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The $M/M/1$ queue with synchronized abandonments

Stella Kapodistria*
stellakap@aegean.gr

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Abstract

In this paper we present a detailed analysis of a single server Markovian queue with impatient customers. Instead of the standard assumption that customers perform independent abandonments, we consider situations where customers abandon the system simultaneously. Moreover, we distinguish two abandonment scenarios, in the first one all present customers become impatient and perform synchronized abandonments, while in the second scenario we exclude the customer in service from the abandonment procedure. Furthermore, we extend our analysis to the $M/M/c$ queue under the second abandonment scenario.

For these models we carry out an extensive analysis including the stationary, the busy period and the conditional sojourn time distributions deriving exact formulas and iterative algorithmic schemes. We, also, obtain explicit results under various limiting regimes that demonstrate the effect of the level of synchronization to the performance of the systems.

Keywords: queueing system, impatient customers, synchronization, basic q -hypergeometric series, markov chain, stationary distribution, mean conditional sojourn time, busy period distribution.

Classification: 90B22, 60K25

1 Introduction

Queueing systems with reneging (i.e., impatient customers) have been studied extensively (see, e.g. Boxma and de Waal (1994), Daley (1965), Palm (1953) and (1957), Takacs (1974)). The main assumption in the literature is that customers perform independent abandonments, that is, each one of them sets an impatience clock and abandons the system as soon as the clock expires. For Markovian models, this type of abandonments introduces state-inhomogeneous transition rate matrices, which complicates the computation of the performance measures. This type of spatial inhomogeneity is usually studied by applying generating function methods. However, such systems are usually intractable or can be solved in terms of hypergeometric series (see e.g. Altman and Yechiali (2006) and (2008), Artalejo and Gomez-Corral (1997), Baykal-Gursoy and Xiao (2004), Keilson and Servi (1993), Krishnamoorthy et al. (2005), Perel and Yechiali (2009), Yechiali (2007)).

In the present paper we concentrate on the study of a single server Markovian queue with impatient customers that perform synchronized abandonments. This basic model is motivated by remote systems where customers have to wait for a certain transport facility to abandon the system. Then, whenever the facility visits the system, the present customers decide whether to leave the system or not. Therefore, we have synchronized departures

*University of the Aegean, Department of Statistics Actuarial-Financial Mathematics, Karlovassi, Samos, Greece

for some of the customers.

We will distinguish two types of abandonment scenarios, in the first we assume that all present customers become impatient and perform synchronized abandonments, while in the second scenario we exclude the customer in service from the abandonment procedure. More specifically, in the first scenario, we suppose that the abandonment epochs occur according to a Poisson process at rate ν , whenever there is at least one customer present in the system. Then, at the abandonment epochs, every present customer remains in the system with probability q or abandons the system with probability $p = 1 - q$, independently of the others. This implies that there exist binomial transition rates of the form $\binom{n}{n'} p^{n-n'} q^{n'} \nu$, from a state n to states n' , for $0 \leq n' \leq n$. In the second scenario, we, again, assume a Poisson abandonment process at rate ν , which is, now, set on whenever there is at least one customer in the waiting line. Then, at the abandonment epochs, every waiting customer remains in the queue with probability q or abandons the queue with probability $p = 1 - q$, independently of the others. This implies that from a state of $n + 1$ present customers, for $n > 0$, there exist binomial transition rates of the form $\binom{n}{n'} p^{n-n'} q^{n'} \nu$ to states $n' + 1$, for $0 \leq n' \leq n$.

Markov chains with binomial type transitions occur in Mathematical Biology in the study of population processes subject to binomial catastrophes (see, e.g. Artalejo et al. (2007), Brockwell et al. (1982), Economou (2004), Economou and Fakinos (2008)). Moreover, Neuts (1994) studied a 1-dimensional discrete-time model with similar dynamics. In the same framework, Economou and Kapodistria (2009) and Economou et al. (2010) studied systems with synchronized services, while Adan et al. (2009) and Economou and Kapodistria (2010) studied systems with server vacations and synchronized abandonments. We have to mention that in all the Markovian models with binomial transitions that have been studied so far in the queueing literature the common factor is that the binomial rates, when appearing, are the only backwards rates affecting the number of customer/units in the system:

- in the case of binomial catastrophes the number of units in the system is only reduced when a catastrophe occurs, since no death is considered in this type of systems
- in the case of synchronized services the number of customers in the system receiving service is reduced according to the binomial distribution at a service completion epoch and no other event occurs that would reduce the number of customers in service
- in the case of synchronized abandonments the authors have only considered systems with server vacations in which abandonments only occur during the absence of the server, hence abandonments and service, or any other event that might reduce the number of customers, don't coexist in the same time period.

The primary aim of this paper is to introduce and study a basic queueing model combining synchronized abandonments. The analysis of this paper extends the analysis of the aforementioned papers since this is a first effort to solve a Markovian model where binomial transition rates coexist with service rates, i.e. we have transition rates of the form $\binom{n}{n'} p^{n-n'} q^{n'} \nu + \mu \delta_{n',n-1}$, $0 \leq n' \leq n$, where $\delta_{n',n-1}$ denotes the Kronecker delta, being equal to 1, if $n' = n - 1$, and equal to 0, otherwise.

The paper is divided in three sections, consisting three parts one for each model we study. In section 2 we study the $M/M/1$ queue under the first abandonment scenario, while in sections 3 and 4 we study the $M/M/1$ and the $M/M/c$ queues, respectively, under the second abandonment scenario. Section 2 is further divided into several subsections: In section 2.1 we present the first model and give the appropriate notation, in the next section, 2.2, we prove the stability of the model and in section 2.3 we study the stationary probability generating function of the queue length. In section 2.4 we present the factorial moments of the queue length. We proceed with section 2.5 where we demonstrate the effect of the level of synchronization to the performance of the system. In sections 2.7 and 2.8 we treat the mean conditional sojourn time of a tagged customer and the system busy period distribution. In section 3 we study the stationary probability generating function (section 3.2), the factorial moments of the queue length (section 3.3), the mean conditional sojourn time (section 3.4)

and the busy period (section 3.5) distribution for the second model. Finally, in section 4 we extend the analysis of the previous section to the $M/M/c$ queue with synchronized abandonments, but not for the customers in service.

2 The $M/M/1$ queue with synchronized abandonments

2.1 Model description and notation

We consider an $M/M/1$ queueing system in which customers arrive according to a Poisson process at rate λ . The service is provided by a single server, who serves the customers according to a FCFS discipline. The successive service times are independent exponentially distributed random variables with rate μ . Customers become impatient and perform synchronized abandonments in the following way: A transportation facility is set on and it arrives at the system according to a Poisson process at rate ν . Every arrival epoch of the transportation facility constitutes an abandonment opportunity for the present customers. We suppose that each one of them decides to abandon the system with probability p or remains in the system with probability $q = 1 - p$, independently of the others.

The system is represented by a continuous time Markov chain $\{Q(t) : t \geq 0\}$, where $Q(t)$ denotes the number of customers in the system at time t , $t \geq 0$. The corresponding transition rate diagram is given in figure 1. From figure 1 it is apparent that the continuous time Markov chain is irreducible and aperiodic. In subsection 2.2, we will show that it is also always positive recurrent.

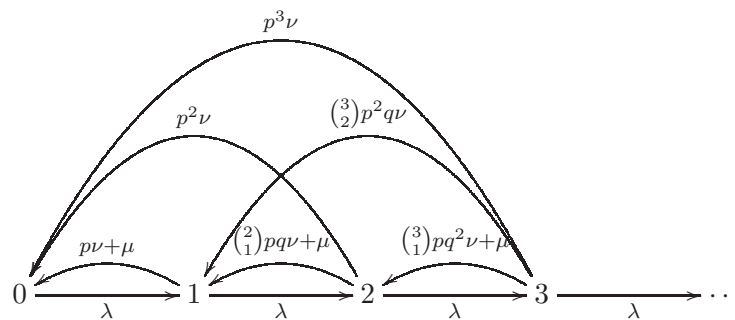


Figure 1: Transition rate diagram of $\{Q(t) : t \geq 0\}$.

2.2 Stability of the Model

Let us consider the embedded Markov chain $\{Q_n : n \geq 0\}$, where Q_n denotes the number of customers in the system left behind by the n th departure. Then, since we have two types of departures those due to service completion and those due to the abandonment process we have that

$$\{Q_{n+1} | Q_n > 0\} = Q_n + M_n - \mathbb{1}_{\{B_n < A_n\}} - \left(\sum_{j=1}^{Q_n + M_n} I_j \right) \cdot \mathbb{1}_{\{B_n \geq A_n\}},$$

where M_n is the number of Poisson arrivals at rate λ during the minimum of a service time, denoted by B_n , and of an inter-abandonment time, denoted by A_n , with $\Pr[B_n < A_n] = \frac{\mu}{\nu + \mu}$, and $\{I_j\}_{j \in \mathbb{N}}$ are identical and independent bernoulli random variables with $P[I_j = 1] = p$, $j \in \mathbb{N}$, independent of the other random variables.

The embedded Markov chain $\{Q_n\}$ can be easily shown to be positive recurrent using Pakes' (1969) theorem: an irreducible and aperiodic Markov chain $\{Q_n\}$ is ergodic provided that for all but finitely many values of i ,

the drift at state i , $E[Q_{n+1} - Q_n | Q_n = i]$, is bounded above by a negative constant. For our model we observe that $\{Q_n\}$ is irreducible, aperiodic and satisfies the condition of stability since

$$\begin{aligned} E[Q_{n+1} - Q_n | Q_n = i, i > 0] &= E[M_n] - \frac{\mu}{\mu + \nu} - \frac{\nu}{\mu + \nu} \left(i + \frac{\lambda}{\nu} \right) p \\ &= \frac{\lambda - \mu - ip\nu - \lambda p}{\mu + \nu} < 0, \quad \forall i \geq \left\lfloor \frac{\lambda q - \mu}{p\nu} \right\rfloor + 1. \end{aligned}$$

Therefore, the continuous time Markov chain $\{Q(t) : t \geq 0\}$ is irreducible and always positive recurrent.

Let $\{\pi_n : n \geq 0\}$ denote the stationary distribution of $\{Q(t) : t \geq 0\}$. We also define the probability generating function $\Pi(z)$ by

$$\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n.$$

In subsection 2.3 we will determine $\Pi(z)$ in terms of q -hypergeometric series (also known as basic hypergeometric series). In subsection 2.4 with the use of some properties of the q -hypergeometric series we will calculate the factorial moments of the queue length. Moreover, in subsections 2.6 and 2.7 we will also see that the study of the mean conditional sojourn time given the state of the system and the busy period distribution of the system are also facilitated using the theory of q -hypergeometric series.

There exists a rich theory for the class of q -hypergeometric series and their q -calculus which enables fast calculations and simplifications. In the queueing theory literature there exist only few papers where this theory has been applied (see e.g. Ismail (1985), Kemp and Newton (1990) and Kemp (2005)). For this reason we will briefly summarize the basic definitions bellow and present some results of the q -calculus in Appendix A. The interested reader can find more details on the definitions and the results (with proofs and extensions) in Gasper and Rahman (2004), Chapters 1-3 and Appendices I-III.

