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# Spinorial Representation of Surfaces into 4-dimensional Space Forms

Pierre Bayard\*, Marie-Amélie Lawn† and Julien Roth‡

## Abstract

In this paper we give a geometrically invariant spinorial representation of surfaces in four-dimensional space forms. In the Euclidean space, we obtain a representation formula which generalizes the Weierstrass representation formula of minimal surfaces. We also obtain as particular cases the spinorial characterizations of surfaces in  $\mathbb{R}^3$  and in  $S^3$  given by T. Friedrich and by B. Morel.

*keywords:* Dirac Operator, Isometric Immersions, Weierstrass Representation.

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## 1 Introduction

The Weierstrass representation describes a conformal minimal immersion of a Riemann surface  $M$  into the three-dimensional Euclidean space  $\mathbb{R}^3$ . Precisely, the immersion is expressed using two holomorphic functions  $f, g : M \rightarrow \mathbb{C}$  by the following integral formula

$$(x_1, x_2, x_3) = \Re e \left( \int f(1 - g^2) dz, \int i f(1 + g^2) dz, \int 2fg dz \right) : M \rightarrow \mathbb{R}^3.$$

On the other hand, the spinor bundle  $\Sigma M$  over  $M$  is a two-dimensional complex vector bundle splitting into

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M = \Lambda^0 M \oplus \Lambda^{0,1} M.$$

Hence, a pair of holomorphic functions  $(g, f)$  can be considered as a spinor field  $\varphi = (g, f dz)$ . Moreover, the Cauchy-Riemann equations satisfied by  $f$  and  $g$  are equivalent to the Dirac equation

$$D\varphi = 0.$$

This representation is still valid for arbitrary surfaces. In the general case, the functions  $f$  and  $g$  are not holomorphic and the Dirac equation becomes

$$D\varphi = H\varphi,$$

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where  $H$  is the mean curvature of the immersion. This fact is well-known and has been studied in the last years by many authors (see [8, 9, 15, 16]).

In [5], T. Friedrich gave a geometrically invariant spinorial representation of surfaces in  $\mathbb{R}^3$ . This approach was generalized to surfaces of other three-dimensional spaces [13, 14] and also in the pseudo-Riemannian case [10, 11].

The aim of the present paper is to extend this approach to the case of codimension 2 and then provide a geometrically invariant representation of surfaces in the 4-dimensional space form  $\mathbb{M}^4(c)$  of sectional curvature  $c$  by spinors solutions of a Dirac equation.

## 2 Preliminaries

### 2.1 The fundamental theorem of surfaces in $\mathbb{M}^4(c)$

Let  $(M^2, g)$  be an oriented surface isometrically immersed into the four-dimensional space form  $\mathbb{M}^4(c)$ . Let us denote by  $E$  its normal bundle and by  $B : TM \times TM \rightarrow E$  its second fundamental form defined by

$$B(X, Y) = \bar{\nabla}_X Y - \nabla_X Y,$$

where  $\nabla$  and  $\bar{\nabla}$  are the Levi-Civita connections of  $M$  and  $\mathbb{M}^4(c)$  respectively. For  $\xi \in \Gamma(E)$ , the shape operator associated to  $\xi$  is defined by

$$S_\xi(X) = -(\bar{\nabla}_X \xi)^T,$$

for all  $X \in \Gamma(TM)$ , where the upper index  $T$  means that we take the component of the vector tangent to  $M$ . Then, the following equations hold:

1.  $K = \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2 + c$ , (Gauss equation)
2.  $K_N = -\langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle$ , (Ricci equation)
3.  $(\nabla'_X B)(Y, Z) - (\nabla'_Y B)(X, Z) = 0$ , (Codazzi equation)

where  $K$  and  $K_N$  are the curvatures of  $(M, g)$  and  $E$ ,  $(e_1, e_2)$  and  $(e_3, e_4)$  are orthonormal and positively oriented bases of  $TM$  and  $E$  respectively, and where  $\nabla'$  is the natural connection induced on the bundle  $T^*M^{\otimes 2} \otimes E$ . Reciprocally, there is the following theorem:

**Theorem** (Tenenblat [17]). *Let  $(M^2, g)$  be a Riemannian surface and  $E$  a vector bundle of rank 2 on  $M$ , equipped with a metric  $\langle \cdot, \cdot \rangle$  and a compatible connection. We suppose that  $M$  and  $E$  are oriented. Let  $B : TM \times TM \rightarrow E$  be a bilinear map satisfying the Gauss, Ricci and Codazzi equations above, where, if  $\xi \in E$ , the shape operator  $S_\xi : TM \rightarrow TM$  is the symmetric operator such that*

$$g(S_\xi(X), Y) = \langle B(X, Y), \xi \rangle$$

*for all  $X, Y \in TM$ . Then, there exists a local isometric immersion  $V \subset M \rightarrow \mathbb{M}^4(c)$  so that  $E$  is identified with the normal bundle of  $M$  into  $\mathbb{M}^4(c)$  and with  $B$  as second fundamental form.*

## 2.2 Twisted spinor bundle

Let  $(M^2, g)$  be an oriented Riemannian surface, with a given spin structure, and  $E$  an oriented and spin vector bundle of rank 2 on  $M$ . We consider the spinor bundle  $\Sigma$  over  $M$  twisted by  $E$  and defined by

$$\Sigma = \Sigma M \otimes \Sigma E.$$

We endow  $\Sigma$  with the spinorial connection  $\nabla$  defined by

$$\nabla = \nabla^{\Sigma M} \otimes \text{Id}_{\Sigma E} + \text{Id}_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

We also define the Clifford product  $\cdot$  by

$$\begin{cases} X \cdot \varphi = (X \cdot_M \alpha) \otimes \bar{\sigma} & \text{if } X \in \Gamma(TM) \\ X \cdot \varphi = \alpha \otimes (X \cdot_E \sigma) & \text{if } X \in \Gamma(E) \end{cases}$$

for all  $\varphi = \alpha \otimes \sigma \in \Sigma M \otimes \Sigma E$ , where  $\cdot_M$  and  $\cdot_E$  denote the Clifford products on  $\Sigma M$  and on  $\Sigma E$  respectively and where  $\bar{\sigma} = \sigma^+ - \sigma^-$ . We finally define the Dirac operator  $D$  on  $\Gamma(\Sigma)$  by

$$D\varphi = e_1 \cdot \nabla_{e_1} \varphi + e_2 \cdot \nabla_{e_2} \varphi,$$

where  $(e_1, e_2)$  is an orthonormal basis of  $TM$ .

We note that  $\Sigma$  is also naturally equipped with a hermitian scalar product  $\langle \cdot, \cdot \rangle$  which is compatible to the connection  $\nabla$ , since so are  $\Sigma M$  and  $\Sigma E$ , and thus also with a compatible real scalar product  $\Re e \langle \cdot, \cdot \rangle$ . We also note that the Clifford product  $\cdot$  of vectors belonging to  $TM \oplus E$  is antihermitian with respect to this hermitian product. Finally, we stress that the four subbundles  $\Sigma^{\pm\pm} := \Sigma^\pm M \otimes \Sigma^\pm E$  are orthogonal with respect to the hermitian product. Throughout the paper we will assume that the hermitian product is  $\mathbb{C}$ -linear w.r.t. the first entry, and  $\mathbb{C}$ -antilinear w.r.t. the second entry.

## 2.3 Spin geometry of surfaces in $\mathbb{M}^4(c)$

It is a well-known fact (see [1, 6, 7]) that there is an identification between the spinor bundle  $\Sigma \mathbb{M}^4(c)|_M$  of  $\mathbb{M}^4(c)$  over  $M$ , and the spinor bundle of  $M$  twisted by the normal bundle  $\Sigma := \Sigma M \otimes \Sigma E$ . Moreover, we have the spinorial Gauss formula: for any  $\varphi \in \Gamma(\Sigma)$  and any  $X \in TM$ ,

$$\tilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi,$$

where  $\tilde{\nabla}$  is the spinorial connection of  $\Sigma \mathbb{M}^4(c)$  and  $\nabla$  is the spinorial connection of  $\Sigma$  defined by

$$\nabla = \nabla^{\Sigma M} \otimes \text{Id}_{\Sigma E} + \text{Id}_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

Here  $\cdot$  is the Clifford product on  $\mathbb{M}^4(c)$ . Therefore, if  $\varphi$  is a Killing spinor of  $\mathbb{M}^4(c)$ , that is satisfying

$$\tilde{\nabla}_X \varphi = \lambda X \cdot \varphi,$$

where the Killing constant  $\lambda$  is 0 for the Euclidean space,  $\pm\frac{1}{2}$  for the sphere and  $\pm\frac{i}{2}$  for the hyperbolic space, that is,  $4\lambda^2 = c$ , then its restriction over  $M$  satisfies

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi + \lambda X \cdot \varphi. \quad (1)$$

Taking the trace in (1), we obtain the following Dirac equation

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi, \quad (2)$$

where we have again  $D\varphi = \sum_{j=1}^2 e_j \cdot \nabla_{e_j} \varphi$  and  $\vec{H} = \frac{1}{2} \sum_{j=1}^2 B(e_j, e_j)$  is the mean curvature vector of  $M$  in  $\mathbb{M}^4(c)$ .

