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# Cartan Subgroups and Generosity in $SL_2(\mathbb{Q}_p)$

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## Abstract

We describe all Cartan subgroups of  $SL_2(\mathbb{Q}_p)$ . We show that the Cartan subgroup consisting of all diagonal matrices is generous and it is the only one up to conjugacy.

*Keywords* p-adic field ; Cartan subgroup ; generosity

*MSC2010* 20G25 ; 20E34 ; 11E57

A subset  $X$  of a group  $G$  is *left-generic* if  $G$  can be covered by finitely many left-translates of  $X$ . We define similar right-genericity. If  $X$  is  $G$ -invariant, then left-genericity is equivalent to right-genericity. This important notion in model theory was particularly developed by B. Poizat for groups in stable theories [3]. For a group of finite Morley-rank and  $X$  a definable subset, generosity is the same as being of maximal dimension [3, lemme 2.5]. The term generous was introduced in [2] to show some conjugation theorem. A definable subset  $X$  of a group  $G$  is *generous* in  $G$  if the union of its  $G$ -conjugates,  $X^G = \{x^g \mid (x, g) \in X \times G\}$ , is generic in  $G$ .

In an arbitrary group  $G$ , we define a *Cartan subgroup*  $H$  as a maximal nilpotent subgroup such that every finite index normal subgroup  $X \triangleleft H$  is of finite index in its normalizer  $N_G(X)$ . In connected reductive algebraic groups over an algebraically closed fields, the maximal torus is typically an example of a Cartan subgroup. Moreover it is the only one up to conjugation and it is generous. It has been remarked in [1] that, in the group  $SL_2(\mathbb{R})$ , the Cartan subgroup consisting of diagonal matrices is also generous. But it has also been remarked that in the case of  $SL_2(\mathbb{R})$ , there exists another Cartan subgroup, namely  $SO_2(\mathbb{R})$ , which is not generous.

We will discuss here some apparently new remarks of the same kind in  $SL_2(\mathbb{Q}_p)$ . First we describe all Cartan subgroups of  $SL_2(\mathbb{Q}_p)$ . After we show that the Cartan

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subgroup consisting of diagonal matrices is generous and it is the only one up to conjugacy.

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## Description of Cartan subgroups up to conjugacy

We note  $v_p : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{+\infty\}$  the p-adic-valuation, and  $ac : \mathbb{Q}_p^\times \rightarrow \mathbb{F}_p$  the angular component defined by  $ac(x) = res(p^{-v_p(x)}x)$  where  $res : \mathbb{Q}_p \rightarrow \mathbb{F}_p$  is the residue map.

With this notations, if  $p \neq 2$ , an element  $x \in \mathbb{Q}_p^\times$  is a square if and only if  $v_p(x)$  is even and  $ac(x)$  is a square in  $\mathbb{F}_p$ . For  $p = 2$ , an element  $x \in \mathbb{Q}_2$  can be written  $x = 2^n u$  with  $n \in \mathbb{Z}$  and  $u \in \mathbb{Z}_2^\times$ , then  $x$  is a square if  $n$  is even and  $u \equiv 1 \pmod{8}$  [4].

**Fact 1** ([4]). *If  $p \neq 2$ , the group  $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , it has for representatives  $\{1, u, p, up\}$ , where  $u \in \mathbb{Z}_p^\times$  is such that  $ac(u)$  is not a square in  $\mathbb{F}_p$*

*The group  $\mathbb{Q}_2^\times / (\mathbb{Q}_2^\times)^2$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , it has for representatives  $\{\pm 1, \pm 2, \pm 5, \pm 10\}$ .*

For any prime  $p$ , and any  $\delta$  in  $\mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ , we put :

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a \in \mathbb{Q}_p^\times \right\}$$

$$Q_\delta = \left\{ \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a, b \in \mathbb{Q}_p \text{ and } a^2 - b^2\delta = 1 \right\}$$

**Lemma 1.**

$$\begin{aligned} \forall x \in Q_1 \setminus \{I, -I\} \quad C_{SL_2(\mathbb{Q}_p)}(x) &= Q_1 \\ \forall x \in Q_\delta \setminus \{I, -I\} \quad C_{SL_2(\mathbb{Q}_p)}(x) &= Q_\delta \end{aligned}$$

The checking of these equalities is left to the reader.

