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ABSTRACT. We give a necessary and sufficient condition for a measure μ in the closed unit disk to be a reverse Carleson measure for Hardy spaces. This extends a previous result of Lefèvre, Li, Queffélec and Rodríguez-Piazza [LLQR]. We also provide a simple example showing that the analogue for the Paley-Wiener space does not hold. This example can be generalised to model spaces associated to one-component inner functions.

1. INTRODUCTION

For $1 \leq p < \infty$ let H^p be the Hardy space on the unit disk $\mathbb{D}$ equipped with its usual norm

$$\|f\|_p = \left(\sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} .$$

Denote by $M_+(\mathbb{D})$ the set of positive, finite Borel measures supported on $\mathbb{D}$, and let $\mu \in M_+(\mathbb{D})$. A well known theorem by Carleson (see [Gar, Chap.I Th. 5.6]) states that H^p embeds into $L^p(\mathbb{D}, \mu)$:

$$\|f\|_{L^p(\mathbb{D},\mu)} \lesssim \|f\|_p, \quad f \in H^p, \tag{1.1}$$

if and only if μ satisfies the Carleson condition: there exists $C > 0$ such that for all arcs I in $\partial\mathbb{D}$

$$\mu(S_I) \leq C|I|, \tag{1.2}$$

where $S_I = \{z \in \overline{\mathbb{D}} : 1 - |I| \leq |z| \leq 1, z/|z| \in I\}$ is the usual Carleson window. This theorem has been extended to several other spaces, like Bergman, Fock, model spaces etc., and we refer the reader to the huge bibliography on this topic for further information.

Note that H^p contains a dense set of continuous functions for which the embedding (1.1) still makes sense when the measure has a part supported on the boundary. Then (1.2) implies that the restriction of the measure μ to the boundary has to be absolutely continuous with respect to Lebesgue measure and with bounded Radon-Nikodym derivative. It is thus possible to consider more generally positive, finite Borel measures supported on the closed unit disk: $M_+(\overline{\mathbb{D}})$.

Here, we are interested in reverse Carleson inequalities $\|f\|_p \lesssim \|f\|_{L^p(\overline{\mathbb{D}},\mu)}$, $f \in C(\overline{\mathbb{D}}) \cap H^p(\mathbb{D})$, $1 < p < \infty$. In [LLQR] Lefèvre et al. proved that when μ is already a Carleson measure

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these hold if and only it there exists $C > 0$ such that for all arcs $I \subset \partial\mathbb{D}$

$$\mu(S_I) \geq C|I|.$$

Our elementary proof actually shows that the reverse inequalities hold without the Carleson condition. It turns out that the interesting part of the measure has to be supported on the boundary, while the part supported in the disk can be dropped.

The embedding problem is closely related with the *reproducing kernel thesis*: if the embedding holds on the reproducing kernels, then it actually holds for every function. We also show that the reproducing kernel thesis holds for the reverse Carleson embedding.

Finally, we provide a simple example showing that the analogous reproducing kernel thesis for the reverse embedding in the Paley-Wiener space does not hold. The construction can be generalised to model spaces associated to one-component inner functions.

We shall use the following standard notation: $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Main result

For $1 < p < \infty$ and $\lambda \in \mathbb{D}$ consider the reproducing kernel in H^p

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}, \quad z \in \mathbb{D},$$

and its normalised companion

$$K_\lambda := \frac{k_\lambda}{\|k_\lambda\|_p} .$$

A standard computation shows that $\|k_\lambda\|_p \simeq (1 - |\lambda|)^{-1/p'}$, where $1/p + 1/p' = 1$.

Our main result reads as follows.