The q -hypergeometric series are series of the form $\sum_{n=0}^{\infty} c_n$ where $c_0 = 1$ and $\frac{c_{n+1}}{c_n}$ is a rational function of q^n for a deformation parameter q , which is taken to satisfy $|q| < 1$. Observing that the ratio $\frac{c_{n+1}}{c_n}$, being rational in q^n , can be written in the form

$$\frac{c_{n+1}}{c_n} = \frac{(1 - a_1 q^n)(1 - a_2 q^n) \cdots (1 - a_r q^n)}{(1 - q^{n+1})(1 - b_1 q^n) \cdots (1 - b_s q^n)} (-q^n)^{1+s-r} z, \quad r, s \in \mathbb{N}_0,$$

we have that every such series assumes the canonical form

$$\begin{aligned} {}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right) &\equiv {}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \end{aligned} \quad (2.1)$$

where $(a; q)_0 = 1$ and $(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})$, $n \geq 1$. The quantity $(a; q)_n$ is referred to as the q -shifted factorial. We also define $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$ and use the abbreviation $(a_1, a_2, \dots, a_r; q)_\infty$ to denote the product $(a_1; q)_\infty (a_2; q)_\infty \cdots (a_r; q)_\infty$. In the definition of a q -series through (2.1), it is assumed that $b_i \neq q^{-m}$ for $m = 0, 1, \dots$ and $i = 1, 2, \dots, s$. For $|q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z when $r \leq s$, and for $|z| < 1$ when $r = s + 1$.

2.3 The steady state distribution

The set of balance equations of the model is given as follows:

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 + \nu p\pi_1 + \nu p^2\pi_2 + \nu p^3\pi_3 + \dots \\ (\lambda + \mu + \nu)\pi_n &= \lambda\pi_{n-1} + \mu\pi_{n+1} + \nu q^n\pi_n + \nu \binom{n+1}{n} pq^n\pi_{n+1} + \dots, \quad n \geq 1. \end{aligned}$$

These equations can be solved efficiently by employing generating function methods and we obtain the following.

Theorem 1 *The probability generating function $\Pi(z)$ is given by*

$$\Pi(z) = \frac{1}{\left(-\frac{1-z}{z_1-1}; q\right)_\infty} \frac{{}_2\phi_1\left(\begin{matrix} -\frac{1-z}{z_1-1}, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)}, \quad |z| < z_1, \quad (2.2)$$

where

$$z_0 = \frac{\lambda + \mu + \nu - \sqrt{(\lambda + \mu + \nu)^2 - 4\lambda\mu}}{2\lambda} \in (0, 1) \quad (2.3)$$

$$z_1 = \frac{\lambda + \mu + \nu + \sqrt{(\lambda + \mu + \nu)^2 - 4\lambda\mu}}{2\lambda} \in (1, \infty). \quad (2.4)$$

The convergence of $\Pi(z)$ is absolute in the corresponding open disks and uniform in every compact subset of them.

Proof. The set of balance equations of the model can be written as:

$$(\lambda + \nu)\pi_0 = \mu\pi_1 + \nu \sum_{j=0}^{\infty} p^j \pi_j \quad (2.5)$$

$$(\lambda + \mu + \nu)\pi_n = \lambda\pi_{n-1} + \mu\pi_{n+1} + \nu \sum_{j=n}^{\infty} \binom{j}{n} p^{j-n} q^n \pi_j, \quad n \geq 1. \quad (2.6)$$

Multiplying both sides of equations (2.5) and (2.6) by z^0 and z^n , respectively, and summing for all $n = 0, 1, 2, \dots$ we obtain

$$\begin{aligned} (\lambda + \mu + \nu)\Pi(z) - \mu\pi_0 &= \lambda z\Pi(z) + \frac{\mu}{z}\Pi(z) - \frac{\mu}{z}\pi_0 + \nu \sum_{j=0}^{\infty} \pi_j \sum_{n=0}^j \binom{j}{n} (qz)^n p^{j-n} \\ &= \lambda z\Pi(z) + \frac{\mu}{z}\Pi(z) - \frac{\mu}{z}\pi_0 + \nu\Pi(p + qz). \end{aligned}$$

Hence

$$\Pi(z) = \frac{(z-1)\mu\pi_0}{(\lambda + \mu + \nu)z - \lambda z^2 - \mu} + \frac{\nu z}{(\lambda + \mu + \nu)z - \lambda z^2 - \mu} \Pi(p + qz). \quad (2.7)$$

We observe that the quadratic polynomial $A(z) = (\lambda + \mu + \nu)z - \lambda z^2 - \mu$ has two real roots given by (2.3) and (2.4) such that $A(z) = \lambda(z - z_0)(z_1 - z)$ and $\lambda(1 - z_0)(z_1 - 1) = \nu$.

Iterating equation (2.7) yields

$$\begin{aligned} \Pi(z) &= \frac{z-1}{A(z)} \mu\pi_0 \sum_{j=0}^n \nu^j q^j \frac{\prod_{k=0}^{j-1} (1 - q^k + q^k z)}{\prod_{k=1}^j A(1 - q^k + q^k z)} \\ &\quad + \Pi(1 - q^{n+1} + q^{n+1} z) \frac{\nu^{n+1}}{A(z)} \frac{\prod_{k=0}^n (1 - q^k + q^k z)}{\prod_{k=1}^n A(1 - q^k + q^k z)}, \quad \forall n \geq 0. \end{aligned} \quad (2.8)$$

Taking into consideration that

$$\begin{aligned} A(1 - q^k + q^k z) &= \lambda(1 - q^k + q^k z - z_0)(z_1 - 1 + q^k - q^k z) \\ &= \nu \left(1 - \frac{1-z}{1-z_0} q^k\right) \left(1 + \frac{1-z}{z_1-1} q^k\right) \end{aligned}$$

equation (2.8) assumes the form

$$\begin{aligned} \Pi(z) &= \frac{z-1}{A(z)} \mu \pi_0 \sum_{j=0}^n q^j \frac{\prod_{k=0}^{j-1} (1 - (1-z)q^k)}{\prod_{k=1}^j \left(1 - \frac{1-z}{1-z_0} q^k\right) \left(1 + \frac{1-z}{z_1-1} q^k\right)} \\ &\quad + \Pi(1 - q^{n+1} + q^{n+1} z) \frac{\nu}{A(z)} \frac{\prod_{k=0}^n (1 - (1-z)q^k)}{\prod_{k=1}^n \left(1 - \frac{1-z}{1-z_0} q^k\right) \left(1 + \frac{1-z}{z_1-1} q^k\right)}, \quad \forall n \geq 0. \end{aligned}$$

Taking in both sides of this last equation the limit as $n \rightarrow \infty$ and using the q -shifted factorial notation, i.e. $(a; q)_j = (1-a)(1-aq) \cdots (1-aq^{j-1})$, we derive

$$\begin{aligned} \Pi(z) &= \frac{z-1}{A(z)} \mu \pi_0 \sum_{j=0}^{\infty} q^j \frac{(1-z; q)_j}{\left(\frac{1-z}{1-z_0} q; q\right)_j \left(-\frac{1-z}{z_1-1} q; q\right)_j} + \frac{\nu}{A(z)} \frac{\prod_{k=0}^{\infty} (1 - (1-z)q^k)}{\prod_{k=1}^{\infty} \left(1 - \frac{1-z}{1-z_0} q^k\right) \left(1 + \frac{1-z}{z_1-1} q^k\right)} \\ &= \frac{z-1}{A(z)} \mu \pi_0 {}_3\phi_2 \left(\begin{matrix} q, 1-z, 0 \\ -\frac{1-z}{z_1-1} q, \frac{1-z}{1-z_0} q \end{matrix}; q, q \right) + \frac{\nu}{A(z)} \frac{(1-z; q)_{\infty}}{\left(\frac{1-z}{1-z_0} q, -\frac{1-z}{z_1-1} q; q\right)_{\infty}}. \end{aligned} \quad (2.9)$$

Multiplying equation (2.9) with $A(z)$ and setting $z = z_0$ we can solve for π_0

$$\pi_0 = \frac{\nu}{(1-z_0)\mu} \frac{(1-z_0; q)_{\infty}}{\left(q, -\frac{1-z_0}{z_1-1} q; q\right)_{\infty}} \frac{1}{{}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1} q \end{matrix}; q, q \right)}. \quad (2.10)$$

Plugging (2.10) into (2.9) we obtain

$$\begin{aligned} \Pi(z) &= \frac{z-1}{A(z)} \frac{\nu}{1-z_0} \frac{(1-z_0; q)_{\infty}}{\left(q, -\frac{1-z_0}{z_1-1} q; q\right)_{\infty}} \frac{{}_3\phi_2 \left(\begin{matrix} q, 1-z, 0 \\ -\frac{1-z}{z_1-1} q, \frac{1-z}{1-z_0} q \end{matrix}; q, q \right)}{{}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1} q \end{matrix}; q, q \right)} \\ &\quad + \frac{\nu}{A(z)} \frac{(1-z; q)_{\infty}}{\left(\frac{1-z}{1-z_0} q, -\frac{1-z}{z_1-1} q; q\right)_{\infty}}. \end{aligned} \quad (2.11)$$

We will now express the ${}_3\phi_2$ series that appear in equation (2.11) as the sum of two ${}_2\phi_1$ series in order to simplify the expression of $\Pi(z)$. To do so, we use Jackson's transformation formula of the ${}_2\phi_1$ series (see Appendix A, formula (A.11)). Equation (A.11) for $a = q$, $b = -\frac{q}{z_1-1}$, $c = -\frac{1-z}{z_1-1} q$ and $x = 1 - z_0$, and observing that for the specific selection of a we have that $abx/c = bxq/c$ and $(bx/c; q)_{\infty} = (1 - bx/c)(abx/c; q)_{\infty}$, assumes the form

$$\begin{aligned} {}_2\phi_1 \left(\begin{matrix} q, -\frac{q}{z_1-1} \\ -\frac{1-z}{z_1-1} q \end{matrix}; q, 1-z_0 \right) &= -\frac{z-1}{z_0-z} {}_3\phi_2 \left(\begin{matrix} q, 1-z, 0 \\ -\frac{1-z}{z_1-1} q, \frac{1-z}{1-z_0} q \end{matrix}; q, q \right) \\ &\quad + \frac{(q, -\frac{1-z_0}{z_1-1} q, 1-z; q)_{\infty}}{\left(-\frac{1-z}{z_1-1} q, -\frac{1-z}{1-z_0}, 1-z_0; q\right)_{\infty}} {}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1} q \end{matrix}; q, q \right). \end{aligned} \quad (2.12)$$