**Proposition 1.** *The groups  $Q_1$  and  $Q_\delta$  are Cartan subgroups of  $SL_2(\mathbb{Q}_p)$*

*Proof.* One checks easily that  $Q_1$  is abelian and the normalizer of  $Q_1$  is :

$$N_{SL_2(\mathbb{Q}_p)}(Q_1) = Q_1 \cdot \langle \omega \rangle \quad \text{where} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For  $X$  a subgroup of  $Q_1$ , if  $g \in N_{SL_2(\mathbb{Q}_p)}(X)$  and  $x \in X$ , then, using lemma 1

$$Q_1 = C_{SL_2(\mathbb{Q}_p)}(x) = C_{SL_2(\mathbb{Q}_p)}(x^g) = C_{SL_2(\mathbb{Q}_p)}(x)^g = Q_1^g$$

It follows that  $N_{SL_2(\mathbb{Q}_p)}(X) = N_{SL_2(\mathbb{Q}_p)}(Q_1) = Q_1 \cdot \langle \omega \rangle$  and if  $X$  of finite index  $k$  in  $Q_1$ , then  $X$  is of index  $2k$  in  $N_{SL_2(\mathbb{Q}_p)}(X)$ . We can see that for  $t$  in  $Q_1$ ,  $t^\omega = \omega^{-1}t\omega = t^{-1}$  thus  $N_{SL_2(\mathbb{Q}_p)}(Q_1)' = Q_1^2$  and  $[Q_1^2, \omega] = Q_1^2$ , in particular  $N_{SL_2(\mathbb{Q}_p)}(Q_1)$  is not nilpotent. By

the normalizer condition for nilpotent groups, if  $Q_1$  is properly contained in a nilpotent group  $K$ , then  $Q_1 < N_K(Q_1) \leq K$ , here  $N_K(Q_1) = Q_1 \cdot \langle \omega \rangle$  which is not nilpotent, a contradiction. It finishes the proof that  $Q_1$  is a Cartan subgroup.

For  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ , we check similiary that the group  $Q_\delta$  is abelian. Since for all subgroups  $X$  of  $Q_\delta$ ,  $C_{SL_2(\mathbb{Q}_p)}(X) = Q_\delta$ , it follows that  $N_{SL_2(\mathbb{Q}_p)}(X) = N_{SL_2(\mathbb{Q}_p)}(Q_\delta) = Q_\delta$ , and if  $X$  is of finite index in  $Q_\delta$  then  $X$  is of finite index in its normalizer. By the normalizer condition for nilpotent groups,  $Q_\delta$  is nilpotent maximal.  $\square$

**Proposition 2.**  $Q_1^{SL_2(\mathbb{Q}_p)} = \{A \in SL_2(\mathbb{Q}_p) \mid \text{tr}(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2\} \cup \{I, -I\}$

$$Q_\delta^{SL_2(\mathbb{Q}_p)} = \{A \in SL_2(\mathbb{Q}_p) \mid \text{tr}(A)^2 - 4 \in \delta \cdot (\mathbb{Q}_p^\times)^2\} \cup \{I, -I\}$$

We put :

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\} \cup \left\{ \begin{pmatrix} -1 & u \\ 0 & -1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\}$$

If  $A \in SL_2(\mathbb{Q}_p)$  satisfies  $\text{tr}(A)^2 - 4 = 0$ , then either  $\text{tr}(A) = 2$  or  $\text{tr}(A) = -2$ , and  $A$  is a conjugate of a element of  $U$ . In this case,  $A$  is said *unipotent*. It follows, from Proposition 2 :

**Corollary 3.** *We have the following partition :*

$$SL_2(\mathbb{Q}_p) \setminus \{I, -I\} = (U \setminus \{I, -I\})^{SL_2(\mathbb{Q}_p)} \sqcup (Q_1 \setminus \{I, -I\})^{SL_2(\mathbb{Q}_p)} \sqcup \bigsqcup_{\delta \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2} (Q_\delta \setminus \{I, -I\})^{SL_2(\mathbb{Q}_p)}$$

**Remark.** *If  $\delta$  and  $\delta'$  in  $\mathbb{Q}_p^\times$  are in the same coset of  $(\mathbb{Q}_p^\times)^2$ , then, by Proposition 2, if  $x' \in Q_{\delta'}$ , then there exists  $x \in Q_\delta$  and  $g \in SL_2(\mathbb{Q}_p)$ , such that  $x' = x^g$ , thus, by lemma 1,  $Q_{\delta'} = C_{SL_2(\mathbb{Q}_p)}(x') = C_{SL_2(\mathbb{Q}_p)}(x)^g = Q_\delta^g$ . Therefore the Corollary 3 makes sense.*

*Proof of Proposition 2.* • If  $A \in Q_1^{SL_2(\mathbb{Q}_p)}$ , then there exists  $P \in SL_2(\mathbb{Q}_p)$  such that

$$A = P \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} P^{-1}$$

with  $a \in \mathbb{Q}_p^\times$ . We have  $\text{tr}(A) = a + a^{-1}$ , so  $\text{tr}(A)^2 - 4 = (a + a^{-1})^2 - 4 = (a - a^{-1})^2$  and  $\text{tr}(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2$ .