Let us consider  $\omega_4 = -e_1 \cdot e_2 \cdot e_3 \cdot e_4$ . We recall that  $\omega_4^2 = 1$  and  $\omega_4$  has two eigenspaces for eigenvalues 1 and  $-1$  of same dimension. We denote by  $\Sigma^+$  and  $\Sigma^-$  these subbundles. They decompose as follows:

$$\begin{cases} \Sigma^+ = (\Sigma^+ M \otimes \Sigma^+ E) \oplus (\Sigma^- M \otimes \Sigma^- E) \\ \Sigma^- = (\Sigma^+ M \otimes \Sigma^- E) \oplus (\Sigma^- M \otimes \Sigma^+ E), \end{cases}$$

where  $\Sigma^\pm M$  and  $\Sigma^\pm E$  are the spaces of half-spinors for  $M$  and  $E$  respectively. In the sequel, for  $\varphi \in \Sigma$ , we will use the following convention:

$$\varphi = \varphi^{++} + \varphi^{--} + \varphi^{+-} + \varphi^{-+},$$

with

$$\begin{cases} \varphi^{++} \in \Sigma^{++} := \Sigma^+ M \otimes \Sigma^+ E, \\ \varphi^{--} \in \Sigma^{--} := \Sigma^- M \otimes \Sigma^- E, \\ \varphi^{+-} \in \Sigma^{+-} := \Sigma^+ M \otimes \Sigma^- E, \\ \varphi^{-+} \in \Sigma^{-+} := \Sigma^- M \otimes \Sigma^+ E. \end{cases}$$

Finally, we set

$$\varphi^+ = \varphi^{++} + \varphi^{--} \quad \text{and} \quad \varphi^- = \varphi^{+-} + \varphi^{-+}.$$

If  $\varphi$  is a Killing spinor of  $\mathbb{M}^4(c)$ , an easy computation yields

$$X|\varphi^+|^2 = 2\Re\langle \lambda X \cdot \varphi^-, \varphi^+ \rangle \quad \text{and} \quad X|\varphi^-|^2 = 2\Re\langle \lambda X \cdot \varphi^+, \varphi^- \rangle.$$

### 3 Main result

**Theorem 1.** *Let  $(M^2, g)$  be an oriented Riemannian surface, with a given spin structure, and  $E$  an oriented and spin vector bundle of rank 2 on  $M$ . Let  $\Sigma = \Sigma M \otimes \Sigma E$  be the twisted spinor bundle and  $D$  its Dirac operator. We assume that  $\lambda$  is a constant belonging to  $\mathbb{R} \cup i\mathbb{R}$  and  $\vec{H}$  is a section of  $E$ . Then the three following statements are equivalent:*

1. *There exists a spinor  $\varphi \in \Gamma(\Sigma)$  solution of the Dirac equation*

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi \quad (3)$$

*such that  $\varphi^+$  and  $\varphi^-$  do not vanish and satisfy*

$$X|\varphi^+|^2 = 2\Re\langle \lambda X \cdot \varphi^-, \varphi^+ \rangle \quad \text{and} \quad X|\varphi^-|^2 = 2\Re\langle \lambda X \cdot \varphi^+, \varphi^- \rangle. \quad (4)$$

2. There exists a spinor  $\varphi \in \Gamma(\Sigma)$  solution of

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi + \lambda X \cdot \varphi,$$

where  $B : TM \times TM \rightarrow E$  is bilinear and  $\frac{1}{2} \text{tr}(B) = \vec{H}$  and such that  $\varphi^+$  and  $\varphi^-$  do not vanish.

3. There exists a local isometric immersion of  $(M, g)$  into  $\mathbb{M}^4(c)$  with normal bundle  $E$ , second fundamental form  $B$  and mean curvature  $\vec{H}$ .

The form  $B$  and the spinor field  $\varphi$  are linked by (6).

In order to prove Theorem 1 we consider the following equivalent technical

**Proposition 3.1.** *Let  $M$ ,  $E$  and  $\Sigma$  as in Theorem 1 and assume that there exists a spinor  $\varphi \in \Gamma(\Sigma)$  solution of*

$$D\varphi = \vec{H} \cdot \varphi - 2\lambda\varphi \quad (5)$$

with  $\varphi^+$  and  $\varphi^-$  nowhere vanishing spinors satisfying (4). Then the symmetric bilinear map

$$B : TM \times TM \rightarrow E$$

defined by

$$\begin{aligned} \langle B(X, Y), \xi \rangle &= \frac{1}{2|\varphi^+|^2} \Re \langle X \cdot \nabla_Y \varphi^+ + Y \cdot \nabla_X \varphi^+ + 2\lambda \langle X, Y \rangle \varphi^-, \xi \cdot \varphi^+ \rangle \\ &+ \frac{1}{2|\varphi^-|^2} \Re \langle X \cdot \nabla_Y \varphi^- + Y \cdot \nabla_X \varphi^- + 2\lambda \langle X, Y \rangle \varphi^+, \xi \cdot \varphi^- \rangle \end{aligned} \quad (6)$$

for all  $X, Y \in \Gamma(TM)$  and all  $\xi \in \Gamma(E)$  satisfies the Gauss, Codazzi and Ricci equations and is such that

$$\vec{H} = \frac{1}{2} \text{tr} B.$$

**Remark 1.** *If  $\lambda = 0$ , and if  $\varphi \in \Gamma(\Sigma)$  is a solution of*

$$D\varphi = \vec{H} \cdot \varphi \quad \text{with} \quad |\varphi^+| = |\varphi^-| = 1,$$

formula (6) simplifies to

$$\begin{aligned} \langle B(X, Y), \xi \rangle &= \frac{1}{2} \Re \langle X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \xi \cdot \varphi \rangle \\ &= \Re \langle X \cdot \nabla_Y \varphi, \xi \cdot \varphi \rangle, \end{aligned} \quad (7)$$

since this last expression is in fact symmetric in  $X$  and  $Y$ .

To prove proposition 3.1 we first state the following lemma.

**Lemma 3.2.** *Assume that  $\varphi$  is a solution of the Dirac equation (5) with  $\varphi^+$  and  $\varphi^-$  non-vanishing spinors satisfying (4). Then, for all  $X \in \Gamma(TM)$ ,*

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \lambda X \cdot \varphi, \quad (8)$$

with

$$\eta(X) = -\frac{1}{2} \sum_{j=1}^2 e_j \cdot B(e_j, X), \quad (9)$$

where the bilinear map  $B$  is defined by (6).

The proof of this lemma will be given in Section 4.

*Proof of Proposition 3.1:* The equations of Gauss, Codazzi and Ricci appear to be the integrability conditions of (8). Indeed computing the spinorial curvature  $\mathcal{R}$  for  $\varphi$ , we first observe that (9) implies

$$X \cdot \eta(Y) - \eta(Y) \cdot X = B(X, Y) = Y \cdot \eta(X) - \eta(X) \cdot Y$$

for all  $X, Y \in TM$ . Then, a direct computation yields

$$\begin{aligned} \mathcal{R}(X, Y)\varphi &= d^\nabla \eta(X, Y) \cdot \varphi + (\eta(Y) \cdot \eta(X) - \eta(X) \cdot \eta(Y)) \cdot \varphi \quad (10) \\ &\quad + \lambda^2(Y \cdot X - X \cdot Y) \cdot \varphi, \end{aligned}$$

where

$$d^\nabla \eta(X, Y) = \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X, Y]).$$

Here we also denote by  $\nabla$  the natural connection on  $Cl(TM \oplus E) \simeq Cl(M) \hat{\otimes} Cl(E)$ , the graded tensor product of the Clifford algebras  $Cl(M)$  and  $Cl(E)$ .

**Lemma 3.3.** *We have:*

1. *The left-hand side of (10) satisfies*

$$\mathcal{R}(e_1, e_2)\varphi = -\frac{1}{2}K e_1 \cdot e_2 \cdot \varphi - \frac{1}{2}K_N e_3 \cdot e_4 \cdot \varphi.$$

2. *The first term of the right-hand side of (10) satisfies*

$$d^\nabla \eta(X, Y) = -\frac{1}{2} \sum_{j=1}^2 e_j \cdot \left( (\nabla'_X B)(Y, e_j) - (\nabla'_Y B)(X, e_j) \right)$$

where  $\nabla'$  stands for the natural connection on  $T^*M \otimes T^*M \otimes E$ .

