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# CYCLIC VECTORS IN KORENBLUM TYPE SPACES

ABDELOUAHAB HANINE

ABSTRACT. In this paper we use the technique of premeasures, introduced by Korenblum in the 1970-s, to give a characterization of cyclic functions in the Korenblum type spaces  $\mathcal{A}_\Lambda^-$ . In particular, we give a positive answer to a conjecture by Deninger [7, Conjecture 42].

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ . Suppose that  $X$  is a topological vector space of analytic functions on  $\mathbb{D}$ , with the property that  $zf \in X$  whenever  $f \in X$ . Multiplication by  $z$  is thus an operator on  $X$ , and if  $X$  is a Banach space, then it is automatically a bounded operator on space  $X$ . A closed subspace  $M \subset X$  (Banach space) is said to be invariant (or  $z$ -invariant) provided that  $zM \subset M$ . For a function  $f \in X$ , the closed linear span in  $X$  of all polynomial multiples of  $f$  is a  $z$ -invariant subspace denoted by  $[f]_X$ ; it is also the smallest closed  $z$ -invariant subspace of  $X$  which contains  $f$ . A function  $f$  in  $X$  is said to be cyclic (or weakly invertible) in  $X$  if  $[f]_X = X$ . For some information on cyclic functions see [3] and the references therein. In the case when  $X = A^2(\mathbb{D})$  is the Bergman space, defined as

$$A^2(\mathbb{D}) = \left\{ f \text{ analytic in } \mathbb{D} : \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty \right\},$$

a singular inner function  $S_\mu$ ,

$$S_\mu(z) := \exp -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta), \quad z \in \mathbb{D},$$

is cyclic in  $A^2(\mathbb{D})$  if and only if its associated positive singular measure  $\mu$  places no mass on any  $\Lambda$ -Carleson set for  $\Lambda(t) = \log(1/t)$ .  $\Lambda$ -Carleson sets constitute a class of thin subsets of  $\mathbb{T}$ , they will be discussed shortly. The necessity of this Carleson set condition was proved by H. S. Shapiro in 1967 [21, Theorem 2], and the sufficiency was proved independently by Korenblum in 1977 [17] and Roberts in 1979 [19, Theorem 2].

In the following a majorant  $\Lambda$  will always denote a positive non-increasing convex differentiable function on  $(0,1]$  such that:

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- $\Lambda(0) = +\infty$
- $t\Lambda(t)$  is a continuous, non-decreasing and concave function on  $[0, 1]$ , and  $t\Lambda(t) \rightarrow 0$  as  $t \rightarrow 0$ .
- There exists  $\alpha \in (0, 1)$  such that  $t^\alpha \Lambda(t)$  is non-decreasing.
- 

$$\Lambda(t^2) \leq C\Lambda(t). \quad (1.1)$$

Typical examples of majorants  $\Lambda$  are  $\log^+ \log^+(1/x)$ ,  $(\log(1/x))^p$ ,  $p > 0$ .

In this work, we shall be interested mainly in studying cyclic vectors in the case  $X = \mathcal{A}_\Lambda^{-\infty}$ , generalizing the theory of premeasures developed by Korenblum; here  $\mathcal{A}_\Lambda^{-\infty}$  is the Korenblum type space associated with the majorant  $\Lambda$ , defined by

$$\mathcal{A}_\Lambda^{-\infty} = \bigcup_{c>0} \mathcal{A}_\Lambda^{-c} = \bigcup_{c>0} \left\{ f \in \text{Hol}(\mathbb{D}) : |f(z)| \leq \exp(c\Lambda(1 - |z|)) \right\}.$$

With the norm

$$\|f\|_{\mathcal{A}_\Lambda^{-c}} = \sup_{z \in \mathbb{D}} |f(z)| \exp(-c\Lambda(1 - r)) < \infty,$$

$\mathcal{A}_\Lambda^{-c}$  becomes a Banach space and for every  $c_2 \geq c_1 > 0$ , the inclusion  $\mathcal{A}_\Lambda^{-c_1} \hookrightarrow \mathcal{A}_\Lambda^{-c_2}$  is continuous. The topology on

$$\mathcal{A}_\Lambda^{-\infty} = \bigcup_{c>0} \mathcal{A}_\Lambda^{-c},$$

is the locally-convex inductive limit topology, i.e. each of the inclusions  $\mathcal{A}_\Lambda^{-c} \hookrightarrow \mathcal{A}_\Lambda^{-\infty}$  is continuous and the topology is the largest locally-convex topology with this property. A sequence  $\{f_n\}_n \in \mathcal{A}_\Lambda^{-\infty}$  converges to  $f \in \mathcal{A}_\Lambda^{-\infty}$  if and only if there exists  $N > 0$  such that all  $f_n$  and  $f$  belong to  $\mathcal{A}_\Lambda^{-N}$ , and  $\lim_{n \rightarrow +\infty} \|f_n - f\|_{\mathcal{A}_\Lambda^{-N}} = 0$ .

The notion of a premeasure (a distribution of the first class) and the definition of the  $\Lambda$ -boundedness property of premeasure was first introduced in [15], for the case of  $\Lambda(t) = \log(1/t)$  in connection with an extension of the Nevanlinna theory (see also [16] and [11, Chapter 7]). Later on, in [18], Korenblum introduced a space of  $\Lambda$ -smooth functions and proved that the so called premeasures of bounded  $\Lambda$ -variation are the bounded linear functionals on this space. Next, he established that any premeasure of bounded  $\Lambda$ -variation is the difference of two  $\Lambda$ -bounded premeasures [18, p.542]. Finally, he described the Poisson integrals of  $\Lambda$ -bounded premeasures.

Our paper is organized as follows: In Section 2, we first introduce the notion of a  $\Lambda$ -bounded premeasure, and we will prove, using some arguments of real-variable theory, a general approximation theorem for  $\Lambda$ -bounded premeasures which will be critical for describing the cyclic vectors in  $\mathcal{A}_\Lambda^{-\infty}$ . Furthermore, this theorem shows that in respect to some general measure-theoretical properties, premeasure with vanishing  $\Lambda$ -singular part (see definition 2.4), behave themselves in some ways like absolutely continuous measures in the classical theory.

In Section 3, we show that every  $\Lambda$ -bounded premeasure  $\mu$  generates a harmonic function  $h(z)$  in  $\mathbb{D}$  (the Poisson integral of  $\mu$ ) such that

$$h(z) = O(\Lambda(1 - |z|)), \quad |z| \rightarrow 1, z \in \mathbb{D}, \quad (1.2)$$

by the formula

$$h(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu.$$

Conversely, every real harmonic function  $h(z)$  in  $\mathbb{D}$ , satisfying  $h(0) = 0$  and (1.2) is the Poisson integral of a  $\Lambda$ -bounded premeasure. (This result is formulated in [18, p.543] without proof, in a more general situation).

Finally, in Section 4 we characterize cyclic vectors in the spaces  $\mathcal{A}_{\Lambda}^{-\infty}$  in terms of vanishing the  $\Lambda$ -singular part of the corresponding premeasure. We prove two results for two different growth ranges of the majorant  $\Lambda$ . At the end we give two examples that show how the cyclicity property of a fixed function changes in a scale of  $\mathcal{A}_{\Lambda_{\alpha}}$  spaces,  $\Lambda_{\alpha}(x) = (\log(1/x))^{\alpha}$ ,  $0 < \alpha < 1$ .

Throughout the paper we use the following notation: given two functions  $f$  and  $g$  defined on  $\Delta$  we write  $f \asymp g$  if for some  $0 < c_1 \leq c_2 < \infty$  we have  $c_1 f \leq g \leq c_2 f$  on  $\Delta$ .

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## 2. $\Lambda$ -BOUNDED PREMEASURES

In this section we extend the results of two papers by Korenblum [15, 16] on  $\Lambda$ -bounded premeasures (see also [11, Chapter 7]) from the case  $\Lambda(t) = \log(1/t)$  to the general case.

Let  $\mathcal{B}(\mathbb{T})$  be the set of all (open, half-open and closed) arcs of  $\mathbb{T}$  including all the single points and the empty set. The elements of  $\mathcal{B}(\mathbb{T})$  will be called intervals.

**Definition 2.1.** *A real function defined on  $\mathcal{B}(\mathbb{T})$  is called a premeasure if the following conditions hold:*

- (1)  $\mu(\mathbb{T}) = 0$
- (2)  $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$  for every  $I_1, I_2 \in \mathcal{B}(\mathbb{T})$  such that  $I_1 \cap I_2 = \emptyset$  and  $I_1 \cup I_2 \in \mathcal{B}(\mathbb{T})$
- (3)  $\lim_{n \rightarrow +\infty} \mu(I_n) = 0$  for every sequence of embedded intervals,  $I_{n+1} \subset I_n$ ,  $n \geq 1$ , such that  $\bigcap_n I_n = \emptyset$ .

Given a premeasure  $\mu$ , we introduce a real valued function  $\hat{\mu}$  on  $(0, 2\pi]$  defined as follows:

$$\hat{\mu}(\theta) = \mu(I_{\theta}),$$

where

$$I_{\theta} = \{\xi \in \mathbb{T} : 0 \leq \arg \xi < \theta\}.$$

The function  $\hat{\mu}$  satisfies the following properties:

- (a)  $\hat{\mu}(\theta^-)$  exists for every  $\theta \in (0, 2\pi]$  and  $\hat{\mu}(\theta^+)$  exists for every  $\theta \in [0, 2\pi)$
- (b)  $\hat{\mu}(\theta) = \lim_{t \rightarrow \theta^-} \hat{\mu}(t)$  for all  $\theta \in (0, 2\pi]$
- (c)  $\hat{\mu}(2\pi) = \lim_{\theta \rightarrow 0^+} \hat{\mu}(\theta) = 0$ .

Furthermore, the function  $\overset{\wedge}{\mu}(\theta)$  has at most countably many points of discontinuity.

**Definition 2.2.** A real premeasure  $\mu$  is said to be  $\Lambda$ -bounded, if there is a positive number  $C_\mu$  such that

$$\mu(I) \leq C_\mu |I| \Lambda(|I|) \quad (2.1)$$

for any interval  $I$ .

The minimal number  $C_\mu$  is called the norm of  $\mu$  and is denoted by  $\|\mu\|_\Lambda^+$ ; the set of all real premeasures  $\mu$  such that  $\|\mu\|_\Lambda^+ < +\infty$  is denoted by  $B_\Lambda^+$ .

**Definition 2.3.** A sequence of premeasures  $\{\mu_n\}_n$  is said to be  $\Lambda$ -weakly convergent to a premeasure  $\mu$  if :

- (1)  $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$ , and
- (2) for every point  $\theta$  of continuity of  $\overset{\wedge}{\mu}$  we have  $\lim_{n \rightarrow \infty} \overset{\wedge}{\mu}_n(\theta) = \overset{\wedge}{\mu}(\theta)$ .

In this situation, the limit premeasure  $\mu$  is  $\Lambda$ -bounded.

Given a closed non-empty subset  $F$  of the unit circle  $\mathbb{T}$ , we define its  $\Lambda$ -entropy as follows:

$$Entr_\Lambda(F) = \sum_n |I_n| \Lambda(|I_n|),$$

where  $\{I_n\}_n$  are the component arcs of  $\mathbb{T} \setminus F$ , and  $|I|$  denotes the normalized Lebesgue measure of  $I$  on  $\mathbb{T}$ . We set  $Entr_\Lambda(\emptyset) = 0$ .