Furthermore, we interchange the arguments of the ${}_2\phi_1$ series that appears on the left hand of equation (2.12) using the first of Heine's transformation formula of the ${}_2\phi_1$ series (see Appendix A, formula (A.8)) Plugging equation (A.8), for $a = q$, $b = -\frac{q}{z_1-1}$, $c = -\frac{1-z}{z_1-1} q$ and $x = 1 - z_0$, into (2.12) and solving for the ${}_3\phi_2$ series we

obtain

$$\begin{aligned}
 {}_3\phi_2 \left(\begin{matrix} q, 1-z, 0 \\ -\frac{1-z}{z_1-1}q, \frac{1-z}{1-z_0}q \end{matrix} ; q, q \right) &= \frac{z-z_0}{z-1} \frac{(q, -\frac{1-z_0}{z_1-1}q; q)_\infty}{(-\frac{1-z}{z_1-1}q, 1-z_0; q)_\infty} {}_2\phi_1 \left(\begin{matrix} -\frac{1-z}{z_1-1}, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix} ; q, q \right) \\
 &\quad - \frac{z-z_0}{z-1} \frac{(q, -\frac{1-z_0}{z_1-1}q, 1-z; q)_\infty}{(-\frac{1-z}{z_1-1}q, \frac{1-z}{1-z_0}, 1-z_0; q)_\infty} {}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix} ; q, q \right). \quad (2.13)
 \end{aligned}$$

Plugging equation (2.13) into (2.11) after some manipulations we obtain

$$\begin{aligned}
 \Pi(z) &= \frac{z-z_0}{A(z)} \frac{\nu}{1-z_0} \frac{1}{(-\frac{1-z}{z_1-1}q; q)_\infty} \frac{{}_2\phi_1 \left(\begin{matrix} -\frac{1-z}{z_1-1}, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix} ; q, q \right)}{{}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix} ; q, q \right)} \\
 &\quad - \frac{\nu}{A(z)} \frac{z-z_0}{1-z_0} \frac{(1-z; q)_\infty}{(\frac{1-z}{1-z_0}, -\frac{1-z}{z_1-1}q; q)_\infty} + \frac{\nu}{A(z)} \frac{(1-z; q)_\infty}{(\frac{1-z}{1-z_0}q, -\frac{1-z}{z_1-1}q; q)_\infty}. \quad (2.14)
 \end{aligned}$$

Observing that $\frac{z-z_0}{A(z)} \frac{\nu}{1-z_0} = \frac{z_1-1}{z_1-z}$ and taking twice into consideration that $(a; q)_\infty = (1-a)(aq; q)_\infty$, once for $a = -\frac{1-z}{z_1-1}$ and once for $a = \frac{1-z}{1-z_0}$, equation (2.14) reduces to (2.2). \blacksquare

2.4 Factorial moments of the queue length

In order to proceed with the calculation of the moments of the queue length distribution in the stationary state we could differentiate every term of equation (2.2). We are allowed to do so because of the uniform convergence we have achieved in theorem 1. However, using the fact that $\Pi(z)$ is by definition analytic (since it is represented by a power series) we can differentiate directly every term of equation (2.7) as stated in the next remark.

Remark 1 Differentiating relation (2.7) and setting $z = 1$ we have that

$$\Pi^{(1)}(1) = \frac{\mu\pi_0 + \lambda - \mu}{\nu(1-q)}.$$

Differentiating relation (2.7) n times and setting $z = 1$ we have that

$$\nu(1-q^n)\Pi^{(n)}(1) + n(\nu(1-q^{n-1}) + \mu - \lambda)\Pi^{(n-1)}(1) - n(n-1)\lambda\Pi^{(n-2)}(1) = 0, \quad n \geq 2.$$

Nevertheless, a closed form for $\Pi^{(n)}(1)$ can be obtained by applying results from the theory of q -series, as can be seen in the following theorem.

Theorem 2 The factorial moments $m_{(n)} = E[Q(Q-1)(Q-2)\cdots(Q-n+1)]$ of the stationary number of customers in the system are given by

$$m_{(n)} = \frac{n!}{(z_1-1)^n(q; q)_n} \frac{{}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix} ; q, q^{n+1} \right)}{{}_2\phi_1 \left(\begin{matrix} 1-z_0, 0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix} ; q, q \right)}, \quad n \geq 1, \quad (2.15)$$

where z_0 and z_1 are given in equations (2.3) and (2.4), respectively.

Proof. The factorial moments exponential generating function $M(z)$ is given by

$$M(z) = \sum_{n=0}^{\infty} m_{(n)} \frac{z^n}{n!} = E \left[\sum_{n=0}^{\infty} \binom{Q}{n} z^n \right] = E[(1+z)^Q] = \Pi(1+z). \quad (2.16)$$

We have already shown in theorem 1 that $\Pi(z)$ converges in a neighborhood of 1, hence $M(z)$ is well defined in a neighborhood of 0. Equation (2.2) assumes the form

$$\begin{aligned} \Pi(1+z) &= \frac{1}{\left(\frac{z}{z_1-1}; q\right)_\infty} \frac{{}_2\phi_1\left(\begin{matrix} \frac{z}{z_1-1}, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)} \\ &= \frac{1}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)} \sum_{k=0}^{\infty} \frac{(1-z_0; q)_k}{(q; q)_k \left(-\frac{1-z_0}{z_1-1}q; q\right)_k} q^k \frac{\left(\frac{z}{z_1-1}; q\right)_k}{\left(\frac{z}{z_1-1}; q\right)_\infty}. \end{aligned} \quad (2.17)$$

Observing that $\left(\frac{z}{z_1-1}; q\right)_k \left(\frac{z}{z_1-1}q^k; q\right)_\infty = \left(\frac{z}{z_1-1}; q\right)_\infty$ equation (2.17) assumes the form

$$\Pi(1+z) = \frac{1}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)} \sum_{k=0}^{\infty} \frac{(1-z_0; q)_k}{(q; q)_k \left(-\frac{1-z_0}{z_1-1}q; q\right)_k} q^k \frac{1}{\left(\frac{z}{z_1-1}q^k; q\right)_\infty}. \quad (2.18)$$

Using now the q -binomial theorem (see Appendix A, formula (A.2)) we have that equation (2.18) can be written as

$$\begin{aligned} \Pi(1+z) &= \frac{1}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)} \sum_{k=0}^{\infty} \frac{(1-z_0; q)_k}{(q; q)_k \left(-\frac{1-z_0}{z_1-1}q; q\right)_k} q^k \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} \left(\frac{z}{z_1-1}q^k\right)^n \\ &= \frac{1}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)} \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} \left(\frac{z}{z_1-1}\right)^n \sum_{k=0}^{\infty} \frac{(1-z_0; q)_k}{(q; q)_k \left(-\frac{1-z_0}{z_1-1}q; q\right)_k} q^{(n+1)k} \\ &= \frac{1}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)} \sum_{n=0}^{\infty} z^n \frac{1}{(q; q)_n (z_1-1)^n} {}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q^{n+1}\right) \end{aligned}$$

which concludes the proof. ■

2.5 Limiting regimes

We now turn our attention to the behavior of the model under certain limiting regimes. To emphasize the dependence on the parameters of the model in the rest of this section, we will denote π_n and $\Pi(z)$ by $\pi(n; \lambda, \mu, \nu, p)$ and $\Pi(z; \lambda, \mu, \nu, p)$, respectively. Note that νp can be thought of as the effective abandonment rate per customer. Indeed the overall abandonment time of a customer is a geometric sum of exponentially distributed random variables with rate ν and so we can easily see that it is also exponentially distributed with parameter νp . Under this perspective, if we have two models with the same parameters λ, μ that differ only in ν and p , but with $\nu p = \nu^*$ fixed, we can think that the models have identical arrival rates λ , service rates μ and effective abandonment rates per customer ν^* and differ only in the ‘level of synchronization’ p . Indeed, the case $p \rightarrow 0^+$ corresponds to no synchronization, since then $\nu = \nu^*/p \rightarrow \infty$, hence the customers depart almost singly at the abandonment epochs. On the contrary, the case $p \rightarrow 1^-$ corresponds to full synchronization, since then $\nu = \nu^*/p \rightarrow \nu^*$, hence almost all present customers depart simultaneously from the system when an abandonment opportunity occurs.

We are interested in studying the stationary behavior of the system for the case where λ, μ and ν^* are kept fixed in the two limiting cases $p \rightarrow 0^+$ ($q \rightarrow 1^-$) and $p \rightarrow 1^-$ ($q \rightarrow 0^+$). The case $p \rightarrow 0^+$ corresponds exactly to

the $M/M/1$ queue with exponentially distributed patience times, where patience refers to the sojourn time, i.e. customers being served may run out of patience and abandon the system. This system following the Kendall notation can be denoted as $M/M/1 + M^s$, where the superscript s refers to the sojourn time. Then for this system the set of balance equations, with patience rate ν^* , is given as follows

$$\lambda p_n = (\mu + (n + 1)\nu^*)p_{n+1}, \quad n \geq 0.$$

These recursive equations can be easily solved. However, note that, it is enough to solve the $M/M/1$ queue, with exponentially distributed patience times, where patience refers to the waiting time, i.e. the $M/M/1 + M$ queue (see, e.g. Palm (1957), Zeltyn (2005)), with balance equations

$$\lambda p_n = (\mu + n\nu^*)p_{n+1}, \quad n \geq 0,$$

and replace μ by $\mu + \nu^*$.

For the limiting case $p \rightarrow 0^+$, where λ , μ and ν^* are kept fixed, we derive the steady state probability generating function in theorem 3 and obtain the factorial moments of the state distribution in theorem 4. One can compare our results with the results presented in Zeltyn (2005) section 6.1, after setting $n = 1$ and replacing μ by $\mu + \nu^*$.

Theorem 3 *For a system with arrival rate λ , service rate μ and effective abandonment rate per customer ν^* the generating function*

$$\Pi^{(I)}(z) = \lim_{q \rightarrow 1^-} \Pi(z; \lambda, \mu, \frac{\nu^*}{1-q}, 1-q),$$

and the steady state probabilities

$$\pi_n^{(I)} = \lim_{q \rightarrow 1^-} \pi(n; \lambda, \mu, \frac{\nu^*}{1-q}, 1-q),$$

in the limiting case of no synchronization are given by

$$\Pi^{(I)}(z) = \frac{{}_1F_1(1; 1 + \frac{\mu}{\nu^*}; \frac{\lambda}{\nu^*} z)}{{}_1F_1(1; 1 + \frac{\mu}{\nu^*}; \frac{\lambda}{\nu^*})} \tag{2.19}$$

$$\pi_n^{(I)} = \left(\frac{\lambda}{\nu^*}\right)^n \frac{\Gamma(1 + \frac{\mu}{\nu^*})}{\Gamma(n + 1 + \frac{\mu}{\nu^*})} \frac{1}{{}_1F_1(1; 1 + \frac{\mu}{\nu^*}; \frac{\lambda}{\nu^*})}, \quad n \geq 0, \tag{2.20}$$

with ${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt$ being the Kummer (confluent hypergeometric) function and $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ the Gamma function.