Conversely , let  $A$  be in  $SL_2(\mathbb{Q}_p)$  with  $\text{tr}(A)^2 - 4$  a square. The characteristic polynomial is  $\chi_A(X) = X^2 - \text{tr}(A)X + 1$  and its discriminant is  $\Delta = \text{tr}(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2$ , so  $\chi_A$  has two distinct roots in  $\mathbb{Q}_p$  and  $A$  is diagonalizable in  $GL_2(\mathbb{Q}_p)$ . There is  $P \in GL_2(\mathbb{Q}_p)$ , and  $D \in SL_2(\mathbb{Q}_p)$  diagonal such that  $A = PDP^{-1}$ . If

$$P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we put

$$\tilde{P} = \begin{pmatrix} \frac{\alpha}{\det(P)} & \beta \\ \frac{\gamma}{\det(P)} & \delta \end{pmatrix}$$

and we have  $\tilde{P} \in SL_2(\mathbb{Q}_p)$  and  $A = \tilde{P}D\tilde{P}^{-1} \in Q_1^{SL_2(\mathbb{Q}_p)}$ .

• If  $A$  is in  $Q_\delta^{SL_2(\mathbb{Q}_p)} \setminus \{I, -I\}$ , then  $\text{tr}(A) = 2a$  and there exists  $b \neq 0$  such that  $a^2 - b^2\delta = 1$ . So  $\text{tr}(A)^2 - 4 = 4a^2 - 4 = 4(b^2\delta + 1) - 4 = (2b)^2\delta \in \delta \cdot (\mathbb{Q}_p^\times)^2$

Conversely we proceed as in the real case and the root  $i \in \mathbb{C}$ . The discriminant of  $\chi_A$ ,  $\Delta = \text{tr}(A)^2 - 4$  is a square in  $\mathbb{Q}_p(\sqrt{\delta})$ , and the characteristic polynomial  $\chi_A$  has two roots in  $\mathbb{Q}_p(\sqrt{\delta})$  :  $\lambda_1 = \alpha + \beta\sqrt{\delta}$  and  $\lambda_2 = \alpha - \beta\sqrt{\delta}$  (with  $\alpha, \beta \in \mathbb{Q}_p$ ). For the two eigen values  $\lambda_1$  and  $\lambda_2$ ,  $A$  has eigen vectors :

$$v_1 = \begin{pmatrix} x + y\sqrt{\delta} \\ x' + y'\sqrt{\delta} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} x - y\sqrt{\delta} \\ x' - y'\sqrt{\delta} \end{pmatrix}$$

In the base  $\{(x, x'), (y, y')\}$ , the matrix  $A$  can be written :

$$\begin{pmatrix} a & b \\ b\delta & a \end{pmatrix}$$

with  $a, b \in \mathbb{Q}_p$ . As above, we can conclude that there exists  $P \in SL_2(\mathbb{Q}_p)$  such that :

$$A = P \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} P^{-1}$$

□

**Theorem 4.** *The subgroups  $Q_1$  and  $Q_\delta$  (for  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ ) are the only Cartan subgroups up to conjugacy of  $SL_2(\mathbb{Q}_p)$*

**Remark.** *By Fact 1 and the previous remark, for  $p \neq 2$  there are four Cartan subgroup up to conjugacy in  $SL_2(\mathbb{Q}_p)$ , and for  $p = 2$ , there are eight.*

*Proof.* For the demonstration we note  $G = SL_2(\mathbb{Q}_p)$  and  $B$  the following subgroup of  $SL_2(\mathbb{Q}_p)$  :

$$B = \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{Q}_p^\times, u \in \mathbb{Q}_p \right\}$$

With these notations, we can easily check for  $g \in U \setminus \{I, -I\}$  that  $C_G(g) = U$  and  $N_G(U) = B$ . Moreover it is known that every  $q \in B$  can be written as  $q = tu$  where  $t \in Q_1$  and  $u \in U$ .