3. *The second term of the right-hand side of (10) satisfies*

$$\begin{aligned} \eta(e_2) \cdot \eta(e_1) - \eta(e_1) \cdot \eta(e_2) &= \frac{1}{2} (|B(e_1, e_2)|^2 - \langle B(e_1, e_1), B(e_2, e_2) \rangle) e_1 \cdot e_2 \\ &\quad + \frac{1}{2} \langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle e_3 \cdot e_4. \end{aligned}$$

*Proof:* First, we compute  $\mathcal{R}(e_1, e_2)\varphi$ . We recall that  $\Sigma = \Sigma M \otimes \Sigma E$  and suppose that  $\varphi = \alpha \otimes \sigma$  with  $\alpha \in \Sigma M$  and  $\sigma \in \Sigma E$ . Thus,

$$\mathcal{R}(e_1, e_2)\varphi = \mathcal{R}^M(e_1, e_2)\alpha \otimes \sigma + \alpha \otimes \mathcal{R}^E(e_1, e_2)\sigma,$$

where  $\mathcal{R}^M$  and  $\mathcal{R}^E$  are the spinorial curvatures on  $M$  and  $E$  respectively. Moreover, by the Ricci identity on  $M$ , we have

$$\mathcal{R}^M(e_1, e_2)\alpha = -\frac{1}{2}K e_1 \cdot e_2 \cdot \alpha,$$

where  $K$  is the Gauss curvature of  $(M, g)$ . Similarly, we have

$$\mathcal{R}^E(e_1, e_2)\sigma = -\frac{1}{2}K_N e_3 \cdot e_4 \cdot \sigma,$$

where  $K_N$  is the curvature of the connection on  $E$ . These last two relations give the first point of the lemma.

For the second point of the lemma, we choose  $e_j$  so that at  $p \in M$ ,  $\nabla e_j|_p = 0$ . Then, we have

$$\begin{aligned}
d^\nabla \eta(X, Y) &= \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X, Y]) \\
&= \sum_{j=1,2} -\frac{1}{2} \nabla_X(e_j \cdot B(Y, e_j)) + \frac{1}{2} \nabla_Y(e_j \cdot B(X, e_j)) + \frac{1}{2} e_j \cdot B([X, Y], e_j) \\
&= \sum_{j=1,2} -\frac{1}{2} e_j \cdot \nabla_X^E(B(Y, e_j)) + \frac{1}{2} e_j \cdot \nabla_Y^E(B(X, e_j)) + \frac{1}{2} e_j \cdot B(\nabla_X Y, e_j) \\
&\quad - \frac{1}{2} e_j \cdot B(\nabla_Y X, e_j) \\
&= -\frac{1}{2} \sum_{j=1,2} e_j \cdot \left( (\nabla'_X B)(Y, e_j) - (\nabla'_Y B)(X, e_j) \right)
\end{aligned}$$

since  $[X, Y] = \nabla_X Y - \nabla_Y X$  and  $(\nabla'_X B)(Y, e_j) = \nabla_X^E(B(Y, e_j)) - B(\nabla_X Y, e_j)$ . Here  $\nabla^E$  stands for the given connection on  $E$ .

We finally prove the third assertion of the lemma. In order to simplify the notation, we set  $B(e_i, e_j) = B_{ij}$ . We have

$$\begin{aligned}
\eta(e_2) \cdot \eta(e_1) - \eta(e_1) \cdot \eta(e_2) &= -\frac{1}{4} \sum_{j,k=1}^2 e_j \cdot B_{1j} \cdot e_k \cdot B_{2k} + \frac{1}{4} \sum_{j,k=1}^2 e_j \cdot B_{2j} \cdot e_k \cdot B_{1k} \\
&= \frac{1}{4} \left[ -e_1 \cdot B_{11} \cdot e_1 \cdot B_{21} - e_1 \cdot B_{11} \cdot e_2 \cdot B_{22} - e_2 \cdot B_{12} \cdot e_1 \cdot B_{21} - e_2 \cdot B_{12} \cdot e_2 \cdot B_{22} \right. \\
&\quad \left. + e_1 \cdot B_{21} \cdot e_1 \cdot B_{11} + e_1 \cdot B_{21} \cdot e_2 \cdot B_{12} + e_2 \cdot B_{22} \cdot e_1 \cdot B_{11} + e_2 \cdot B_{22} \cdot e_2 \cdot B_{12} \right] \\
&= \frac{1}{2} \left[ |B_{12}|^2 - \langle B_{11}, B_{22} \rangle \right] e_1 \cdot e_2 + \frac{1}{4} \left[ -B_{11} \cdot B_{21} + B_{21} \cdot B_{11} - B_{12} \cdot B_{22} + B_{22} \cdot B_{12} \right].
\end{aligned}$$

Now, if we write  $B_{ij} = B_{ij}^3 e_3 + B_{ij}^4 e_4$ , we have

$$-B_{11} \cdot B_{21} + B_{21} \cdot B_{11} = 2(-B_{11}^3 B_{21}^4 + B_{21}^3 B_{11}^4) e_3 \cdot e_4$$

and

$$-B_{12} \cdot B_{22} + B_{22} \cdot B_{12} = 2(-B_{12}^3 B_{22}^4 + B_{12}^4 B_{22}^3) e_3 \cdot e_4.$$

Moreover

$$-B_{11}^3 B_{21}^4 + B_{21}^3 B_{11}^4 - B_{12}^3 B_{22}^4 + B_{12}^4 B_{22}^3 = \langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle$$

since for  $j \in \{1, 2\}$  and  $k \in \{3, 4\}$ , we have  $S_{e_k} e_j = B_{j1}^k e_1 + B_{j2}^k e_2$ . The formula follows.  $\square$

Now, we give this final lemma

**Lemma 3.4.** *If  $T$  is an element of  $Cl(M) \hat{\otimes} Cl(E)$  of order 2, that is of*

$$\Lambda^2 M \otimes 1 \oplus TM \otimes E \oplus 1 \otimes \Lambda^2 E,$$

so that

$$T \cdot \varphi = 0,$$

where  $\varphi$  is a spinor field of  $\Sigma$  such that  $\varphi^+$  and  $\varphi^-$  do not vanish, then  $T = 0$ .

*Proof:* We have

$$\mathcal{Cl}_2 \hat{\otimes} \mathcal{Cl}_2 \simeq \mathcal{Cl}_4 \simeq \mathbb{H}(2),$$

where  $\mathbb{H}(2)$  is the set of  $2 \times 2$  matrices with quaternionic coefficients. The spinor bundle  $\Sigma$  and the Clifford product come from the representation

$$\mathbb{H}(2) \longrightarrow \text{End}_{\mathbb{H}}(\mathbb{H} \oplus \mathbb{H}).$$

The first factor of  $\mathbb{H} \oplus \mathbb{H}$  correspond to  $\Sigma^+$  and the second to  $\Sigma^-$ . Moreover, elements of order 2 of  $\mathcal{Cl}_4$  are matrices

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

where  $p, q$  are purely imaginary quaternions. Hence  $T \cdot \varphi = 0$  is equivalent to

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} \alpha \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with  $\alpha, \sigma$  non zero quaternions. Thus  $p = q = 0$ , and so  $T$  vanishes identically.  $\square$

We deduce from (10) and Lemma 3.4 and comparing terms, that

$$\begin{cases} K = \langle B(e_1, e_1), B(e_2, e_2) \rangle - |B(e_1, e_2)|^2 + 4\lambda^2, \\ K_N = -\langle (S_{e_3} \circ S_{e_4} - S_{e_4} \circ S_{e_3})(e_1), e_2 \rangle, \\ (\nabla'_X B)(Y, e_j) - (\nabla'_Y B)(X, e_j) = 0, \quad \forall j = 1, 2, \end{cases}$$

which are respectively the Gauss, Ricci and Codazzi equations.  $\square$

From Proposition 3.1 and the fundamental theorem of submanifolds, we deduce that a spinor field solution of (5) such that (4) holds defines a local isometric immersion of  $M$  into  $\mathbb{M}^4(c)$  with normal bundle  $E$  and second fundamental form  $B$ . This implies the equivalence between assertions 1 and 3 in Theorem 1. The equivalence between assertions 1 and 2 is given by Lemma 3.2 and will be proven in the next section.

