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CONTROL OF DISCRETIZATION ERROR AND ADAPTATIVITY FOR FINITE ELEMENT PROBLEMS SOLVED BY DOMAIN DECOMPOSITION

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Abstract. This paper presents a strategy for adaptative verification of computations driven in a sub-structured framework arising from the use of non-overlapping domain decomposition methods. We currently restrict ourselves to global estimates of discretization error in the context of linear mechanical problems. The key point of our method which is based on the use of constitutive relation error is a parallel recovery of admissible interface fields. This allows to obtain relevant estimates far before the domain decomposition solver reaches convergence.

1 INTRODUCTION

For the last decades, three trends have grown and reinforced each other in computational mechanics: the fast growth of hardware computational capacities, the requirement of finer and larger finite element models for industrial simulations and the development of efficient multiscale computational strategies amongst which non-overlapping domain decomposition (DD) methods (like FETI [4] and BDD [11, 12]) are very popular since they have proved to be scalable in many applications. Unfortunately, the quality of the discretized models is not always checked in spite of the development of verification techniques and error estimators [8, 2, 14, 17].

Our objective is to warranty the quality of computations of large finite element models by providing a strategy to control discretization error in large FE problems solved by non-overlapping domain decomposition methods.

We first focus on global a posteriori error estimates and associated element-wise contributions obtained through the error in constitutive relation [8] for linear mechanical problems. Our method [15] has been developed within a generic framework for domain decomposition [6]. It computes error estimate in a fully parallel way for both primal

(BDD) and dual (FETI) approaches whatever the state (converged or not) of the associated iterative solver. To do so, our strategy consists in building, in parallel and during the iterations, displacement and stress fields which are kinematically admissible (KA) and statically admissible (SA) on the whole structure. The numerical costs remain low thanks to the exploitation of classical domain decomposition preconditioners to build in parallel continuous displacement and balanced traction fields independently on each subdomain. It comes out that the study convergence of the discretization error estimator with respect to the interface error of the domain decomposition method (displacement gap and lack of balance of interface forces) enables to propose new stopping criteria for the DD solver.

Though our method has been developed for both dual and primal approach of domain decomposition, this paper aims to present our strategy especially in the framework of the FETI method. We thus invite the interested reader to refer to [15] for more details about the specific setup related to the BDD method. Section 2 recalls the general framework related to substructuring and the formulation of FETI method. After introducing the concept of error in constitutive relation, section 3 presents its formulation within domain decomposition. Then, section 4 details the specific implementation in the context of the FETI method. Finally, section 5 presents numerical assessments, especially to show that a good estimation can be obtained far earlier than the solver converged (in the sense of domain decomposition iterative solver).

2 FRAMEWORK AND BASICS

2.1 Reference mechanical problem

Let us consider the static equilibrium of a structure which occupies the open domain $\Omega \subset \mathbb{R}^d$ and which is submitted to given body forces f, to given traction forces g on $\partial_f\Omega$ and to given displacements u_d on the complementary part $\partial_u\Omega \neq \varnothing$. We assume the structure undergoes small perturbations and that the material is linear elastic, characterized by the Hooke's tensor $\mathbb{H}$.

For an open subset $\omega \subset \Omega$, and denoting by u the unknown displacement field, $\varepsilon(u)$ the symmetric part of the gradient, σ the Cauchy stress tensor, we introduce the following subspaces $\mathcal{K}a(\Omega)$ of *kinematically admissible* and $\mathcal{S}a(\Omega)$ *statically admissible* fields:

$$\mathcal{K}a(\omega) = \left\{ v \in \left[H^1(\omega)\right]^d \text{ such that } \operatorname{tr}(v)_{|\partial_u\omega} = u_d \right\} \tag{1a}$$

$$\mathcal{S}a(\omega) = \left\{ \tau \in \left[L^2(\omega)\right]^{d^2}_{sym} \text{ such that } \int_\omega \tau : \varepsilon(v)\, d\omega = \int_\omega f.v\, d\omega + \int_{\partial_f\omega} g.v\, d\Upsilon, \forall\ v \in \mathcal{K}a^0(\omega) \right\} \tag{1b}$$

where tr is the trace operator and $KA^0(\omega)$ is the vector space associated to $\mathcal{K}a(\omega)$.

The mechanical problem set on Ω can then be formulated as:

$$\text{Find } (u,\sigma) \in \mathcal{C}a(\Omega) \times \mathcal{S}a(\Omega) \text{ verifying: } \sigma = \mathbb{H} : (u) \tag{2}$$

2.2 Substructured formulation

Let us consider a decomposition of domain Ω in open subsets $(\Omega^{(s)})_{1\leqslant s\leqslant N_{sd}}$ (N_{sd} is the number of subdomains) so that $\Omega^{(s)} \cap \Omega^{(s')} = \varnothing$ for $s \neq s'$ and $\bar{\Omega} = \cup_s \bar{\Omega}^{(s)}$. For any N_{sd}-turple $(v^{(s)})_{1\leqslant s\leqslant N_{sd}}$ of local fields denoted by $v^{\square}$, we define the global assembling operator $\mathcal{A}$ by:

$$v = \mathcal{A}(v^{\square}) \Leftrightarrow v_{|\Omega^{(s)}} = v^{(s)} \tag{3}$$