Consider  $K$  a Cartan subgroup of  $SL_2(\mathbb{Q}_p)$ . We will show that  $K$  is a conjugate of  $Q_1$  or  $Q_\delta$  (for  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ ). First we prove  $K$  cannot contain a unipotent element other than  $I$  or  $-I$ . Since a conjugate of a Cartan subgroup is still a Cartan subgroup, it suffice to show that  $K \cap U = \{I, -I\}$ .

In order to find a contradiction, let  $u \in K$  be a element of  $U$  different of  $I$  or  $-I$ ,  $u$  is in  $K \cap B$ . If  $\alpha \in N_G(K \cap B)$ , then we have that  $u^\alpha \in K \cap B$ , and since  $\text{tr}(u^\alpha) = \text{tr}(u) = \pm 2$ ,  $u^\alpha$  is still in  $U$ . Therefore  $U = C_G(u) = C_G(u^\alpha) = C_G(u)^\alpha = U^\alpha$  and so  $\alpha$  is in  $N_G(U) = B$ . It follows  $N_K(K \cap B) \leq B$  and finally  $N_K(K \cap B) = K \cap B$ . By the normalizer condition  $K \cap B$  cannot be proper in  $K$ , then  $K \leq B$ .

It is known (see for example [5, Lemma 0.1.10]) that if  $K$  is a nilpotent group and  $H \trianglelefteq K$  a non trivial normal subgroup, then  $H \cap Z(K)$  is not trivial. If we assume that  $K \not\leq U$ , since  $K \leq B = N_G(U)$ ,  $K \cap U$  is normal in  $K$ , and so  $K \cap U$  contains an element  $x$  of the center  $Z(K)$ . For  $q \in K \setminus U$ , there are  $t \in Q_1 \setminus \{I, -I\}$  and  $u \in U$  such that  $q = tu$ . We have  $[x, q] = I$  so  $[x, t] = I$ , that is impossible because  $C_G(x) = U$ . Therefore  $K \leq U$ . Since  $K$  is maximal nilpotent and  $U$  abelian,  $K = U$ . But  $U$  is not a cartan subgroup, because it is of infinite index in its normalizer  $B$ . A contradiction.

Since  $K$  does not contain a unipotent element,  $K$  intersects a conjugate of  $Q_1$  or  $Q_\delta$  (for  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ ) by Corollary 3, we note  $Q$  this subgroup. Let us show that  $K = Q$ . Let be  $x$  in  $K \cap Q$ , and  $\alpha \in N_K(K \cap Q)$ , then  $x^\alpha \in Q$ , and, by lemma 1,  $Q = C_G(x^\alpha) = C_G(x)^\alpha = Q^\alpha$ . Thus  $\alpha \in N_G(Q)$ , and  $N_K(K \cap Q) \leq N_G(Q)$ .

**1rst case**  $Q$  is a conjugate of  $Q_1$ , then  $N_G(Q) = Q \cdot \langle w' \rangle$  where  $w' = w^g$  if  $Q = Q_1^g$ . We have also  $w'^2 \in Q$  and  $t^{w'} = t^{-1}$  for  $t \in Q$ . If  $w' \in K$  then  $P = Q \cdot \langle w' \rangle \cap K$  is a subgroup of  $K$ , but  $P' = P^2$  which is  $K$ -invariant and  $P$  is not nilpotent. A contradiction, so  $w' \notin K$ . Then  $N_K(Q \cap K) \leq Q \cap K$ , it follows by normalizer condition that  $K \leq Q$ , and by maximality of  $K$ ,  $K = Q$ .

**2nd case**  $Q$  is a conjugate of  $Q_\delta$  (for  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ ), then  $N_G(Q) = Q$ . It follows similarly that  $K = Q$ .

□

## Generosity of the Cartan subgroups

Our purpose is now to show the generosity of the Cartan subgroup  $Q_1$ . It follows from the next more general proposition :

**Proposition 5.** 1. *The set  $W = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(\text{tr}(A)) < 0\}$  is generic in  $SL_2(\mathbb{Q}_p)$ .*

2. *The set  $W' = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(\text{tr}(A)) \geq 0\}$  is not generic in  $SL_2(\mathbb{Q}_p)$ .*

*Proof.* 1. We consider the matrices :

$$A_1 = I, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & -b^{-1} \\ b & 0 \end{pmatrix}$$

with  $v_p(a) > 0$  and  $v_p(b) > 0$ .

We show that  $SL_2(\mathbb{Q}_p) = \cup_{i=1}^4 A_i W$ . Suppose there exists

$$M = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in SL_2(\mathbb{Q}_p)$$

such that  $M \notin \cup_{i=1}^4 A_i W$ .