**Remark 2.** *If in Theorem 1 we assume moreover that the manifold is simply connected, the spinor field solution of (5) defines a global isometric immersion of  $M$  into  $\mathbb{M}^4(c)$ .*

## 4 Proof of Lemma 3.2

In order to prove Lemma 3.2, we need some preliminary results. First, we remark that

$$\begin{cases} D\varphi^{--} = \vec{H} \cdot \varphi^{++} - 2\lambda\varphi^{+-}, \\ D\varphi^{++} = \vec{H} \cdot \varphi^{--} - 2\lambda\varphi^{-+}, \\ D\varphi^{+-} = \vec{H} \cdot \varphi^{-+} - 2\lambda\varphi^{--}, \\ D\varphi^{-+} = \vec{H} \cdot \varphi^{+-} - 2\lambda\varphi^{++}. \end{cases}$$

We fix a point  $p \in M$ , and consider  $e_3$  a unit vector in  $E_p$  so that  $\vec{H} = |\vec{H}|e_3$  at  $p$ . We complete  $e_3$  by  $e_4$  to get a positively oriented and orthonormal frame of  $E_p$ . We first assume that  $\varphi^{--}, \varphi^{++}, \varphi^{+-}$  and  $\varphi^{-+}$  do not vanish at  $p$ . We see easily that

$$\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|} \right\}$$

is an orthonormal frame of  $\Sigma^{++}$  for the real scalar product  $\Re \langle \cdot, \cdot \rangle$ . Indeed, we have

$$\begin{aligned} \Re \langle e_1 \cdot e_3 \cdot \varphi^{--}, e_2 \cdot e_3 \cdot \varphi^{--} \rangle &= \Re \langle \varphi^{--}, e_3 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\ &= \Re (i|\varphi^{--}|^2) = 0. \end{aligned}$$

Analogously,

$$\begin{aligned} &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{++}}{|\varphi^{++}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{++}}{|\varphi^{++}|} \right\}, \\ &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{-+}}{|\varphi^{-+}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{-+}}{|\varphi^{-+}|} \right\}, \\ &\left\{ e_1 \cdot e_3 \cdot \frac{\varphi^{+-}}{|\varphi^{+-}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{+-}}{|\varphi^{+-}|} \right\} \end{aligned}$$

are orthonormal frames of  $\Sigma^{--}, \Sigma^{+-}$  and  $\Sigma^{-+}$  respectively. We define the following bilinear forms

$$\begin{aligned} F_{++}(X, Y) &= \Re \langle \nabla_X \varphi^{++}, Y \cdot e_3 \cdot \varphi^{--} \rangle, \\ F_{--}(X, Y) &= \Re \langle \nabla_X \varphi^{--}, Y \cdot e_3 \cdot \varphi^{++} \rangle, \\ F_{+-}(X, Y) &= \Re \langle \nabla_X \varphi^{+-}, Y \cdot e_3 \cdot \varphi^{-+} \rangle, \\ F_{-+}(X, Y) &= \Re \langle \nabla_X \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{+-} \rangle, \end{aligned}$$

and

$$\begin{aligned} B_{++}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{--} \rangle, \\ B_{--}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{+-}, Y \cdot e_3 \cdot \varphi^{++} \rangle, \\ B_{+-}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{--}, Y \cdot e_3 \cdot \varphi^{-+} \rangle, \\ B_{-+}(X, Y) &= -\Re \langle \lambda X \cdot \varphi^{++}, Y \cdot e_3 \cdot \varphi^{+-} \rangle. \end{aligned}$$

We have this first lemma:

**Lemma 4.1.** *We have*

1.  $\text{tr}(F_{++}) = -|\vec{H}||\varphi^{--}|^2 + 2\Re \langle \lambda \varphi^{-+}, e_3 \cdot \varphi^{--} \rangle,$
2.  $\text{tr}(F_{--}) = -|\vec{H}||\varphi^{++}|^2 + 2\Re \langle \lambda \varphi^{+-}, e_3 \cdot \varphi^{++} \rangle,$
3.  $\text{tr}(F_{+-}) = -|\vec{H}||\varphi^{-+}|^2 + 2\Re \langle \lambda \varphi^{--}, e_3 \cdot \varphi^{-+} \rangle,$
4.  $\text{tr}(F_{-+}) = -|\vec{H}||\varphi^{+-}|^2 + 2\Re \langle \lambda \varphi^{++}, e_3 \cdot \varphi^{+-} \rangle.$

This second lemma gives the defect of symmetry:

**Lemma 4.2.** *We have*

1.  $F_{++}(e_1, e_2) = F_{++}(e_2, e_1) - 2\Re \langle \lambda \varphi^{-+}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle,$
2.  $F_{--}(e_1, e_2) = F_{--}(e_2, e_1) - 2\Re \langle \lambda \varphi^{+-}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{++} \rangle,$
3.  $F_{+-}(e_1, e_2) = F_{+-}(e_2, e_1) - 2\Re \langle \lambda \varphi^{--}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{-+} \rangle,$
4.  $F_{-+}(e_1, e_2) = F_{-+}(e_2, e_1) - 2\Re \langle \lambda \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{+-} \rangle.$

For sake of brevity, we only prove Lemma 4.2. The proof of Lemma 4.1 is very similar.

*Proof:* We have

$$\begin{aligned}
F_{++}(e_1, e_2) &= \Re \langle \nabla_{e_1} \varphi^{++}, e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \langle e_1 \cdot \nabla_{e_1} \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \left\langle \vec{H} \cdot \varphi^{--} - 2\lambda \varphi^{-+} - e_2 \cdot \nabla_{e_2} \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle.
\end{aligned}$$

The first term is

$$\begin{aligned}
\Re \left\langle \vec{H} \cdot \varphi^{--}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle &= -\Re \left\langle \varphi^{--}, i\vec{H} \cdot e_3 \cdot \varphi^{--} \right\rangle \\
&= -\Re \left( i|\vec{H}| |\varphi^{--}|^2 \right) = 0.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
F_{++}(e_1, e_2) + 2\Re \langle \lambda \varphi^{-+}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle &= -\Re \langle e_2 \cdot \nabla_{e_2} \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \langle \nabla_{e_2} \varphi^{++}, e_2 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= \Re \langle \nabla_{e_2} \varphi^{++}, e_1 \cdot e_3 \cdot \varphi^{--} \rangle \\
&= F_{++}(e_2, e_1).
\end{aligned}$$

The proof is similar for the three other forms.  $\square$

By analogous computations, we also get the following lemmas:

**Lemma 4.3.** *We have*

1.  $tr(B_{++}) = -2\Re \langle \lambda \varphi^{-+}, e_3 \cdot \varphi^{--} \rangle,$
2.  $tr(B_{--}) = -2\Re \langle \lambda \varphi^{+-}, e_3 \cdot \varphi^{++} \rangle,$
3.  $tr(B_{+-}) = -2\Re \langle \lambda \varphi^{--}, e_3 \cdot \varphi^{-+} \rangle,$
4.  $tr(B_{-+}) = -2\Re \langle \lambda \varphi^{++}, e_3 \cdot \varphi^{+-} \rangle.$

**Lemma 4.4.** *We have*

1.  $B_{++}(e_1, e_2) = B_{++}(e_2, e_1) + 2\Re \langle \lambda \varphi^{-+}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \rangle,$
2.  $B_{--}(e_1, e_2) = B_{--}(e_2, e_1) + 2\Re \langle \lambda \varphi^{+-}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{++} \rangle,$
3.  $B_{+-}(e_1, e_2) = B_{+-}(e_2, e_1) + 2\Re \langle \lambda \varphi^{--}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{-+} \rangle,$
4.  $B_{-+}(e_1, e_2) = B_{-+}(e_2, e_1) + 2\Re \langle \lambda \varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{+-} \rangle.$

Now, we set

$$\begin{cases} A_{++} := F_{++} + B_{++}, \\ A_{--} := F_{--} + B_{--}, \\ A_{+-} := F_{+-} + B_{+-}, \\ A_{-+} := F_{-+} + B_{-+}, \end{cases}$$

and

$$F_+ = \frac{A_{++}}{|\varphi^{--}|^2} - \frac{A_{--}}{|\varphi^{++}|^2} \quad \text{and} \quad F_- = \frac{A_{+-}}{|\varphi^{-+}|^2} - \frac{A_{-+}}{|\varphi^{+-}|^2}.$$

From the last four lemmas we deduce immediately that  $F_+$  and  $F_-$  are symmetric and trace-free. Moreover, by a direct computation using the conditions (4) on the norms of  $\varphi^+$  and  $\varphi^-$ , we get the following lemma:

**Lemma 4.5.** *The symmetric operators  $F^+$  and  $F^-$  of  $TM$  associated to the bilinear forms  $F_+$  and  $F_-$ , defined by*

$$F^+(X) = F_+(X, e_1)e_1 + F_+(X, e_2)e_2 \quad \text{and} \quad F^-(X) = F_-(X, e_1)e_1 + F_-(X, e_2)e_2$$

for all  $X \in TM$ , satisfy

1.  $\Re \langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle = 0,$
2.  $\Re \langle F^-(X) \cdot e_3 \cdot \varphi^{-+}, \varphi^{+-} \rangle = 0.$