Theorem 2.1. *Let $1 < p < \infty$ and let $\mu \in M_+(\overline{\mathbb{D}})$. Then the following assertions are equivalent:*

(1) *There exists $C_1 > 0$ such that for every function $f \in H^p \cap C(\overline{\mathbb{D}})$,*

$$\int_{\overline{\mathbb{D}}} |f|^p d\mu \geq C_1 \|f\|_p^p ,$$

(2) *There exists $C_2 > 0$ such that for every $\lambda \in \mathbb{D}$,*

$$\int_{\overline{\mathbb{D}}} |K_\lambda|^p d\mu \geq C_2 ,$$

(3) *There exists $C_3 > 0$ such that for every arc $I \subset \partial\mathbb{D}$,*

$$\mu(S_I) \geq C_3|I| .$$

(4) *There exists $C_4 > 0$ such that the Radon-Nikodym derivative of $\mu|_{\partial\mathbb{D}}$ with respect to the length measure is bounded below by C_4.*

Observe that in this theorem we do not require absolute continuity of the restriction $\mu|_{\partial\mathbb{D}}$. Still, if we want to extend (1) to the entire H^p-space, then, in order that $\int_{\overline{\mathbb{D}}} |f|^p d\mu$ makes sense for every function in H^p, we need to impose absolute continuity on $\mu|_{\partial\mathbb{D}}$. Note that the integral

$\int_{\overline{\mathbb{D}}} |f|^p d\mu$ can be infinite for certain $f \in H^p$ when the Radon-Nikodym derivative of $\mu|_{\partial\mathbb{D}}$ is not bounded.

Proof. (1) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (4). Take $h > 0$ so that $|I|/h$ is a large integer N and consider the modified Carleson window

$$S_{I,h} = \{z \in \overline{\mathbb{D}} : 1 - h \leq |z| \leq 1,\ z/|z| \in I\}\ .$$

Split I into N subarcs I_k such that $|I_k| = h$ (and hence $S_{I_k,h} = S_{I_k}$). Then

$$\mu(S_{I,h}) = \mu(\bigcup_{k=1}^{N} S_{I_k,h}) = \sum_{k=1}^{N} \mu(S_{I_k,h}) \geq C_3 \sum_{k=1}^{N} |I_k| = C_3|I|.$$

Now, for every open set O in $\overline{\mathbb{D}}$ for which $I \subset O$ there exists $h > 0$ such that $S_{I,h} \subset O$. Since $\mu \in M_+(\mathbb{D}^-)$ is outer regular (see [Ru, Theorem 2.18]) we thus have

$$\mu(I) = \inf_{I \subset O \text{ open in } \overline{\mathbb{D}}} \mu(O) \geq \inf_{h>0} \mu(S_{I,h}) \geq C_3|I|.$$

We deduce that the Lebesgue measure on $\partial\mathbb{D}$ denoted by m is absolutely continuous with respect to the restriction of μ to $\partial\mathbb{D}$ and that the corresponding Radon-Nikodym derivative of μ is bounded below by C_3. In particular one can choose $C_4 = C_3$.

(4) $\Rightarrow$ (1) Clearly, for all $f \in H^p$,

$$\int_{\overline{\mathbb{D}}} |f|^p d\mu \geq \int_{\partial\mathbb{D}} |f|^p d\mu \geq C_4 \int_{\partial\mathbb{D}} |f|^p dm = C_4 \|f\|_p^p$$

(in particular, one can choose $C_1 = C_4$).

(2) $\Rightarrow$ (3). By hypothesis, integrating over $S_{I,h}$ with respect to area measure dA on $\overline{\mathbb{D}}$ we get

$$C_2|I| \times h \leq \int_{S_{I,h}} \int_{\overline{\mathbb{D}}} |K_\lambda|^p d\mu dA(\lambda) \simeq \int_{\overline{\mathbb{D}}} \int_{S_{I,h}} \frac{(1-|\lambda|^2)^{p/p'}}{|1-\overline{\lambda}z|^p} dA(\lambda) d\mu(z).$$