We say that a closed set  $F$  is a  $\Lambda$ -Carleson set if  $F$  is non-empty, has Lebesgue measure zero (i.e.  $|F| = 0$ ), and  $Entr_\Lambda(F) < +\infty$ .

Denote by  $\mathcal{C}_\Lambda$  the set of all  $\Lambda$ -Carleson sets and by  $\mathcal{B}_\Lambda$  the set of all Borel sets  $B \subset \mathbb{T}$  such that  $\overline{B} \in \mathcal{C}_\Lambda$ .

**Definition 2.4.** A function  $\sigma : \mathcal{B}_\Lambda \rightarrow \mathbb{R}$  is called a  $\Lambda$ -singular measure if

- (1)  $\sigma$  is a finite Borel measure on every set in  $\mathcal{C}_\Lambda$  (i.e.  $\sigma|_F$  is a Borel measure on  $\mathbb{T}$ ).
- (2) There is a constant  $C > 0$  such that

$$|\sigma(F)| \leq C Entr_\Lambda(F)$$

for all  $F \in \mathcal{C}_\Lambda$ .

Given a premeasure  $\mu$  in  $B_\Lambda^+$ , its  $\Lambda$ -singular part is defined by :

$$\mu_s(F) = - \sum_n \mu(I_n), \quad (2.2)$$

where  $F \in \mathcal{C}_\Lambda$  and  $\{I_n\}_n$  is the collection of complementary intervals to  $F$  in  $\mathbb{T}$ . Using the argument in [15, Theorem 6] one can see that  $\mu_s$  extends to a  $\Lambda$ -singular measure on  $\mathcal{B}_\Lambda$ .

**Proposition 2.5.** If  $\mu$  is a  $\Lambda$ -bounded premeasure,  $F \in \mathcal{C}_\Lambda$ , then  $\mu_s|_F$  is finite and non-positive.

*Proof.* Let  $F \in \mathcal{C}_\Lambda$ . We are to prove that  $\mu_s(F) \leq 0$ .

Let  $\{I_n\}_n$  be the (possibly finite) sequence of the intervals complementary to  $F$  in  $\mathbb{T}$ . For  $N \geq 1$ , we consider the disjoint intervals  $\{J_n^N\}_{1 \leq n \leq N}$  such that  $\mathbb{T} \setminus \bigcup_{n=1}^N I_n = \bigcup_n J_n^N$ . Then

$$-\sum_{n=1}^N \mu(I_n) = \sum_{n=1}^N \mu(J_n^N) \leq \|\mu\|_\Lambda^+ \sum_{n=1}^N |J_n^N| \Lambda(|J_n^N|).$$

Furthermore, each interval  $J_n^N$  is covered by intervals  $I_m \subset J_n^N$  up to a set of measure zero, and  $\max_{1 \leq n \leq N} |J_n^N| \rightarrow 0$  as  $N \rightarrow \infty$  (If the sequence  $\{I_n\}_n$  is finite, then all  $J_n^N$  are single points for the corresponding  $N$ ). Therefore,

$$-\sum_{n=1}^N \mu(I_n) \leq \|\mu\|_\Lambda^+ \sum_{n=1}^N \sum_{I_m \subset J_n^N} |I_m| \Lambda(|I_m|) \leq \|\mu\|_\Lambda^+ \sum_{n > N} |I_n| \Lambda(|I_n|).$$

Since  $F$  is a  $\Lambda$ -Carleson set,

$$-\lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(I_n) \leq 0.$$

Thus,  $\mu_s|_F \leq 0$ . □

Given a closed subset  $F$  of  $\mathbb{T}$ , we denote by  $F^\delta$  its  $\delta$ -neighborhood:

$$F^\delta = \{\zeta \in \mathbb{T} : d(\zeta, F) \leq \delta\}.$$

**Proposition 2.6.** *Let  $\mu$  be a  $\Lambda$ -bounded premeasure and let  $\mu_s$  be its  $\Lambda$ -singular part. Then for every  $F \in \mathcal{C}_\Lambda$  we have*

$$\mu_s(F) = \lim_{\delta \rightarrow 0} \mu(F^\delta). \quad (2.3)$$

*Proof.* Let  $F \in \mathcal{C}_\Lambda$ , and let  $\{I_n\}_n$ ,  $|I_1| \geq |I_2| \geq \dots$ , be the intervals of the complement to  $F$  in  $\mathbb{T}$ . We set

$$I_n^{(\delta)} = \{e^{i\theta} : \text{dist}(e^{i\theta}, \mathbb{T} \setminus I_n) > \delta\}.$$

Then for  $|I_n| \geq 2\delta$ , we have

$$I_n = I_n^1 \sqcup I_n^{(\delta)} \sqcup I_n^2$$

with  $|I_n^1| = |I_n^2| = \delta$ . We see that

$$\mu(F^\delta) = - \sum_{|I_n| > 2\delta} \mu(I_n^{(\delta)}).$$

Using relation (2.2) we obtain that

$$\begin{aligned} -\mu_s(F) &= \sum_n \mu(I_n) \\ &= \sum_{|I_n| \leq 2\delta} \mu(I_n) + \sum_{|I_n| > 2\delta} [\mu(I_n^1) + \mu(I_n^{(\delta)}) + \mu(I_n^2)] \\ &= \sum_{|I_n| \leq 2\delta} \mu(I_n) - \mu(F^\delta) + \sum_{|I_n| > 2\delta} [\mu(I_n^1) + \mu(I_n^2)]. \end{aligned}$$

Therefore,

$$\mu(F^\delta) - \mu_s(F) = \sum_{|I_n| \leq 2\delta} \mu(I_n) + \sum_{|I_n| > 2\delta} [\mu(I_n^1) + \mu(I_n^2)]$$

The first sum tends to zero as  $\delta \rightarrow 0$ , and it remains to prove that

$$\lim_{\delta \rightarrow 0} \sum_{|I_n| > 2\delta} \mu(I_n^1) = 0. \quad (2.4)$$

We have

$$\sum_{|I_n| > 2\delta} \mu(I_n^1) \leq C \sum_{|I_n| > \delta} \delta \Lambda(\delta) = C \sum_{|I_n| > \delta} \frac{\delta \Lambda(\delta)}{|I_n| \Lambda(|I_n|)} \cdot |I_n| \Lambda(|I_n|).$$

Since the function  $t \mapsto t\Lambda(t)$  does not decrease, we have

$$\frac{\delta \Lambda(\delta)}{|I_n| \Lambda(|I_n|)} \leq 1, \quad |I_n| > \delta.$$

Furthermore,

$$\lim_{\delta \rightarrow 0} \frac{\delta \Lambda(\delta)}{|I_n| \Lambda(|I_n|)} = 0, \quad n \geq 1.$$

Since

$$\sum_{n \geq 1} |I_n| \Lambda(|I_n|) < \infty,$$

we conclude that (2.4), and, hence, (2.3) hold.  $\square$

**Definition 2.7.** A premeasure  $\mu$  in  $B_\Lambda^+$  is said to be  $\Lambda$ -absolutely continuous if there exists a sequence of  $\Lambda$ -bounded premeasures  $(\mu_n)_n$  such that:

- (1)  $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$ .
- (2)  $\sup_{I \in \mathcal{B}(\mathbb{T})} |(\mu + \mu_n)(I)| \rightarrow 0$  as  $n \rightarrow +\infty$ .

**Theorem 2.8.** Let  $\mu$  be a premeasure in  $B_\Lambda^+$ . Then  $\mu$  is  $\Lambda$ -absolutely continuous if and only if its  $\Lambda$ -singular part  $\mu_s$  is zero.

The only if part holds in a more general situation considered by Korenblum, [18, Corollary, p.544]. On the other hand, the if part does not hold for differences of  $\Lambda$ -bounded premeasures (premeasures of  $\Lambda$ -bounded variation), see [18, Remark, p.544].

To prove this theorem we need several lemmas. The first one is a linear programming lemma from [11, Chapter 7].

**Lemma 2.9.** Consider the following system of  $N(N+1)/2$  linear inequalities in  $N$  variables  $x_1, \dots, x_N$

$$\sum_{j=k}^l x_j \leq b_{k,l}, \quad 1 \leq k \leq l \leq N,$$

subject to the constraint:  $x_1 + x_2 + \cdots + x_N = 0$ . This system has a solution if and only if

$$\sum_n b_{k_n, l_n} \geq 0$$

for every simple covering  $\mathcal{P} = \{[k_n, l_n]\}_n$  of  $[1, N]$ .

The following lemma gives a necessary and sufficient conditions for a premeasure in  $B_\Lambda^+$  to be  $\Lambda$ -absolutely continuous.

**Lemma 2.10.** *Let  $\mu$  be a  $\Lambda$ -bounded premeasure. Then  $\mu$  is  $\Lambda$ -absolutely continuous if and only if there is a positive constant  $C > 0$  such that for every  $\varepsilon > 0$  there exists a positive  $M$  such that the system*

$$\begin{cases} x_{k,l} & \leq M|I_{k,l}|\Lambda(|I_{k,l}|) \\ \mu(I_{k,l}) + x_{k,l} & \leq \min\{C|I_{k,l}|\Lambda(|I_{k,l}|), \varepsilon\} \\ x_{k,l} & = \sum_{s=k}^{l-1} x_{s,s+1} \\ x_{0,N} & = 0 \end{cases} \quad (2.5)$$

in variables  $x_{k,l}$ ,  $0 \leq k < l \leq N$ , has a solution for every positive integer  $N$ . Here  $I_{k,l}$  are the half-open arcs of  $\mathbb{T}$  defined by

$$I_{k,l} = \left\{ e^{i\theta} : 2\pi \frac{k}{N} \leq \theta < 2\pi \frac{l}{N} \right\}.$$

*Proof.* Suppose that  $\mu$  is  $\Lambda$ -absolutely continuous and denote by  $\{\mu_n\}$  a sequence of  $\Lambda$ -bounded premeasures satisfying the conditions of Definition 2.7. Set

$$C = \sup_n \|\mu + \mu_n\|_\Lambda^+, \quad M = \sup_n \|\mu_n\|_\Lambda^+,$$

and let  $\varepsilon > 0$ . For large  $n$ , the numbers  $x_{k,l} = \mu_n(I_{k,l})$ ,  $0 \leq k < l \leq N$ , satisfy relations (2.5) for all  $N$ .

Conversely, suppose that for some  $C > 0$  and for every  $\varepsilon > 0$  there exists  $M = M(\varepsilon) > 0$  such that for every  $N$  there are  $\{x_{k,l}\}_{k,l}$  (depending on  $N$ ) satisfying relations (2.5). We consider the measures  $d\mu_N$  defined on  $I_{s,s+1}$ ,  $0 \leq s < N$ , by

$$d\mu_N(\xi) = \frac{x_{s,s+1}}{|I_{s,s+1}|} |d\xi|,$$

where  $|d\xi|$  is normalized Lebesgue measure on the unit circle  $\mathbb{T}$ . To show that  $\mu_N \in B_\Lambda^+$ , it suffices to verify that the quantity  $\sup_I \frac{\mu(I)}{|I|\Lambda(|I|)}$  is finite for every interval  $I \in \mathcal{B}(\mathbb{T})$ . Fix  $I \in \mathcal{B}(\mathbb{T})$  such that  $1 \notin I$ .