*Proof.* Since

$$A_{++}(X, Y) = \Re \langle \nabla_X \varphi^{++} - \lambda X \cdot \varphi^{-+}, Y \cdot e_3 \cdot \varphi^{--} \rangle,$$

and since  $(e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|})$  is an orthonormal frame of  $\Sigma^{++}$ , we have

$$\begin{aligned} & \Re \langle \nabla_X \varphi^{++} - \lambda X \cdot \varphi^{-+}, \varphi^{++} \rangle \\ = & \frac{A_{++}}{|\varphi^{--}|^2}(X, e_1) \Re \langle e_1 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle + \frac{A_{++}}{|\varphi^{--}|^2}(X, e_2) \Re \langle e_2 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} & \Re \langle \nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-}, \varphi^{--} \rangle \\ = & \frac{A_{--}}{|\varphi^{++}|^2}(X, e_1) \Re \langle e_1 \cdot e_3 \cdot \varphi^{++}, \varphi^{--} \rangle + \frac{A_{--}}{|\varphi^{++}|^2}(X, e_2) \Re \langle e_2 \cdot e_3 \cdot \varphi^{++}, \varphi^{--} \rangle \\ = & -\frac{A_{--}}{|\varphi^{++}|^2}(X, e_1) \Re \langle e_1 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle - \frac{A_{--}}{|\varphi^{++}|^2}(X, e_2) \Re \langle e_2 \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle. \end{aligned}$$

These two formulas imply that

$$\Re \langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \rangle = \Re \langle \nabla_X \varphi^+ - \lambda X \cdot \varphi^-, \varphi^+ \rangle;$$

by the first condition in (4), this last expression is zero.  $\square$

Hence, the operators  $F^+$  and  $F^-$  are of rank at most  $\leq 1$ . Since they are symmetric and trace-free, they vanish identically.

Using again that  $(e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}, e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|})$  is an orthonormal frame of  $\Sigma^{++}$ , we have

$$\nabla_X \varphi^{++} = F_{++}(X, e_1) e_1 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|} + F_{++}(X, e_2) e_2 \cdot e_3 \cdot \frac{\varphi^{--}}{|\varphi^{--}|}.$$

Since  $F_{++} = A_{++} - B_{++}$  and denoting by  $A^{++}$  and  $B^{++}$  the symmetric operators of  $TM$  associated to  $A_{++}$  and  $B_{++}$  and defined by

$$A^{++}(X) = A_{++}(X, e_1) e_1 + A_{++}(X, e_2) e_2, \quad B^{++}(X) = B_{++}(X, e_1) e_1 + B_{++}(X, e_2) e_2,$$

we get

$$\nabla_X \varphi^{++} = \frac{1}{|\varphi^{--}|^2} [A^{++}(X) \cdot e_3 \cdot \varphi^{--} - B^{++}(X) \cdot e_3 \cdot \varphi^{--}]. \quad (11)$$

Similarly, if  $A^{--}$  and  $B^{--}$  denote the symmetric operators of  $TM$  associated to  $A_{--}$  and  $B_{--}$ , we have

$$\nabla_X \varphi^{--} = \frac{1}{|\varphi^{++}|^2} [A^{--}(X) \cdot e_3 \cdot \varphi^{++} - B^{--}(X) \cdot e_3 \cdot \varphi^{++}]. \quad (12)$$

Moreover, we easily get

$$B^{++}(X) \cdot e_3 \cdot \varphi^{--} = -|\varphi^{--}|^2 \lambda X \cdot \varphi^{-+} \quad \text{and} \quad B^{--}(X) \cdot e_3 \cdot \varphi^{++} = -|\varphi^{++}|^2 \lambda X \cdot \varphi^{+-}.$$

Thus

$$\begin{aligned} \nabla_X \varphi^+ &= \frac{1}{|\varphi^{--}|^2} A^{++}(X) \cdot e_3 \cdot \varphi^{--} + \lambda X \cdot \varphi^{-+} \\ &\quad + \frac{1}{|\varphi^{++}|^2} A^{--}(X) \cdot e_3 \cdot \varphi^{++} + \lambda X \cdot \varphi^{+-}. \end{aligned}$$

Setting  $A^+ = A^{++} + A^{--}$  we get from the definition of  $A^{++}$  and  $A^{--}$  and from  $F^+ = 0$  that  $\frac{A^+}{|\varphi^+|^2} = \frac{A^{++}}{|\varphi^{++}|^2}$ . Bearing in mind that  $|\varphi^+|^2 = |\varphi^{++}|^2 + |\varphi^{--}|^2$ , we get finally

$$\frac{A^+}{|\varphi^+|^2} = \frac{A^{++}}{|\varphi^{--}|^2} = \frac{A^{--}}{|\varphi^{++}|^2}. \quad (13)$$

Thus

$$\nabla_X \varphi^+ = \frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \cdot \varphi^+ + \lambda X \cdot \varphi^-. \quad (14)$$

Similarly, denoting by  $A^{+-}$  and  $A^{-+}$  the symmetric operators of  $TM$  associated to  $A_{+-}$  and  $A_{-+}$ , setting  $A^- = A^{+-} + A^{-+}$  and using  $F^- = 0$  we get

$$\begin{aligned} \nabla_X \varphi^- &= \frac{1}{|\varphi^{+-}|^2} A^{-+}(X) \cdot e_3 \cdot \varphi^{+-} + \lambda X \cdot \varphi^{++} \\ &\quad + \frac{1}{|\varphi^{-+}|^2} A^{+-}(X) \cdot e_3 \cdot \varphi^{-+} + \lambda X \cdot \varphi^{--} \\ &= \frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3 \cdot \varphi^- + \lambda X \cdot \varphi^+. \end{aligned} \quad (15)$$

We now observe that formulas (14) and (15) also hold if  $\varphi^{++}$  or  $\varphi^{--}$ , (resp.  $\varphi^{+-}$  or  $\varphi^{-+}$ ) vanishes at  $p$ : indeed, assuming for instance that  $\varphi^{++}(p) = 0$ ,

and thus that  $\varphi^{--}(p) \neq 0$  since  $\varphi^+(p) \neq 0$ , equation (11) holds, and, from the first condition in (4),

$$\Re \langle \nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-}, \varphi^{--} \rangle = 0.$$

Since  $\left( \frac{\varphi^{--}}{|\varphi^{--}|}, i \frac{\varphi^{--}}{|\varphi^{--}|} \right)$  is an orthonormal basis of  $\Sigma^{--}$ , we deduce that

$$\nabla_X \varphi^{--} - \lambda X \cdot \varphi^{+-} = i\alpha(X) \frac{\varphi^{--}}{|\varphi^{--}|}$$

for some real 1-form  $\alpha$ . Since  $D\varphi^{--} + 2\lambda\varphi^{+-} = 0$  ( $\varphi^{++} = 0$  at  $p$ ), this implies that

$$(\alpha(e_1)e_1 + \alpha(e_2)e_2) \cdot \frac{\varphi^{--}}{|\varphi^{--}|} = 0,$$

and thus that  $\alpha = 0$ . We thus get  $\nabla_X \varphi^{--} = \lambda X \cdot \varphi^{+-}$  instead of (12), which, together with (11), easily implies (14).

Now, we set

$$\eta^+(X) = \left( \frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \right)^+ \quad \text{and} \quad \eta^-(X) = \left( \frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3 \right)^-$$

where, if  $\sigma$  belongs to  $\mathcal{C}l^0(TM \oplus E)$ , we denote by  $\sigma^+ := \frac{1+\omega_4}{2} \cdot \sigma$  and by  $\sigma^- := \frac{1-\omega_4}{2} \cdot \sigma$  the parts of  $\sigma$  acting on  $\Sigma^+$  and on  $\Sigma^-$  only, i.e., such that

$$\sigma^+ \cdot \varphi = \sigma \cdot \varphi^+ \in \Sigma^+ \quad \text{and} \quad \sigma^- \cdot \varphi = \sigma \cdot \varphi^- \in \Sigma^-.$$

Setting  $\eta = \eta^+ + \eta^-$  we thus get

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \lambda X \cdot \varphi,$$

as claimed in Lemma 3.2.