Since  $M \notin A_1W \cup A_2W$ , we have  $x+v = \varepsilon$  and  $y-u = \delta$  with  $v_p(\varepsilon) \geq 0$  and  $v_p(\delta) \geq 0$ . Since  $M \notin A_3W$ , we have  $ax + a^{-1}v = \eta$  with  $v_p(\eta) \geq 0$ . We deduce  $a(\varepsilon - v) + a^{-1}v = \eta$  and  $v = \frac{\eta - a\varepsilon}{a^{-1} - a}$ . Similarly, it follows from  $M \notin A_4W$  that  $u = \frac{\theta - b\delta}{b^{-1} - b}$  with some  $\theta$  such that  $v_p(\theta) \geq 0$ .

Since  $v_p(a) > 0$ , we have  $v_p(a + a^{-1}) < 0$ . From  $v_p(\eta - a\varepsilon) \geq \min\{v_p(\eta); v_p(a\varepsilon)\} \geq 0$ , we deduce that  $v_p(v) = v_p(\frac{\eta - a\varepsilon}{a + a^{-1}}) = v_p(\eta - a\varepsilon) - v_p(a + a^{-1}) > 0$ . Similarly  $v_p(u) > 0$ . It follows that  $v_p(x) = v_p(\varepsilon - v) \geq 0$  and  $v_p(y) \geq 0$ .

Therefore  $v_p(\det(M)) = v_p(xv - uy) \geq \min\{v_p(xv), v_p(uy)\} > 0$  and thus  $\det(M) \neq 1$ , a contradiction .

2. We show that the family of matrices  $(M_x)_{x \in \mathbb{Q}_p^\times}$  cannot be covered by finitely many  $SL_2(\mathbb{Q}_p)$ -translates of  $W'$ , where :

$$M_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}_p)$ . Then  $\text{tr}(A^{-1}M_x) = dx + ax^{-1}$ . If  $v_p(x) > \max\{|v_p(a)|, |v_p(d)|\}$  then  $v_p(\text{tr}(A^{-1}M_x)) < 0$  and  $M_x \notin AW'$ .

Therefore for every finite family  $\{A_j\}_{j=1}^n$ , there exist  $x \in \mathbb{Q}_p$  such that  $M_x \notin \bigcup_{j=1}^n A_jW'$ .  $\square$

**Remark.** We remark that the sets  $W$  and  $W'$  form a partition of  $SL_2(\mathbb{Q}_p)$ . There are both definable because the valuation  $v_p$  is definable in  $\mathbb{Q}_p$ .

**Lemma 2.**  $W \subseteq Q_1^{SL_2(\mathbb{Q}_p)}$  and for  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ ,  $Q_\delta^{SL_2(\mathbb{Q}_p)} \subseteq W'$

*Proof.* Let be  $A \in SL_2(\mathbb{Q}_p)$  with  $v_p(\text{tr}(A)) < 0$ .

For  $p \neq 2$ , since  $v_p(\text{tr}(A)) < 0$ ,  $v_p(\text{tr}(A)^2 - 4) = 2v_p(\text{tr}(A))$  and  $ac(\text{tr}(A)^2 - 4) = ac(\text{tr}(A)^2)$ , so  $\text{tr}(A)^2 - 4$  is a square in  $\mathbb{Q}_p$ .

For  $p = 2$ , we can write  $\text{tr}(A) = 2^n u$  with  $n \in \mathbb{Z}$  and  $u \in \mathbb{Z}_p^\times$ . Then  $\text{tr}(A)^2 - 4 = 2^{2n}(u^2 - 4 \cdot 2^{-2n})$ . Since  $n \leq -1$ ,  $u^2 - 4 \cdot 2^{-2n} \equiv u^2 \equiv 1 \pmod{8}$ , so  $\text{tr}(A)^2 - 4 \in (\mathbb{Q}_2^\times)^2$ .

In all cases, by the proposition 2,  $W \subseteq Q_1^{SL_2(\mathbb{Q}_p)}$  and ,by complementarity,  $Q_\delta^{SL_2(\mathbb{Q}_p)} \subseteq W'$ .  $\square$

We can now conclude with the following corollary, similar to [1, Remark 9.8] :

**Corollary 6.** 1. The Cartan subgroup  $Q_1$  is generous in  $SL_2(\mathbb{Q}_p)$ .

2. The Cartan subgroups  $Q_\delta$  (for  $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$ ) are not generous in  $SL_2(\mathbb{Q}_p)$ .

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