Under this framework, kinematic and static admissibility on the whole structure can be restricted to each sub-structure $\Omega^{(s)}$ provided the verification of interface conditions, namely displacements continuity and tractions balance (action-reaction principle):

$$\mathcal{A}(u^{\square}) \in \mathcal{K}a(\Omega) \Leftrightarrow \begin{cases} u^{(s)} \in \mathcal{K}a(\Omega^{(s)}), \ \forall s \\ \mathrm{tr}(u^{(s)}) = \mathrm{tr}(u^{(s')}) \text{ on } \Upsilon^{(ss')}, \ \forall (s,s') \end{cases} \tag{4a}$$

$$\mathcal{A}(\sigma^{\square}) \in \mathcal{S}a(\Omega) \Leftrightarrow \begin{cases} \sigma^{(s)} \in \mathcal{S}a(\Omega^{(s)}), \ \forall s \\ \sigma^{(s)}.n^{(s)} + \sigma^{(s')}.n^{(s')} = 0 \text{ on } \Upsilon^{(ss')}, \ \forall (s,s') \end{cases} \tag{4b}$$

The sub-structured problem may then be formulated as:

$$\text{Find } (\mathcal{A}(u^{\square}), \mathcal{A}(\sigma^{\square})) \in \mathcal{C}a(\Omega) \times \mathcal{S}a(\Omega) \text{ verifying: } \sigma^{(s)} = \mathbb{H} : \left(u^{(s)}\right) \quad (1 \leq s \leq \mathcal{N}_{\mathrm{sd}}) \tag{5}$$

2.3 Finite element approximation

Let Ω_h be a tessellation of $\bar{\Omega}$ associated to a finite dimensional subspace $\mathcal{K}a_h(\Omega)$ of $\mathcal{K}a(\Omega)$ and leading to a conforming substructuring so that (i) each element only belongs to one subdomain and (ii) nodes are matching on the interfaces. Under those assumptions, each degree of freedom is either located inside a subdomain (subscript i) or on its boundary $\Upsilon^{(s)} = \cup_{s'} \Upsilon^{(ss')}$ (subscript b) where it is shared with at least one neighboring subdomain. Approximation $u_h \in \mathcal{K}a_h(\Omega)$ of u may then be expressed with the $d \times N_{dof}$ matrix $\boldsymbol{\varphi}$ of shape functions (which form a basis of $\mathcal{K}a_h(\Omega)$) and the vector of nodal unknowns $\mathbf{u}$ as $u_h = \boldsymbol{\varphi}\mathbf{u}$.

Introducing $\boldsymbol{\lambda}_b^{(s)}$, vector of unknown efforts imposed on the interface of a subdomain $\Omega^{(s)}$ by its neighbors, the finite element problem can be written on each subdomain separating internal and boundary degrees of freedom (dof):

$$\begin{pmatrix} \mathbf{K}_{ii}^{(s)} & \mathbf{K}_{ib}^{(s)} \\ \mathbf{K}_{bi}^{(s)} & \mathbf{K}_{bb}^{(s)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_b^{(s)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i^{(s)} \\ \mathbf{f}_b^{(s)} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_i^{(s)} \\ \boldsymbol{\lambda}_b^{(s)} \end{pmatrix} \tag{6}$$

where $\mathbf{K}^{(s)}$ is the (symmetric positive definite) stiffness matrix of domain $\Omega_h^{(s)}$ and $\mathbf{f}^{(s)}$ is the vector of generalized forces. Eq. (6) then allow to express equilibrium on sub-domains in a condensed form function of the dof located on $\Upsilon^{(s)}$ (eq. 7a). In addition to that, the discretization of interface conditions included in (4a) and (4b) can be expressed as dof-to-dof relationships thanks to the conforming feature of the decomposition. As a result, finite

element approximation of the substructured reference problem leads to the following interface system:

$$\mathbf{S}^{(s)}\mathbf{u}_b^{(s)} = \mathbf{b}^{(s)} + \boldsymbol{\lambda}_b^{(s)} \quad (7a) \qquad \text{with:}$$

$$\sum_s \underline{\mathbf{A}}^{(s)}\mathbf{u}_b^{(s)} = \mathbf{0} \quad (7b) \qquad \mathbf{S}^{(s)} = \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)}\mathbf{K}_{ii}^{(s)^{-1}}\mathbf{K}_{ib}^{(s)}$$

$$\sum_s \mathbf{A}^{(s)}\boldsymbol{\lambda}_b^{(s)} = \mathbf{0} \quad (7c) \qquad \mathbf{b}^{(s)} = \mathbf{f}_b^{(s)} - \mathbf{K}_{bi}^{(s)}\mathbf{K}_{ii}^{(s)^{-1}}\mathbf{f}_i^{(s)}$$