Explicitely, setting  $A_+(X, Y) := \langle A^+(X), Y \rangle$  and  $A_-(X, Y) := \langle A^-(X), Y \rangle$ , the form  $\eta$  is given by

$$\begin{aligned} \eta(X) &= \frac{1}{2|\varphi^+|^2} [A_+(X, e_1)(e_1 \cdot e_3 - e_2 \cdot e_4) + A_+(X, e_2)(e_2 \cdot e_3 + e_1 \cdot e_4)] \\ &\quad + \frac{1}{2|\varphi^-|^2} [A_-(X, e_1)(e_1 \cdot e_3 + e_2 \cdot e_4) + A_-(X, e_2)(e_2 \cdot e_3 - e_1 \cdot e_4)] \end{aligned}$$

with

$$A_+(X, Y) = \Re \langle \nabla_X \varphi^+ - \lambda X \cdot \varphi^-, Y \cdot e_3 \cdot \varphi^+ \rangle$$

and

$$A_-(X, Y) = \Re \langle \nabla_X \varphi^- - \lambda X \cdot \varphi^+, Y \cdot e_3 \cdot \varphi^- \rangle.$$

By direct computations we get that

$$B(X, Y) := X \cdot \eta(Y) - \eta(Y) \cdot X$$

is a vector belonging to  $E$  which is such that

$$\begin{aligned} \langle B(X, Y), \xi \rangle &= \frac{1}{|\varphi^+|^2} \Re \langle X \cdot \nabla_Y \varphi^+ - \lambda X \cdot Y \cdot \varphi^-, \xi \cdot \varphi^+ \rangle \\ &\quad + \frac{1}{|\varphi^-|^2} \Re \langle X \cdot \nabla_Y \varphi^- - \lambda X \cdot Y \cdot \varphi^+, \xi \cdot \varphi^- \rangle \end{aligned}$$

for all  $\xi \in E$ . This last expression appears to be symmetric in  $X, Y$  (the proof is analogous to the proof of the symmetry of  $A_{++} = F_{++} + B_{++}$  above). Computing

$$\langle B(X, Y), \xi \rangle = \frac{1}{2} (\langle B(X, Y), \xi \rangle + \langle B(Y, X), \xi \rangle)$$

we finally obtain that  $B$  is given by formula (6).

Since  $B(e_j, X) = e_j \cdot \eta(X) - \eta(X) \cdot e_j$ , we obtain

$$\sum_{j=1,2} e_j \cdot B(e_j, X) = -2\eta(X) - \sum_{j=1,2} e_j \cdot \eta(X) \cdot e_j.$$

Writing  $\eta(X)$  in the form  $\sum_k e_k \cdot n_k$  for some vectors  $n_k$  belonging to  $E$ , we easily get that  $\sum_j e_j \cdot \eta(X) \cdot e_j = 0$ . Thus

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(e_j, X).$$

The last claim in Lemma 3.2 is proved.  $\square$

## 5 Weierstrass representation of surfaces in $\mathbb{R}^4$

We are interested here in isometric immersions in euclidean space  $\mathbb{R}^4$  (thus  $c = \lambda = 0$ ); we obtain that the immersions are given by a formula which generalizes the representation formula given by T. Friedrich in [5]. Such a formula was also found in [4] using a different method involving twistor theory.

We consider the scalar product  $\langle\langle \cdot, \cdot \rangle\rangle$  defined on  $\Sigma^+$  by

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \Sigma^+ \times \Sigma^+ &\rightarrow \mathbb{H} \\ (\varphi^+, \psi^+) &\mapsto \overline{[\psi^+]}[\varphi^+], \end{aligned}$$

where  $[\varphi^+]$  and  $[\psi^+] \in \mathbb{H}$  represent the spinors  $\varphi^+$  and  $\psi^+$  in some frame, and where, if  $q = q_1 I + q_2 I + q_3 J + q_4 K$  belongs to  $\mathbb{H}$ ,

$$\bar{q} = q_1 1 - q_2 I - q_3 J - q_4 K.$$

We also define the product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\Sigma^-$  by an analogous formula:

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : \Sigma^- \times \Sigma^- &\rightarrow \mathbb{H} \\ (\varphi^-, \psi^-) &\mapsto \overline{[\psi^-]}[\varphi^-]. \end{aligned}$$

The following properties hold: for all  $\varphi, \psi \in \Sigma$  and all  $X \in TM \oplus E$ ,

$$\langle\langle \varphi^+, \psi^+ \rangle\rangle = \overline{\langle\langle \psi^+, \varphi^+ \rangle\rangle}, \quad \langle\langle \varphi^-, \psi^- \rangle\rangle = \overline{\langle\langle \psi^-, \varphi^- \rangle\rangle} \quad (16)$$

and

$$\langle\langle X \cdot \varphi^+, \psi^- \rangle\rangle = -\langle\langle \varphi^+, X \cdot \psi^- \rangle\rangle. \quad (17)$$

Assume that we have a spinor  $\varphi$  solution of the Dirac equation  $D\varphi = \vec{H} \cdot \varphi$  so that  $|\varphi^+| = |\varphi^-| = 1$ , and define the  $\mathbb{H}$ -valued 1-form  $\xi$  by

$$\xi(X) = \langle\langle X \cdot \varphi^-, \varphi^+ \rangle\rangle \in \mathbb{H}.$$

**Proposition 5.1.** *The form  $\xi \in \Omega^1(M, \mathbb{H})$  is closed.*

*Proof:* By a straightforward computation, we get

$$d\xi(e_1, e_2) = \langle \langle e_2 \cdot \nabla_{e_1} \varphi^-, \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \nabla_{e_2} \varphi^-, \varphi^+ \rangle \rangle + \langle \langle e_2 \cdot \varphi^-, \nabla_{e_1} \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \varphi^-, \nabla_{e_2} \varphi^+ \rangle \rangle.$$

First observe that

$$\begin{aligned} \langle \langle e_2 \cdot \nabla_{e_1} \varphi^-, \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \nabla_{e_2} \varphi^-, \varphi^+ \rangle \rangle &= -\langle \langle e_1 \cdot \nabla_{e_1} \varphi^-, e_1 \cdot e_2 \cdot \varphi^+ \rangle \rangle + \langle \langle e_2 \cdot \nabla_{e_2} \varphi^-, e_2 \cdot e_1 \cdot \varphi^+ \rangle \rangle \\ &= -\langle \langle D\varphi^-, e_1 \cdot e_2 \cdot \varphi^+ \rangle \rangle \end{aligned}$$

and similarly that

$$\begin{aligned} \langle \langle e_2 \cdot \varphi^-, \nabla_{e_1} \varphi^+ \rangle \rangle - \langle \langle e_1 \cdot \varphi^-, \nabla_{e_2} \varphi^+ \rangle \rangle &= \langle \langle e_1 \cdot e_2 \cdot \varphi^-, e_1 \cdot \nabla_{e_1} \varphi^+ \rangle \rangle - \langle \langle e_2 \cdot e_1 \cdot \varphi^-, e_2 \cdot \nabla_{e_2} \varphi^+ \rangle \rangle \\ &= \langle \langle e_1 \cdot e_2 \cdot \varphi^-, D\varphi^+ \rangle \rangle. \end{aligned}$$

Thus

$$d\xi(e_1, e_2) = \langle \langle e_1 \cdot e_2 \cdot D\varphi^-, \varphi^+ \rangle \rangle + \langle \langle e_1 \cdot e_2 \cdot \varphi^-, D\varphi^+ \rangle \rangle.$$

Since  $D\varphi = \vec{H} \cdot \varphi$ , then  $D\varphi^+ = \vec{H} \cdot \varphi^+$  and  $D\varphi^- = \vec{H} \cdot \varphi^-$ , which implies

$$d\xi(e_1, e_2) = \langle \langle (e_1 \cdot e_2 \cdot \vec{H} - \vec{H} \cdot e_1 \cdot e_2) \cdot \varphi^-, \varphi^+ \rangle \rangle = 0.$$

□

Assuming that  $M$  is simply connected, there exists a function  $F : M \rightarrow \mathbb{H}$  so that  $dF = \xi$ . We now identify  $\mathbb{H}$  to  $\mathbb{R}^4$  in the natural way.