Set

$$\varphi_h(z) = \frac{1}{h} \int_{S_{I,h}} \frac{(1-|\lambda|^2)^{p/p'}}{|1-\overline{\lambda}z|^p} dA(\lambda) = \frac{1}{h} \int_{S_{I,h}} \frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p} dA(\lambda),$$

so that the previous estimate becomes

$$\int_{\overline{\mathbb{D}}} \varphi_h(z) d\mu(z) \gtrsim |I|\ . \tag{2.1}$$

We claim that

$$\lim_{h \to 0} \varphi_h(z) \begin{cases} \simeq 1 & \text{if } z \in \overline{I}, \\ = 0 & \text{otherwise.} \end{cases}$$

Indeed, if $z \notin \overline{I}$, then there are $\delta, h_0 > 0$ such that for every $0 < h < h_0$ and for every $\lambda \in S_{I,h}$, we have $|1 - \overline{\lambda}z| \geq \delta > 0$, and the result follows from the estimate

$$0 \leq \varphi_h(z) = \frac{1}{h} \int_{S_{I,h}} \frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p} dA(\lambda) \leq \frac{1}{\delta^p} \frac{|I| \times h}{h} \times (2h)^{p-1} \lesssim h^{p-1}.$$

Suppose now that $z = e^{i\theta_0} \in \overline{I}$. Let $h \leq |I|$, then setting $\lambda = (1-t)e^{i\theta}$ for $\lambda \in S_{I,h}$ we have

$$\begin{aligned}\varphi_h(z) &= \frac{1}{h}\int_{S_{I,h}} \frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p} dA(\lambda) \geq \frac{1}{h}\int_{e^{i\theta}\in I}\int_0^h \frac{t^{p-1}}{|e^{i\theta_0}-(1-t)e^{i\theta}|^p}(1-t)dtd\theta \\ &\gtrsim \frac{1}{h}\int_0^h \int_{|\theta-\theta_0|\leq t, e^{i\theta}\in I} \frac{t^{p-1}}{|\theta-\theta_0|^p + t^p} d\theta dt \\ &\geq \frac{1}{h}\int_0^h \int_{|\theta-\theta_0|\leq t, e^{i\theta}\in I} \frac{t^{p-1}}{2t^p} d\theta dt.\end{aligned}$$

Since $0 \leq t \leq h \leq |I|$ and $z = e^{it} \in \overline{I}$, the set $\{e^{i\theta} : |\theta - \theta_0| \leq t, e^{i\theta} \in I\}$ contains an interval of length at least $t/2$, we get

$$\varphi_h(z) \gtrsim \frac{1}{h}\int_0^h \frac{t}{2} \times \frac{t^{p-1}}{2t^p} dt \simeq 1.$$

On the other hand, integrating in polar coordinates, we get

$$\begin{aligned}\varphi_h(z) &= \frac{1}{h}\int_{S_{I,h}} \frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p} dA(\lambda) = \frac{1}{h}\int_{1-h}^1 (1-r^2)^{p-1}\int_I \frac{1}{|1-re^{i(\theta-\theta_0)}|^p} d\theta r dr \\ &\lesssim \frac{1}{h}\int_0^h t^{p-1}\frac{1}{t^{p/p'}} dt \simeq 1.\end{aligned}$$

Hence φ_h converges pointwise to a function comparable to $\chi_{\overline{I}}$, and φ_h is uniformly bounded in h. Now, from (2.1) and by dominated convergence we finally deduce that

$$\mu(\overline{I}) = \int_{\mathbb{D}^-} \chi_{\overline{I}} d\mu \simeq \int_{\overline{\mathbb{D}}} \lim_{h\to 0} \varphi_h(z) d\mu(z) = \lim_{h\to 0}\int_{\overline{\mathbb{D}}} \varphi_h(z) d\mu(z) \gtrsim |I|\,.$$

■

Remark. The following example shows that the reproducing kernel thesis fails for the reverse Carleson inequalities in the Paley-Wiener space PW_π, the space of Fourier transforms of square integrable functions on $[-\pi, \pi]$. In Section 2 we will show how it can be adapted to any model space associated to a one-component inner function.