If  $I \subset I_{k,k+1}$ , then

$$\mu_N(I) = \frac{x_{k,k+1}}{|I_{k,k+1}|} |I| \leq \frac{x_{k,k+1}}{|I_{k,k+1}|\Lambda(|I_{k,k+1}|)} |I|\Lambda(|I|) \leq M|I|\Lambda(|I|).$$

If  $I = I_{k,l}$ , then

$$\mu_N(I_{k,l}) = \sum_{s=k}^{l-1} \mu_N(I_{s,s+1}) = \sum_{s=k}^{l-1} x_{s,s+1} = x_{k,l} \leq M|I_{k,l}|\Lambda(|I_{k,l}|).$$



Otherwise, denote by  $I_{k,l}$  the largest interval such that  $I_{k,l} \subset I$ . We have

$$\begin{aligned} \mu_N(I) &= \mu_N(I_{k,l}) + \mu_N(I \setminus I_{k,l}) \\ &\leq M|I_{k,l}|\Lambda(|I_{k,l}|) + \max(x_{k-1,k}, 0) + \max(x_{l,l+1}, 0) \\ &\leq 3M|I_{k,l}|\Lambda(|I_{k,l}|) \leq 3M|I|\Lambda(|I|). \end{aligned}$$

Thus,  $\mu_N$  is a  $\Lambda$ -bounded premeasure. Next, using a Helly-type selection theorem for premeasures due to Cyphert and Kelingos [6, Theorem 2], we can find a  $\Lambda$ -bounded premeasure  $\nu$  and a subsequence  $\mu_{N_k} \in B_\Lambda^+$  such that  $\{\mu_{N_k}\}_k$  converge  $\Lambda$ -weakly to  $\nu$ . Furthermore,  $\nu$  satisfies the following conditions:

$$\nu(J) \leq 3M|J|\Lambda(|J|) \text{ and } \mu(J) + \nu(J) \leq \min\{C|J|\Lambda(|J|), \varepsilon\} \text{ for every interval } J \subset \mathbb{T} \setminus \{1\}.$$

Now, if  $I$  is an interval containing the point 1, we can represent it as  $I = I_1 \sqcup \{1\} \sqcup I_2$ , for some (possibly empty) intervals  $I_1$  and  $I_2$ . Then

$$\begin{aligned} \mu(I) + \nu(I) &= (\mu + \nu)(I_1) + (\mu + \nu)(I_2) + (\mu + \nu)(\{1\}) \\ &\leq (\mu + \nu)(I_1) + (\mu + \nu)(I_2). \end{aligned}$$

Therefore, for every  $I \in \mathcal{B}(\mathbb{T})$  we have  $\mu(J) + \nu(J) \leq 2\varepsilon$ . Since  $(\mu + \nu)(\mathbb{T} \setminus I) = -\mu(I) - \nu(I)$ , we have

$$|\mu(J) + \nu(J)| \leq 2\varepsilon.$$

Thus  $\mu$  is  $\Lambda$ -absolutely continuous.  $\square$

**Lemma 2.11.** *Let  $\mu \in B_\Lambda^+$  be not  $\Lambda$ -absolutely continuous. Then for every  $C > 0$  there is  $\varepsilon > 0$  such that for all  $M > 0$ , there exists a simple covering of  $\mathbb{T}$  by a finite number of half-open intervals  $\{I_n\}_n$ , satisfying the relation*

$$\sum_n \min\{\mu(I_n) + M|I_n|\Lambda(|I_n|), C|I_n|\Lambda(|I_n|), \varepsilon\} < 0.$$

*Proof.* By Lemma 2.10, for every  $C > 0$  there exists a number  $\varepsilon > 0$  such that for all  $M > 0$ , the system (2.5) has no solutions for some  $N \in \mathbb{N}$ . In other words, there are no  $\{x_{k,l}\}_{k,l}$  such that:

$$\sum_{s=k}^{l-1} \mu(I_{s,s+1}) + x_{s,s+1} \leq \min\{\mu(I_{k,l}) + M|I_{k,l}|\Lambda(|I_{k,l}|), C|I_{k,l}|\Lambda(|I_{k,l}|), \varepsilon\} \quad (2.6)$$

with  $x_{k,l} = \sum_{s=k}^{l-1} x_{s,s+1}$  and  $x_{0,N} = 0$ .

We set  $X_j = \mu(I_{j,j+1}) + x_{j,j+1}$ , and

$$b_{k,l} = \min\{\mu(I_{k,l+1}) + M|I_{k,l+1}|\Lambda(|I_{k,l+1}|), C|I_{k,l+1}|\Lambda(|I_{k,l+1}|), \varepsilon\}.$$

Then relations (2.6) are rewritten as

$$\sum_{j=k}^l X_j \leq b_{k,l}, \quad 0 \leq k < l \leq N - 1.$$

Therefore, we are in the conditions of Lemma 2.9 with variables  $X_j$ . We conclude that there is a simple covering of the circle  $\mathbb{T}$  by a finite number of half-open intervals  $\{I_n\}$  such that

$$\sum_n \min \{ \mu(I_n) + M|I_n|\Lambda(|I_n|), C|I_n|\Lambda(|I_n|), \varepsilon \} < 0.$$

□

In the following lemma we give a normal families type result for the  $\Lambda$ -Carleson sets.

**Lemma 2.12.** *Let  $\{F_n\}_n$  be a sequence of sets on the unit circle, and let each  $F_n$  be a finite union of closed intervals. We assume that*

- (i)  $|F_n| \rightarrow 0, \quad n \rightarrow \infty,$
- (ii)  $\text{Entr}_\Lambda(F_n) = O(1), \quad n \rightarrow \infty.$

*Then there exists a subsequence  $\{F_{n_k}\}_k$  and a  $\Lambda$ -Carleson set  $F$  such that: for every  $\delta > 0$  there is a natural number  $N$  with*

- (a)  $F_{n_k} \subset F^\delta$
- (b)  $F \subset F_{n_k}^\delta.$

*for all  $k \geq N$ .*

*Proof.* Let  $\{I_{k,n}\}_k$  be the complementary arcs to  $F_n$  such that  $|I_{1,n}| \geq |I_{2,n}| \geq \dots$ . We show first that the sequence  $\{|I_{1,n}|\}_n$  is bounded away from zero. Since the function  $\Lambda$  is non-increasing, we have

$$\text{Entr}_\Lambda(F_n) = \sum_k |I_{k,n}| \Lambda(|I_{k,n}|) \geq |\mathbb{T} \setminus F_n| \Lambda(|I_{1,n}|),$$

and therefore,

$$\frac{\text{Entr}_\Lambda(F_n)}{|\mathbb{T} \setminus F_n|} \geq \Lambda(|I_{1,n}|).$$

Now the conditions (i) and (ii) of lemma and the fact that  $\Lambda(0^+) = +\infty$  imply that the sequence  $\{|I_{1,n}|\}_n$  is bounded away from zero.

Given a subsequence  $\{F_k^{(m)}\}_k$  of  $F_n$ , we denote by  $(I_{j,k}^{(m)})_j$  the complementary arcs to  $F_k^{(m)}$ . Let us choose a subsequence  $\{F_k^{(1)}\}_k$  such that

$$I_{1,k}^{(1)} = (a_k^{(1)}, b_k^{(1)}) \rightarrow (a^1, b^1) = J_1$$

as  $k \rightarrow +\infty$ , where  $J_1$  is a non-empty open arc.

If  $|J_1| = 1$ , then  $F = \mathbb{T} \setminus J_1$  is a  $\Lambda$ -Carleson set, and we are done: we can take  $\{F_{n_k}\}_k = \{F_k^{(1)}\}_k$ .

Otherwise, if  $|J_1| < 1$ , then, using the above method we show that

$$\Lambda(|I_{2,k}^{(1)}|) \leq \frac{\text{Entr}_\Lambda(F_k^{(1)})}{|\mathbb{T} \setminus F_k^{(1)}| - |I_{1,k}^{(1)}|}.$$

Since  $\lim_{k \rightarrow +\infty} |\mathbb{T} \setminus F_k^{(1)}| - |I_{1,k}^{(1)}| = 1 - |J_1| > 0$ , the sequence  $\Lambda(|I_{2,k}^{(1)}|)$  is bounded, and hence, the sequence  $|I_{2,k}^{(1)}|$  is bounded away from zero. Next we choose a subsequence

$\{F_k^{(2)}\}_k$  of  $\{F_k^{(1)}\}_k$  such that the arcs  $I_{2,k}^{(2)} = (a_k^2, b_k^2)$  tend to  $(a^{(2)}, b^{(2)}) = J_2$ , where  $J_2$  is a non-empty open arc. Repeating this process we can have two possibilities. First, suppose that after a finite number of steps we have  $|J_1| + \dots + |J_m| = 1$ , and then we can take  $\{F_{n_k}\}_k = \{F_k^{(m)}\}_k$ ,

$$I_{j,k}^{(m)} \rightarrow J_j, \quad 1 \leq j \leq m,$$

as  $k \rightarrow +\infty$ , and  $F = \mathbb{T} \setminus \bigcup_{j=1}^m J_j$  is  $\Lambda$ -Carleson.

Now, if the number of steps is infinite, then using the estimate

$$\Lambda(|J_l|) \leq \frac{\sup_n \{Entr_\Lambda(F_n)\}}{1 - \sum_{k=1}^{l-1} |J_k|},$$

and the fact  $|J_m| \rightarrow 0$  as  $m \rightarrow \infty$ , we conclude that

$$\sum_{j=1}^{\infty} |J_j| = 1.$$

We can set  $\{F_{n_k}\}_k = \{F_m^{(m)}\}_m$ ,  $F = \mathbb{T} \setminus \bigcup_{j \geq 1} J_j$ .

In all three situations the properties (a) and (b) follow automatically.  $\square$

**Proof of Theorem 2.8.** First we suppose that  $\mu$  is  $\Lambda$ -absolutely continuous, and prove that  $\mu_s = 0$ . Choose a sequence  $\mu_n$  of  $\Lambda$ -bounded premeasures satisfying the properties (1) and (2) of Definition 2.7. Let  $F$  be a  $\Lambda$ -Carleson set and let  $(I_n)_n$  be the sequence of the complementary arcs to  $F$ . Denote by  $(\mu + \mu_n)_s$  the  $\Lambda$ -singular part of  $\mu + \mu_n$ . Then

$$\begin{aligned} -(\mu + \mu_n)_s(F) &= \sum_k (\mu + \mu_n)(I_k) \\ &= \sum_{k \leq N} (\mu + \mu_n)(I_k) + \sum_{k > N} (\mu + \mu_n)(I_k) \\ &\leq \sum_{k \leq N} (\mu + \mu_n)(I_k) + C \sum_{k > N} |I_k| \Lambda(|I_k|) \end{aligned}$$

Using the property (2) of Definition 2.7 we obtain that

$$-\liminf_{n \rightarrow \infty} (\mu + \mu_n)_s(F) \leq C \sum_{k > N} |I_k| \Lambda(|I_k|).$$

Since  $F \in \mathcal{C}_\Lambda$ , we have  $\sum_{k > N} |I_k| \Lambda(|I_k|) \rightarrow 0$  as  $N \rightarrow +\infty$ , and hence  $\liminf_{n \rightarrow \infty} (\mu + \mu_n)_s(F) \geq 0$ . Since  $(\mu + \mu_n) \in B_\Lambda^+$ , by Proposition 2.5 its  $\Lambda$ -singular part is non-positive. Thus  $\lim_{n \rightarrow \infty} (\mu + \mu_n)_s(F) = 0$  for all  $F \in \mathcal{C}_\Lambda$ , which proves that  $\mu_s = 0$ .