where $\mathbf{A}^{(s)}$ and $\underline{\mathbf{A}}^{(s)}$ are assembling operators [1] so that $\mathbf{A}^{(s)}$ enables to formulate the mechanical equilibrium of interfaces (4b) and $\underline{\mathbf{A}}^{(s)}$ enables to formulate the continuity of displacements (4a) (see [6] for more an extensive description of all operators). Let's bring back the fundamental orthogonality property verified by assembling operators:

$$\sum_s \mathbf{A}^{(s)}\underline{\mathbf{A}}^{(s)^T} = \mathbf{0} \tag{8}$$

2.4 FETI approach

The starting point of the Finite Element Tearing and Interconnecting domain decomposition [4], is to introduce a unique interface traction unknown $\boldsymbol{\lambda}_b$ so that interface equilibrium is always insured when expressing $\boldsymbol{\lambda}_b^{(s)}$:

$$\boldsymbol{\lambda}_b^{(s)} = \underline{\mathbf{A}}^{(s)^T}\boldsymbol{\lambda}_b \Longrightarrow \sum_s \mathbf{A}^{(s)}\boldsymbol{\lambda}_b^{(s)} = \mathbf{0} \tag{9}$$

Displacements can yet be deduced from $\boldsymbol{\lambda}_b$ (10a) if it satisfies Fredholm's alternative (10b) on each substructure:

$$\mathbf{u}_b^{(s)} = \mathbf{S}^{(s)^+}\left(\mathbf{b}^{(s)} + \underline{\mathbf{A}}^{(s)^T}\boldsymbol{\lambda}_b\right) + \mathbf{R}_b^{(s)}\boldsymbol{\alpha}^{(s)} \tag{10a}$$

$$\mathbf{0} = \mathbf{R}_b^{(s)^T}\left(\mathbf{b}^{(s)} + \underline{\mathbf{A}}^{(s)^T}\boldsymbol{\lambda}_b\right) \tag{10b}$$

where $\boldsymbol{\alpha}^{(s)}$ is the unknown magnitude of rigid body motions and superscript $^+$ denotes the generalised inverse. The FETI solver then consists in iteratively finding an interface effort $\boldsymbol{\lambda}_b$ (under the previous constraint) insuring the continuity of interface displacement $\mathbf{0} = \sum_s \underline{\mathbf{A}}^{(s)}\mathbf{u}_b^{(s)}$, which leads to the well-known interface problem:

$$\begin{pmatrix} \underline{\mathbf{S}}_d & \underline{\mathbf{G}} \\ \underline{\mathbf{G}}^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{\lambda}_b \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{b}}_d \\ \mathbf{e} \end{pmatrix} \tag{11}$$

where:

$$\underline{\mathbf{S}}_d = \sum_{s=1}^{\mathcal{N}_{\text{sd}}} \underline{\mathbf{A}}^{(s)}\mathbf{S}^{+(s)}\underline{\mathbf{A}}^{(s)^T} \quad \underline{\mathbf{G}} = \left[\underline{\mathbf{A}}^{(1)}\mathbf{R}_b^{(1)} \dots \underline{\mathbf{A}}^{(\mathcal{N}_{\text{sd}})}\mathbf{R}_b^{(\mathcal{N}_{\text{sd}})}\right] \qquad \boldsymbol{\alpha} = \left[\boldsymbol{\alpha}^{(1)^T} \dots \boldsymbol{\alpha}^{(\mathcal{N}_{\text{sd}})^T}\right]^T$$

$$\underline{\mathbf{b}}_d = -\sum_{s=1}^{\mathcal{N}_{\text{sd}}} \underline{\mathbf{A}}^{(s)}\mathbf{S}^{(s)^+}\mathbf{b}_p^{(s)} \quad \mathbf{e} = \left[-\mathbf{b}_p^{(1)^T}\mathbf{R}_b^{(1)} \cdots -\mathbf{b}_p^{(\mathcal{N}_{\text{sd}})^T}\mathbf{R}_b^{(\mathcal{N}_{\text{sd}})}\right]^T$$

[1] In the case of two subdomains, one has $\sum_s \mathbf{A}^{(s)}\boldsymbol{\lambda}_b = \boldsymbol{\lambda}_b^{(1)} + \boldsymbol{\lambda}_b^{(2)} = \mathbf{0}$ and $\sum_s \underline{\mathbf{A}}^{(s)}\mathbf{u}_b^{(s)} = \mathbf{u}_b^{(1)} - \mathbf{u}_b^{(2)} = \mathbf{0}$.