Consider the sequence $S = \{x_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, where

$$x_n = \begin{cases} n + 1/8 & \text{if } n \text{ is even} \\ n - 1/8 & \text{if } n \text{ is odd.}\end{cases}$$

By the Kadets-Ingham theorem (see e.g. [Nik, Theorem D4.1.2]) S would be a minimal sampling sequence if we added the point 0. Since S is not sampling the discrete measure $\mu := \sum_{n\neq 0}\delta_{x_n}$ does not satisfy the reverse inequality $\|f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mu)}$, $f \in PW_\pi$.

Let us see that, on the other hand, the μ-norm of the normalised reproducing kernels

$$K_\lambda(z) = c_\lambda \operatorname{sinc}(\pi(z-\lambda)) = c_\lambda \frac{\sin(\pi(z-\lambda))}{\pi(z-\lambda)}, \qquad c_\lambda^2 \simeq (1 + |\operatorname{Im}\lambda|)e^{-2\pi|\operatorname{Im}\lambda|},$$

is uniformly bounded from below. If λ is such that $|\operatorname{Im}\lambda| > 1$ then $|\sin(\pi(x_n - \lambda))| \simeq e^{\pi|\operatorname{Im}\lambda|}$, and hence

$$\int_{\mathbb{C}} |K_\lambda(x)|^2 d\mu(x) = \sum_{n\neq 0} c_\lambda^2 \left|\frac{\sin(\pi(x_n-\lambda))}{\pi(x_n-\lambda)}\right|^2 \simeq \sum_{n\neq 0} \frac{|\operatorname{Im}\lambda|}{|x_n-\lambda|^2} \simeq 1.$$

It is thus enough to consider points $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| \leq 1$. Let x_{n_0} be the point of S closest to λ; then there is $\delta > 0$, independent of λ, such that

$$\int_{\mathbb{C}} |K_\lambda(x)|^2 d\mu(x) = \sum_{n\neq 0} |K_\lambda(x_n)|^2 \geq \left|\frac{\sin(\pi(x_{n_0}-\lambda))}{\pi(x_{n_0}-\lambda)}\right|^2 \geq \delta\,.$$

It is interesting to point out that μ is a Carleson measure for PW_π, since S is in a strip and separated.

3. Failure in other model spaces

The previous construction can be generalised to certain model spaces in the disk. The model space associated to an inner function Θ is $K_\Theta = H^2 \ominus \Theta H^2$, and the reproducing kernel corresponding to $\lambda \in \mathbb{D}$ is given by

$$k_\lambda^\Theta(z) = \frac{1-\overline{\Theta(\lambda)}\Theta(z)}{1-\overline{\lambda}z}, \quad z \in \mathbb{D}.$$

A particular class of model spaces is given by the so-called *one-component* inner functions, those for which the sub-level set $\{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}$ is connected for some $0 < \varepsilon < 1$.

The Paley-Wiener space corresponds, after a conformal mapping of $\mathbb{D}$ into the upper half-plane, to the inner function $\Theta_{2\pi}(z) = e^{i2\pi z}$. More precisely $K_{\Theta_{2\pi}} = e^{i\pi z} PW_\pi$.

Here we show the following result.

Theorem 3.1. *If Θ is a one-component inner function, then the reverse reproducing kernel thesis does not hold in K_Θ.*

We refer the reader to [BFGHR] for sufficient conditions for reverse Carleson measures in model spaces.