Now, let us suppose that  $\mu$  is not  $\Lambda$ -absolutely continuous. We apply Lemma 2.11 with  $C = 4\|\mu\|_\Lambda^+$  and find  $\varepsilon > 0$  such that for all  $M > 0$ , there is a simple covering of circle  $\mathbb{T}$  by a half-open intervals  $\{I_1, I_2, \dots, I_N\}$  such that

$$\sum_n \min \{ \mu(I_n) + M|I_n| \Lambda(|I_n|), 4\|\mu\|_\Lambda^+ |I_n| \Lambda(|I_n|), \varepsilon \} < 0. \quad (2.7)$$

Let us fix a number  $\rho > 0$  satisfying the inequality  $\rho\Lambda(\rho) \leq \varepsilon/4\|\mu\|_\Lambda^+$ . We divide the intervals  $\{I_1, I_2, \dots, I_N\}$  into two groups. The first group  $\{I_n^{(1)}\}_n$  consists of intervals  $I_n$  such that

$$\min\{\mu(I_n) + M|I_n|\Lambda(|I_n|), 4\|\mu\|_\Lambda^+|I_n|\Lambda(|I_n|), \varepsilon\} = \mu(I_n) + M|I_n|\Lambda(|I_n|), \quad (2.8)$$

and the second one is  $\{I_n^{(2)}\}_n = \{I_n\}_n \setminus \{I_n^{(1)}\}_n$ .

Using these definitions and the fact that  $\Lambda$  is non-increasing, we rewrite inequality (2.7) as

$$\begin{aligned} \sum_n \mu(I_n^{(1)}) + M \sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) \\ < -4\|\mu\|_\Lambda^+ \sum_{n: |I_n^{(2)}| < \rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) - \varepsilon \text{Card}\{n : |I_n^{(2)}| \geq \rho\}. \end{aligned} \quad (2.9)$$

Next we establish three properties of these families of intervals. From now on we assume that  $M > 4\|\mu\|_\Lambda^+$ .

(1) We have  $\{I_n^{(2)} : |I_n^{(2)}| \geq \rho\} \neq \emptyset$ . Otherwise, by (2.9), we would have

$$\begin{aligned} 0 &= \mu(\mathbb{T}) = \sum_n \mu(I_n^{(1)}) + \sum_n \mu(I_n^{(2)}) \\ &\leq -M \sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) - 4\|\mu\|_\Lambda^+ \sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) + \|\mu\|_\Lambda^+ \sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \\ &\leq -M \sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) - 3\|\mu\|_\Lambda^+ \sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) < 0. \end{aligned}$$

(2) We have  $\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \leq 2\Lambda(\rho)$ . To prove this relation, we notice first that for every simple covering  $\{J_n\}_n$  of  $\mathbb{T}$ , we have

$$0 = \mu(\mathbb{T}) = \sum_n \mu(J_n) = \sum_n \mu(J_n)^+ - \sum_n \mu(J_n)^-,$$

and hence,

$$\sum_n |\mu(J_n)| = \sum_n \mu(J_n)^+ + \sum_n \mu(J_n)^- = 2 \sum_n \mu(J_n)^+ \leq 2\|\mu\|_\Lambda^+ \sum_n |J_n|\Lambda(|J_n|).$$

Applying this to our simple covering, we get

$$\sum_n |\mu(I_n^{(1)})| + \sum_n |\mu(I_n^{(2)})| \leq 2\|\mu\|_\Lambda^+ \sum_n \left[ |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \right],$$

and hence,

$$-\sum_n \mu(I_n^{(1)}) \leq 2\|\mu\|_\Lambda^+ \sum_n \left[ |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \right].$$

Now, using (2.9) we obtain that

$$M \sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + 4\|\mu\|_\Lambda^+ \sum_{|I_n^{(2)}| < \rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \leq 2\|\mu\|_\Lambda^+ \sum_n \left[ |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \right],$$

and hence,

$$\left(M - 2\|\mu\|_{\Lambda}^{\dagger}\right) \sum_n |I_n^{(1)}| \Lambda(|I_n^{(1)}|) \leq 2\|\mu\|_{\Lambda}^{\dagger} \left[ \sum_{|I_n^{(2)}| \geq \rho} |I_n^{(2)}| \Lambda(|I_n^{(2)}|) - \sum_{|I_n^{(2)}| < \rho} |I_n^{(2)}| \Lambda(|I_n^{(2)}|) \right]. \quad (2.10)$$

As a consequence, we have

$$\sum_{|I_n^{(2)}| < \rho} |I_n^{(2)}| \Lambda(|I_n^{(2)}|) \leq \sum_{|I_n^{(2)}| \geq \rho} |I_n^{(2)}| \Lambda(|I_n^{(2)}|),$$

and, finally,

$$\sum_n |I_n^{(2)}| \Lambda(|I_n^{(2)}|) \leq 2 \sum_{|I_n^{(2)}| \geq \rho} |I_n^{(2)}| \Lambda(|I_n^{(2)}|) \leq 2 \sum_n |I_n^{(2)}| \Lambda(\rho) \leq 2\Lambda(\rho).$$

(3) We have

$$\sum_n |I_n^{(1)}| \Lambda(|I_n^{(1)}|) \leq \frac{2\|\mu\|_{\Lambda}^{\dagger}}{M - 2\|\mu\|_{\Lambda}^{\dagger}} \cdot \Lambda(\rho).$$

This property follows immediately from (2.10).

We set  $F_M = \bigcup_n \overline{I_n^{(1)}}$ . Inequality (2.9) and the properties (1)–(3) show that

- (i)  $\text{Entr}_{\Lambda}(F_M) = O(1)$ ,  $M \rightarrow \infty$ ,
- (ii)  $|F_M| \Lambda(|F_M|) \leq \frac{2\|\mu\|_{\Lambda}^{\dagger}}{M - 2\|\mu\|_{\Lambda}^{\dagger}} \cdot \Lambda(\rho)$ ,
- (iii)  $\mu(F_M) \leq -4\|\mu\|_{\Lambda}^{\dagger} \left[ \sum_n |I_n^{(1)}| \Lambda(|I_n^{(1)}|) + \sum_{n: |I_n^{(2)}| < \rho} |I_n^{(2)}| \Lambda(|I_n^{(2)}|) \right] - \varepsilon$ .

By Lemma 2.12 there exists a subsequence  $M_n \rightarrow +\infty$  such that  $F_n^* := F_{M_n}$  (composed of a finite number of closed arcs) converge to a  $\Lambda$ -Carleson set  $F$ . More precisely,  $F \subset F_n^{*\delta}$  and  $F_n^* \subset F^{\delta}$  for every fixed  $\delta > 0$  and for sufficiently large  $n$ . Furthermore, (iii) yields

$$\mu(F_n^*) \leq -4\|\mu\|_{\Lambda}^{\dagger} \left[ \sum_k |R_{k,n}| \Lambda(|R_{k,n}|) + \sum_{k: |L_{k,n}| < \rho} |L_{k,n}| \Lambda(|L_{k,n}|) \right] - \varepsilon, \quad (2.11)$$

where  $F_n^* = \bigsqcup_k R_{k,n}$  and  $\mathbb{T} \setminus F_n^* = \bigsqcup_k L_{k,n}$ .

It remains to show that

$$\mu_s(F) < 0.$$

Otherwise, if  $\mu_s(F) = 0$ , then by Proposition 2.6 we have

$$\lim_{\delta \rightarrow 0} \mu(F^{\delta}) = 0.$$

Modifying a bit the set  $F_n^*$ , if necessary, we obtain  $\lim_{\delta \rightarrow 0} \mu(F_n^* \cap F^{\delta}) = 0$ . Now we can choose a sequence  $\delta_n > 0$  rapidly converging to 0 and a sequence  $\{k_n\}$  rapidly converging to  $\infty$  such that the sets  $F_n$  defined by

$$F_n = F_{k_n}^* \setminus F^{\delta_{n+1}} \subset F^{\delta_n} \setminus F^{\delta_{n+1}},$$

and consisting of a finite number of intervals  $\{I_{k,n}\}_k$  satisfy the inequalities

$$\mu(F_n) \leq -4\|\mu\|_\Lambda^+ \left[ \sum_k |I_{k,n}| \Lambda(|I_{k,n}|) + \sum_k |J_{n,k}| \Lambda(|J_{n,k}|) \right] - \varepsilon/2, \quad (2.12)$$

where  $\bigsqcup_k J_{n,k} = (F^{\delta_n} \setminus F^{\delta_{n+1}}) \setminus F_n =: G_n$ .

We denote by  $\mathcal{I}_n$ ,  $\mathcal{J}_n$ , and  $\mathcal{K}_n$  the systems of intervals that form  $F_n$ ,  $G_n$ , and  $F^{\delta_n}$ , respectively. Furthermore, we denote by  $\mathcal{I}_0$  be the system of intervals complementary to  $F^{\delta_1}$ , and we put  $\mathcal{S}_n = (\bigcup_{k=1}^n \mathcal{I}_k) \cup (\bigcup_{k=1}^n \mathcal{J}_k) \cup \mathcal{K}_{n+1}$ . Summing up the estimates on  $\mu(F_n)$  in (2.12) we obtain

$$\begin{aligned} \sum_{I \in \mathcal{I}_0} |\mu(I)| + \sum_{I \in \mathcal{S}_n} |\mu(I)| &\geq \sum_{i=1}^n |\mu(F_i)| \\ &\geq 4\|\mu\|_\Lambda^+ \sum_{i=1}^n \left[ \sum_k |I_{i,k}| \Lambda(|I_{i,k}|) + \sum_k |J_{i,k}| \Lambda(|J_{i,k}|) \right] + n\varepsilon/2 \\ &= 4\|\mu\|_\Lambda^+ \sum_{I \in \mathcal{S}_n} |I| \Lambda(|I|) - 4\|\mu\|_\Lambda^+ \sum_{I \in \mathcal{K}_{n+1}} |I| \Lambda(|I|) + n\varepsilon/2 \\ &= 4\|\mu\|_\Lambda^+ \left[ \sum_{I \in \mathcal{S}_n \cup \mathcal{I}_0} |I| \Lambda(|I|) - \sum_{I \in \mathcal{K}_{n+1}} |I| \Lambda(|I|) \right] \\ &\quad - 4\|\mu\|_\Lambda^+ \sum_{I \in \mathcal{I}_0} |I| \Lambda(|I|) + n\varepsilon/2. \end{aligned} \quad (2.13)$$