**Theorem 2.** 1. *The map  $F = (F_1, F_2, F_3, F_4) : M \rightarrow \mathbb{R}^4$  is an isometry.*

2. *The map*

$$\begin{aligned} \Phi_E : E &\longrightarrow M \times \mathbb{R}^4 \\ X \in E_m &\longmapsto (F(m), \xi_1(X), \xi_2(X), \xi_3(X), \xi_4(X)) \end{aligned}$$

*is an isometry between  $E$  and the normal bundle  $N(F(M))$  of  $F(M)$  into  $\mathbb{R}^4$ , preserving connections and second fundamental forms.*

*Proof.* Note first that the euclidean norm of  $\xi \in \mathbb{R}^4 \simeq \mathbb{H}$  is

$$|\xi|^2 = \langle \xi, \xi \rangle = \bar{\xi} \xi \in \mathbb{R},$$

and more generally that the real scalar product  $\langle \xi, \xi' \rangle$  of  $\xi, \xi' \in \mathbb{R}^4 \simeq \mathbb{H}$  is the component of 1 in  $\langle \langle \xi, \xi' \rangle \rangle = \bar{\xi}' \xi \in \mathbb{H}$ . We first compute, for all  $X, Y$  belonging to  $E \cup TM$ ,

$$\begin{aligned} \overline{\xi(Y)} \xi(X) &= \overline{\langle \langle Y \cdot \varphi^-, \varphi^+ \rangle \rangle} \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle = \overline{([\varphi^+][Y \cdot \varphi^-])} ([\varphi^+][X \cdot \varphi^-]) \\ &= \overline{[Y \cdot \varphi^-]} [X \cdot \varphi^-] \end{aligned}$$

since  $[\varphi^+][\overline{[\varphi^+]}] = 1$  ( $|\varphi^+| = 1$ ). Here and below the brackets  $[\cdot]$  stand for the components ( $\in \mathbb{H}$ ) of the spinor fields in some local frame. Thus

$$\overline{\xi(Y)} \xi(X) = \langle \langle X \cdot \varphi^-, Y \cdot \varphi^- \rangle \rangle, \quad (18)$$

which in particular implies (considering the components of  $I$  of these quaternions)

$$\langle \xi(X), \xi(Y) \rangle = \Re \langle X \cdot \varphi^-, Y \cdot \varphi^- \rangle. \quad (19)$$

This last identity easily gives

$$\langle \xi(X), \xi(Y) \rangle = 0 \quad \text{and} \quad |\xi(Z)|^2 = |Z|^2 \quad (20)$$

for all  $X \in TM$ ,  $Y \in E$  and  $Z \in E \cup TM$ . Thus  $F = \int \xi$  is an isometry, and  $\xi$  maps isometrically the bundle  $E$  into the normal bundle of  $F(M)$  in  $\mathbb{R}^4$ .

We now prove that  $\xi$  preserves the normal connection and the second fundamental form: let  $X \in TM$  and  $Y \in \Gamma(E) \cup \Gamma(TM)$ ; then  $\xi(Y)$  is a vector field normal or tangent to  $F(M)$ . Considering  $\xi(Y)$  as a map  $M \rightarrow \mathbb{R}^4 \simeq \mathbb{H}$ , we have

$$\begin{aligned} d(\xi(Y))(X) &= d\langle Y \cdot \varphi^-, \varphi^+ \rangle(X) \\ &= \langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle + \langle Y \cdot \nabla_X \varphi^-, \varphi^+ \rangle + \langle Y \cdot \varphi^-, \nabla_X \varphi^+ \rangle \end{aligned} \quad (21)$$

where the connection  $\nabla_X Y$  denotes the connection on  $E$  (if  $Y \in \Gamma(E)$ ) or the Levi-Civita connection on  $TM$  (if  $Y \in \Gamma(TM)$ ). We will need the following formulas:

**Lemma 5.2.** *We have*

$$\begin{aligned} \langle \langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle, \xi(\nu) \rangle &= \Re \langle \nabla_X Y \cdot \varphi^-, \nu \cdot \varphi^- \rangle \\ &= \Re \langle \nabla_X Y \cdot \varphi^+, \nu \cdot \varphi^+ \rangle, \\ \langle \langle Y \cdot \nabla_X \varphi^-, \varphi^+ \rangle, \xi(\nu) \rangle &= \Re \langle Y \cdot \nabla_X \varphi^-, \nu \cdot \varphi^- \rangle \end{aligned}$$

and

$$\langle \langle Y \cdot \varphi^-, \nabla_X \varphi^+ \rangle, \xi(\nu) \rangle = \Re \langle Y \cdot \nabla_X \varphi^+, \nu \cdot \varphi^+ \rangle.$$

In the expressions above,  $\langle \cdot, \cdot \rangle$  defined on  $\mathbb{H}$  for the left-hand side and  $\Re \langle \cdot, \cdot \rangle$  defined on  $\Sigma$  for the right-hand side of each identity, stand for the natural real scalar products.

*Proof.* The first identity is a consequence of (19) and the second identity may be obtained by a very similar computation observing that, by (16)-(17),

$$\langle \langle \nabla_X Y \cdot \varphi^-, \varphi^+ \rangle, \xi(\nu) \rangle = \langle \langle \nabla_X Y \cdot \varphi^+, \varphi^- \rangle, \langle \nu \cdot \varphi^+, \varphi^- \rangle \rangle.$$

The last two identities may be obtained by very similar computations.  $\square$

From (21) and the lemma, we readily get the formula

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \frac{1}{2} \Re \langle \nabla_X Y \cdot \varphi, \nu \cdot \varphi \rangle + \Re \langle Y \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle. \quad (22)$$

We first suppose that  $X, Y \in \Gamma(TM)$ . The first term in the right-hand side of the equation above vanishes in that case since  $\nabla_X Y \in \Gamma(TM)$ ,  $\nu \in \Gamma(E)$ . Recalling (7), we get then

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \Re \langle Y \cdot \nabla_X \varphi, \nu \cdot \varphi \rangle = \langle B(X, Y), \nu \rangle = \langle \xi(B(X, Y)), \xi(\nu) \rangle.$$

Hence the component of  $d(\xi(Y))(X)$  normal to  $F(M)$  is given by

$$(d(\xi(Y))(X))^N = \xi(B(X, Y)). \quad (23)$$

We now suppose that  $X \in \Gamma(TM)$  and  $Y \in \Gamma(E)$ . We first observe that the second term in the right-hand side of equation (22) vanishes. Indeed, if  $(e_3, e_4)$  stands for an orthonormal basis of  $E$ , for all  $i, j \in \{3, 4\}$  we have

$$\Re \langle e_i \cdot \nabla_X \varphi, e_j \cdot \varphi \rangle = -\Re \langle \nabla_X \varphi, e_i \cdot e_j \cdot \varphi \rangle = -\Re \langle \eta(X) \cdot \varphi, e_i \cdot e_j \cdot \varphi \rangle,$$

which is a sum of terms of the form  $\Re \langle e \cdot \varphi, e' \cdot \varphi \rangle$  with  $e$  and  $e'$  belonging to  $TM$  and  $E$  respectively; these terms are therefore all equal to zero. Thus, (22) simplifies to

$$\langle d(\xi(Y))(X), \xi(\nu) \rangle = \frac{1}{2} \Re \langle \nabla_X Y \cdot \varphi, \nu \cdot \varphi \rangle = \langle \xi(\nabla_X Y), \xi(\nu) \rangle.$$

Hence

$$(d(\xi(Y))(X))^N = \xi(\nabla_X Y). \quad (24)$$

Equations (23) and (24) mean that  $\Phi_E = \xi$  preserves the second fundamental form and the normal connection respectively.  $\square$

**Remark 3.** The immersion  $F : M \rightarrow \mathbb{R}^4$  given by the fundamental theorem is thus

$$F = \int \xi = \left( \int \xi_1, \int \xi_2, \int \xi_3, \int \xi_4 \right).$$

This formula generalizes the classical Weierstrass representation: let  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  be the  $\mathbb{C}$ -linear forms defined by

$$\alpha_k(X) = \xi_k(X) - i\xi_k(JX),$$

for  $k = 1, 2, 3, 4$ , where  $J$  is the natural complex structure of  $M$ . Let  $z$  be a conformal parameter of  $M$ , and let  $\psi_1, \psi_2, \psi_3, \psi_4 : M \rightarrow \mathbb{C}$  be such that

$$\alpha_1 = \psi_1 dz, \quad \alpha_2 = \psi_2 dz, \quad \alpha_3 = \psi_3 dz, \quad \alpha_4 = \psi_4 dz.$$

By an easy computation using  $D\varphi = \vec{H} \cdot \varphi$ , we see that  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are holomorphic forms if and only if  $M$  is a minimal surface ( $\vec{H} = \vec{0}$ ). Then if  $M$  is minimal,

$$\begin{aligned} F &= \operatorname{Re} \left( \int \alpha_1, \int \alpha_2, \int \alpha_3, \int \alpha_4 \right) \\ &= \operatorname{Re} \left( \int \psi_1 dz, \int \psi_2 dz, \int \psi_3 dz, \int \psi_4 dz \right) \end{aligned}$$

where  $\psi_1, \psi_2, \psi_3, \psi_4$  are holomorphic functions. This is the Weierstrass representation of minimal surfaces.

**Remark 4.** Theorem 2 also gives a spinorial proof of the fundamental theorem. We may integrate the Gauss, Ricci and Codazzi equations in two steps:

1- first solving

$$\nabla_X \varphi = \eta(X) \cdot \varphi, \quad (25)$$

where

$$\eta(X) = -\frac{1}{2} \sum_{j=1,2} e_j \cdot B(e_j, X)$$

(there is a unique solution in  $\Gamma(\Sigma)$ , up to the natural right-action of  $Spin(4)$ );

2- then solving

$$dF = \xi$$

where  $\xi(X) = \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle$  (the solution is unique, up to translations).