Proof. We will write $\Pi(z)$ in terms of the q -integral and take advantage of the convergence of the q -integral to the usual integral, as $q \rightarrow 1^-$ (see Appendix, formula (A.3)). More concretely, we have that the two ${}_2\phi_1$ series, that appear in the expression of $\Pi(z)$, as given in equation (2.2), can be written as q -integrals, using the ${}_{r+1}\phi_r$ series transformation formula to the q -integral (see Appendix A, formula (A.12)) and obtain

$${}_2\phi_1 \left(\begin{matrix} -\frac{1-z}{z_1-1}, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q \right) = \frac{(-\frac{1-z}{z_1-1}, 1-z_0; q)_\infty}{(1-q)(q, -\frac{1-z_0}{z_1-1}q; q)_\infty} \int_0^1 \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{(-\frac{1-z}{z_1-1}t, (1-z_0)t; q)_\infty} d_q t \tag{2.21}$$

$${}_2\phi_1 \left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q \right) = \frac{(1-z_0; q)_\infty}{(1-q)(q, -\frac{1-z_0}{z_1-1}q; q)_\infty} \int_0^1 \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{((1-z_0)t; q)_\infty} d_q t. \tag{2.22}$$

Plugging (2.21) and (2.22) into (2.2) and simplifying several terms we have that (2.2) assumes the form

$$\Pi(z) = \frac{\int_0^1 \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{(-\frac{1-z}{z_1-1}t, (1-z_0)t; q)_\infty} d_q t}{\int_0^1 \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{((1-z_0)t; q)_\infty} d_q t}. \tag{2.23}$$

In order to obtain the limit of $\Pi(z)$ as $q \rightarrow 1^-$, under the condition $\nu(1-q) = \nu^*$ is kept fixed, we need to calculate the limit of the terms $\frac{(qt;q)_\infty}{((1-z_0)t;q)_\infty}$, $(-\frac{1-z}{z_1-1}t;q)_\infty$ and $(-\frac{1-z_0}{z_1-1}qt;q)_\infty$ that appear in expression (2.23). On what the first of the terms is concerned using formula (A.5), we obtain that

$$\lim_{q \rightarrow 1^-} \frac{(qt;q)_\infty}{((1-z_0)t;q)_\infty} = \lim_{q \rightarrow 1^-} \frac{(\frac{q}{1-z_0}(1-z_0)t;q)_\infty}{((1-z_0)t;q)_\infty} = (1 - \lim_{q \rightarrow 1^-} (1-z_0)t)^{-\lim_{q \rightarrow 1^-} \frac{d}{dq} \frac{q}{1-z_0}}. \quad (2.24)$$

Replacing ν by $\frac{\nu^*}{1-q}$ and taking the limit as $q \rightarrow 1^-$ we have that

$$\lim_{q \rightarrow 1^-} z_0 = 0 \quad (2.25)$$

$$\lim_{q \rightarrow 1^-} \frac{d}{dq} \frac{q}{1-z_0} = 1 - \frac{\mu}{\nu^*}. \quad (2.26)$$

Under the light of these two results equation (2.24) becomes

$$\lim_{q \rightarrow 1^-} \frac{(qt;q)_\infty}{((1-z_0)t;q)_\infty} = (1-t)^{\frac{\mu}{\nu^*}-1}. \quad (2.27)$$

Moreover, to calculate the limit of the second and third term we will use formula (A.4) and obtain that

$$\lim_{q \rightarrow 1^-} \left(-\frac{1-z}{z_1-1}t;q\right)_\infty = e^{\lim_{q \rightarrow 1^-} \frac{(1-z)t}{(z_1-1)(1-q)}} \quad (2.28)$$

$$\lim_{q \rightarrow 1^-} \left(-\frac{1-z_0}{z_1-1}qt;q\right)_\infty = e^{\lim_{q \rightarrow 1^-} \frac{(1-z_0)qt}{(z_1-1)(1-q)}}. \quad (2.29)$$

Replacing ν by $\frac{\nu^*}{1-q}$ and taking the limit as $q \rightarrow 1^-$ we have that

$$\lim_{q \rightarrow 1^-} (z_1-1)(1-q) = \frac{\nu^*}{\lambda}. \quad (2.30)$$

Under the light of equations (2.25) and (2.30) we have that (2.28) and (2.29) assume the form

$$\lim_{q \rightarrow 1^-} \left(-\frac{1-z}{z_1-1}t;q\right)_\infty = e^{\frac{\lambda}{\nu^*}(1-z)t} \quad (2.31)$$

$$\lim_{q \rightarrow 1^-} \left(-\frac{1-z_0}{z_1-1}qt;q\right)_\infty = e^{\frac{\lambda}{\nu^*}t}. \quad (2.32)$$

Taking, now, the limit as $q \rightarrow 1^-$ in (2.23), taking into account equations (2.27), (2.31) and (2.32) yields

$$\Pi^{(I)}(z) = \frac{\int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} e^{\frac{\lambda}{\nu^*}zt} dt}{\int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} e^{\frac{\lambda}{\nu^*}t} dt}. \quad (2.33)$$

Using, now, the Kummer function notation yields (2.19). Furthermore, from equation (2.33) after expanding $e^{\frac{\lambda}{\nu^*}zt}$, that appears in the numerator, as series we have that

$$\begin{aligned} \Pi^{(I)}(z) &= \frac{1}{\int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} e^{\frac{\lambda}{\nu^*}t} dt} \int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{\nu^*}zt\right)^n}{n!} dt \\ &= \frac{1}{\int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} e^{\frac{\lambda}{\nu^*}t} dt} \sum_{n=0}^{\infty} z^n \frac{\left(\frac{\lambda}{\nu^*}\right)^n}{n!} \int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} t^n dt \\ &= \frac{1}{\int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} e^{\frac{\lambda}{\nu^*}t} dt} \sum_{n=0}^{\infty} z^n \left(\frac{\lambda}{\nu^*}\right)^n \frac{\Gamma(\mu/\nu^*)}{\Gamma(n+1+\mu/\nu^*)}. \end{aligned}$$

Equating the coefficients of z^n and using the Kummer function notation yields (2.20). ■

Theorem 4 *The factorial moments $m_{(n)}^{(I)} = E[Q(Q-1)(Q-2)\cdots(Q-n+1)]$ of the stationary number of customers in the system are given by*

$$m_{(n)}^{(I)} = \left(\frac{\lambda}{\nu^*}\right)^n \frac{n! \Gamma(1 + \frac{\mu}{\nu^*}) {}_1F_1(n+1; n+1 + \frac{\mu}{\nu^*}; \frac{\lambda}{\nu^*})}{\Gamma(n+1 + \frac{\mu}{\nu^*}) {}_1F_1(1; 1 + \frac{\mu}{\nu^*}; \frac{\lambda}{\nu^*})}, \quad (2.34)$$

with ${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt$ being the Kummer (confluent hypergeometric) function and $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ the Gamma function.

Proof. As in the proof of the previous theorem we will express the two ${}_2\phi_1$ series that appear in the formula of the factorial moments $m_{(n)}$, as given by (2.15), in terms of the q -integral. More concretely using (A.12) we can easily see that

$${}_2\phi_1 \left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q^{n+1} \right) = \frac{1}{1-q} \frac{(1-z_0; q)_\infty}{(q, -\frac{1-z_0}{z_1-1}q; q)_\infty} \int_0^1 t^n \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{((1-z_0)t; q)_\infty} d_q t. \quad (2.35)$$

Plugging (2.22) and (2.35) into (2.15) and simplifying several terms we have that (2.15) assumes the form

$$m_{(n)} = \frac{n!}{(z_1-1)^n (q; q)_n} \frac{\int_0^1 t^n \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{((1-z_0)t; q)_\infty} d_q t}{\int_0^1 \frac{(qt, -\frac{1-z_0}{z_1-1}qt; q)_\infty}{((1-z_0)t; q)_\infty} d_q t}. \quad (2.36)$$

We have already calculated the limit of the terms $\frac{(qt; q)_\infty}{((1-z_0)t; q)_\infty}$ and $(-\frac{1-z_0}{z_1-1}qt; q)_\infty$ in equations (2.27) and (2.32), respectively. So, we only need to calculate the limit of the expression $(z_1-1)^n (q; q)_n$, as $q \rightarrow 1^-$, under the condition that $\nu(1-q) = \nu^*$ is kept fixed. Replacing ν by $\frac{\nu^*}{1-q}$ and taking the limit as $q \rightarrow 1^-$ we have that

$$\lim_{q \rightarrow 1^-} (z_1-1)^n (q; q)_n = n! \left(\frac{\nu^*}{\lambda}\right)^n. \quad (2.37)$$

Taking the limit as $q \rightarrow 1^-$ in (2.36), taking into account equations (2.27), (2.32) and (2.37) yields

$$m_{(n)}^{(I)} = \left(\frac{\lambda}{\nu^*}\right)^n \frac{\int_0^1 t^n (1-t)^{\frac{\mu}{\nu^*}-1} e^{-\frac{\lambda}{\nu^*}t} dt}{\int_0^1 (1-t)^{\frac{\mu}{\nu^*}-1} e^{-\frac{\lambda}{\nu^*}t} dt}.$$

Using the Kummer function notation yields (2.34). ■

We now derive the corresponding results for the other extreme case of full synchronization, i.e. when $p \rightarrow 1^-$. In this case our model converges to the $M/M/1$ queue with total catastrophes (see, e.g. Di Crescenzo et al. (2003)). In theorem 5 we derive the steady state probability generating function, while in theorem 6 we obtain the factorial moments of the state distribution. One can easily compare our results to the results presented in the paper of Di Crescenzo et al. (2003) p. 330 formula (3) and p. 331 formula (5).

Theorem 5 *For a system with arrival rate λ , service rate μ and effective abandonment rate per customer ν^* , the steady state distribution $\pi_n^{(C)} = \lim_{q \rightarrow 0^+} \pi(n; \lambda, \mu, \frac{\nu^*}{1-q}, 1-q)$ in the limiting case of full synchronization is given by*

$$\pi_n^{(C)} = \left(1 - \frac{1}{z_1^*}\right) \left(\frac{1}{z_1^*}\right)^n, \quad n \geq 0, \quad (2.38)$$

where

$$z_1^* = \frac{\lambda + \mu + \nu^* + \sqrt{(\lambda + \mu + \nu^*)^2 - 4\lambda\mu}}{2\lambda} \in (1, \infty). \quad (2.39)$$

Proof. In order to calculate the limit of $\Pi(z)$ as $q \rightarrow 0^+$, we will take advantage of the convergence of the ${}_2\phi_1(a, b; c; q, q)$ series to 1, as $q \rightarrow 0^+$, which can be easily proven by using the definition of the ${}_2\phi_1$ series. Taking the limit as $q \rightarrow 0^+$ in (2.2), yields

$$\Pi^{(C)}(z) = \lim_{q \rightarrow 0^+} \frac{1}{\left(-\frac{1-z}{z_1-1}; q\right)_\infty} \frac{{}_2\phi_1\left(\begin{matrix} -\frac{1-z}{z_1-1}, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)}{{}_2\phi_1\left(\begin{matrix} 0, 1-z_0 \\ -\frac{1-z_0}{z_1-1}q \end{matrix}; q, q\right)}. \quad (2.40)$$

Using that the q -shifted factorial $(a; q)_\infty$ converges to $1-a$ and that the two ${}_2\phi_1$ series converge to 1, as $q \rightarrow 0^+$, we obtain that

$$\Pi^{(C)}(z) = \frac{1}{1 + \frac{1-z}{z_1^*-1}} = \frac{z_1^* - 1}{z_1^* - z}, \quad (2.41)$$

where z_1^* denotes the limit of z_1 , as $q \rightarrow 0^+$, under the condition that $\nu(1-q) = \nu^*$ is kept fixed. Expanding $(z_1^* - z)^{-1}$ in power series and equating the coefficients of z^n yields (2.38). ■

Theorem 6 *The factorial moments $m_{(n)}^{(C)} = E[Q(Q-1)(Q-2)\cdots(Q-n+1)]$ of the stationary number of customers in the system are given by*

$$m_{(n)}^{(C)} = \frac{n!}{(z_1^* - 1)^n}, \quad n \geq 1, \quad (2.42)$$

where z_1^* is given in equation (2.39).