Notice that

$$\sum_{I \in \mathcal{K}_{n+1}} |I| \Lambda(|I|) \leq \sum_{|J_k| < 2\delta_{n+1}} |J_k| \Lambda(|J_k|) + 2\delta_{n+1} \Lambda(\delta_{n+1}) \cdot \text{Card}\{k : |J_k| \geq 2\delta_{n+1}\},$$

where  $\{J_k\}_k$ ,  $|J_1| \geq |J_2| \geq \dots$  are the complementary arcs to the  $\Lambda$ -Carleson set  $F$ . Since  $\lim_{t \rightarrow 0} t\Lambda(t) = 0$ , we obtain that

$$\lim_{n \rightarrow +\infty} \sum_{I \in \mathcal{K}_{n+1}} |I| \Lambda(|I|) = 0.$$

Thus for sufficiently large  $n$ , (2.13) gives us the following relation

$$\sum_{I \in \mathcal{S}_n \cup \mathcal{I}_0} |\mu(I)| \geq 4\|\mu\|_\Lambda^+ \sum_{I \in \mathcal{S}_n \cup \mathcal{I}_0} |I| \Lambda(|I|)$$

where  $\mathcal{S}_n \cup \mathcal{I}_0$  is a simple covering of the unit circle. However, since  $\mu \in B_\Lambda^+$ , we have

$$\sum_{I \in \mathcal{S}_n \cup \mathcal{I}_0} |\mu(I)| = 2 \sum_{I \in \mathcal{S}_n \cup \mathcal{I}_0} \max(\mu(I), 0) \leq 2\|\mu\|_\Lambda^+ \sum_{I \in \mathcal{S}_n \cup \mathcal{I}_0} |I| \Lambda(|I|).$$

This contradiction completes the proof of the theorem.  $\square$

## 3. HARMONIC FUNCTIONS OF RESTRICTED GROWTH

Every bounded harmonic function can be represented via the Poisson integral of its boundary values. In the following theorem we show that a large class of real-valued harmonic functions in the unit disk  $\mathbb{D}$  can be represented as the Poisson integrals of  $\Lambda$ -bounded premeasures. Before formulating the main result of this section, let us introduce some notations.

**Definition 3.1.** *Let  $f$  be a function in  $C^1(\mathbb{T})$  and let  $\mu \in B_\Lambda^+$ . We define the integral of the function  $f$  with respect to  $\mu$  by the formula*

$$\int_{\mathbb{T}} f d\mu = \int_0^{2\pi} f(e^{it}) d\hat{\mu}^\Lambda(t).$$

In particular, we have

$$\int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu(\theta) = - \int_0^{2\pi} \left( \frac{\partial}{\partial \theta} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \right) \hat{\mu}^\Lambda(\theta) d\theta.$$

Given a  $\Lambda$ -bounded premeasure  $\mu$  we denote by  $P[\mu]$  its Poisson integral:

$$P[\mu](z) = \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu(\theta).$$

**Proposition 3.2.** *Let  $\mu \in B_\Lambda^+$ . The Poisson integral  $P[\mu]$  satisfies the estimate*

$$P[\mu](z) \leq 10 \|\mu\|_\Lambda^+ \Lambda(1 - |z|), \quad z \in \mathbb{D}.$$

*Proof.* It suffices to verify the estimate on the interval  $(0, 1)$ . Let  $0 < r < 1$ . Then

$$\begin{aligned} P[\mu](r) &= \int_0^{2\pi} \frac{1 - r^2}{|e^{i\theta} - r|^2} d\mu(\theta) \\ &= - \int_0^{2\pi} \left[ \frac{\partial}{\partial \theta} \left( \frac{1 - r^2}{|e^{i\theta} - r|^2} \right) \right] \hat{\mu}^\Lambda(\theta) d\theta \\ &= \int_0^{2\pi} \frac{2r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^2} \mu(I_\theta) d\theta \\ &= \int_0^\pi \frac{2r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^2} \mu(I_\theta) d\theta - \int_\pi^0 - \frac{2r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^2} \mu(I_{2\pi - \theta}) d\theta \\ &= \int_0^\pi \frac{2r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^2} \left[ \mu(I_\theta) + \mu([- \theta, 0]) \right] d\theta \\ &= \int_0^\pi \frac{2r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^2} \mu([- \theta, \theta]) d\theta. \end{aligned}$$

Integrating by parts and using the fact that  $\Lambda$  is decreasing and  $t\Lambda(t)$  is increasing we get

$$\begin{aligned} P[\mu](r) &\leq \|\mu\|_{\Lambda}^+ \Lambda(1-r) \left[ (1-r) \int_0^{\frac{1-r}{2}} \frac{2r(1-r^2) \sin \theta}{(1-2r \cos \theta + r^2)^2} d\theta - \int_{\frac{1-r}{2}}^{\pi} 2\theta \left[ \frac{\partial}{\partial \theta} \left( \frac{1-r^2}{|e^{i\theta} - r|^2} \right) \right] d\theta \right] \\ &\leq \|\mu\|_{\Lambda}^+ \Lambda(1-r) \left[ 2(1-r)^3 \int_0^{\frac{1-r}{2}} \frac{d\theta}{(1-r)^4} + \frac{(1-r)(1-r^2)}{(1-r)^2} + 2 \int_0^{\pi} \frac{1-r^2}{|e^{i\theta} - r|^2} d\theta \right] \\ &\leq 10 \|\mu\|_{\Lambda}^+ \Lambda(1-r). \end{aligned}$$

□

The following theorem is stated by Korenblum in [18, Theorem 1, p. 543] without proof, in a more general situation.

**Theorem 3.3.** *Let  $h$  be a real-valued harmonic function on the unit disk such that  $h(0) = 0$  and*

$$h(z) = O(\Lambda(1 - |z|)), \quad |z| \rightarrow 1, z \in \mathbb{D}.$$

*Then the following statements hold.*

- (1) *For every open arc  $I$  of the unit circle  $\mathbb{T}$  the following limit exists:*

$$\mu(I) = \lim_{r \rightarrow 1^-} \mu_r(I) = \lim_{r \rightarrow 1^-} \int_I h(r\xi) |d\xi| < \infty.$$

- (2)  *$\mu$  is a  $\Lambda$ -bounded premeasure.*

- (3) *The function  $h$  is the Poisson integral of the premeasure  $\mu$ :*

$$h(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu(\theta), \quad z \in \mathbb{D}.$$

*Proof.* Let

$$h(re^{i\theta}) = \sum_{n=-\infty}^{+\infty} a_n r^{|n|} e^{in\theta}.$$

Since  $a_0 = h(0) = 0$ , we have

$$\int_0^{2\pi} h^+(re^{i\theta}) d\theta = \int_0^{2\pi} h^-(re^{i\theta}) d\theta = \frac{1}{2} \int_0^{2\pi} |h(re^{i\theta})| d\theta.$$



Furthermore,

$$\begin{aligned}
|a_n| &= \left| \frac{r^{-|n|}}{2\pi} \int_0^{2\pi} h(re^{i\theta}) e^{-in\theta} d\theta \right| \\
&\leq \frac{r^{-|n|}}{2\pi} \int_0^{2\pi} |h(re^{i\theta})| d\theta = \frac{r^{-|n|}}{\pi} \int_0^{2\pi} h^+(re^{i\theta}) d\theta \\
&\leq Cr^{-|n|}\Lambda(1-r) \\
&\leq C_1\Lambda\left(\frac{1}{|n|}\right), \quad \frac{1}{|n|} = 1-r, \quad n \in \mathbb{Z} \setminus \{-1, 0, 1\}.
\end{aligned} \tag{3.1}$$

Let  $I = \{e^{i\theta} : \alpha \leq \theta \leq \beta\}$  be an arc of  $\mathbb{T}$ ,  $\tau = \beta - \alpha$ . For  $\theta \in [\alpha, \beta]$  we define

$$t(\theta) = \min\{\theta - \alpha, \beta - \theta\}, \quad \eta(\theta) = \frac{1}{\tau}(\beta - \theta)(\theta - \alpha).$$

Then

$$\frac{1}{2}t(\theta) \leq \eta(\theta) \leq t(\theta), \quad |\eta'(\theta)| \leq 1, \quad \eta''(\theta) = \frac{-2}{\tau}, \quad \theta \in [\alpha, \beta].$$

Given  $p > 2$  we introduce the function  $q(\theta) = 1 - \eta(\theta)^p$  satisfying the following properties:

$$|q'(\theta)| \leq p\eta(\theta)^{p-1}, \quad |q''(\theta)| \leq p^2\eta(\theta)^{p-2}, \quad \theta \in (\alpha, \beta).$$

Integrating by parts we obtain for  $|n| \geq 1$  and  $\tau < 1$  that

$$\begin{aligned}
\left| \int_{\alpha}^{\beta} (1 - q(\theta))^{|n|} e^{in\theta} d\theta \right| &= \frac{1}{|n|} \left| \int_{\alpha}^{\beta} |n|q(\theta)^{|n|-1} q'(\theta) e^{in\theta} d\theta \right| \\
&\leq \frac{|n|-1}{|n|} \int_{\alpha}^{\beta} q(\theta)^{|n|-2} |q'(\theta)|^2 d\theta + \frac{1}{|n|} \int_{\alpha}^{\beta} q(\theta)^{|n|-1} |q''(\theta)| d\theta \\
&\leq 2p^2 \int_0^{\tau/2} \left(1 - \left[\frac{t}{2}\right]^p\right)^{|n|-2} t^{2p-2} dt + \frac{2p^2}{|n|} \int_0^{\tau/2} \left(1 - \left[\frac{t}{2}\right]^p\right)^{|n|-1} t^{p-2} dt \\
&\leq C_p \left[ \int_0^{\tau/4} (1 - t^p)^{|n|-2} t^{2p-2} dt + \frac{1}{|n|} \int_0^{\tau/4} (1 - t^p)^{|n|-1} t^{p-2} dt \right],
\end{aligned}$$

and, hence,

$$\begin{aligned}
\left| \int_{\alpha}^{\beta} (1 - q(\theta))^{|n|} e^{in\theta} d\theta \right| &\leq C_{1,p}\tau \max_{0 \leq t \leq 1} \left\{ (1 - t^p)^{|n|-2} t^{2p-2} + \frac{1}{|n|} (1 - t^p)^{|n|-1} t^{p-2} \right\} \\
&\leq C_{2,p}\tau |n|^{-2(1-\frac{1}{p})}.
\end{aligned}$$

On the other hand, we have

$$\frac{1}{2\pi} \int_I h(r\xi) |d\xi| = \frac{1}{2\pi} \int_{\alpha}^{\beta} h(rq(\theta)e^{i\theta}) d\theta + \frac{1}{2\pi} \int_{\alpha}^{\beta} [h(re^{i\theta}) - h(rq(\theta)e^{i\theta})] d\theta.$$

By (3.1), we obtain

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_{\alpha}^{\beta} [h(re^{i\theta}) - h(rq(\theta)e^{i\theta})] d\theta \right| &\leq \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |a_n| \left| \int_{\alpha}^{\beta} r^{|n|} (1 - q(\theta)^{|n|}) e^{in\theta} d\theta \right| \\
&\leq C_{3,p}\tau \sum_{n \in \mathbb{Z}} |a_n| (|n| + 1)^{-2(1-\frac{1}{p})} \\
&\leq C_{4,p}\tau \sum_{n \in \mathbb{Z}} \Lambda\left(\frac{1}{\max(|n|, 1)}\right) (|n| + 1)^{-2(1-\frac{1}{p})}.
\end{aligned}$$

Therefore, if  $t \mapsto t^{\alpha}\Lambda(t)$  increase, and

$$\alpha + \frac{2}{p} < 1, \quad (3.2)$$

then

$$\left| \frac{1}{2\pi} \int_{\alpha}^{\beta} [h(re^{i\theta}) - h(rq(\theta)e^{i\theta})] d\theta \right| \leq C_{5,p}\tau.$$

Since  $\Lambda(x^p) \leq C_p\Lambda(x)$ , we obtain

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_{\alpha}^{\beta} h(rq(\theta)e^{i\theta}) d\theta \right| &\leq C \int_{\alpha}^{\beta} \Lambda(1 - q(\theta)) d\theta \\
&\leq C \int_{\alpha}^{\beta} \Lambda\left(\frac{t(\theta)}{2}\right) d\theta \\
&\leq C_1 \int_0^{\tau/4} \Lambda(t) dt \\
&= C_1 \int_0^{\tau/4} t^{-\alpha} t^{\alpha} \Lambda(t) dt \\
&\leq C_2 \tau^{\alpha} \Lambda(\tau) \int_0^{\tau/4} t^{-\alpha} dt \\
&= C_3 \tau \Lambda(\tau).
\end{aligned}$$

Hence,

$$\mu_r(I) \leq C|I|\Lambda(|I|)$$

for some  $C$  independent of  $I$ .

