C_1 + 75A & C_1 + 100A & C_1 & C_1 + 25A \end{pmatrix}$$

And

$$C'_2 = \begin{pmatrix} C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \\ C'_1 & C'_1 & C'_1 & C'_1 & C'_1 \end{pmatrix}$$

The recurrence is

$$C_{i+1} = \begin{pmatrix} C_i & C_i + 5^i A & C_i + 2.5^i A & C_i + 3.5^i A & C_i + 4.5^i A \\ C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A \\ C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 \\ C_1 + 4.5^i A & C_1 & C_1 + 5^i A & C_1 + 2.5^i A & C_1 + 3.5^i A \\ C_1 + 2.5^i A & C_1 + 3.5^i A & C_1 + 4.5^i A & C_1 & C_1 + 5^i A \end{pmatrix}$$

And

$$C'_{i+1} = \begin{pmatrix} C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \\ C'_i & C'_i & C'_i & C'_i & C'_i \end{pmatrix}$$

And $C_i + C'_i$ is a magic square containing all the numbers from 1 to

$$\begin{aligned} 5 + 20(5 + 5^2 + \dots + 5^{i-1}) + 20 &= 5 + 100(1 + 5 + 5^2 + \dots + 5^{i-2}) + 20 \\ &= 5 + 100\left(\frac{5^{i-1} - 1}{4}\right) + 20 = 5 + 25(5^{i-1} - 1) + 20 = 5^{i+1} \end{aligned}$$

The sum of C_{i+1} is $S_{i+1} = 5S_i + 10.5^i + 5S'_i$ ($S_1 = 65$, $S'_i = 50.5^{i-1}$). Hence

$$S_{i+1} = 5S_i + 60.5^i$$

$$5S_i = 5^2 S_{i-1} + 60.5^i$$

$$\begin{aligned}
& \dots \\
& 5^{i-1}S_2 = 5^i S_1 + 60.5^i \\
& 5^i S_1 = 5^i(65) = 13.5^{i+1} \\
& S_{i+1} = (60i + 65)5^i = (12i + 13)5^{i+1}
\end{aligned}$$

If M is the greatest number of the square, S the sum, the value of the integer $\pi.10^{p_i}$ that approaches π in the square (example between 1 and 25, $\pi.10^{p_i} = 3$ and between 1 and 125, it is 31). The number $\pi(p_i)$ depends of M and S . This square contains p_i first digit of π without the dot, we note it : $\pi.10^{p_i}$. It contains also $e.10^{p_i}$ and $C_n.10^{p_i}$ (and also $\pi + e$, πe , $\pi + C_n$, $\pi C_n \dots$). For $i + 1$, the greatest number in the square is $M = 5^{i+2}$ which has $E(\frac{(i+2)\log 5}{\log 10}) = i + 1$ ciphers and begins by 1, 2, 3 and then $p_i = i$, or by a number geater and then $p_i = i + 1$. The sum is $S_{i+1} = (12i + 13)5^{i+1}$ which has less than $p_i + 2$ ciphers. The sum $S_i.10^{-p_i}$ tends, in the infinity, to a number S which has less than three ciphers ! But S_i is rational, thus S is rational and algebraic. By another way, we pose

$$\left\{ \begin{array}{l} S = 20 \\ A = S - \pi \\ B = S - e \\ C = S - C_n \\ A_1 = S - \pi - e \\ A_2 = S - \pi + e \\ A_3 = S - \pi e \\ A_4 = S - \pi^2 \end{array} \right.$$

We know that

$$S = \pi + A = e + B = \pi + e + A_1 = \pi - e + A_2 = \pi e + A_3 = \pi^2 + A_4$$

But

$$\pi^2 - n\pi - (\pi - n)\pi = 0$$

As π is transcendental, it can not be solution of a polynomial equation, thus $(\pi - n)\pi$ is transcendental $\forall n$ algebraic. Also if $e\pi$ is algebraic, as e is transcendental, $e(\pi + n)$ is transcendental, $\forall n$ algebraic, thus $e\pi$ or (and) $e(\pi \pm n)$ is transcendental. Also $e\pi$ or (and) $\pi(e \pm n)$ is transcendental, $\forall n$ algebraic. Let us prove now that $e\pi$ is transcendental. Let us suppose that $\exists n_0, m_0$ integers for which $e\pi^{n_0}$ and πe^{m_0} are algebraics for $m_0, n_0 \neq 0$ and $e\pi^n$ and πe^m are transcendentals for $n \neq n_0, m \neq m_0$, thus

$$e^{m_0+n_0}\pi^{n_0^2+1} = (e\pi^{\frac{n_0^2+1}{m_0+n_0}})^{m_0+n_0}$$

is algebraic it means that

$$e\pi^{n_0}\pi^{\frac{n_0^2+1}{n_0+m_0}-n_0} = e\pi^{n_0}\pi^{\frac{1-n_0m_0}{n_0+m_0}}$$

is algebraic and $\pi^{(1-n_0m_0)}$ is algebraic then $n_0 = m_0 = \pm 1$ and $e\pi$ and $\frac{e}{\pi}$ are algebraics! But

$$n\pi^2 - \left(\frac{e}{\pi} + n\right)\pi^2 + e\pi = 0$$

And

$$n\pi^{-2} - (e\pi + n)\pi^{-2} + \frac{e}{\pi} = 0$$

π and π^{-1} are transcendentals, they can not be solutions of polynomial equations, thus two at least of the numbers $\frac{e}{\pi}$, $e\pi$, $\frac{\pi}{e}$ are transcendentals: let us suppose $e\pi = A_A$ algebraic, we have