Proof. We replace ν by $\frac{\nu^*}{1-q}$ in (2.15) and we take the limit as $q \rightarrow 0^+$. ■

2.6 Sojourn time

Let Y denote the unconditional total sojourn time of an arbitrary customer in the system, regardless of whether he completes service or not. Moreover, let Y_n denote the conditional total sojourn time of a tagged customer in the system, given that upon arrival he finds the system in state n .

We employ first-step analysis excluding arrivals, because future arrivals do not influence the tagged customer. Indeed, by conditioning on whether the next transition is a service completion or an abandonment opportunity we obtain the equation

$$E[Y_n] = \frac{1}{\mu + \nu} + \frac{\mu}{\mu + \nu} E[Y_{n-1}] + \frac{\nu q}{\mu + \nu} \sum_{i=0}^n \binom{n}{i} p^{n-i} q^i E[Y_i], \quad n \geq 0, \quad (2.43)$$

where we have assumed that $E[Y_{-1}] \equiv 0$. The system of recursive relation (2.43) can be solved explicitly employing a generating function approach and using the theory of q -hypergeometric series. The solution is summarized in the following theorem.

Theorem 7 *The conditional expected total sojourn time of a tagged customer in the system Y_n , given that upon his arrival he finds the system in state n , is given as follows*

$$E[Y_n] = \frac{1}{\mu + \nu(1-q)} \sum_{i=0}^n \left(-\frac{\nu}{\mu + \nu}\right)^i \frac{(q^2; q)_i}{\left(\frac{\nu q^2}{\mu + \nu}; q\right)_i} \sum_{j=i}^n \binom{j}{i}, \quad n \geq 0. \quad (2.44)$$

Proof. We define the generating function of the mean conditional expected total sojourn time $Y(z)$ given as

$$Y(z) = \sum_{n=0}^{\infty} E[Y_n]z^n, |z| < 1.$$

The generating function $Y(z)$ do converges for $|z| < 1$, since $|Y(z)| \leq \sum_{n=0}^{\infty} E[Y_n]|z|^n \leq \frac{1}{\nu p} \sum_{n=0}^{\infty} |z|^n < \infty$, where we have used that $E[Y_n]$ is less or equal to the mean overall abandonment time of a customer.

Multiplying equation (2.43) with $(\mu + \nu)z^n$ and adding for all $n \geq 0$ results to

$$\begin{aligned} (\mu + \nu)Y(z) &= \frac{1}{1-z} + \mu z Y(z) + \nu q \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} p^{n-i} q^i E[Y_i] z^n \\ &= \frac{1}{1-z} + \mu z Y(z) + \nu q \sum_{i=0}^{\infty} E[Y_i] \left(\frac{q}{p}\right)^i \sum_{n=i}^{\infty} \binom{n}{i} (pz)^n. \end{aligned} \quad (2.45)$$

Using now the binomial theorem (see Gasper and Rahman 2004, p. 8 formula (1.3.1))

$$\sum_{n=i}^{\infty} \binom{n}{i} x^n = \frac{x^i}{(1-x)^{i+1}}, |x| < 1, \quad (2.46)$$

we have that (2.45) assumes the form

$$(\mu(1-z) + \nu)Y(z) = \frac{1}{1-z} + \frac{\nu q}{1-(1-q)z} Y\left(\frac{qz}{1-(1-q)z}\right). \quad (2.47)$$

We observe that equation (2.47) can be put in the form

$$Y(z) = \frac{H(z)}{G(z)} Y(T(z)) + \frac{K(z)}{G(z)}, \quad (2.48)$$

where

$$T(z) = \frac{qz}{1-(1-q)z} \quad (2.49)$$

and

$$G(z) = \mu(1-z) + \nu, \quad H(z) = \nu \frac{T(z)}{z}, \quad K(z) = \frac{1}{1-z}.$$

The solution of (2.48) can be done by iteration. To this end it seems convenient to introduce here an operator notation: The transformation $T(z)$ defined by (2.49) is a linear fractional transformation and therefore its k -th compositions defined by $T_0(z) = z$ and $T_k(z) = T(T_{k-1}(z))$, $k \geq 1$, can be computed in closed form. Indeed, it can be proved inductively that

$$T_k(z) = \frac{q^k z}{1-(1-q^k)z}, \quad k \geq 0.$$

By iterating (2.48) n times we obtain

$$Y(z) = \sum_{k=0}^n \frac{K(T_k(z))}{H(T_k(z))} \prod_{i=0}^k \frac{H(T_i(z))}{G(T_i(z))} + Y(T_{n+1}(z)) \prod_{i=0}^n \frac{H(T_i(z))}{G(T_i(z))}. \quad (2.50)$$

However, note that

$$\frac{H(T_i(z))}{G(T_i(z))} = \frac{\nu T_{i+1}(z)}{T_i(z)} \frac{1}{\mu(1-T_i(z)) + \nu} = \frac{\nu T_{i+1}(z)}{T_i(z)} \frac{1-(1-q^i)z}{(\mu + \nu)(1-z) + \nu z q^i}$$

and

$$\frac{K(T_k(z))}{H(T_k(z))} = \left(\frac{1}{1-T_k(z)} \right) \frac{T_k(z)}{\nu T_{k+1}(z)} = \frac{1-(1-q^k)z}{1-z} \frac{T_k(z)}{\nu T_{k+1}(z)}$$

so (2.50) assumes the form

$$Y(z) = \frac{1}{(1-z)((\mu+\nu)(1-z)+\nu z)} \sum_{k=0}^n \left(\frac{\nu q}{\mu+\nu}\right)^k \frac{(-\frac{z}{1-z}q; q)_k}{(-\frac{z}{1-z}\frac{\nu}{\mu+\nu}q; q)_k} + Y(T_{n+1}(z)) \left(\frac{\nu q}{\mu+\nu}\right)^{n+1} \frac{1}{1-(1-q^{n+1})z} \frac{(-\frac{z}{1-z}; q)_{n+1}}{(-\frac{z}{1-z}\frac{\nu}{\mu+\nu}; q)_{n+1}}$$

and by taking the limit as $n \rightarrow \infty$ we obtain

$$Y(z) = \frac{1}{(1-z)((\mu+\nu)(1-z)+\nu z)} \sum_{k=0}^{\infty} \left(\frac{\nu q}{\mu+\nu}\right)^k \frac{(-\frac{z}{1-z}q; q)_k}{(-\frac{z}{1-z}\frac{\nu}{\mu+\nu}q; q)_k} = \frac{1}{(1-z)((\mu+\nu)(1-z)+\nu z)} {}_2\phi_1 \left(\begin{matrix} -\frac{z}{1-z}q, q \\ -\frac{z}{1-z}\frac{\nu}{\mu+\nu}q \end{matrix}; q, \frac{\nu}{\mu+\nu}q \right). \quad (2.51)$$

Interchanging the arguments of the ${}_2\phi_1$ series, using the second of Heine's transformation formula of the ${}_2\phi_1$ series (see Appendix A, formula (A.9)) for $a = -\frac{z}{1-z}q$, $b = q$, $c = -\frac{z}{1-z}\frac{\nu}{\mu+\nu}q$ and $x = \frac{\nu}{\mu+\nu}q$, and substituting the result into (2.51) we have that

$$Y(z) = \frac{1}{(1-z)((\mu+\nu)(1-z)+\nu z)} \frac{(-\frac{z}{1-z}\frac{\nu}{\mu+\nu}, \frac{\nu}{\mu+\nu}q^2; q)_{\infty}}{(-\frac{z}{1-z}\frac{\nu}{\mu+\nu}q, \frac{\nu}{\mu+\nu}q; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} q^2, q \\ \frac{\nu}{\mu+\nu}q^2 \end{matrix}; q, -\frac{z}{1-z}\frac{\nu}{\mu+\nu} \right). \quad (2.52)$$

Using twice that $(a; q)_{\infty} = (1-a)(aq; q)_{\infty}$ and expanding the ${}_2\phi_1$ series, equation (2.52) assumes the form

$$Y(z) = \frac{1}{\mu+\nu(1-q)} \sum_{i=0}^{\infty} \left(-\frac{\nu}{\mu+\nu}\right)^i \frac{(q^2; q)_i}{(\frac{\nu q^2}{\mu+\nu}; q)_i} \frac{z^i}{(1-z)^{i+1}} \frac{1}{1-z}. \quad (2.53)$$

Using (2.46) equation (2.53) assumes the form

$$Y(z) = \frac{1}{\mu+\nu(1-q)} \sum_{i=0}^{\infty} \left(-\frac{\nu}{\mu+\nu}\right)^i \frac{(q^2; q)_i}{(\frac{\nu q^2}{\mu+\nu}; q)_i} \sum_{n=i}^{\infty} \sum_{j=i}^n \binom{j}{i} z^n = \frac{1}{\mu+\nu(1-q)} \sum_{n=0}^{\infty} z^n \sum_{i=0}^n \left(-\frac{\nu}{\mu+\nu}\right)^i \frac{(q^2; q)_i}{(\frac{\nu q^2}{\mu+\nu}; q)_i} \sum_{j=i}^n \binom{j}{i}. \quad (2.54)$$

Now (2.54) implies readily (2.44). ■

In the two limiting regimes that we have considered in the previous section, where λ , μ and ν^* are kept fixed, we can proceed a bit further and give the results for the conditional expected total sojourn times in the case of no synchronization ($p \rightarrow 0^+$) and full synchronization ($p \rightarrow 1^-$). The results are immediate by taking the appropriate limits in (2.44). More specifically we have the following theorems.

Theorem 8 *Consider a system with arrival rate λ , service rate μ and effective abandonment rate per customer ν^* . In the limiting case of no synchronization ($\nu = \frac{\nu^*}{1-q}$, $q \rightarrow 1^-$), the conditional expected total sojourn time of a tagged customer in the system $Y_n^{(I)}$, given that upon his arrival he finds the system in state n , is given as follows*

$$E[Y_n^{(I)}] = \frac{1}{\mu+\nu^*} \sum_{i=0}^n (-1)^i \frac{(i+1)!}{\prod_{k=2}^{i+1} (\frac{\mu}{\nu^*} + k)} \sum_{j=i}^n \binom{j}{i}, \quad n \geq 0. \quad (2.55)$$

Proof. We replace ν by $\frac{\nu^*}{1-q}$ in (2.44) and observing that

$$\lim_{q \rightarrow 1^-} \frac{1-q^k}{1 - \frac{\nu^*}{\mu(1-q)+\nu^*} q^k} = \frac{k}{\frac{\mu}{\nu^*} + k} \quad \lim_{q \rightarrow 1^-} \frac{(q^2; q)_i}{(\frac{\nu q^2}{\mu+\nu}; q)_i} = \frac{(i+1)!}{\prod_{k=2}^{i+1} (\frac{\mu}{\nu^*} + k)} \quad (2.56)$$

yields easily (2.55). ■

Theorem 9 Consider a system with arrival rate λ , service rate μ and effective abandonment rate per customer ν^* . In the limiting case of full synchronization ($\nu = \frac{\nu^*}{1-q}$, $q \rightarrow 0^+$), the conditional expected total sojourn time of a tagged customer in the system $Y_n^{(C)}$, given that upon his arrival he finds the system in state n , is given as follows

$$E[Y_n^{(C)}] = \frac{1}{\mu + \nu^*} \sum_{i=0}^n \left(-\frac{\nu^*}{\mu + \nu^*}\right)^i \sum_{j=i}^n \binom{n}{j}, \quad n \geq 0. \quad (2.57)$$

2.7 Busy period

We define L_n to be the first passage time to state 0 starting from n . Then the busy period is given by L_1 . We consider the Laplace-Stieltjes transformation, which we denote by $\theta_n(s) = E[e^{-sL_n}]$, $s \geq 0$. By first step analysis we obtain that

$$\theta_0(s) = 1, \quad (2.58)$$

$$\begin{aligned} \theta_n(s) = & \frac{\lambda}{\lambda + \mu + \nu + s} \theta_{n+1}(s) + \frac{\mu}{\lambda + \mu + \nu + s} \theta_{n-1}(s) \\ & + \frac{\nu}{\lambda + \mu + \nu + s} \sum_{j=0}^n \binom{n}{n-j} p^{n-j} q^j \theta_j(s), \quad n \geq 1. \end{aligned} \quad (2.59)$$

Solving the recursive scheme of equations (2.58) and (2.59) we obtain the expression for $\theta_1(s)$ as seen in the following theorem.