Given  $r \in (0, 1)$ , we define  $h_r(z) = h(rz)$ . The  $h_r$  is the Poisson integral of  $d\mu_r = h_r(e^{i\theta}) d\theta$ :

$$h_r(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu_r(\theta)$$

The set  $\{\mu_r : r \in (0, 1)\}$  is a uniformly  $\Lambda$ -bounded family of premeasures. Using a Helly-type selection theorem [15, Theorem 1, p. 204], we can find a sequence of premeasures

$\mu_{r_n} \in B_\Lambda^+$  converging weakly to a  $\Lambda$ -bounded premeasure  $\mu$  as  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} r_n = 1$ . Then

$$\mu(I) \leq C|I|\Lambda(|I|)$$

for every arc  $I$ , and

$$h_{r_n}(z) = - \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \right) \mu_n^\lambda(\theta) d\theta.$$

Passing to the limit we conclude that

$$h(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu(\theta).$$

□

#### 4. CYCLIC VECTORS

Given a  $\Lambda$ -bounded premeasure  $\mu$ , we consider the corresponding analytic function

$$f_\mu(z) = \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta). \quad (4.1)$$

If  $\tilde{\mu}$  is a positive singular measure on the circle  $\mathbb{T}$ , we denote by  $S_{\tilde{\mu}}$  the associated singular inner function. Notice that in this case  $\mu = \tilde{\mu}(\mathbb{T})m - \tilde{\mu}$  is a premeasure, and we have  $S_{\tilde{\mu}} = f_\mu/S_{\tilde{\mu}}(0)$ ;  $m$  is (normalized) Lebesgue measure.

Let  $f$  be a zero-free function in  $\mathcal{A}_\Lambda^{-\infty}$  such that  $f(0) = 1$ . According to Theorem 3.3, there is a premeasure  $\mu_f \in B_\Lambda^+$  such that

$$f(z) = \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_f(\theta).$$

The following result follows immediately from Theorem 2.8.

**Theorem 4.1.** *Let  $f \in \mathcal{A}_\Lambda^{-\infty}$  be a zero-free function such that  $f(0) = 1$ . If  $(\mu_f)_s \equiv 0$ , then  $f$  is cyclic in  $\mathcal{A}_\Lambda^{-\infty}$ .*

*Proof.* Suppose that  $(\mu_f)_s \equiv 0$ . By theorem 2.8,  $\mu_f$  is  $\Lambda$ -absolutely continuous. Let  $\{\mu_n\}_{n \geq 1}$  be a sequence of  $\Lambda$ -bounded premeasures from Definition 2.7. We set

$$g_n(z) = \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_n(\theta), \quad z \in \mathbb{D}.$$

By Proposition 3.2,  $g_n \in \mathcal{A}_\Lambda^{-\infty}$ , and

$$\begin{aligned} f(z)g_n(z) &= \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d(\mu_f + \mu_n)(\theta) \\ &= \exp \left[ - \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) [\mu_n^\lambda(\theta) - \mu^\lambda(\theta)] d\theta \right] \\ &= \exp \left[ - \int_0^{2\pi} \frac{\partial}{\partial \theta} \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} \right) [\mu(I_\theta) + \mu_n(I_\theta)] d\theta \right]. \end{aligned}$$

Again by Definition 2.7, we obtain that  $f(z)g_n(z) \rightarrow 1$  uniformly on compact subsets of unit disk  $\mathbb{D}$ . This yields that  $fg_n \rightarrow 1$  in  $\mathcal{A}_\Lambda^{-\infty}$  as  $n \rightarrow \infty$ .  $\square$

From now on, we deal with the statements converse to Theorem 4.1. We'll establish two results valid for different growth ranges of the majorant  $\Lambda$ . More precisely, we consider the following growth and regularity assumptions:

$$\text{for every } c > 0, \text{ the function } x \mapsto \exp[c\Lambda(1/x)] \text{ is concave for large } x, \quad (\text{C1})$$

$$\lim_{t \rightarrow 0} \frac{\Lambda(t)}{\log(1/t)} = \infty. \quad (\text{C2})$$

Examples of majorants  $\Lambda$  satisfying condition (C1) include

$$(\log(1/x))^p, \quad 0 < p < 1, \quad \text{and} \quad \log(\log(1/x)), \quad x \rightarrow 0.$$

Examples of majorants  $\Lambda$  satisfying condition (C2) include

$$(\log(1/x))^p, \quad p > 1.$$

Thus, we consider majorants which grow less rapidly than the Korenblum majorant ( $\Lambda(x) = \log(1/x)$ ) in Case 1 or more rapidly than the Korenblum majorant in Case 2.

**4.1. Weights  $\Lambda$  satisfying condition (C1).** We start with the following observation:

$$\Lambda(t) = o(\log 1/t), \quad t \rightarrow 0.$$

Next we pass to some notations and auxiliary lemmas. Given a function  $f$  in  $L^1(\mathbb{T})$ , we denote by  $P[f]$  its Poisson transform,

$$P[f](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|} f(e^{i\theta}) d\theta, \quad z \in \mathbb{D}.$$

Denote by  $A(\mathbb{D})$  the disk-algebra, i.e., the algebra of functions continuous on the closed unit disk and holomorphic in  $\mathbb{D}$ . A positive continuous increasing function  $\omega$  on  $[0, \infty)$  is said to be a modulus of continuity if  $\omega(0) = 0$ ,  $t \mapsto \omega(t)/t$  decreases near 0, and  $\lim_{t \rightarrow 0} \omega(t)/t = \infty$ . Given a modulus of continuity  $\omega$ , we consider the Lipschitz space  $\text{Lip}_\omega(\mathbb{T})$  defined by

$$\text{Lip}_\omega(\mathbb{T}) = \{f \in C(\mathbb{T}) : |f(\xi) - f(\zeta)| \leq C(f)\omega(|\xi - \zeta|)\}.$$

Since the function  $t \mapsto \exp[2\Lambda(1/t)]$  is concave for large  $t$ , and  $\Lambda(t) = o(\log(1/t))$ ,  $t \rightarrow 0$ , we can apply a result of Kellay [12, Lemma 3.1], to get a non-negative summable function  $\Omega_\Lambda$  on  $[0, 1]$  such that

$$e^{2\Lambda(\frac{1}{n+1})} - e^{2\Lambda(\frac{1}{n})} \asymp \int_{1-\frac{1}{n}}^1 \Omega_\Lambda(t) dt, \quad n \geq 1.$$

Next we consider the Hilbert space  $L^2_{\Omega_\Lambda}(\mathbb{T})$  of the functions  $f \in L^2(\mathbb{T})$  such that

$$\|f\|_{\Omega_\Lambda}^2 = |P[f](0)|^2 + \int_{\mathbb{D}} \frac{P[|f|^2](z) - |P[f](z)|^2}{1 - |z|^2} \Omega_\Lambda(|z|) dA(z) < \infty,$$

where  $dA$  denote the normalized area measure. We need the following lemma.

**Lemma 4.2.** *Under our conditions on  $\Lambda$  and  $\Omega_\Lambda$ , we have*

- (1)  $\|f\|_{\Omega_\Lambda}^2 \asymp \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 e^{2\Lambda(1/n)}$ ,  $f \in L_{\Omega_\Lambda}^2(\mathbb{T})$ ,
- (2) the functions  $\exp(-c\Lambda(t))$  are moduli of continuity for  $c > 0$ ,
- (3) for some positive  $a$ , the function  $\rho(t) = \exp(-\frac{3}{2a}\Lambda(t))$  satisfies the property

$$\text{Lip}_\rho(\mathbb{T}) \subset L_{\Omega_\Lambda}^2(\mathbb{T}).$$

For the first statement see [5, Lemma 6.1] (where it is attributed to Aleman [1]); the second statement is [5, Lemma 8.4]; the third statement follows from [5, Lemmas 6.2 and 6.3].

Recall that

$$\mathcal{A}_\Lambda^{-1} = \{f \in \text{Hol}(\mathbb{D}) : |f(z)| \leq C(f) \exp(\Lambda(1 - |z|))\}.$$

**Lemma 4.3.** *Under our conditions on  $\Lambda$ , there exists a positive number  $c$  such that*

$$P_+ \text{Lip}_{e^{-c\Lambda}}(\mathbb{T}) \subset (\mathcal{A}_\Lambda^{-1})^*$$

via the Cauchy duality

$$\langle f, g \rangle = \sum_{n \geq 0} a_n \overline{\hat{g}(n)},$$

where  $f(z) = \sum_{n \geq 0} a_n z^n \in \mathcal{A}_\Lambda^{-1}$ ,  $g \in \text{Lip}_{e^{-c\Lambda}}(\mathbb{T})$ , and  $P_+$  is the orthogonal projector from  $L^2(\mathbb{T})$  onto  $H^2(\mathbb{D})$ .