Let $\sigma(\Theta)$ denote the spectrum of Θ, that is, the set of $\zeta \in \overline{\mathbb{D}}$ such that $\liminf_{z\to\zeta, z\in\mathbb{D}} |\Theta(z)| = 0$. For one-component inner functions the set $\partial\mathbb{D} \setminus \sigma(\Theta)$ is a countable union of arcs where Θ is analytic (and on which the argument of Θ increases by 2π). Moreover, for any $|\alpha| = 1$,

$$E_\alpha := \{\zeta \in \partial\mathbb{D} \setminus \sigma(\Theta) : \Theta(\zeta) = \alpha\}$$

is countable and the system $(K_{\zeta_n}^\Theta)_{\zeta_n \in E_\alpha}$ is an orthonormal basis of K_Θ, a so-called Clark basis (see [Cl], and [BaDy, Section 4] for the material needed here). For such $\zeta \in \partial\mathbb{D} \setminus \sigma(\Theta)$ the reproducing kernel is defined as

$$k_\zeta^\Theta(z) = \frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\overline{\zeta}z} = \zeta\overline{\Theta(\zeta)}\frac{\Theta(\zeta)-\Theta(z)}{\zeta - z}, \quad z \in \mathbb{D}.$$

Its norm is $\sqrt{|\Theta'(\zeta)|}$, so that the corresponding normalised reproducing kernel is

$$K_\zeta^\Theta := \frac{k_\zeta^\Theta}{\|k_\zeta^\Theta\|_2} = \frac{k_\zeta^\Theta}{\sqrt{|\Theta'(\zeta)|}}\,.$$

With these elements we follow the scheme of the Paley-Wiener case to prove Theorem 3.1.

Proof. Pick the Clark basis $(K_{\zeta_n}^\Theta)_{n\geq 0}$ for $\alpha = 1$ and set

$$\xi_n = \begin{cases} \zeta_n & \text{if } n \neq 1 \\ \xi_1 & \text{if } n = 1, \end{cases}$$

where we choose ξ_1 sufficiently close to ζ_1 (and in particular different from ζ_n, $n \neq 1$) but different from ζ_1, implying in particular $\langle K_{\xi_1}^\Theta, K_{\zeta_n}^\Theta\rangle \neq 0$ for every n, so that $(K_{\xi_n}^\Theta)_{n\geq 0}$ is an unconditional basis (see [BaDy]; it is actually not far from being orthogonal). It will be clear from the proof below how close to ζ_1 we have to choose ξ_1.

We now consider the measure

$$\mu := \sum_{n>0} \|k_{\xi_n}^\Theta\|_2^{-2}\delta_{\xi_n} = \sum_{n>0} |\Theta'(\xi_n)|^{-1}\delta_{\xi_n}$$

where we have taken away the very first point ξ_0, so that $(K_{\xi_n}^\Theta)_{n>0}$ is an incomplete family. Notice that this is a perturbation of the Clark measure $\sigma = \sum_{n\geq 0} \|k_{\zeta_n}^\Theta\|^{-2}\delta_{\zeta_n}$ with one mass point deleted. Thus μ is not a reverse Carleson measure since there are functions vanishing in all the points ξ_n, $n > 0$, but not in ξ_0.

Let us check that the reverse reproducing kernel thesis fails, which, in view of the above, amounts to find a $\delta > 0$ such that $\|K_z^\Theta\|_{L^2(\mu)} \geq \delta$ for every $z \in \mathbb{D}$. Note that

$$\|K_z^\Theta\|_{L^2(\mu)}^2 = \sum_{n\geq 1} \frac{1}{|\Theta'(\xi_n)|}|K_z^\Theta(\xi_n)|^2 = \sum_{n\geq 1} |\langle K_z^\Theta, K_{\xi_n}^\Theta\rangle|^2, \tag{3.1}$$

which are just the generalised Fourier coefficients of K_z^Θ in $K_{\xi_n}^\Theta$, $n \geq 1$.

Let us introduce the following function

$$\varphi(z) := |\langle K_{\zeta_0}^\Theta, K_z^\Theta\rangle|^2 = \left|\frac{\Theta(\zeta_0) - \Theta(z)}{\zeta_0 - z}\right|^2 \frac{1}{|\Theta'(\zeta_0)|}\frac{1-|z|^2}{1-|\Theta(z)|^2}, \quad z \in \mathbb{D}.$$

By the Cauchy-Schwarz inequality $\varphi(z) \leq 1$ for all $z \in \mathbb{D}$. Also, since $\|K_{\zeta_0}^\Theta\|_2 = \|K_z^\Theta\|_2 = 1$, the only way to get $\varphi(z) = 1$ is that $K_z^\Theta = \alpha K_{\zeta_0}^\Theta$, $|\alpha| = 1$, i.e. $z = \zeta_0$.