Theorem 10 The LST of the busy period of the system is given as

$$\theta_1(s) = \frac{\lambda + \mu + s}{\lambda} - \frac{1}{z_0(s)} \frac{{}_1\phi_1 \left(\begin{matrix} -\frac{z_0(s)}{1-z_0(s)}q \\ \frac{z_0(s)}{1-z_0(s)} \frac{1-z_1(s)}{z_1(s)}q \end{matrix}; q, \frac{\nu z_0(s)}{\lambda(1-z_0(s))} \right)}{{}_1\phi_1 \left(\begin{matrix} -\frac{z_0(s)}{1-z_0(s)} \\ \frac{z_0(s)}{1-z_0(s)} \frac{1-z_1(s)}{z_1(s)}q \end{matrix}; q, \frac{\nu z_0(s)}{\lambda(1-z_0(s))}q \right)}, \quad s \geq 0, \quad (2.60)$$

where

$$z_0(s) = \frac{\lambda + \mu + \nu + s - \sqrt{(\lambda + \mu + \nu + s)^2 - 4\lambda\mu}}{2\mu} \quad \text{and } |z_0(s)| < 1 \quad \forall s > 0 \quad (2.61)$$

$$z_1(s) = \frac{\lambda + \mu + \nu + s + \sqrt{(\lambda + \mu + \nu + s)^2 - 4\lambda\mu}}{2\mu} \quad \text{and } |z_1(s)| > 1 \quad \forall s > 0. \quad (2.62)$$

Proof. We define the generating function of $\theta_n(s)$ as

$$\Theta(s, z) = \sum_{n=0}^{\infty} \theta_n(s) z^n, \quad s \geq 0, \quad |z| < 1.$$

This mixed transform $\Theta(s, z)$ do converges for $s \geq 0$ and $|z| < 1$. Indeed the LST $\theta_n(s) = E[e^{-sL_n}]$ are well-defined for $s \geq 0$. Moreover, for $s \geq 0$ we have that $|\theta_n(s)| \leq 1$ and hence $|\Theta(s, z)| \leq \sum_{n=0}^{\infty} |\theta_n(s)| |z|^n \leq \sum_{n=0}^{\infty} |z|^n < \infty$.

First of all through equations (2.58) and (2.59) we will determine the generating function $\Theta(s, z)$ which will lead us to the calculation of $\theta_1(s)$.

Multiplying both sides of the equations (2.58) and (2.59) by z^0 and z^n , respectively, and summing them for all $n = 0, 1, \dots$ we obtain

$$\begin{aligned} \Theta(s, z) &= 1 + \frac{\lambda}{z(\lambda + \mu + \nu + s)} (\Theta(s, z) - \theta_1(s)z - 1) + \frac{\mu z}{\lambda + \mu + \nu + s} \Theta(s, z) \\ &+ \frac{\nu}{(1 - pz)(\lambda + \mu + \nu + s)} \Theta\left(s, \frac{qz}{1 - pz}\right) - \frac{\nu}{\lambda + \mu + \nu + s}. \end{aligned}$$

So we conclude that

$$B(s, z)\Theta(s, z) = z(\lambda + \mu + s) - \lambda z\theta_1(s) - \lambda + \frac{\nu z}{1 - pz} \Theta\left(s, \frac{qz}{1 - pz}\right) \quad (2.63)$$

where $B(s, z) = -\mu z^2 + (\lambda + \mu + \nu + s)z - \lambda$ and we can easily check that $B(s, z)$ has two roots given by (2.61) and (2.62), such that $B(s, z) = -\mu(z - z_0(s))(z - z_1(s))$ and $\mu z_0(s)z_1(s) = \lambda$.

By setting

$$B(s, z)\Theta(s, z) = \Theta^*(s, z)$$

we can conclude that

$$\Theta^*(s, z) = z(\lambda + \mu + s) - \lambda z\theta_1(s) - \lambda + \frac{\nu T(z)}{qB(s, T(z))} \Theta^*(s, T(z)), \quad (2.64)$$

where $T(z)$ is the operator introduced in (2.49). Iterating (2.64) as in the proof of theorem 7 we obtain that

$$\begin{aligned} \Theta^*(s, z) &= z(\lambda + \mu + s) - \lambda z\theta_1(s) - \lambda \\ &+ \sum_{k=1}^n \prod_{j=1}^k \frac{\nu T_j(z)}{qB(s, T_j(z))} [T_k(z)(\lambda + \mu + s) - \lambda T_k(z)\theta_1(s) - \lambda] \\ &+ \prod_{k=1}^{n+1} \frac{\nu T_j(z)}{qB(s, T_j(z))} \Theta^*(s, T_{n+1}(z)). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in this last expression, and keeping in mind that

$$\begin{aligned} \lim_{n \rightarrow \infty} T_n(z) &= 0 & \lim_{n \rightarrow \infty} B(s, T_n(z)) &= -\lambda \\ \lim_{n \rightarrow \infty} \frac{\nu T_n(z)}{qB(s, T_n(z))} &= 0 & \lim_{n \rightarrow \infty} \Theta^*(s, T_{n+1}(z)) &= 1, \end{aligned}$$

we obtain that

$$\Theta^*(s, z) = \sum_{k=0}^{\infty} \prod_{j=1}^k \frac{\nu T_j(z)}{qB(s, T_j(z))} [T_k(z)(\lambda + \mu + s) - \lambda T_k(z)\theta_1(s) - \lambda]. \quad (2.65)$$

We can easily verify that

$$\frac{T_i(z)}{B(s, T_i(z))} = \frac{(-1)q^i z}{\lambda(1-z)} \frac{1 + \frac{z}{1-z}q^i}{\left(1 - \frac{1-z_0(s)}{z_0(s)} \frac{z}{1-z}q^i\right) \left(1 - \frac{1-z_1(s)}{z_1(s)} \frac{z}{1-z}q^i\right)}, \quad i \geq 1$$

consequently,

$$\prod_{i=1}^k \frac{\nu T_i(z)}{qB(s, T_i(z))} = (-1)^k \left(\frac{\nu z}{\lambda(1-z)} \right)^k q^{\binom{k}{2}} \frac{\left(-\frac{z}{1-z}q; q\right)_k}{\left(\frac{1-z_0(s)}{z_0(s)} \frac{z}{1-z}q; q\right)_k \left(\frac{1-z_1(s)}{z_1(s)} \frac{z}{1-z}q; q\right)_k}.$$

Hence, equation (2.65) assumes the form

$$B(s, z)\Theta(s, z) = (\lambda + \mu + s - \lambda\theta_1(s)) \sum_{k=0}^{\infty} T_k(z) \prod_{j=1}^k \frac{\nu T_j(z)}{qB(s, T_j(z))} - \lambda \sum_{k=0}^{\infty} \prod_{j=1}^k \frac{\nu T_j(z)}{qB(s, T_j(z))}, \quad (2.66)$$

where the two series can be written in terms of q -series as follows

$$\begin{aligned} \sum_{k=0}^{\infty} \prod_{j=1}^k \frac{\nu T_j(z)}{qB(s, T_j(z))} &= {}_2\phi_2 \left(\begin{matrix} q, -\frac{z}{1-z}q \\ \frac{1-z_0(s)}{z_0(s)} \frac{z}{1-z}q, \frac{1-z_1(s)}{z_1(s)} \frac{z}{1-z}q \end{matrix}; q, \frac{\nu z}{\lambda(1-z)} \right) \\ \sum_{k=0}^{\infty} T_k(z) \prod_{j=1}^k \frac{\nu T_j(z)}{qB(s, T_j(z))} &= z {}_2\phi_2 \left(\begin{matrix} q, -\frac{z}{1-z} \\ \frac{1-z_0(s)}{z_0(s)} \frac{z}{1-z}q, \frac{1-z_1(s)}{z_1(s)} \frac{z}{1-z}q \end{matrix}; q, \frac{\nu zq}{\lambda(1-z)} \right). \end{aligned}$$

In order to calculate $\theta_1(s)$ we set $z = z_0(s)$ in (2.66) and since the left side of the equation is equal to zero we can solve to obtain $\theta_1(s)$

$$\theta_1(s) = \frac{\lambda + \mu + s}{\lambda} - \frac{\sum_{k=0}^{\infty} \prod_{j=1}^k \frac{\nu T_j(z_0(s))}{qB(s, T_j(z_0(s)))}}{\sum_{k=0}^{\infty} T_k(z_0(s)) \prod_{j=1}^k \frac{\nu T_j(z_0(s))}{qB(s, T_j(z_0(s)))}}.$$

Writing the two series of the last expression in the canonical form of q -series and observing that for $z = z_0(s)$ the term $\frac{1-z_0(s)}{z_0(s)} \frac{z}{1-z}q$ reduces to q we obtain equation (2.60). ■

3 The $M/M/1$ queue with synchronized abandonments but not for the customer in service

We proceed with the study of the second abandonment scenario, where we have excluded the customer in service from the abandonment procedure.

3.1 Model description and notation

We consider an $M/M/1$ queueing system in which customers arrive according to a Poisson process at rate λ . The service is provided by a single server, who serves the customers according to a FCFS discipline. The successive service times are independent exponentially distributed random variables with rate μ . Customers waiting in the queue become impatient and perform synchronized abandonments at the epochs of a Poisson abandonment process at rate ν . Every waiting customer at the epochs of the Poisson process decides to abandon the system with probability p or remains in the system with probability $q = 1 - p$, independently of the others.

The system is represented by a continuous time Markov chain $\{\tilde{Q}(t) : t \geq 0\}$, where $\tilde{Q}(t)$ is the number of customers in the system at time t , $t \geq 0$. The corresponding transition rate diagram is given in figure 2.