In order to iteratively solve problem (11) in an efficient way, the best preconditioner $\underline{\tilde{\mathbf{S}}}_d^{-1}$ for operator $\underline{\mathbf{S}}_d$ is built as an assembling of inverses of local operators $\mathbf{S}^{(s)^+}$ previously involved:

$$\underline{\tilde{\mathbf{S}}}_d^{-1} = \sum_{s=1}^{\mathcal{N}_{\text{sd}}} \underline{\tilde{\mathbf{A}}}^{(s)} \mathbf{S}^{(s)} \underline{\tilde{\mathbf{A}}}^{(s)T} \tag{12}$$

where $(\underline{\tilde{\mathbf{A}}}^{(s)})_s$ are scaled assembling operators verifying the fundamental property (13), whose typical example is the multiplicity scaling (14)[2]:

$$\sum_s \underline{\mathbf{A}}^{(s)} \, \underline{\tilde{\mathbf{A}}}^{(s)^T} = \mathbf{I} \tag{13}$$

$$\underline{\tilde{\mathbf{A}}}^{(s)^T} = \underline{\mathbf{A}}^{(s)^T} \left(\sum_j \underline{\mathbf{A}}^{(j)} \, \underline{\mathbf{A}}^{(j)^T} \right)^{-1} \tag{14}$$

Let us note that more elaborate operators taking the heterogeneity into account may be used in the case of heterogeneous structures [16].

3 A POSTERIORI ERROR ESTIMATION

3.1 Error in constitutive relation

Let us come back to the continuous level and introduce the functional *error in constutive relation* measuring the non-verification of constitutive relation between a given pair of displacement and stress (v, τ):

$$\boldsymbol{e}_{\text{rdc},\omega}(v,\tau) = \left\| \tau - \mathbb{H} : \varepsilon(v) \right\|_{\mathbb{H}^{-1},\omega} \qquad \text{with } \| \bullet \|_{\mathbb{H}^{-1},\omega} = \left[\int_\omega \bullet : \mathbb{H}^{-1} : \bullet \, d\omega \right]^{1/2} \tag{15}$$

Recall that given the tessellation Ω_h and its associated subspace of admissible displacements $\mathcal{K}a_h(\Omega) \subset \mathcal{K}a(\Omega)$, the usual finite element approximation of reference problem (2) writes:

$$\begin{gathered} \text{Find } u_h \in \mathcal{C}a_h(\Omega) \text{ such that } \sigma_h = \mathbb{H} : \varepsilon(u_h) \text{ satisfy:} \\ \int_\Omega \sigma(u_h) : \varepsilon(v_h) \, d\Omega = \int_\Omega f.v_h \, d\Omega + \int_{\partial_f \Omega} g.v_h \, d\Upsilon, \ \forall \ v_h \in \mathcal{C}a_h^0(\Omega) \end{gathered} \tag{16}$$

Then, on the basis of the solution (u_h, σ_h) of (16) satisfying the classical properties $u_h \in \mathcal{K}a(\Omega)$ and $\boldsymbol{e}_{\text{cr},\Omega}(u_h, \sigma_h) = 0$ but $\sigma_h \notin \mathcal{S}a(\Omega)$, the basic principle of the error in constitutive relation consists in deducing an admissible displacement-stress pair $(\hat{u}_h, \hat{\sigma}_h) \in \mathcal{K}a(\Omega) \times \mathcal{S}a(\Omega)$ in order to measure the residual on the constitutive equation (15). Hence, the evaluation of $\boldsymbol{e}_{\text{cr},\Omega}(\hat{u}_h, \hat{\sigma}_h) \geq 0$ for any admissible pair $(\hat{u}_h, \hat{\sigma}_h)$ provides a guaranteed upper bound of the global error thanks to the *Prager-Synge* theorem:

$$\|e\|_{\mathbb{H},\Omega} = \|u - u_h\|_{\mathbb{H},\Omega} \leq \left[\boldsymbol{e}_{\text{rdc},\Omega}(\hat{u}_h, \hat{\sigma}_h) \right]^{1/2} \qquad \text{with } \| \bullet \|_{\mathbb{H},\omega} = \left[\int_\omega \varepsilon(\bullet) : \mathbb{H} : \varepsilon(\bullet) \, d\omega \right]^{1/2} \tag{17}$$

[2] In the case of two subdomains, one has $\sum_s \underline{\tilde{\mathbf{A}}}^{(s)} \boldsymbol{\lambda}_b^{(s)} = \frac{1}{2} \left(\boldsymbol{\lambda}_b^{(1)} - \boldsymbol{\lambda}_b^{(2)} \right)$.

$\mathcal{K}a_h(\Omega)$ being a subspace of $\mathcal{K}a(\Omega)$, the construction of an admissible displacement field $\hat{u}_h$ is straightforward since it can be taken equal to u_h. On the other hand, as σ_h is not statically admissible, the construction of an admissible stress field $\hat{\sigma}_h \in \mathcal{S}a(\Omega)$ is a crucial point which has already been widely studied in the literature. In addition to the use of dual formulation [7], classical methods consist in post-processing a statically admissible field from Field σ_h (obtained by a displacement formulation) through the element equilibration techniques [8, 9], which have been improved by the use of the concept of partition of unity [10] or flux-free-based methods [14, 5, 13]. In most cases they involve the computation of efforts on "star-patches" which are the set of elements sharing one node, for each node of the mesh. Though rather simple, these computations are in great number and thus expensive.