Since ζ_0 is not in the spectrum, there is a closed neighbourhood C of ζ_0 in $\overline{\mathbb{D}}$ on which Θ is analytic, which implies that φ is continuous on C. We suppose C small enough that it does not contain any other ζ_k, $k \neq 0$, nor ξ_1.

Introduce the sets

$$U_\delta := \{z \in C : |z - \zeta_0| < \delta\}$$

and define

$$\psi(\delta) := \sup_{z\notin U_\delta} \varphi(z)$$

Claim: For sufficiently small δ the function $\psi(\delta)$ is decreasing, with $\psi(0) = 1$ and $\psi(\delta) < 1$ for $\delta > 0$.

We postpone the proof of the claim and proceed now to prove that $\|K_z^\Theta\|_{L^2(\mu)} \gtrsim 1$. Pick $\delta > 0$ sufficiently small such that $\psi(\delta) < 1$. We consider two cases.

Assume first that $z \notin U_\delta$. Pick $0 < \varepsilon < 1 - \psi(\delta)$. Since $\{\zeta_0\} \cup \{\xi_k\}_{k\geq 1}$ gives rise to a perturbation of the orthonormal Clark basis $(K_{\zeta_n})_{n\geq 0}$, it suffices to choose ξ_1 close enough to ζ_1 so that there is $0 < \eta < \epsilon$ such that for every $f \in K_\Theta$ (see [BaDy])

$$(1-\eta)\|f\|_2^2 \leq |\langle f, K_{\zeta_0}\rangle|^2 + \sum_{n\geq 1} |\langle f, K_{\xi_n}\rangle|^2 \leq (1+\eta)\|f\|_2^2.$$

Then, by (3.1)

$$\begin{aligned}\|K_z^\Theta\|_{L^2(\mu)}^2 &= \sum_{n\geq 1} |\langle K_z^\Theta, K_{\xi_n}^\Theta\rangle|^2 = |\langle K_z^\Theta, K_{\zeta_0}^\Theta\rangle|^2 + \sum_{n\geq 1} |\langle K_z^\Theta, K_{\xi_n}^\Theta\rangle|^2 - |\langle K_z^\Theta, K_{\zeta_0}^\Theta\rangle|^2 \\ &\geq (1-\eta)\|K_z^\Theta\|_2^2 - \varphi(z) \geq (1-\eta) - (1-\epsilon) = \epsilon - \eta > 0\end{aligned}$$

Assume now that $z \in U_\delta \subset C$. We will check that on this set it suffices to consider only two terms of the sum $\varphi_1(z) = |\langle K_z^\Theta, K_{\xi_1}^\Theta\rangle|^2$ and $\varphi_2(z) = |\langle K_z^\Theta, K_{\zeta_2}^\Theta\rangle|^2$. It is here that we need that ξ_1 is a small perturbation of ζ_1 which is "not harmonic" with ζ_1, meaning that $|\langle K_{\zeta_2}^\Theta, K_{\xi_1}^\Theta\rangle|^2 \neq 0$. Indeed φ_1 and φ_2 are continuous functions on the compact set $\overline{U}_\delta$. Since $U_\delta \subset C$, we have $\varphi_2(z) = 0$, $z \in \overline{U}_\delta$, if and only if $z = \zeta_0$. Now $\varphi_1(\zeta_0) > 0$ so that by a continuity argument we conclude that $\varphi_1(z) + \varphi_2(z)$ is strictly bounded away from 0 for $z \notin U_\delta$, which concludes the proof.

Proof of the Claim. It is clear that $\psi(\delta)$ is decreasing and $\psi(0) = 1$.