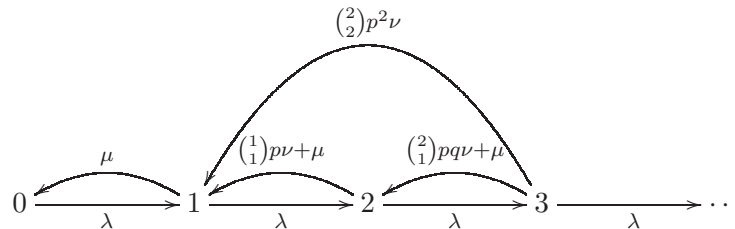


Figure 2: Transition rate diagram of $\{\tilde{Q}(t) : t \geq 0\}$.

Let $\{p_n : n \geq 0\}$ denote the equilibrium distribution of $\{\tilde{Q}(t) : t \geq 0\}$. We also define the probability generating function $P(z)$ by

$$P(z) = \sum_{n=0}^{\infty} p_n z^n.$$

3.2 The steady state distribution

We partition the set space $S = \{0\} \cup \{1, 2, \dots\}$ and we set $A = \{1, 2, \dots\}$. We will calculate the stationary distribution of the censored Markov chain over the set A . Let $\{p_n^A\}_{n \in A}$ denote the steady-state distribution of the censored Markov chain and $P_A(z)$ the corresponding probability generating function. We observe then that the generator of the censored Markov chain, denoted by $G(A)$, is given as

$$\begin{aligned} G(A) &= \begin{bmatrix} -(\lambda + \mu) & \lambda & 0 & \cdots \\ \mu + \nu p & -(\lambda + \mu + p\nu) & \lambda & \cdots \\ \nu p^2 & \mu + \nu \binom{2}{1} pq & -(\lambda + \mu + \nu p^2 + \nu \binom{2}{1} pq) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \\ \vdots \end{bmatrix} (\lambda)^{-1} \begin{bmatrix} \lambda & 0 & 0 & \cdots \end{bmatrix} \\ &= \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu + \nu p & -(\lambda + \mu + \nu(1 - q)) & \lambda & \cdots \\ \nu p^2 & \mu + \nu \binom{2}{1} pq & -(\lambda + \mu + \nu(1 - q^2)) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{aligned}$$

Thus the generator of the censored Markov chain is identical to the generator of the first model we have studied and so we can directly write down the probability generating function for the censored queue from theorem 1. We also know that the censored queue and the original queue are connected via the following relation

$$p_n^A = \frac{p_n}{1 - p_0}, \quad \forall n \geq 1. \quad (3.1)$$

Taking into consideration equation (3.1) and that $p_n^A = \pi_{n-1}$, for all $n \geq 1$, we obtain that

$$P(z) = (1 - p_0)z\Pi(z) + p_0, \quad (3.2)$$

where $\Pi(z)$ is given in equation (2.2) and p_0 can be calculated from equation (3.1) and the balance equation for state 0, $\lambda p_0 = \mu p_1$, so we conclude that

$$\pi_0 = \frac{\lambda}{\mu} \frac{p_0}{1 - p_0} \Rightarrow p_0 = \frac{\mu \pi_0}{\lambda + \mu \pi_0},$$

where $\pi_0 = \Pi(0)$.

3.3 Factorial moments of the queue length

Differentiating n times equation (3.2) and setting $z = 1$ we have that

$$P^{(n)}(1) = (1 - p_0)[m_{(n)} + nm_{(n-1)}], \quad n \geq 1,$$

where $m_{(n)}$ is given in equation (2.15).

3.4 Sojourn time

Let S denote the unconditional total sojourn time of an arbitrary customer in the system, regardless of whether he completes service or not. Moreover, let S_n denote the conditional total sojourn time of a tagged customer in

the system, given that upon arrival he finds the system in state n .

By conditioning on whether the next transition is a service completion or an abandonment opportunity we obtain the equations

$$E[S_0] = \frac{1}{\mu}, \tag{3.3}$$

$$E[S_{n+1}] = \frac{1}{\mu + \nu} + \frac{\mu}{\mu + \nu} E[S_n] + \frac{\nu q}{\mu + \nu} \sum_{i=0}^n \binom{n}{i} p^{n-i} q^i E[S_{i+1}], \quad n \geq 0. \tag{3.4}$$

We observe that if we replace $E[S_{n+1}]$ by $E[Y_n]$ in equation (3.4) we obtain equation (2.43), and therefore we have an ‘identical’ recursive scheme to solve with a different initial condition. Based on this observation the solution of the recursive relations (3.3) and (3.4) is summarized in the following theorem.

Theorem 11 *The conditional expected total sojourn time of a tagged customer in the system S_n , given that upon his arrival he finds the system in state n , is given as follows*

$$E[S_n] = E[Y_{n-1}] + \frac{1}{\mu}(1 - \nu p E[Y_{n-1}]), \quad n \geq 0, \tag{3.5}$$

where $E[Y_{-1}] = 0$ and $E[Y_n]$, $n \geq 0$, are given in equation (2.44).

Proof. Let us first define the homogeneous recursive scheme of equation (2.43), this is of the form

$$x_n = \frac{\mu}{\mu + \nu} x_{n-1} + \frac{\nu q}{\mu + \nu} \sum_{i=0}^n \binom{n}{i} p^{n-i} q^i x_i, \quad n \geq 0, \tag{3.6}$$

and select initial condition $x_{-1} = 1$. Then the solution of the non-homogeneous recursive scheme of equation (3.4), with initial condition (3.3), can be obtained in terms of the solution of the homogeneous equation, x_n , and the non-homogeneous one, $E[Y_n]$, in which the initial condition is set to 0. For more details on this theory the interested reader is referred to the book of Elaydi (2005), chapter 6, p. 306, theorem 6.27. More specifically, the solution of the non-homogeneous equation (3.4) with non-zero initial condition is given as

$$E[S_n] = x_{n-1} E[S_0] + E[Y_{n-1}], \quad n \geq 0. \tag{3.7}$$

Furthermore, we need to connect the two solutions x_n and $E[Y_n]$. To this end we observe that the equilibrium point of the non-homogeneous recursion given by equation (2.43) is $1/\nu p$, hence, the homogeneous recursion, x_n , and the non-homogeneous one, $E[Y_n]$, after changing the coordinates to match the initial conditions, are connected via the following relationship

$$E[Y_{n-1}] = (1 - x_{n-1}) \frac{1}{\nu p}, \quad n \geq 0. \tag{3.8}$$

Solving for x_{n-1} and substituting in (3.7) we obtain the result. ■

Remark 2 *We observe that the sequence $x_n = 1 - \nu p E[Y_n]$, that satisfies the recursive scheme (3.6), with initial condition $x_{-1} = 1$, has a probabilistic interpretation, x_{n-1} can be seen as the probability that a tagged customer, who upon arrival finds n , $n \geq 0$, customers in the system, never abandons the system and stays until his service starts. Then $E[S_n] = E[Y_{n-1}] + \frac{1}{\mu}(1 - \nu p E[Y_{n-1}])$ can be analyzed as the sojourn time from state $n + 1$ to state 1, which is equivalent to the sojourn time of the first model given that upon arrival there were $n - 1$ customers in front of our customer, plus the service time of the tagged customer given that he has not abandon the system till his service starts.*

3.5 Busy period

Let $J_{(n,i)}$ be the first passage time to state i starting from n and let $\varphi_n(s)$ denote the Laplace-Stieltjes transformation of $J_{(n,0)}$. Then $J_{(n,0)}$, for $n \geq 2$, can be analyzed as the first passage time from n to state 1, $J_{(n,1)}$, plus the time starting from state 1 until the first time we enter state 0, $J_{(1,0)}$. However the time $J_{(n,1)}$ is equal to the first passage time from state $n - 1$ to state 0 of the first model denoted by L_{n-1} . Thus, in terms of Laplace-Stieltjes transforms we have that $\varphi_n(s) = \theta_{n-1}(s)\varphi_1(s)$, for all $n \geq 2$. Furthermore, by first step analysis we obtain that

$$\begin{aligned} \varphi_1(s) &= \frac{\lambda}{\lambda + \mu + s}\varphi_2(s) + \frac{\mu}{\lambda + \mu + s}\varphi_0(s) \\ &= \frac{\lambda}{\lambda + \mu + s}\varphi_2(s) + \frac{\mu}{\lambda + \mu + s} \\ &= \frac{\lambda}{\lambda + \mu + s}\theta_1(s)\varphi_1(s) + \frac{\mu}{\lambda + \mu + s}. \end{aligned}$$

Consequently,

$$\varphi_1(s) = \frac{\mu}{\lambda + \mu + s - \lambda\theta_1(s)}.$$

Substituting $\theta_1(s)$ from equation (2.60) we obtain that

$$\varphi_1(s) = z_0(s)\frac{\mu}{\lambda} \frac{{}_1\phi_1\left(\begin{matrix} -\frac{z_0(s)}{1-z_0(s)} \\ \frac{z_0(s)}{1-z_0(s)} \frac{1-z_1(s)}{z_1(s)} q \end{matrix}; q, \frac{\nu z_0(s)}{\lambda(1-z_0(s))} q\right)}{{}_1\phi_1\left(\begin{matrix} -\frac{z_0(s)}{1-z_0(s)} q \\ \frac{z_0(s)}{1-z_0(s)} \frac{1-z_1(s)}{z_1(s)} q \end{matrix}; q, \frac{\nu z_0(s)}{\lambda(1-z_0(s))}\right)}, \quad s > 0, \quad (3.9)$$

where $z_0(s)$ and $z_1(s)$ are given in equations (2.61) and (2.62), respectively.

4 The $M/M/c$ queue with synchronized abandonments but not for the customers in service

We consider a multi-server $M/M/c$ queue with c identical servers, where customers arrive according to a Poisson process with rate λ and the service times of the customers are independent exponentially distributed random variables with common mean $1/c\mu$ and we assume that customers in the waiting line perform synchronized abandonments, in the same manner as in section 3. We have rescaled the maximum service rate to μ to match the results of this section to the previous ones.

For this model we have produced similar results following the methodology of the $M/M/1$ queue with synchronized abandonments for only the waiting customers presented in section 3. More concretely, let p_n^c , $n \geq 0$, denote the steady state distribution of the number of customers in the system and $P_c(z) = \sum_{n=0}^{\infty} p_n^c z^n$ the corresponding probability generating function, for the $M/M/c$ model under investigation.

Partitioning the set space $S = \{0, 1, \dots, c - 1\} \cup \{c, c + 1, \dots\}$ and observing that the generator of the censored Markov chain over the set $\{c, c + 1, \dots\}$ is identical to the generator of the first model, we can connect the censored queue and the model under consideration and have that $P_c(z)$ is given as

$$P_c(z) = (1 - p_0^c \sum_{i=0}^{c-1} \frac{(c\lambda)^i}{i!\mu^i}) z^c \Pi(z) + p_0^c \sum_{i=0}^{c-1} \frac{(c\lambda)^i}{i!\mu^i} z^i, \quad (4.1)$$

where $\Pi(z)$ is given in equation (2.2) and

$$p_0^c = \frac{\pi_0}{\frac{(c\lambda)^c}{c!\mu^c} + \pi_0 \sum_{i=0}^{c-1} \frac{(c\lambda)^i}{i!\mu^i}}$$

and $\pi_0 = \Pi(0)$.