Indeed, equation (25) is solvable, since its conditions of integrability are exactly the Gauss, Ricci and Codazzi equations; see the proof of Theorem 1. Moreover, the multiplication of  $\varphi$  on the right by a constant belonging to  $Spin(4)$  in the first step, and the addition to  $F$  of a constant belonging to  $\mathbb{R}^4$  in the second step, correspond to a rigid motion in  $\mathbb{R}^4$ .

## 6 Surfaces in $\mathbb{R}^3$ and $S^3$ .

The aim of this section is to obtain as particular cases the spinor characterizations of T. Friedrich [5] and B. Morel [13] of surfaces in  $\mathbb{R}^3$  and  $S^3$ . Assume that  $M^2 \subset \mathcal{H}^3 \subset \mathbb{R}^4$ , where  $\mathcal{H}^3$  is a hyperplane, or a sphere of  $\mathbb{R}^4$ . Let  $N$  be a unit vector field such that

$$T\mathcal{H} = TM \oplus_{\perp} \mathbb{R}N.$$

The intrinsic spinors of  $M$  identify with the spinors of  $\mathcal{H}$  restricted to  $M$ , which in turn identify with the positive spinors of  $\mathbb{R}^4$  restricted to  $M$  :

**Proposition 6.1.** *There is an identification*

$$\begin{aligned} \Sigma M &\xrightarrow{\sim} \Sigma_M^+ \\ \psi &\mapsto \psi^* \end{aligned}$$

such that

$$(\nabla\psi)^* = \nabla(\psi^*)$$

and such that the Clifford actions are linked by

$$(X \cdot_M \psi)^* = N \cdot X \cdot \psi^*$$

for all  $X \in TM$  and all  $\psi \in \Sigma M$ .

Using this identification, the intrinsic Dirac operator on  $M$  defined by

$$D_M\psi := e_1 \cdot_M \nabla_{e_1}\psi + e_2 \cdot_M \nabla_{e_2}\psi$$

is linked to  $D$  by

$$(D_M\psi)^* = N \cdot D\psi^*.$$

If  $\varphi \in \Gamma(\Sigma)$  is a solution of

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad |\varphi^+| = |\varphi^-| = 1$$

then  $\varphi^+ \in \Sigma^+$  may be considered as belonging to  $\Sigma M$ ; it satisfies

$$D_M\varphi^+ = N \cdot D\varphi^+ = N \cdot \vec{H} \cdot \varphi^+. \quad (26)$$

We examine separately the case of a surface in a hyperplane, and in a 3-dimensional sphere:

1. If  $\mathcal{H}$  is a hyperplane, then  $\vec{H}$  is of the form  $HN$ , and (26) reads

$$D_M\varphi^+ = -H\varphi^+. \quad (27)$$

This is the equation considered by T. Friedrich in [5].

2. If  $\mathcal{H} = S^3$ , then  $\vec{H}$  is of the form  $HN - \nu$ , where  $\nu$  is the outer unit normal of  $S^3$ , and (26) reads

$$D_M\varphi^+ = -H\varphi^+ - i\overline{\varphi^+}. \quad (28)$$

This equation is obtained by B. Morel in [13].

Conversely, we now suppose that  $\psi$  is an intrinsic spinor field on  $M$  solution of (27) or (28). The aim is to construct a spinor field  $\varphi$  in dimension 4 which induces an immersion in a hyperplane, or in a 3-sphere. Define  $E = M \times \mathbb{R}^2$ , with its natural metric  $\langle \cdot, \cdot \rangle$  and its trivial connection  $\nabla'$ , and consider  $\nu, N \in \Gamma(E)$  such that

$$|\nu| = |N| = 1, \quad \langle \nu, N \rangle = 0 \quad \text{and} \quad \nabla'\nu = \nabla'N = 0.$$

We first consider the case of an hyperplane:

**Proposition 6.2.** *Let  $\psi \in \Gamma(\Sigma M)$  be a solution of*

$$D_M\psi = -H\psi$$

*of constant length  $|\psi| = 1$ . There exists  $\varphi \in \Gamma(\Sigma)$  solution of*

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad |\varphi^+| = |\varphi^-| = 1, \quad (29)$$

*with  $\vec{H} = HN$ , such that*

$$\varphi^+ = \psi$$

*and the normal vector field*

$$\xi(\nu) = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle$$

*has a fixed direction in  $\mathbb{H}$ . In particular, the immersion given by  $\varphi$  belongs to the hyperplane  $\xi(\nu)^\perp$  of  $\mathbb{H}$ . The spinor field  $\varphi$  is unique, up to the natural right-action of  $S^3$  on  $\varphi^-$ .*

*Proof:* define  $\varphi = (\varphi^+, \varphi^-)$  by

$$\varphi^+ = \psi, \quad \varphi^- = -\nu \cdot \psi.$$

We compute:

$$D\varphi^- = \nu \cdot D\varphi^+ = \nu \cdot \vec{H} \cdot \varphi^+ = \vec{H} \cdot \varphi^-,$$

$$\xi(\nu) = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle = 1,$$

and, for all  $X \in TM$ ,

$$\xi(X) = \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle = -\langle \langle X \cdot \nu \cdot \psi, \psi \rangle \rangle = \langle \langle \psi, X \cdot \nu \cdot \psi \rangle \rangle = \overline{\langle \langle X \cdot \nu \cdot \psi, \psi \rangle \rangle} = -\overline{\xi(X)},$$

that is  $\xi(X) \in \Im m(\mathbb{H})$ , the hyperplane of pure imaginary quaternions. Thus  $F = \int \xi$  also belongs to the hyperplane  $\Im m(\mathbb{H})$ . Uniqueness is straightforward.  $\square$

We now consider the case of the 3-sphere:

**Proposition 6.3.** *Let  $\psi \in \Gamma(\Sigma M)$  be a solution of*

$$D_M \psi = -H\psi - i\bar{\psi}$$

*of constant length  $|\psi| = 1$ . There exists  $\varphi \in \Gamma(\Sigma)$  solution of*

$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad |\varphi^+| = |\varphi^-| = 1, \quad (30)$$

*with  $\vec{H} = HN - \nu$ , such that*

$$\varphi^+ = \psi$$

*and the immersion  $F$  defined by  $\varphi$  is given by the unit normal vector field  $\xi(\nu)$ :*

$$F = \xi(\nu) = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle.$$

*In particular  $F(M)$  belongs to the sphere  $S^3 \subset \mathbb{H}$ . The spinor field  $\varphi$  is unique, up to the natural right-action of  $S^3$  on  $\varphi^-$ .*

*Proof:* The system

$$\begin{cases} F = \langle \langle \nu \cdot \varphi^-, \varphi^+ \rangle \rangle \\ dF(X) = \langle \langle X \cdot \varphi^-, \varphi^+ \rangle \rangle \end{cases}$$

is equivalent to

$$\varphi^- = -\nu \cdot \varphi^+ \cdot F$$

where  $F : M \rightarrow \mathbb{H}$  solves the equation

$$dF(X) = \beta(X)F \quad (31)$$

in  $\mathbb{H}$ , with

$$\beta(X) = -\langle \langle X \cdot \nu \cdot \varphi^+, \varphi^+ \rangle \rangle.$$

By a direct computation, the compatibility equation

$$d\beta(X, Y) = \beta(X)\beta(Y) - \beta(Y)\beta(X)$$

of (31) is satisfied, and equation (31) is solvable. Uniqueness is straightforward.  $\square$

**Remark 5.** *Let  $M$  be a minimal surface in  $S^3$  and  $N$  be such that*

$$TM \oplus_{\perp} \mathbb{R}N = TS^3.$$

*For any  $x \in S^3$ , denote by  $\vec{x} = \overrightarrow{0x}$  the position vector of  $x$ . At  $x \in M$ ,  $\vec{H} = -\vec{x}$ . Thus,  $M \subset S^3$  is represented by a solution  $\varphi \in \Gamma(\Sigma)$  of*

$$D\varphi = -\vec{x} \cdot \varphi.$$

*The spinor field*

$$\tilde{\varphi} := (\varphi^+, N \cdot \varphi^+)$$

*defines a surface of constant mean curvature  $H = -1$  in  $\Im m(\mathbb{H}) \simeq \mathbb{R}^3$ . This is a classical transformation, described by B. Lawson in [12], and by T. Friedrich using spinors in dimension 3 in [5].*

**Remark 6.** In [13] B. Morel also obtained a spinor characterization of an isometric immersion of a surface in three-dimensional hyperbolic space  $\mathbb{H}^3$ . Very similarly to the case of a surface in  $\mathbb{R}^3$  or in  $S^3$  described above, such a characterization may be naturally obtained from a spinor characterization of a spacelike surface in four-dimensional Minkowski space; see [3].

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