We prove now that $\psi(\delta) < 1$ for $\delta > 0$. Indeed, suppose not, then there is a sequence $(z_n)_n \subset \mathbb{D} \setminus U_\delta$ such that $\varphi(z_n) = |\langle K_{\zeta_0}^\Theta, K_{z_n}^\Theta\rangle|^2 \to 1$ as $n \to \infty$. We can also assume that $z_n \to \zeta \in \operatorname{clos}(\partial\mathbb{D} \setminus U_\delta)$. Now $(K_{z_n}^\Theta)_n$ is a bounded family, and by the Alaoglu theorem it admits a weak convergent subsequence, which in order not to overcharge notation, we can suppose to be also indexed by n. Let f be a weak limit of this sequence so that $|\langle K_{\zeta_0}^\Theta, f\rangle| = 1$. It is also clear that $\|f\|_2 = 1$. From the same observation as above we can deduce $f = \alpha K_{\zeta_0}^\Theta$, $|\alpha| = 1$ (in fact, every weak convergent subsequence has $K_{\zeta_0}^\Theta$ as weak limit). In particular, by the weak convergence, for every $f \in K_\Theta$,

$$f(z_n)\sqrt{\frac{1-|z_n|^2}{1-|\Theta(z_n)|^2}} = \langle f, K_{z_n}^\Theta\rangle \to \langle f, K_{\zeta_0}^\Theta\rangle = \frac{f(\zeta_0)}{\sqrt{|\Theta'(\zeta_0)|}}. \tag{3.2}$$

Observe that K_Θ contains continuous functions (by a result of Aleksandrov continuous functions in K_Θ form actually a dense set in K_Θ, see [CMR, p.186]).

Now, if there are two continuous functions f_1 and f_2 in K_Θ such that the vectors $(f_1(\zeta), f_1(\zeta_0))$ and $(f_2(\zeta), f_2(\zeta_0))$ are linearly independent, then we can deduce from (3.2) that necessarily, first

$$\frac{1-|z_n|^2}{1-|\Theta(z_n)|^2} \to \frac{1}{|\Theta'(\zeta_0)|}$$

and then

$$f_1(\zeta) = f_1(\zeta_0) \quad \text{and} \quad f_2(\zeta) = f_2(\zeta_0)$$

which is not possible unless $\zeta = \zeta_0$.

Let us prove that if $\zeta \neq \zeta_0$ then there are two such functions f_1, f_2. We start by taking two linearly independent continuous functions $h_1, h_2 \in K_\Theta$. It may happen that $(h_1(\zeta), h_1(\zeta_0))$ and $(h_2(\zeta), h_2(\zeta_0))$ are linearly independent and then we are done. If they are linearly dependent, then we can find a linear combination f of h_1 and h_2 which is not identically 0 and such that $f(\zeta) = f(\zeta_0) = 0$. Consider the backward shift operator $S^*f(z) = \frac{f(z)-f(0)}{z}$ and recall that $S^*K_\Theta \subset K_\Theta$. Observe that if moreover $f(0) = 0$ then also $S^*f(\zeta) = S^*f(\zeta_0) = 0$. Hence, after sufficiently many applications of S^* we can suppose that $f(0) \neq 0$, $f(\zeta) = f(\zeta_0) = 0$, and, renormalising, that $f(0) = 1$.

Then $g = S^*f$ is continuous in K_θ and takes two different values $g(\zeta) = -\overline{\zeta}$ and $g(\zeta_0) = -\overline{\zeta_0}$. Set now $h = S^{*2}f$ which takes the values $h(\zeta) = -\overline{\zeta}^2 - \overline{\zeta}h'(0)$ and $h(\zeta_0) = -\overline{\zeta_0}^2 - \overline{\zeta_0}h'(0)$. Then either the vectors $(g(\zeta), g(\zeta_0))$ and $(h(\zeta), h(\zeta_0))$ are linearly independent (and we are done) or they are not, in which case the solution of the linear dependence gives $\zeta = \zeta_0$. ■

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