In the following, we note by $\mathcal{F}_h$ the algorithm which has been chosen to build an admissible stress field $\hat{\sigma}_h$. Whatever the choice, the algorithm takes as input not only the finite element stress field σ_h but also the continuous representation of the imposed forces (f, g).

$$\hat{\sigma}_h = \mathcal{F}_h(\sigma_h, f, g) \in \mathcal{S}a\Omega$$

The algorithm we have used for our applications is the one proposed in [8] using a three degrees higher polynomial basis when solving the local problems on elements [3].

3.2 Formulation within a generic domain decomposition

Following the previously presented idea of error in constitutive relation in a sequential framework, the key point for an efficient evaluation in a substructured context (without overlapping) is to define admissible pairs $(\hat{u}_h^{(s)}, \hat{\sigma}_h^{(s)}) \in \mathcal{K}a(\Omega^{(s)}) \times \mathcal{S}a(\Omega^{(s)})$ on each subdomain so that the associated assembled pair is admissible for the reference problem $(\mathcal{A}(\hat{u}_h^{\square}), \mathcal{A}(\hat{\sigma}_h^{\square})) \in \mathcal{K}a(\Omega) \times \mathcal{S}a(\Omega)$. Due to the absence of overlap, the additive structure of the associated error in constitutive relation leads to a fully parallel evaluation of the a posteriori error estimator:

$$\boldsymbol{e}_{\mathrm{cr},\Omega}(\mathcal{A}(\hat{u}_h^{\square}), \mathcal{A}(\hat{\sigma}_h^{\square}))^2 = \sum_{s=1}^{\mathcal{N}_{\mathrm{sd}}} \left(\boldsymbol{e}_{\mathrm{cr},\Omega^{(s)}}(\hat{u}_h^{(s)}, \hat{\sigma}_h^{(s)}) \right)^2$$

However, the application of a classical recovery strategy to compute admissible fields raises two difficulties in a substructured context. First, the star-patches can not be employed on the boundary nodes without assuming communication between subdomains. Though these exchanges would remain limited, we propose an alternate strategy to achieve full parallelism without impairing the properties of the error in constitutive relation. Second, recall that the FETI method used on the discrete substructured problem (7) consists in iteratively solving eq. (11) so that displacement continuity (4a) is only verified (up to a certain precision) once the solver converged. Thus recovering strategies need to be adapted so that the local fields $(\hat{u}_h^{(s)}, \hat{\sigma}_h^{(s)})$ satisfy the interface conditions.

4 PARALLEL RECOVERY OF ADMISSIBLE FIELDS

4.1 Kinematically admissible fields

In order to ensure interface Condition (4a) when building $\hat{u}_h^{(s)} \in \mathcal{K}a(\Omega^{(s)})$ so that $\mathcal{A}(\hat{u}_h^{\square}) \in \mathcal{K}a(\Omega)$, we start off by introducing on each sub-structure continuous interface displacement fields $\hat{u}_{bh}^{(s)}$ verifying $\hat{u}_{bh}^{(s)} = \hat{u}_{bh}^{(s')}$ on $\Upsilon^{(ss')}$. Denoting by $\hat{\mathbf{u}}_b^{(s)}$ the nodal value $\hat{u}_{bh}^{(s)}$, the last condition can be directly transposed into a discrete counterpart thanks to the matching discretizations on the interface:

$$\hat{\mathbf{u}}_b^{(s)} = \hat{\mathbf{u}}_b^{(s')}, \ \forall (s, s') \qquad \Longleftrightarrow \qquad \sum_s \underline{\mathbf{A}}^{(s)} \hat{\mathbf{u}}_b^{(s)} = 0 \tag{18}$$

However, displacement continuity (7b) is only verified upon convergence in the FETI method, but local displacements on each substructure may be computed at each iteration through (10a) since condition (10b) is always satisfied. The associated gap of interface displacement is given by:

$$[\![\mathbf{u}_b]\!] = \sum_s \underline{\mathbf{A}}^{(s)} \mathbf{u}_b^{(s)} \tag{19}$$

Then, we choose to use the scaled assembly operators introduced for the preconditionning step (12), and define $\hat{\mathbf{u}}_b^{(s)}$ as:

$$\hat{\mathbf{u}}_b^{(s)} = \mathbf{u}_b^{(s)} - \underline{\tilde{\mathbf{A}}}^{(s)^T} [\![\mathbf{u}_b]\!] \tag{20}$$

Thereafter, the property (13) satisfied by operators $\underline{\tilde{\mathbf{A}}}^{(s)}$ ensures the verification of the discrete continuity condition (18) on $\hat{\mathbf{u}}_b^{(s)}$.