On what the factorial moments of the queue length are concerned we obtain that by differentiating n times equation (4.1) and setting $z = 1$ we have that

$$P_c^{(n)}(1) = \begin{cases} (1 - p_0^c \sum_{i=0}^{c-1} \frac{(c\lambda)^i}{i!\mu^i}) \sum_{k=0}^n \binom{n}{k} \frac{c!}{(c-k)!} m_{(n-k)} + p_0^c \sum_{i=n}^{c-1} \frac{(c\lambda)^i}{(i-n)!\mu^i}, & n < c \\ (1 - p_0^c \sum_{i=0}^{c-1} \frac{(c\lambda)^i}{i!\mu^i}) \sum_{k=0}^c \binom{n}{k} \frac{c!}{(c-k)!} m_{(n-k)}, & n \geq c, \end{cases} \quad (4.2)$$

where $m_{(n)}$ is given in equation (2.15).

Furthermore, the conditional expected total sojourn time of a tagged customer in the system S_n^c , given that upon his arrival he finds the system in state n , can be analyzed, for the case $n \geq c$, as the time from state $n+1$ to state c , which is equivalent to $E[Y_{n-c}]$, plus his service time given he has not abandon the system till his service starts, while for the case $n < c$, it is just the service time. Hence, $E[S_n^c]$ is given as follows

$$E[S_n^c] = \frac{c}{\mu}, \quad n < c, \quad (4.3)$$

$$E[S_n^c] = E[Y_{n-c}] + \frac{c}{\mu}(1 - \nu p E[Y_{n-c}]), \quad n \geq c, \quad (4.4)$$

where $E[Y_n]$ is given in equation (2.44).

Finally, let $\varphi_n^c(s)$ denote the Laplace-Stieltjes transformation of the first passage time to state 0 starting from state n , $n \geq 0$. It can be proven using first step analysis that $\varphi_n^c(s)$, for $1 \leq n \leq c$, satisfy the following recursive scheme

$$\varphi_n^c(s) = \frac{c\lambda}{c\lambda + n\mu + cs} \varphi_{n+1}^c(s) + \frac{n\mu}{c\lambda + n\mu + cs} \varphi_{n-1}^c(s), \quad 1 \leq n \leq c. \quad (4.5)$$

With conditions $\varphi_0^c(s) = 1$ and $\varphi_{c+1}^c(s) = \varphi_c^c(s)\theta_1(s)$, where $\theta_1(s)$ is given in equation (2.60).

Corollary 1 *An algorithmic solution of the recursive scheme presented in (4.5) can be obtain by observing that if we divide all terms of (4.5) by $\varphi_n^c(s)$ and set $u_n(s) = \varphi_n^c(s)/\varphi_{n-1}^c(s)$ we obtain the following first order recursion*

$$u_n(s) = \frac{\frac{n\mu}{c\lambda}}{1 + \frac{s}{\lambda} + \frac{n\mu}{c\lambda} - u_{n+1}(s)}, \quad 1 \leq n \leq c, \quad (4.6)$$

with conditions $u_1(s) = \varphi_1^c(s)$ and $u_{c+1}(s) = \theta_1(s)$. Hence,

$$\varphi_1^c(s) = \frac{\frac{\mu}{c\lambda}}{1 + \frac{s}{\lambda} + \frac{\mu}{c\lambda} - \frac{\frac{2\mu}{c\lambda}}{1 + \frac{s}{\lambda} + \frac{2\mu}{c\lambda} - \dots - \frac{\frac{\mu}{\lambda}}{1 + \frac{s}{\lambda} + \frac{\mu}{\lambda} - \theta_1(s)}}, \quad (4.7)$$

where $\theta_1(s)$ is given in equation (2.60).

Artalejo and Lopez-Herrero (2001), p. 211, Corollary 2.1, propose the following algorithmic approach for the solution of the recursive scheme of equation (4.5)

$$\begin{aligned} \varphi_i^c(s) &= - \sum_{j=1}^{i-1} \frac{j!}{(c\lambda/\mu)^j} + \varphi_1^c(s) \left(1 + \left(1 + \frac{cs}{\mu}\right) \sum_{j=1}^{i-1} \frac{j!}{(c\lambda/\mu)^j} \right) \\ &\quad + \frac{s}{\lambda} \sum_{k=2}^{i-1} \varphi_k^c(s) \frac{(c\lambda/\mu)^k}{k!} \sum_{j=k}^{i-1} \frac{j!}{(c\lambda/\mu)^j}, \quad 2 \leq i \leq c+1. \end{aligned} \quad (4.8)$$

Keeping in mind that $\varphi_{c+1}^c(s) = \varphi_c^c(s)\theta_1(s)$, where $\theta_1(s)$ is known, the problem reduces to solving a simple recursive system on the unknowns $\varphi_i^c(s)$, $1 \leq i \leq c+1$.

However, we must note that the homogeneous second order linear difference equation with polynomial coefficients of equation (4.5) can be explicitly solved (see Jagerman (2000), p. 230, formula 2.229a) and we obtain that

$$\varphi_n^c(s) = C_0(s)a_n(s) + C_1(s)b_n(s), \quad 0 \leq n \leq c+1, \quad (4.9)$$

where

$$\begin{aligned} a_n(s) &= \sum_{k=0}^n \binom{n}{k} \left(\frac{\Gamma(\frac{\mu}{c} + 1)}{\Gamma(\lambda + 1)\Gamma(\frac{\mu}{c} - \lambda + 1)} \right)^k \frac{\Gamma(k + \frac{cs}{\mu})}{\Gamma(\frac{cs}{\mu})}, \quad n \geq 0 \\ b_n(s) &= \frac{\Gamma(n+1)}{\Gamma(n+1 + \frac{cs}{\mu})} {}_1F_1\left(\frac{cs}{\mu}; n+1 + \frac{cs}{\mu}; \frac{c\lambda}{\mu}\right), \quad n \geq 0, \end{aligned}$$

with ${}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt$ being the Kummer (confluent hypergeometric) function and $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ the Gamma function. Furthermore, $C_0(s)$ and $C_1(s)$ can be determined from the two conditions $\varphi_0^c(s) = 1$ and $\varphi_{c+1}^c(s) = \varphi_c^c(s)\theta_1(s)$, more explicitly

$$\begin{aligned} C_0(s) &= 1 - C_1(s)b_0(s) \\ C_1(s) &= \frac{a_{c+1}(s) - \theta_1(s)a_c(s)}{b_0(s)a_{c+1}(s) - b_{c+1}(s) - \theta_1(s)(b_0(s)a_c(s) - b_c(s))}. \end{aligned}$$

Setting, now $n = 1$ in equation (4.9) we can obtain the Laplace-Stieltjes transform of the busy period.

Appendix A

Definitions and results of q -hypergeometric series

The q -hypergeometric series are defined as

$$\begin{aligned} {}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, x \right) &\equiv {}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, x) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q; q)_n (b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n, \end{aligned} \quad (A.1)$$

where r and s are non-negative integers, $(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n$ and, $(a; q)_0 = 1$ and $(a; q)_n = (1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-1})$, $n \geq 1$. The product $(a; q)_n$ is referred to as q -shifted factorial. We also define $(a; q)_\infty = \prod_{k=0}^{\infty} (1-aq^k)$ and use the abbreviation $(a_1, a_2, \dots, a_r; q)_\infty$ to denote the product $(a_1; q)_\infty (a_2; q)_\infty \dots (a_r; q)_\infty$. In the definition of a q -series through (A.1), it is assumed that $b_i \neq q^{-m}$ for $m = 0, 1, \dots$ and $i = 1, 2, \dots, s$. For $|q| < 1$, the ${}_r\phi_s$ series converges absolutely for all x when $r \leq s$ and for

$|x| < 1$ when $r = s + 1$.

A q -calculus has been developed that parallels the theory of hypergeometric functions. The most important summation formula for the q -hypergeometric series is given by the q -binomial theorem (see Gasper and Rahman (2004) p. 8, formula (1.3.2))

$${}_1\phi_0(a; -; q, x) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1. \quad (\text{A.2})$$

The q -binomial theorem enables to express the q -shifted factorials in the form of q -series.

A very important feature of the q -calculus is the definition of the q -integral on an interval $[0, a]$ (see Gasper and Rahman (2004) section 1.11) as

$$\int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n. \quad (\text{A.3})$$

As $q \rightarrow 1^-$ the q -analogues reduce to their standard counterparts. In particular we have the relationships:

$$\lim_{q \rightarrow 1^-} (x(1-q); q)_{\infty} = e^{-x}, \quad (\text{A.4})$$

$$\lim_{q \rightarrow 1^-} \frac{(q^a x; q)_{\infty}}{(x; q)_{\infty}} = (1-x)^{-a}, \quad (\text{A.5})$$

$$\lim_{q \rightarrow 1^-} \int_0^a f(t) d_q t = \int_0^a f(t) dt. \quad (\text{A.6})$$

As $q \rightarrow 0^+$ we can easily see that

$$\lim_{q \rightarrow 0^+} (x; q)_{\infty} = 1-x, \quad \lim_{q \rightarrow 0^+} {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, q \right) = 1. \quad (\text{A.7})$$

Finally some very important transformation formulas of q -hypergeometric series are listed beneath. The first ones are Heine's transformations formula of the ${}_2\phi_1$ series (see Gasper and Rahman (2004) p.359, formulas (III.1)-(III.3)). It enables one to interchange the arguments of a ${}_2\phi_1$ function in certain ways. More concretely, it states that

$${}_2\phi_1(a, b; c; q, x) = \frac{(b, ax; q)_{\infty}}{(c, x; q)_{\infty}} {}_2\phi_1(c/b, x; ax; q, b) \quad (\text{A.8})$$

$${}_2\phi_1(a, b; c; q, x) = \frac{(c/b, bx; q)_{\infty}}{(c, x; q)_{\infty}} {}_2\phi_1(abx/c, b; bx; q, c/b) \quad (\text{A.9})$$

$${}_2\phi_1(a, b; c; q, x) = \frac{(abx/c; q)_{\infty}}{(x, q)_{\infty}} {}_2\phi_1(c/a, c/b; c; q, abz/c). \quad (\text{A.10})$$

The second one is Jackson's three term transformation formula of the ${}_2\phi_1$ series (see Gasper and Rahman (2004) p. 359, formula (III.5))

$${}_2\phi_1(a, b; c; q, x) = \frac{(abx/c; q)_{\infty}}{(bx/c; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} a, c/b, 0 \\ c, cq/bx \end{matrix}; q, q \right) + \frac{(a, bx, c/b; q)_{\infty}}{(c, x, c/bx; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} x, abx/c, 0 \\ bx, bxq/c \end{matrix}; q, q \right). \quad (\text{A.11})$$

The third one enables the transformation of a ${}_{r+1}\phi_r$ function to a q -integral (see Gasper and Rahman (2004) p. 26, exercise 1.4 (iii)):

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, q^x \right) = \frac{(a_1, \dots, a_{r+1}; q)_{\infty}}{(1-q)(q, b_1, \dots, b_r; q)_{\infty}} \int_0^1 t^{x-1} \frac{(qt, b_1 t, \dots, b_r t; q)_{\infty}}{(a_1 t, \dots, a_{r+1} t; q)_{\infty}} d_q t. \quad (\text{A.12})$$

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