*Proof.* Denote

$$L_\Lambda^2(\mathbb{D}) = \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 |\Lambda'(1 - |z|)| e^{-2\Lambda(1 - |z|)} dA(z) < +\infty \right\},$$

and

$$\mathcal{B}_\Lambda^2 = \left\{ f(z) = \sum_{n \geq 0} a_n z^n : |a_0|^2 + \sum_{n > 0} |a_n|^2 e^{-2\Lambda(1/n)} < \infty \right\}.$$

Let us prove that

$$L_\Lambda^2(\mathbb{D}) = \mathcal{B}_\Lambda^2. \tag{4.2}$$

To verify this equality, it suffices sufficient to check that

$$e^{-2\Lambda(1/n)} \asymp \int_0^1 r^{2n+1} |\Lambda'(1 - r)| e^{-2\Lambda(1-r)} dr.$$

In fact,

$$\int_{1-1/n}^1 r^{2n+1} |\Lambda'(1 - r)| e^{-2\Lambda(1-r)} dr \asymp \int_{1-1/n}^1 |\Lambda'(1 - r)| e^{-2\Lambda(1-r)} dr \asymp e^{-2\Lambda(\frac{1}{n})}, \quad n \geq 1.$$

On the other hand,

$$\begin{aligned} \int_0^{1-1/n} r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)} dr &= - \int_0^{1-1/n} r^{2n+1} d e^{-2\Lambda(1-r)} \\ &\asymp -e^{-2\Lambda(1/n)} + (2n+1) \int_0^{1-1/n} r^{2n} e^{-2\Lambda(1-r)} dr \\ &\asymp n \sum_{k=1}^n e^{-2n/k} e^{-2\Lambda(1/k)} \frac{1}{k^2}. \end{aligned}$$

Since the function  $\exp[2\Lambda(1/x)]$  is concave, we have  $e^{2\Lambda(1/k)} \geq \frac{k}{n} e^{2\Lambda(1/n)}$ , and hence,

$$e^{-2\Lambda(1/k)} \leq \frac{n}{k} e^{-2\Lambda(1/n)}.$$

Therefore,

$$\int_0^{1-1/n} r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)} dr \leq C n^2 e^{-2\Lambda(1/n)} \sum_{k=1}^n e^{-2n/k} \frac{1}{k^3} \asymp e^{-2\Lambda(1/n)},$$

and (4.2) follows.

Since  $\mathcal{A}_\Lambda^{-1} \subset L_\Lambda^2(\mathbb{D})$ , we have  $(\mathcal{B}_\Lambda^2)^* \subset (\mathcal{A}_\Lambda^{-1})^*$ . By Lemma 4.2, we have  $P_+ \text{Lip}_\rho(\mathbb{T}) \subset (\mathcal{B}_\Lambda^2)^*$ . Thus,

$$P_+ \text{Lip}_\rho(\mathbb{T}) \subset (\mathcal{A}_\Lambda^{-1})^*.$$

□

**Lemma 4.4.** *Let  $f \in \mathcal{A}_\Lambda^{-n}$  for some  $n > 0$ . The function  $f$  is cyclic in  $\mathcal{A}_\Lambda^{-\infty}$  if and only if there exists  $m > n$  such that  $f$  is cyclic in  $\mathcal{A}_\Lambda^{-m}$ .*

*Proof.* Notice that the space  $\mathcal{A}_\Lambda^{-\infty}$  is endowed with the inductive limit topology induced by the spaces  $\mathcal{A}_\Lambda^{-N}$ . A sequence  $\{f_n\}_n \in \mathcal{A}_\Lambda^{-\infty}$  converges to  $g \in \mathcal{A}_\Lambda^{-\infty}$  if and only if there exists  $N > 0$  such that all  $f_n$  and  $g$  belong to  $\mathcal{A}_\Lambda^{-N}$ , and  $\lim_{n \rightarrow +\infty} \|f_n - g\|_{\mathcal{A}_\Lambda^{-N}} = 0$ . The statement of the lemma follows. □

**Theorem 4.5.** *Let  $\mu \in B_\Lambda^+$ , and let the majorant  $\Lambda$  satisfy condition (C1). Then the function  $f_\mu$  is cyclic in  $\mathcal{A}_\Lambda^{-\infty}$  if and only if  $\mu_s \equiv 0$ .*

*Proof.* Suppose that the  $\Lambda$ -singular part  $\mu_s$  of  $\mu$  is non-trivial. There exists a  $\Lambda$ -Carleson set  $F \subset \mathbb{T}$  such that  $-\infty < \mu_s(F) < 0$ . We set  $\nu = -\mu_s|_F$ . By a theorem of Shirokov [22, Theorem 9, pp.137,139], there exists an outer function  $\varphi$  such that

$$\varphi \in \text{Lip}_\rho(\mathbb{T}) \cap H^\infty(\mathbb{D}), \quad \varphi S_\nu \in \text{Lip}_\rho(\mathbb{T}) \cap H^\infty(\mathbb{D}),$$

and the zero set of the function  $\varphi$  coincides with  $F$ . Next, for  $\xi, \theta \in [0, 2\pi]$  we have

$$\begin{aligned} |\varphi \overline{S_\nu}(e^{i\xi}) - \varphi \overline{S_\nu}(e^{i\theta})| &= |\varphi(e^{i\xi}) S_\nu(e^{i\theta}) - \varphi(e^{i\theta}) S_\nu(e^{i\xi})| \\ &\leq |(\varphi(e^{i\xi}) - \varphi(e^{i\theta})) S_\nu(e^{i\theta})| + |(\varphi(e^{i\theta}) - \varphi(e^{i\xi})) S_\nu(e^{i\xi})| \\ &\quad + |(\varphi S_\nu)(e^{i\theta}) - (\varphi S_\nu)(e^{i\xi})|, \end{aligned}$$

and hence,

$$\varphi \overline{S_\nu} \in \text{Lip}_\rho(\mathbb{T}).$$

Set  $g = P_+(\overline{z\varphi}S_\nu)$ . Since  $\varphi \overline{S_\nu} \in \text{Lip}_\rho(\mathbb{T})$ , we have  $g \in (\mathcal{A}_\Lambda^{-1})^*$ . Consider the following linear functional on  $\mathcal{A}_\Lambda^{-1}$ :

$$L_g(f) = \langle f, g \rangle = \sum_{n \geq 0} a_n \overline{\widehat{g}(n)}, \quad f(z) = \sum_{n \geq 0} a_n z^n \in \mathcal{A}_\Lambda^{-1}.$$

Suppose that  $L_g = 0$ . Then, for every  $n \geq 0$  we have

$$\begin{aligned} 0 &= L_g(z^n) \\ &= \int_0^{2\pi} e^{in\theta} \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{i(n+1)\theta} \frac{\varphi(e^{i\theta})}{S_\nu(e^{i\theta})} \frac{d\theta}{2\pi}. \end{aligned}$$

We conclude that  $\varphi/S_\nu \in H^\infty(\mathbb{D})$ , which is impossible. Thus,  $L_g \neq 0$ .

On the other hand we have, for every  $n \geq 0$ ,

$$\begin{aligned} L_g(z^n S_\nu) &= \int_0^{2\pi} e^{in\theta} S_\nu(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{in\theta} S_\nu(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{i(n+1)\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= 0. \end{aligned}$$

Thus,  $g \perp [f_\mu]_{\mathcal{A}_\Lambda^{-1}}$  which implies that the function  $f_\mu$  is not cyclic in  $\mathcal{A}_\Lambda^{-1}$ . By Lemma 4.4,  $f_\mu$  is not cyclic in  $\mathcal{A}_\Lambda^{-\infty}$ .  $\square$

**4.2. Weights  $\Lambda$  satisfying condition (C2).** We start with an elementary consequence of the Cauchy formula.

**Lemma 4.6.** *Let  $f(z) = \sum_{n \geq 0} a_n z^n$  be an analytic function in  $\mathbb{D}$ . If  $f \in \mathcal{A}_\Lambda^{-\infty}$ , then there exists  $C > 0$  such that*

$$|a_n| = O(\exp[C\Lambda(\frac{1}{n})]) \quad \text{as } n \rightarrow +\infty.$$

**Theorem 4.7.** *Let  $\mu \in B_\Lambda^+$ , and let the majorant  $\Lambda$  satisfy condition (C2). Then the function  $f_\mu$  is cyclic in  $\mathcal{A}_\Lambda^{-\infty}$  if and only if  $\mu_s \equiv 0$ .*

*Proof.* We define

$$\mathcal{A}_\Lambda^\infty = \bigcap_{c < \infty} \left\{ g \in \text{Hol}(\mathbb{D}) \cap C^\infty(\overline{\mathbb{D}}) : |\widehat{f}(n)| = O(\exp[-c\Lambda(\frac{1}{n})]) \right\},$$

and, using Lemma 4.6, we obtain that  $\mathcal{A}_\Lambda^\infty \subset (\mathcal{A}_\Lambda^{-\infty})^*$  via the Cauchy duality

$$\langle f, g \rangle = \sum_{n \geq 0} \widehat{f}(n) \overline{\widehat{g}(n)} = \lim_{r \rightarrow 1} \int_0^{2\pi} f(r\xi) \overline{g(\xi)} d\xi, \quad f \in \mathcal{A}_\Lambda^{-\infty}, \quad g \in \mathcal{A}_\Lambda^\infty.$$

Suppose that the  $\Lambda$ -singular part  $\mu_s$  of  $\mu$  is nonzero. Then there exists a  $\Lambda$ -Carleson set  $F \subset \mathbb{T}$  such that  $-\infty < \mu_s(F) < 0$ . We set  $\sigma = \mu_s|_F$ . By a theorem of Bourhim, El-Fallah, and Kellay [5, Theorem 5.3] (extending a result of Taylor and Williams), there exist an outer function  $\varphi \in \mathcal{A}_\Lambda^\infty$  such that the zero set of  $\varphi$  and of all its derivatives coincides exactly with the set  $F$ , a function  $\tilde{\Lambda}$  such that

$$\Lambda(t) = o(\tilde{\Lambda}(t)), \quad t \rightarrow 0, \quad (4.3)$$

and a positive constant  $B$  such that

$$|\varphi^{(n)}(z)| \leq n! B^n e^{\tilde{\Lambda}^*(n)}, \quad n \geq 0, \quad z \in \mathbb{D}, \quad (4.4)$$

where  $\tilde{\Lambda}^*(n) = \sup_{x>0} \{nx - \tilde{\Lambda}(e^{-x/2})\}$ .

We set

$$\Psi = \varphi \overline{S_\sigma}.$$

For some positive  $D$  we have

$$|S_\sigma^{(n)}(z)| \leq \frac{D^n n!}{\text{dist}(z, F)^{2n}}, \quad z \in \mathbb{D}, \quad n \geq 0. \quad (4.5)$$

By the Taylor formula, for every  $n, k \geq 0$ , we have

$$|\varphi^{(n)}(z)| \leq \frac{1}{k!} \text{dist}(z, F)^k \max_{w \in \mathbb{D}} |\varphi^{(n+k)}(w)|, \quad z \in \mathbb{D}. \quad (4.6)$$

Next, integrating by parts, for every  $n \neq 0, k \geq 0$  we obtain

$$|\widehat{\Psi}(n)| = |\widehat{(\varphi \overline{S_\sigma})}(n)| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{(\varphi \overline{S_\sigma})^{(k)}(e^{it})}{n^k} e^{-int} dt \right|.$$



Applying the Leibniz formula and estimates (4.4)–(4.6), we obtain for  $n \geq 1$  that

$$\begin{aligned}
|\widehat{\Psi}(n)| &\leq \inf_{k \geq 0} \left\{ \frac{1}{n^k} \max_{t \in [0, 2\pi]} |(\varphi \overline{S_\sigma})^{(k)}(e^{it})| \right\} \\
&\leq \inf_{k \geq 0} \left\{ \frac{1}{n^k} \sum_{s=0}^k C_k^s \max_{t \in [0, 2\pi]} |S_\sigma^{(s)}(e^{it})| \max_{t \in [0, 2\pi]} |\varphi^{(k-s)}(e^{it})| \right\} \\
&\leq \inf_{k \geq 0} \left\{ \frac{1}{n^k} \sum_{s=0}^k C_k^s D^s s! \frac{1}{(2s)!} (k+s)! B^{k+s} e^{\tilde{\Lambda}^*(k+s)} \right\} \\
&\leq \inf_{k \geq 0} \left\{ e^{\tilde{\Lambda}^*(2k)} \left( \frac{B^2 D}{n} \right)^k \sum_{s=0}^k \frac{(k+s)! k!}{(2s)! (k-s)!} \right\} \\
&\leq \inf_{k \geq 0} \left\{ k! e^{\tilde{\Lambda}^*(2k)} \left( \frac{4B^2 D}{n} \right)^k \right\} \\
&\leq \inf_{k \geq 0} k! \left\{ \left( \frac{4B^2 D}{n} \right)^k \sup_{0 < t < 1} \left\{ e^{-\tilde{\Lambda}(t^{1/4})} t^{-k} \right\} \right\}.
\end{aligned}$$