Finally, one finite element problem solved independently on each subdomain with imposed Dirichlet conditions on the interface given by $\hat{\mathbf{u}}_b^{(s)}$ enables to deduce the kinematically admissible displacement field $\hat{u} = \mathcal{A}(\hat{u}_h^{\square}) \in \mathcal{K}a(\Omega)$:

$$\begin{aligned} \hat{\mathbf{u}}_i^{(s)} &= \mathbf{K}_{ii}^{(s)^{-1}} \left(\mathbf{f}_i^{(s)} - \mathbf{K}_{ib}^{(s)} \mathbf{A}^{(s)^T} \hat{\mathbf{u}}_b^{(s)} \right) \\ \hat{u}_h^{(s)} &= \boldsymbol{\varphi}^{(s)} \hat{\mathbf{u}}^{(s)} = \begin{pmatrix} \boldsymbol{\varphi}_i^{(s)} & \boldsymbol{\varphi}_b^{(s)} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}}_i^{(s)} \\ \hat{\mathbf{u}}_b^{(s)} \end{pmatrix} \end{aligned} \tag{21}$$

Let us note a key feature of this approach which comes from the fact that all the involved operations are already realized during the steps related to the Dirichlet's preconditioner (12), so that all finite element quantities (even the internal ones) are available at no cost: the quantity $[\![\mathbf{u}_b]\!]$ is directly available during the classical solution procedure (without computing any $\boldsymbol{\alpha}^{(j)}$) which is based on an initialization/projection algorithm [4], and the displacement field $\mathbf{u}^{(s)}$ can be defined up to an element of the kernel (a rigid body motion) since only its symmetric gradient is used during the computation of the error.

4.2 Statically admissible fields

In order to ensure interface Condition (4b) when building $\hat{\sigma}_h^{(s)} \in \mathcal{S}a(\Omega^{(s)})$ so that $\mathcal{A}(\hat{\sigma}_h^{\square}) \in \mathcal{S}a(\Omega)$, a generic way consists in introducing on each subdomain a continuous balanced interface traction field $\hat{F}_{bh}^{(s)}$ defined on $\Upsilon^{(s)}$ which satisfy:

$$\hat{F}_{bh}^{(s)} + \hat{F}_{bh}^{(s')} = 0 \quad \text{sur } \Gamma^{(ss')} \tag{22a}$$

$$\int_{\Omega^{(s)}} f^{(s)}.\rho d\Omega + \int_{\partial_f \Omega^{(s)}} g^{(s)}.\rho dS + \int_{\Gamma^{(s)}} \hat{F}_{bh}^{(s)}.\rho dS = 0 \quad \forall \rho \in \mathcal{K}aR(\Omega^{(s)}) \tag{22b}$$

where $\mathcal{K}aR(\Omega^{(s)})$ is the set of rigid body motions which are compatible with Dirichlet conditions imposed on $\partial_u \Omega^{(s)}$, and (22b) stands for the translation of Fredholm's alternative. We then associate $\hat{F}_{bh}^{(s)}$ to a finite element nodal reaction field $\hat{\boldsymbol{\lambda}}_b^{(s)}$ through an algorithm $\hat{F}_{bh}^{(s)} = \mathcal{G}_h\left(\hat{\boldsymbol{\lambda}}_b^{(s)}\right)$ requiring that the discrete field $\hat{\boldsymbol{\lambda}}_b^{(s)}$ and the continuous field $\hat{F}_{bh}^{(s)}$ develop the same virtual work in any finite element displacement field:

$$\int_{\Upsilon^{(ss')}} \hat{F}_{bh}^{(s)} \cdot \varphi_{j\,|\Upsilon^{(ss')}}^{(s)} dS = \hat{\boldsymbol{\lambda}}_{b,j}^{(s)} \tag{23}$$

where j denotes a node of the interface, $\varphi_j^{(s)}$ its associated shape function and $\hat{\boldsymbol{\lambda}}_{b,j}^{(s)}$ the corresponding nodal component of $\hat{\boldsymbol{\lambda}}_b^{(s)}$. We get the following discrete counterparts for $\hat{\boldsymbol{\lambda}}_b^{(s)}$:

$$\sum_s \mathbf{A}^{(s)} \hat{\boldsymbol{\lambda}}_b^{(s)} = \mathbf{0} \qquad\qquad \mathbf{R}_b^{(s)T}\left(\hat{\boldsymbol{\lambda}}_b^{(s)} + \mathbf{b}^{(s)}\right) = \mathbf{0} \tag{24}$$

In the FETI solver, the nodal interface fields are by construction (9) always balanced at the interface and associated to well-posed discrete Neumann problems on each substructure thanks to (10b), allowing us to directly set $\hat{\boldsymbol{\lambda}}_b^{(s)} = \boldsymbol{\lambda}_b^{(s)}$. Setting up the algorithm $\mathcal{G}_h$ in order to get $\hat{F}_{bh}^{(s)}$ from $\hat{\boldsymbol{\lambda}}_b^{(s)}$ is strongly linked to a choice of interpolation basis. A simple choice is then to use the basis of finite element shape functions on $\Upsilon^{(s)}$: $\hat{F}_{bh} = \boldsymbol{\varphi}_{|\Upsilon^{(s)}}^{(s)} \hat{\mathbf{F}}_b^{(s)}$, which, though leading to a coarse interpolation, doesn't alter the global performance of the method [3].