$$\begin{aligned} & e\pi - (e + \pi)\pi + \pi^2 \\ &= \pi^2 - (e + \pi)\pi + A_A = 0 \\ &= (S - A)^2 - (e + \pi)\pi + A_A = 0 \\ &= A^2 - 2AS - (e + \pi)\pi + A'_A = 0 \\ &A = S \pm \sqrt{A^2 - A'_A + (e + \pi)\pi} \\ &S^2 - 2AS + A'_A - (e + \pi)\pi = 0 \\ &= -2AS - A''_A - (e + \pi)\pi = 0 \\ &= -2(S - \pi)S - A''_A - (e + \pi)\pi = 0 \\ &= A''_A - \pi(-2S + e + \pi) = 0 \\ &= A''_A - (-2S + \pi)\pi - e\pi = A'''_A + 2S\pi - \pi^2 = 0 \end{aligned}$$

It means that π is solution of a polynomial equation: it is impossible, $e\pi$ can not be algebraic, it is transcendental! We prove by the same way that $\frac{\pi}{e}$ and $\frac{e}{\pi}$ are transcendentals. The initial hypothesis is false: m_0, n_0 do not exist, $e\pi^n$ or πe^m is transcendental for all n, m ! Then $m = 1$ or $n = 1$ implies that $e\pi$ is transcendental! Let us prove now that $e + \pi$ is transcendental, we suppose it algebraic: $e + \pi = A_A$, it means that $A + B = 2S - \pi - e = A'_A$ is algebraic, but

$$\pi e = (S - A)(S - B) = A''_A + AB$$

But

$$\begin{aligned} & \pi(e + \pi) = A_A\pi \\ \Rightarrow & e\pi = (e + \pi)\pi - \pi^2 = A_A\pi - \pi^2 = A''_A + AB \\ & \pi^2 - A_A\pi + A''_A + AB = 0 \\ &= (S - A)^2 - A_A(S - A) + A''_A + AB = 0 \\ &= S^2 - 2AS + A^2 - A_AS + A_AA + A''_A + AB = 0 \end{aligned}$$

And

$$A^2 + (B + A'''_A)A + A''''_A = 0$$

Or

$$A = \frac{-B - A''' \pm \sqrt{(B + A''')^2 - 4A''''}}{2}$$

Or

$$(2A + B + A''')^2 = (A + A''''')^2 = B^2 + 2A''''B + A''''''$$

And

$$\begin{aligned} A^2 - B^2 &= A'_A(A - B) = 2A''''B - 2A''''''A + A'''''''' \\ &= A'_A(2A - A_A) = 2A''''(A_A - A) - 2A''''''A + A'''''''' \end{aligned}$$

And

$$A''''''''A = A'''''''''' \Rightarrow A = S - \pi = A''''''''''$$

It means that π is algebraic and it is impossible, thus $e + \pi$ is transcendental! Also, by the same way, we prove that π^n , e^m , C_n^r , $\pi^n e^m$, $C_n^r \pi^n$ and $C_n^r e^m$ are transcendentals for all n, m, p integers! More generally, if T_1, T_2 are transcendentals, and $T_1 T_2$ is irrational. If $T_1 T_2$ is algebraic, $T_1(T_2 \pm n)$ is transcendental, $\forall n$ algebraic, thus $T_1 T_2$ or (and) $T_1(T_2 \pm n)$ is transcendental. Thus $T_1 T_2$ or (and) $T_2(T_1 \pm n)$ is transcendental and a Liouville number, $\forall n$ algebraic. But

$$|T_1(T_2 \pm n)| > |T_1 T_2 - \frac{p}{q}| > \frac{C}{q^m}$$

And $T_1(T_2 \pm n)$ is algebraic and we have supposed it transcendental it means that $T_1 T_2$ can not be algebraic, it is transcendental! Hence $\pi + A$ and $C_n + C$ have the same nature : they are algebraics, with A and C transcendentals (because $\pi = S - A$ and $C_n = S - C$ are!), also $\pi + C$ and $C_n + A$ have the same nature, thus $2\pi + A + C = 4S - 2C_n - (A + C) = \pi + 2S - C_n$ has the same nature than $2C_n + A + C$. If $A + C$ is algebraic, then $\pi + C_n$ will be algebraic and $\pi - C_n = 2S - (A + C) - 2C_n$ will be transcendental! If $A + C$ is transcendental, then $\pi + C_n$ is! By the same reasoning, if $A + B$ is transcendental, $\pi + e = 2S - (A + B)$ will be transcendental and if $A + B$ is algebraic, then $\pi + e$ is algebraic and $\pi - e = 2S - (A + B) - 2e$ is transcendental. The nature of $e + C_n$ depends of $B + C$. Let $\pi + e + C_n + A + B + C = 3S$, it is algebraic : $\pi + e + A + B$ has the same nature than $C_n + C$ (they are algebraics) and $\pi + e + C$ has the same nature than $C_n + A + B$ hence, by the same reasoning, if $A + B + C$ is transcendental then $\pi + e + C_n$ is transcendental, if $A + B + C$ is algebraic, then $\pi + e$, $\pi + C_n$ and $e + C_n$ are transcendentals.

Conclusion

We did not presented like this, but we have given a method to find the nature of several numbers, we have shown the nature of $\pi^n e^m$, $\pi^n C_n^p$, $e^m C_n^p$, π^n , e^m and C_n^p ...

Références

- [1] Alan Baker, *Transcendental number theory Cambridge University Press*, (1975).