By property (4.3), for every  $C > 0$  there exists a positive number  $K$  such that

$$e^{-\tilde{\Lambda}(t^{1/4})} \leq K e^{-\Lambda(Ct)}, \quad t \in (0, 1).$$

We take  $C = \frac{1}{8B^2 D}$ , and obtain for  $n \neq 0$  that

$$\begin{aligned}
|\widehat{\Psi}(n)| &\leq K \inf_{k \geq 0} \left\{ \left( \frac{4B^2 D}{n} \right)^k k! \sup_{0 < t < 1} \frac{e^{-\Lambda(Ct)}}{t^k} \right\} \\
&\leq K_1 \inf_{k \geq 0} \left\{ (2n)^{-k} k! \sup_{0 < t < 1} \frac{e^{-\Lambda(t)}}{t^k} \right\}.
\end{aligned}$$

Finally, using [14, Lemma 6.5] (see also [5, Lemma 8.3]), we get

$$|\widehat{\Psi}(n)| = O(e^{-\Lambda(1/n)}), \quad |n| \rightarrow \infty.$$

Thus, the function  $g = P_+(\overline{z\varphi} S_\sigma)$  belongs to  $(\mathcal{A}_\Lambda^{-1})^*$ . Now we obtain that  $f_\mu$  is not cyclic using the same argument as that at the end of Case 1. This concludes the proof of the theorem.  $\square$

Theorems 4.5 and 4.7 together give a positive answer to a conjecture by Deninger [7, Conjecture 42].

We complete this section by two examples that show how the cyclicity property of a fixed function changes in a scale of  $\mathcal{A}_\Lambda^{-\infty}$  spaces.

**Example 4.8.** Let  $\Lambda_\alpha(x) = (\log(1/x))^\alpha$ ,  $0 < \alpha < 1$ , and let  $0 < \alpha_0 < 1$ . There exists a singular inner function  $S_\mu$  such that

$$S_\mu \text{ is cyclic in } \mathcal{A}_{\Lambda_\alpha}^{-\infty} \iff \alpha > \alpha_0.$$

*Construction.* We start by defining a Cantor type set and the corresponding canonical measure. Let  $\{m_k\}_{k \geq 1}$  be a sequence of natural numbers. Set  $M_k = \sum_{1 \leq s \leq k} m_s$ , and assume that

$$M_k \asymp m_k, \quad k \rightarrow \infty. \quad (4.7)$$

Consider the following iterative procedure. Set  $\mathcal{I}_0 = [0, 1]$ . On the step  $n \geq 1$  the set  $\mathcal{I}_{n-1}$  consist of several intervals  $I$ . We divide each  $I$  into  $2^{m_n+1}$  equal subintervals and replace it by the union of every second interval in this division. The union of all such groups is  $\mathcal{I}_n$ . Correspondingly,  $\mathcal{I}_n$  consists of  $2^{M_n}$  intervals; each of them is of length  $2^{-n-M_n}$ . Next, we consider the probabilistic measure  $\mu_n$  equidistributed on  $\mathcal{I}_n$ . Finally, we set  $E = \bigcap_{n \geq 1} \mathcal{I}_n$ , and define by  $\mu$  the weak limit of the measures  $\mu_n$ .

Now we estimate the  $\Lambda_\alpha$ -entropy of  $E$ :

$$\text{Entr}_{\Lambda_\alpha}(\mathcal{I}_n) \asymp \sum_{1 \leq k \leq n} 2^{M_k} \cdot 2^{-k-M_k} \cdot \Lambda_\alpha(2^{-k-M_k}) \asymp \sum_{1 \leq k \leq n} 2^{-k} \cdot m_k^\alpha, \quad n \rightarrow \infty.$$

Thus, if

$$\sum_{n \geq 1} 2^{-n} \cdot m_n^{\alpha_0} < \infty, \quad (4.8)$$

then  $\text{Entr}_{\Lambda_{\alpha_0}}(E) < \infty$ . By Theorem 4.5,  $S_\mu$  is not cyclic in  $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$  for  $\alpha \leq \alpha_0$ .

Next we estimate the modulus of continuity of the measure  $\mu$ ,

$$\omega_\mu(t) = \sup_{|I|=t} \mu(I).$$

Assume that

$$A_{j+1} = 2^{-(j+1)-M_{j+1}} \leq |I| < A_j = 2^{-j-M_j},$$

and that  $I$  intersects with one of the intervals  $I_j$  that constitute  $\mathcal{I}_j$ . Then

$$\mu(I) \leq 4 \frac{|I|}{A_j} \mu(I_j) = 4|I| 2^{j+M_j} 2^{-M_j} = 4|I| 2^j.$$

Thus, if

$$2^j \leq C(\log(1/A_j))^\alpha \asymp m_j^\alpha, \quad j \geq 1, \quad \alpha_0 < \alpha < 1, \quad (4.9)$$

then

$$\omega_\mu(t) \leq Ct(\log(1/t))^\alpha.$$

By [2, Corollary B], we have  $\mu(F) = 0$  for any  $\Lambda_\alpha$ -Carleson set  $F$ ,  $\alpha_0 < \alpha < 1$ . Again by Theorem 4.5,  $S_\mu$  is cyclic in  $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$  for  $\alpha > \alpha_0$ . It remains to fix  $\{m_k\}_{k \geq 1}$  satisfying (4.7)–(4.9). The choice  $m_k = 2^{k/\alpha_0} k^{-2/\alpha_0}$  works.  $\square$

Of course, instead of Theorem 4.5 we could use here [5, Theorem 7.1].

**Example 4.9.** Let  $\Lambda_\alpha(x) = (\log(1/x))^\alpha$ ,  $0 < \alpha < 1$ , and let  $0 < \alpha_0 < 1$ . There exists a premeasure  $\mu$  such that  $\mu_s$  is infinite,

$$f_\mu \text{ is cyclic in } \mathcal{A}_{\Lambda_\alpha}^{-\infty} \iff \alpha > \alpha_0,$$

where  $f_\mu$  is defined by (4.1).

It looks like the subspaces  $[f_\mu]_{\mathcal{A}_{\Lambda_\alpha}^{-\infty}}$ ,  $\alpha \leq \alpha_0$ , contain no nonzero Nevanlinna class functions. For a detailed discussion on Nevanlinna class generated invariant subspaces in the Bergman space (and in the Korenblum space) see [10].

For  $\alpha \leq \alpha_0$ , instead of Theorem 4.5 we could once again use here [5, Theorem 7.1].

*Construction.* We use the measure  $\mu$  constructed in Example 4.8. Choose a decreasing sequence  $u_k$  of positive numbers such that

$$\sum_{k \geq 1} u_k = 1, \quad \sum_{k \geq 1} v_k = +\infty,$$

where  $v_k = u_k \log \log(1/u_k) > 0$ ,  $k \geq 1$ .

Given a Borel set  $B \subset B^0 = [0, 1]$ , denote

$$B_k = \{u_k t + \sum_{j=1}^{k-1} u_j : t \in B\} \subset [0, 1],$$

and define measures  $\nu_k$  supported by  $B_k^0$  by

$$\nu_k(B_k) = \frac{v_k}{u_k} m(B_k) - v_k \mu(B),$$

where  $m(B_k)$  is Lebesgue measure of  $B_k$ .

We set

$$\nu = \sum_{k \geq 1} \nu_k.$$

Then  $\nu(B_k^0) = \nu_k(B_k^0) = 0$ ,  $k \geq 1$ , and  $\nu$  is a premeasure.

Since

$$v_k \leq C(\alpha) u_k \Lambda_\alpha(u_k), \quad 0 < \alpha < 1,$$

$\nu$  is a  $\Lambda_\alpha$ -bounded premeasure for  $\alpha \in (0, 1)$ .

Furthermore, as above, by Theorem 4.5,  $f_\nu$  is not cyclic in  $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$  for  $\alpha \leq \alpha_0$ .

Next, we estimate

$$\omega_\nu(t) = \sup_{|I|=t} |\nu(I)|.$$

As in Example 4.8, if  $j, k \geq 1$  and

$$u_k A_{j+1} \leq |I| < u_k A_j,$$

then

$$\frac{|\nu(I)|}{|I|} \leq C \cdot 2^j \cdot \frac{v_k}{u_k}. \quad (4.10)$$

Now we verify that

$$\omega_\nu(t) \leq C t (\log(1/t))^\alpha, \quad \alpha_0 < \alpha < 1. \quad (4.11)$$

Fix  $\alpha \in (\alpha_0, 1)$ , and use that

$$\left(\log \frac{1}{A_j}\right)^\alpha \geq C \cdot 2^{(1+\varepsilon)j}, \quad j \geq 1,$$

for some  $C, \varepsilon > 0$ . By (4.10), it remains to check that

$$2^j \log \log \frac{1}{u_k} \leq C(2^{(1+\varepsilon)j} + (\log \frac{1}{u_k})^\alpha).$$

Indeed, if

$$\log \log \frac{1}{u_k} > 2^{\varepsilon j},$$

then

$$C(\log \frac{1}{u_k})^\alpha > 2^j \log \log \frac{1}{u_k}.$$

Finally, we fix  $\alpha \in (\alpha_0, 1)$  and a  $\Lambda_\alpha$ -Carleson set  $F$ . We have

$$\mathbb{T} \setminus F = \sqcup_s L_s^*$$

for some intervals  $L_s^*$ . By [2, Theorem B], there exist disjoint intervals  $L_{n,s}$  such that

$$F \subset \sqcup_s L_{n,s}, \quad \sum_s |L_{n,s}| \Lambda_\alpha(|L_{n,s}|) < \frac{1}{n}, \quad n \geq 1.$$

Then by (4.11),

$$\sum_s |\nu(L_{n,s})| < \frac{c}{n}.$$

Set

$$\mathbb{T} \setminus \sqcup_s L_{n,s} = \sqcup_s L_{n,s}^*.$$

Then

$$|\sum_s \nu(L_{n,s}^*)| < \frac{c}{n}.$$

Since  $F$  is  $\Lambda_\alpha$ -Carleson, we have

$$\sum_s |L_s^*| \Lambda_\alpha(|L_s^*|) < \infty,$$

and hence,

$$\sum_s \nu(L_{n,s}^*) \rightarrow \sum_s \nu(L_s^*)$$

as  $n \rightarrow \infty$ . Thus,

$$\sum_s \nu(L_s^*) = 0,$$

and hence,  $\nu(F) = 0$ . Again by Theorem 4.5,  $f_\nu$  is cyclic in  $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$  for  $\alpha > \alpha_0$ .  $\square$

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