Once the traction field $\hat{F}_{bh}^{(s)}$ has been recovered, we get the statically admissible field $\hat{\sigma}_h = \mathcal{A}(\hat{\sigma}_h^{\square}) \in \mathcal{S}a(\Omega)$ through a two-step process. First one is to solve a finite element problem (25a) on each subdomain with imposed Neumann conditions on the interface given by $\hat{F}_{bh}^{(s)}$. The next one is to apply a sequential recovery algorithm $\mathcal{F}_h$ on each finite

[3] Let us note that more consistent choices needs to allow discontinuities at nodes, a counterpart of which is to use an additional minimization problem. Some alternative choices are currently studied and will be discussed.

element stress field $\mathbb{H} : \varepsilon(\boldsymbol{\varphi}^{(s)}\tilde{\mathbf{u}}^{(s)})$ previously obtained (25a) taking into account $\hat{F}_{bh}^{(s)}$ as an imposed force on $\Upsilon^{(s)}$:

$$\tilde{\mathbf{u}}^{(s)} = \mathbf{K}^{(s)^+}\left(\mathbf{f}^{(s)} + \mathbf{t}^{(s)^T}\hat{\boldsymbol{\lambda}}_b^{(s)}\right) \tag{25a}$$

$$\hat{\sigma}_h^{(s)} = \mathcal{F}_{\text{eq}}\left(\mathbb{H} : \varepsilon(\boldsymbol{\varphi}^{(s)}\tilde{\mathbf{u}}^{(s)}), f^{(s)}, \left\{g^{(s)}, \mathcal{G}_h(\hat{\boldsymbol{\lambda}}_b^{(s)})\right\}\right) \tag{25b}$$

$$\tag{25c}$$

It has to be observed that the fully parallel procedure $\mathcal{G}_h$ proposed above leads to a different admissible traction field than the one which would have been obtained using standard patch-technique [9] (referred in the sequel as the sequential approach) so that the parallel error estimation is different from the standard sequential one. Finally, let us note that the only extra operations are due to the use of algorithms $\mathcal{G}_h$ (to compute $\hat{F}_{bh}$) and $\mathcal{F}_h$ (to compute $\hat{\sigma}_h$).

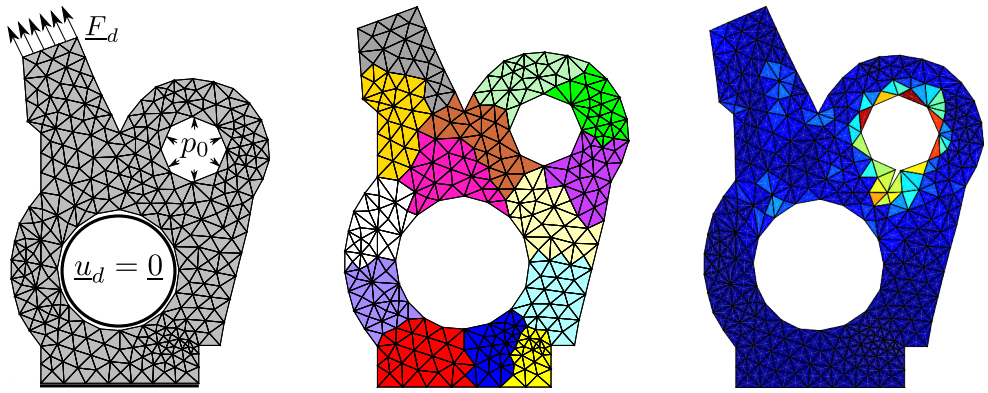
5 NUMERICAL ASSESSMENT

The performances of our parallel error estimator has been assessed on a 2D toy problem of a Γ-shape structure clamped on its basis and submitted to traction and shear on its upper-right side, with an isotropic linear elastic material behavior. A set of computations with increasing number of subdomains has been driven on a sequence of regular meshes generated with a increasing refinement. In addition to the new parallel error estimator $\boldsymbol{e}_{\text{cr}}^{\text{ddm}}$, the standard sequential one $\boldsymbol{e}_{\text{cr}}^{\text{seq}}$ and the true error e_h (using a reference field u_{ex} on a very fine mesh) has been computed:

$$\boldsymbol{e}_{\text{cr}}^{\text{seq}} = \boldsymbol{e}_{\text{cr},\Omega}(\hat{u}_h, \hat{\sigma}_h) \quad \boldsymbol{e}_{\text{cr}}^{\text{ddm}} = \boldsymbol{e}_{\text{cr},\Omega}(\mathcal{A}(\hat{u}_h^{\square}), \mathcal{A}(\hat{\sigma}_h^{\square})) \quad e_h = \|u_{ex} - \hat{u}_h\|_{\mathbb{H},\Omega} = \sqrt{\|u_{ex}\|_{\mathbb{H},\Omega}^2 - \|\hat{u}_h\|_{\mathbb{H},\Omega}^2} \tag{26}$$

At last, let us note that all our computations are driven in the ZeBuLoN finite element code [1], using elements of polynomial degree $p = 1$. The results when convergence of the domain decomposition solver is reached, which can be found in the paper [15], show that $\boldsymbol{e}_{\text{cr}}^{\text{ddm}}$ barely depends on the substructuring so that the estimates are quite similar whether they are conducted on a single domain or on N_{sd} subdomains. As a conclusion, the parallel error estimator $\boldsymbol{e}_{\text{cr}}^{\text{ddm}}$ enables to recover the same efficiency factor as the standard sequential one, while the CPU-time is divided by N_{sd}.

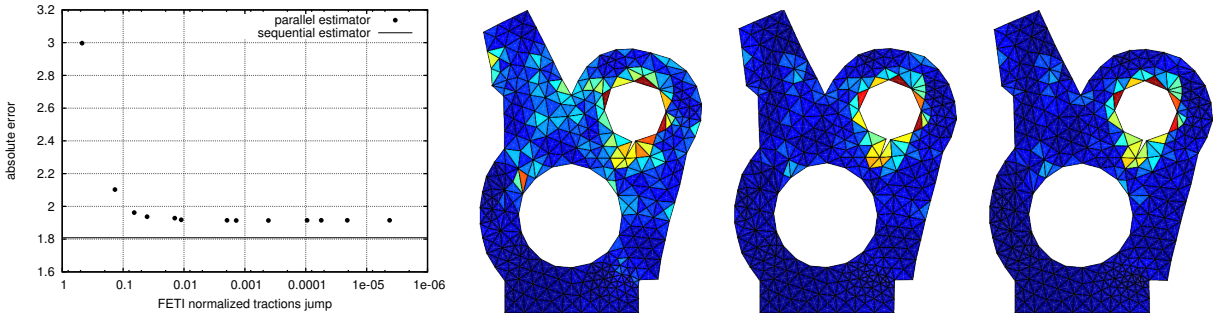
Another interesting feature associated to the use of an iterative solver for the domain decomposition (DD) problem is that the discretization error estimation can be conducted before DD convergence is reached, that is in presence of displacement discontinuity at the interface when using FETI. We illustrate this point on a small crack opening problem (fig. 1) already used in other papers [14, 10]. The behavior is supposed to be isotropic linear elastic ($E = 2000$ Pa and $\mu = 0.3$) and plane stress hypothesis is assumed. Beside, the FETI algorithm used on the 14-sub-domain decomposition is equipped with the Dirichlet



(a) Finite-element problem (b) Substructuring ($N_{sd} = 14$) (c) Elementary contributions $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr},E}$ at convergence

Figure 1: Crack opening problem

preconditioner (12) while its convergence criterion (which stands here for the interface displacement gap) is set to 10^{-6}. Figure 2 shows the convergence curve of $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr}}$ during the FETI iterations, plotted as a function of the corresponding residual that is the normalized displacement gap at the interface.



Convergence of $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr}}$ $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr},E}$: iter. 1 $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr},E}$: iter. 2 $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr},E}$: iter. 3

Figure 2: Convergence of $\mathbf{e}^{\mathrm{ddm}}_{\mathrm{cr}}$ vs. interface residual and elementary contributions to $\mathbf{e}^{\mathrm{ddm}}_{\mathrm{cr}}$

The curve show a rapid convergence of the parallel error estimator along iterations of the solver, so that $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr}}$ can be considered as converged when the residual reaches an order of magnitude of 5.10^{-2} which corresponds to at most 4 iterations whereas the solver convergence is achieved in about 15 iterations. The "L"-shaped curves show that the impact of residual of the DD solver is preponderant only at the first iterations (when interface fields are very poorly estimated), afterwards $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr}}$ stagnates at a value close to $\boldsymbol{e}^{\mathrm{seq}}_{\mathrm{cr}}$ which is only associated to the discretization error. The maps of the elementary contributions $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr},E}$ to the parallel error estimator $\boldsymbol{e}^{\mathrm{ddm}}_{\mathrm{cr}}$ at iterations 1, 2 and 3 corroborate the previous observation. An interesting fallout of the foregoing is that when willing to carry out remeshing procedures, the maps obtained after few iterations of the solver are

sufficient to define correct refinement instructions.

6 CONCLUSIONS

In this paper, we presented a new method to handle error estimation within domain decomposition approaches. Our approach relies on the construction of admissible interface fields followed by a fully parallel construction of a displacement-stress pair that is kinematically and statically admissible on the whole structure. We then obtain a simple and efficient extension to error in constitutive relation in a substructured context. Moreover, this construction can be realized at every iteration of the domain decomposition solver, at a reduced numerical cost when taking advantage of the associated preconditionners. As a result, not only our method enables to divide the huge CPU cost associated to error estimation, but a satisfactory estimate can also be accessed far before the solver reaches convergence. Indeed, results show that even roughly estimated interface fields enable to obtain a good estimation of the discretization error and correct maps of elementary contributions. This last feature let us envisage, on the one hand, the development of a new convergence criterion for the iterative solver, and, on the other hand, the set up of efficient mesh adaptation procedures relying on estimates obtained after only few iterations. Such procedures are currently studied and leads to different approaches whether the remeshing step may preserve or not the topology of the sub-structuring. Other current studies deals with heterogeneous problems, whose results will be presented during the conference.

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